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Research article

Exploratory mean-variance portfolio selection with constant elasticity of variance models in regime-switching markets

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Appendix

A. Proof of Equation (2.9)

According to Guo et al. [27], we derive the exploratory wealth process under stochastic policy Π. Assume the investor selects portfolios according to strategy Π, which is independent of the driving Brownian motion. By the Central Limit Theorem, the limiting distribution depends only on the first and second moments. Consider N independent Brownian sample paths $\{W_t^n, n = 1, 2, \cdots, N\}$. For a uniform partition of $[0, T]$ with mesh size Δt, the corresponding wealth increment on $[t, t + \Delta t]$ takes the form

$$\Delta X^n(t) \approx (\mu(t,\alpha_t) - r)u(t)\Delta t + \sigma(t,\alpha_t)\,(S^n(t))^{\beta(t,\alpha_t)}\,u(t)\Delta W_t^n, \tag{A.1}$$

and the corresponding risky asset is

$$\Delta S^n(t) \approx \mu(t,\alpha_t)S^n(t)\Delta t + \sigma(t,\alpha_t)(S^n(t))^{\beta(t,\alpha_t)+1}\Delta W_t^n. \tag{A.2}$$

The law of large numbers yields

$$\frac{1}{N}\sum_{n=1}^{N}\Delta X^n(t) \approx \frac{1}{N}\sum_{n=1}^{N}\Big[(\mu(t,\alpha_t)-r)u(t,\alpha_t)\Delta t+\sigma(t,\alpha_t)(S^n(t))^{\beta(t,\alpha_t)}u(t,\alpha_t)\Delta W_t^n\Big]$$
$$\xrightarrow{\text{a.s.}}\left((\mu(t,\alpha_t)-r)\int_R u(t,\alpha_t)d\Pi_t(u)\right)\Delta t,$$
$$\frac{1}{N}\sum_{n=1}^{N}(\Delta X^n(t))^2 \approx \frac{1}{N}\sum_{n=1}^{N}\sigma^2(t,\alpha_t)(S^n(t))^{2\beta(t,\alpha_t)}u^2(t,\alpha_t)\Delta t \xrightarrow{\text{a.s.}}\left(\sigma^2(t,\alpha_t)S(t)^{2\beta(t,\alpha_t)}\int_R u^2(t,\alpha_t)d\Pi_t(u)\right)\Delta t,$$
$$\frac{1}{N}\sum_{n=1}^{N}\Delta X^n(t)\Delta S^n(t) \approx \frac{1}{N}\sum_{n=1}^{N}\sigma^2(t,\alpha_t)(S^n(t))^{2\beta(t,\alpha_t)+1}u(t,\alpha_t)\Delta t \xrightarrow{\text{a.s.}}\left(\sigma^2(t,\alpha_t)S(t)^{2\beta(t,\alpha_t)+1}\int_R u(t,\alpha_t)d\Pi_t(u)\right)\Delta t, \tag{A.3}$$

on the other hand

$$\frac{1}{N}\sum_{n=1}^{N}\Delta X^n(t)\xrightarrow{\text{a.s.}}E[\Delta X_t],\quad \frac{1}{N}\sum_{n=1}^{N}(\Delta X^n(t))^2\xrightarrow{\text{a.s.}}E[(\Delta X_t)^2],\quad \frac{1}{N}\sum_{n=1}^{N}\Delta X^n(t)\Delta S^n(t)\xrightarrow{\text{a.s.}}E[\Delta X(t)\Delta S(t)].$$

We replace (2.4) with the following process linked to the randomized strategy, which serves as the basis for the EMV formulation

$$dX^{\Pi}(t)=(\mu(t,\alpha_t)-r)M(t,\alpha_t)dt+\sigma(t,\alpha_t)S_t^{\beta(t,\alpha_t)}\left(M(t,\alpha_t)dW_t+N(t,\alpha_t)d\widetilde{W}_t\right). \tag{A.4}$$

$\widetilde{W}_t$ is the Brownian motion independent of W_t and

$$M(t,\alpha_t):=\int_{\mathbb{R}}u(t,\alpha_t)d\Pi_t(u)\ \ ,\ \ N^2(t,\alpha_t):=\int_{\mathbb{R}}u^2(t,\alpha_t)d\Pi_t(u)-M^2(t,\alpha_t).$$

