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Research article

Production and inventory rationing in an unreliable Make-to-Stock System under preventive maintenance policy

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Appendix

.1. Proof of Proposition 3.2

Proof. To show that $D_{11}TV(x,i) \geq 0$, we begin by expressing $D_1TV(x,i)$ as a sum of terms:

$$\begin{aligned}
D_1TV(x,i) &= TV(x+1,i) - TV(x,i) \\
&= h(x+1) + w(s) + \mu_i T^i V(x+1,i) + \xi_i V(x+1,i-1) \\
&\quad + \sum_{k=1}^{n} \lambda_k T_k V(x+1,i) + \left(\sum_{l\neq 1}^{m} \mu_l + \sum_{l\neq 1}^{m} \xi_l \right) V(x+1,i) \\
&\quad - h(x) - w(s) - \mu_i T^i V(x,i) + \xi_i V(x,i-1) \\
&\quad - \sum_{k=1}^{n} \lambda_k T_k V(x,i) - \left(\sum_{l\neq 1}^{m} \mu_l + \sum_{l\neq 1}^{m} \xi_l \right) V(x,i) \\
&= [h(x+1) - h(x)] + \xi_i [V(x+1,i-1) - V(x,i-1)] \\
&\quad + \left(\sum_{l\neq 1}^{m} \mu_l + \sum_{l\neq 1}^{m} \xi_l \right) \cdot [V(x+1,i) - V(x,i)] \\
&\quad + \sum_{k=1}^{n} \lambda_k [T_k V(x+1,i) - T_k V(x,i)] + \mu_i \left[T^i V(x+1,i) - T^i V(x,i) \right]
\end{aligned} \tag{.1}$$

Here, $h(x)$ is a monotonically increasing function of x, and $V(x,i)$ satisfies property **P2**, which states that it is also monotonically increasing with respect to x. Therefore, the first three terms in the above expression are also increasing functions of x. We now focus on the last two terms, whose non-negativity depends on whether below two statements hold true:

$$D_{11}T_kV(x,i) = D_1T_kV(x+1,i) - D_1T_kV(x,i) \geq 0 \tag{.2}$$
$$D_{11}T^iV(x,i) = D_1T^iV(x+1,i) - D_1T^iV(x,i) \geq 0 \tag{.3}$$

We begin by establishing the proof of the statement (.2), which can be rewrote as

$$\begin{aligned} D_1TV(x+1,i) &= TV(x+1,i) - TV(x,i) \\ &= \min\{V(x+1,i), V(x+2,i)+c_k\} - \min\{V(x,i), V(x+1,i)+c_k\} \\ &\geq \min\{V(x,i), V(x+1,i)+c_k\} - \min\{V(x-1,i), V(x,i)+c_k\} \\ &= D_1TV(x,i) \end{aligned}$$

Consider that $D_1V(x-1,i) \leq D_1V(x,i) \leq D_1V(x+1,i)$, there are three different cases for $x \neq 0$. When $D_1V(x+1,i) \geq D_kV(x-1,i) \geq -c_k$, there is

$$\begin{aligned} D_1TV(x+1,i) &= V(x+1,i) - V(x,i) \geq V(x,i) - V(x-1,i) \\ &= D_1TV(x,i) \end{aligned}$$

When $D_1V(x+1,i) \geq -c_k \geq D_kV(x-1,i)$, there is

$$\begin{aligned} D_1TV(x+1,i) &\geq V(x+1,i) - V(x,i) = [V(x+1,i)+C_k] - [V(x-1,i)+C_k] \\ &= D_1TV(x,i) \end{aligned}$$

When $-c_k \geq D_1V(x+1,i) \geq D_kV(x-1,i)$, there is

$$\begin{aligned} [D_1TV(x+1,i) &= [V(x+2,i)+C_k] - [V(x+1,i)+C_k] \\ &\geq [V(x+1,i)+C_k] - [V(x,i)+C_k] \\ &= D_1TV(x,i) \end{aligned}$$