B. Proof of Theorem 3.2

Recall the classical MV problem with regime switching (2.8). By applying the dynamic programming principle, we derive the HJB equation

$$\begin{aligned}&\min_u\Bigg\{(\mu_i-r)u(t,e_i)V_x(t,s,x,e_i)+\frac{1}{2}\sigma_i^2 s^{2\beta_i}u^2(t,e_i)V_{xx}(t,s,x,e_i)+\sigma_i^2 s^{2\beta_i+1}u(t,e_i)V_{sx}(t,s,x,e_i)\\&+\sum_{j=1}^{n}q_{ij}V\left(t,x,s,e_j\right)\Bigg\}+V_t(t,s,x,e_i)+\mu_i sV_s(t,s,x,e_i)+\frac{1}{2}\sigma_i^2 s^{2\beta_i+2}V_{ss}(t,s,x,e_i)=0.\end{aligned} \tag{B.1}$$

The optimal solution $u^*(t,s,x,e_i)$ of the HJB equation is given by

$$u^*(t,s,x,e_i)=-\frac{(\mu_i-r)V_x(t,s,x,e_i)+\sigma_i^2 s^{2\beta_i+1}V_{sx}(t,s,x,e_i)}{\sigma_i^2 s^{2\beta_i}V_{xx}(t,s,x,e_i)}. \tag{B.2}$$

Assuming the value function is

$$V(t,s,x,e_i)=a(t,s,e_i)(x-w)^2-(w-z)^2+b(t,s,e_i). \tag{B.3}$$

Substituting the optimal strategy $u^*(t,s,x,e_i)$ and $V(t,s,x,e_i)$ into the HJB equation yields

$$
\begin{aligned}
&a_t(t,s,e_i)(x-w)^2+(\mu_i-r)\left(-\frac{(\mu_i-r)2a(t,s,e_i)+\sigma_i^2 s^{2\beta_i+1}\cdot 2a_s(t,s,e_i)}{\sigma_i^2\cdot s^{2\beta_i}\cdot 2a}\right)2a(t,s,e_i)(x-w)^2\\
&+b_t(t,s,e_i)+\frac{1}{2}\sigma_i^2 s^{2\beta_i}\left[\left(\frac{(\mu_i-r)2a(t,s,e_i)(x-w)+\sigma_i^2 s^{2\beta_i+1}2a_s(t,s,e_i)(x-w)}{\sigma_i^2 s^{2\beta_i}2a(t,s,e_i)}\right)^2\right]2a(t,s,e_i)\\
&+\frac{1}{2}\sigma_i^2 s^{2\beta_i+2}\left[a_{ss}(t,s,e_i)(x-w)^2+b_{ss}(t,s,e_i)\right]+\sum_{j=1}^{n}q_{ij}\Big(a\big(t,s,e_j\big)(x-w)^2+b\big(t,s,e_j\big)\Big)\\
&+\sigma_i^2 s^{2\beta_i+1}\left(-\frac{(\mu_i-r)2a(t,s,e_i)+2\sigma_i^2 s^{2\beta_i+1}a_s(t,s,e_i)}{2\sigma_i^2 s^{2\beta_i}a}\right)\big(2a_s(t,s,e_i)(x-w)^2\big)\\
&+\mu_i s\Big(a_s(t,s,e_i)(x-w)^2+b_s(t,s,e_i)\Big)=0.
\end{aligned}
\tag{B.4}
$$

Setting the coefficient of $(x-w)^2$to zero, we obtain

$$
\begin{aligned}
&a_t(t,s,e_i)-\left(\frac{(\mu_i-r)^2}{\sigma_i^2 s^{2\beta_i}}\right)a(t,s,e_i)+(-2(\mu_i-r)+\mu_i)sa_s(t,s,e_i)-\sigma_i^2 s^{2\beta_i+2}\frac{a_s^2(t,s,e_i)}{a(t,s,e_i)}\\
&+\frac{1}{2}\sigma_i^2 s^{2\beta_i+2}a_{ss}(t,s,e_i)+\sum_{j=1}^{n}q_{ij}a\big(t,s,e_j\big)=0.
\end{aligned}
\tag{B.5}
$$