If $x = 0$, then check for two cases for $D_kTV(1,i)$. When $D_kV(1,i) \geq -c_k$, there is

$$\begin{aligned} [D_1TV(1,i) &\geq V(1,i) - [V(1,i)+C_k] \\ &\geq V(0,i) - [V(0,i)+C_k] \\ &= D_1TV(0,i) \end{aligned}$$

When $D_kV(1,i) \leq -c_k$, there is

$$\begin{aligned} [D_1TV(x+1,i) &= [V(2,i)+C_k] - [V(1,i)+C_k] \\ &\geq [V(1,i)+C_k] - [V(0,i)+C_k] \\ &= D_1TV(0,i) \end{aligned}$$

To sum up, statement (.2) is true. The second statement (.3) can be proved similarly as a special case when $c_k = q$, which will not be restated here for brevity.

The proof is complete. □

.2. Proof of Proposition 3.3

Proof. If $D_{12}T^iV(x,i) \geq 0$ holds true, then $D_2V(x,i)$ to be an increasing function of x. Therefore, We first express that $D_2V(x,i) = T^{i+1}V(x,i+1) - T^iV(x,i) = I_1 + I_2$, where I_1 and I_2 are defined as

$$\begin{aligned} I_1 &= \mu_{i+1}\min(V(x+1,i+1)+q, V(x,i+1)) - \mu_i \min(V(x+1,i)+q, V(x,i)) \\ &\quad +[\mu_iV(x,i+1) - \mu_{i+1}V(x,i)] - \mu_{i+1}\min(V(x,i+1)+q, V(x-1,i+1)) \\ &\quad +\mu_i\min(V(x,i)+q, V(x-1,i)) - [\mu_iV(x-1,i+1) - \mu_{i+1}V(x-1,i)] \end{aligned}$$

$$\begin{aligned} I_2 &= r(i+1)(1-s)V(x,i) + \sum_{k=1}^{n}\lambda_k \min(V(x,i+1)+c_k, V(x-1,i+1)) \\ &\quad +V(x,i+1)\sum_{l\neq i+1,i,0}^{m} r(l)(1-s) + V(x,i+1)r(i)(1-s) \\ &\quad -r(i)(1-s)V(x,i-1) - \sum_{k=1}^{n}\lambda_k\min(V(x,i)-c_k, V(x-1,i)) \\ &\quad -V(x,i)\sum_{l\neq i+1,i,0}^{m} r(l)(1-s) + V(x,i)r(i+1)(1-s) \end{aligned}$$

We will first show that I_1 is an increasing function of x. There are three cases to consider.

If $D_1V(x,i+1) < -q$, then

$$\begin{aligned} I_1 &= \mu_{i+1}V(x+1,i+1) + \mu_iV(x,i+1) - \mu_iV(x+1,i) - \mu_{i+1}V(x,i) + (\mu_{i+1}-\mu_i)\cdot q \\ &= (\mu_{i+1}-\mu_i)V(x+1,i+1) - (\mu_{i+1}-\mu_i)V(x,i) + (\mu_{i+1}-\mu_i)\cdot q \\ &\quad +\mu_i[V(x+1,i+1) - V(x+1,i)] + \mu_i[V(x,i+1) - V(x,i)] \\ &= \mu_i[D_2V(x+1,i) + D_2V(x,i)] + (\mu_{i+1}-\mu_i)[D_2V(x+1,i) + D_1V(x,i)] \end{aligned}$$

Since $D_1V(x,i)$ and $D_2V(x,i)$ both increase with x and $\mu_i > 0$, it follows that I_1 is an increasing function of x.