Setting the constant term to zero, the differential equation becomes

$$
b_t(t,s,e_i)+\mu_i s b_s(t,s,e_i)+\frac{1}{2}\sigma_i^2 s^{2\beta_i+2}b_{ss}(t,s,e_i)+\sum_{j=1}^{n}q_{ij}\Big(b\big(t,s,e_j\big)\Big)=0.
\tag{B.6}
$$

The solution to differential equation (B.5) can be obtained using the same method as for Equation (3.7), Equation (B.6) has a particular solution of $b(t,s,e_i)=0$. We finally obtain the value of $\mathbf{a}(t,s)$ as

$$
\mathbf{a}(t,s)=\mathbf{H}(t)\exp(\mathbf{K}(t)\mathbf{Y}),
\tag{B.7}
$$

where $h_i(t)$ and $k_i(t)$ satisfy the equations (3.16) and (3.17).

C. Proof of Theorem 4.1

Fix $(t,x,s,e_i)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathcal{N}$. Suppose the policy $\Pi=\{\Pi_v,v\in[t,T]\}$ is admissible, and let $\{X_s^\Pi,s\in[t,T]\}$ denote the corresponding wealth process. $\widetilde{\Pi}$ is obtained from (4.1). Applying Itô's lemma, we obtain

$$
\begin{aligned}
V^{\Pi}(h,X_h^{\widetilde{\Pi}},S_h,e_i) &= V^{\Pi}(t,x,s,e_i)+\int_t^h V_t^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)dv+\int_t^h V_x^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)dX_v^{\widetilde{\Pi}} \\
&+\int_t^h V_s^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)dS_v+\frac{1}{2}\int_t^h V_{xx}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)d\langle X^{\widetilde{\Pi}},X^{\widetilde{\Pi}}\rangle_v \\
&+\frac{1}{2}\int_t^h V_{ss}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)d\langle S,S\rangle_v+\int_t^h V_{sx}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)d\langle X^{\widetilde{\Pi}},S\rangle_v \\
&+\int_t^h \sum_{j=1}^n q_{\alpha_v j}V(v,X_v^{\widetilde{\Pi}},S_v,e_j)dv \\
&= V^{\Pi}(t,x,s,e_i)+\int_t^h (\sigma_i S_v^{\beta_i}M_i^{\widetilde{\Pi}}V_x^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)+\sigma_i S_v^{\beta_i+1}V_s^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)dW_v \\
&+\int_t^h \sigma_i S_v^{\beta_i}N_i^{\widetilde{\Pi}}V_x^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)d\widetilde{W}_v \\
&+\int_t^h \Big(V_t^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,e_i)+(\mu_i-r)M_i^{\widetilde{\Pi}}V_x^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)+\mu_i S_v V_s^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v) \\
&+\frac{1}{2}\sigma_i^2 S_v^{2\beta_i}[(M_i^{\widetilde{\Pi}})^2+(N_i^{\widetilde{\Pi}})^2]V_{xx}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)+\frac{1}{2}\sigma_i^2 S_v^{2\beta_i+2}V_{ss}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v) \\
&+\sigma_i^2 S_v^{2\beta_i+1}M_i^{\widetilde{\Pi}}V_{sx}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)+\sum_{j=1}^n q_{\alpha_v j}V(v,X_v^{\widetilde{\Pi}},S_v,e_j)\Big)dv.
\end{aligned}
\tag{C.1}
$$

Define the stopping time as

$$
\tau_n := \inf\left\{h\geq t:\int_t^h \sigma_i^2 S_v^{2\beta_i+2}(V_S^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v))^2+\sigma_i^2 S_v^{2\beta_i}\Big((N_i^{\widetilde{\Pi}})^2+(M_i^{\widetilde{\Pi}})^2\Big)(V_x^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v))^2dv\geq n\right\},
$$

where $n\geq 1$, we obtain the following results

$$
\begin{aligned}
V^{\Pi}(t,x,s,e_i) &= E\Big[V^{\Pi}(h\wedge\tau_n,X_{h\wedge\tau_n}^{\widetilde{\Pi}},S_{h\wedge\tau_n},\alpha_{h\wedge\tau_n}) \\
&-\int_t^{h\wedge\tau_n}\Big(V_t^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)+(\mu_i-r)M_i^{\widetilde{\Pi}}V_x^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v) \\
&+\mu_i S_v V_s^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)+\frac{1}{2}\sigma_i^2 S_v^{2\beta_i}[(M_i^{\widetilde{\Pi}})^2+(N_i^{\widetilde{\Pi}})^2]V_{xx}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v) \\
&+\frac{1}{2}\sigma_i^2 S_v^{2\beta_i+2}V_{ss}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v)+\sigma_i^2 S_v^{2\beta_i+1}M_i^{\widetilde{\Pi}}V_{sx}^{\Pi}(v,X_v^{\widetilde{\Pi}},S_v,\alpha_v) \\
&+\sum_{j=1}^n q_{\alpha_v j}V(v,X_v^{\widetilde{\Pi}},S_v,e_j)\Big)dv\Big].
\end{aligned}
\tag{C.2}
$$

Under standard verification arguments and assuming V^{Π} is sufficiently smooth, we have

$$\begin{aligned}&V_t^{\Pi}(t,x,s,e_i)+(\mu_i-r)M_i^{\Pi}V_x^{\Pi}(t,x,s,e_i)+\frac{1}{2}\sigma_i^2 s^{2\beta_i}\left((M_i^{\Pi})^2+(N_i^{\Pi})^2\right)V_{xx}^{\Pi}(t,x,s,e_i)+\mu_i s V_s^{\Pi}(t,x,s,e_i)\\&+\frac{1}{2}\sigma_i^2 s^{2\beta_i+2}V_{ss}^{\Pi}(t,x,s,e_i)+\sigma_i^2 s^{2\beta_i+1}M_i^{\Pi}V_{sx}^{\Pi}(t,x,s,e_i)-\lambda(t)\Phi(\Pi)+\sum_{j=i}^{n}q_{ij}V\left(t,x,s,e_j\right)=0,\end{aligned} \tag{C.3}$$

for any $(t,x,s,e_i)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathcal{N}$, we have

$$\begin{aligned}&V_t^{\Pi}(t,x,s,e_i)+\min_{\Pi'}\Bigg((\mu_i-r)M_i^{\Pi'}V_x^{\Pi}(t,x,s,e_i)+\frac{1}{2}\sigma_i^2 s^{2\beta_i}\left((M_i^{\Pi'})^2+(N_i^{\Pi'})^2\right)V_{xx}^{\Pi}(t,x,s,e_i)+\mu_i s V_s^{\Pi}(t,x,s,e_i)\\&+\frac{1}{2}\sigma_i^2 s^{2\beta_i+2}V_{ss}^{\Pi}(t,x,s,e_i)+\sigma_i^2 s^{2\beta_i+1}M_i^{\Pi'}V_{sx}^{\Pi}(t,x,s,e_i)-\lambda(t)\Phi(\Pi')+\sum_{j=1}^{n}q_{ij}V\left(t,x,s,e_j\right)\Bigg)\le 0.\end{aligned} \tag{C.4}$$

The infimum in the Hamiltonian operator of the HJB equation $\widetilde{\Pi}$ is explicitly attained by the feedback strategy derived from the regularizer's optimality condition in (C.4) . Specifically

$$V^{\Pi}(t,x,s,e_i)\ge E\left[V^{\Pi}(h\wedge\tau_n,X_{h\wedge\tau_n}^{\widetilde{\Pi}},S_{h\wedge\tau_n}^{\widetilde{\Pi}},\alpha_{h\wedge\tau_n})-\int_t^{h\wedge\tau_n}\lambda(v)\Phi(\widetilde{\Pi}_v)dv\Big|X_t^{\widetilde{\Pi}}=x\right]. \tag{C.5}$$

For any fixed $(t,x,s,e_i)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathcal{N}$, let $h=T$. Given the terminal condition $V^{\Pi}(T,x,s,e_i)=V^{\widetilde{\Pi}}(T,x,s,e_i)=(x-w)^2-(w-z)^2$ and the admissibility of $\widetilde{\Pi}$, we apply the Dominated Convergence Theorem as $n\to\infty$ to obtain

$$V^{\Pi}(t,x,s,e_i)\ge E\left[V^{\widetilde{\Pi}}(T,X_T^{\widetilde{\Pi}},S_T^{\widetilde{\Pi}},\alpha_T)-\int_t^{T}\lambda(v)\Phi(\widetilde{\Pi}_v)dv\Big|X_t^{\widetilde{\Pi}}=x\right]=V^{\widetilde{\Pi}}(t,x,s,e_i). \tag{C.6}$$