Similarly, if $-q < D_1V(x-1,i)$, then

$$\begin{aligned} I_1 &= \mu_{i+1}V(x,i+1) - \mu_iV(x,i) + \mu_iV(x,i+1) - \mu_{i+1}V(x,i) \\ &= \mu_{i+1}D_2V(x,i) + \mu_iD_2(x,i). \end{aligned}$$

If $D_1V(x,i+1) > -q > D_1V(x-1,i)$, there is

$$\begin{aligned} I_1 &= \mu_{i+1}V(x,i+1) - \mu_i\min(V(x+1,i)+q, V(x,i)) \\ &\quad +[\mu_iV(x,i+1) - \mu_{i+1}V(x,i)] - [\mu_iV(x-1,i+1) - \mu_{i+1}V(x-1,i)] \\ &\geq \mu_{i+1}V(x,i+1) - \mu_iV(x,i) + \mu_iV(x,i+1) - \mu_{i+1}V(x,i) \\ &\quad -\mu_iV(x-1,i+1) + \mu_{i+1}V(x-1,i) \\ &= \mu_{i+1}V(x,i+1) - \mu_iV(x,i) + \mu_iD_1V(x-1,i+1) - \mu_{i+1}D_1V(x-1,i), \end{aligned}$$

while

$$\begin{aligned} &\mu_{i+1}V(x,i+1) - \mu_iV(x,i) + (\mu_{i+1}-\mu_i)\cdot q \\ &\geq \mu_{i+1}\min(V(x,i+1)+q, V(x-1,i+1)) - \mu_iV(x,i) - \mu_iq \end{aligned}$$

Since $D_1V(x,i+1) > D_1V(x-1,i)$ and $-q > D_1V(x-1,i)$, we can write that

$$\mu_i(D_1V(x-1,i+1)+q) \geq \mu_{i+1}(D_1V(x-1,i)+q),$$

that is

$$I_1 \geq \mu_{i+1}V(x,i+1) - \mu_iV(x,i) + (\mu_{i+1}-\mu_i)q$$

Hence, it can be proved that I_1 in an increasing function about x.

Next, we prove that the I_2 is also increasing on x, which can be expressed as

$$\begin{aligned} I_2 &= D_2\sum_{k=1}^{n}\lambda_k\min(V(x,i)+c_k, V(x-1,i)) + D_2V(x,i)\sum_{l\neq i+1,i,0}^{m} r(l)(1-s) \\ &\quad +[r(i)(1-s)V(x,i+1) - r(i)(1-s)V(x,i-1)] \end{aligned}$$

Therein, the second term is clearly an increasing function of x; the third term can be simplified to $r(i)(1-s)[D_2V(x,i)+D_2V(x,i-1)]$, which is obviously an increasing function of x. Thus prove the monotonicity of $D_2\min(V(x,i)+c_k, V(x-1,i))$, which can be written as

$$\begin{aligned} D_2\min(V(x,i)+c_k, V(x-1,i)) &= \min(V(x+1,i+1)+c_k, V(x,i+1)) \\ &\quad -\min(V(x+1,i)+c_k, V(x,i)) \\ &\quad -\min(V(x,i+1)+c_k, V(x-1,i+1)) \\ &\quad +\min(V(x,i)+c_k, V(x-1,i)) \end{aligned}$$

Three cases are discussed.

If $-c_k > D_1V(x,i+1)$, then

$$\begin{aligned} D_2\min(V(x,i)+c_k, V(x-1,i)) &= V(x+1,i+1) - V(x+1,i) \\ &\quad -V(x,i+1) + V(x,i) \\ &= D_{12}V(x,i) \end{aligned}$$

If $D_1V(x-1,i) > -C_k$, then

$$\begin{aligned} D_2\min(V(x,i)+c_k, V(x-1,i)) &= V(x,i+1) - V(x,i) \\ &\quad -V(x-1,i+1) + V(x-1,i) \\ &= D_{12}V(x-1,i) \end{aligned}$$

If $D_1V(x,i+1) > -c_k > D_1V(x-1,i)$, then

$$\begin{aligned} D_2\min(V(x,i)+c_k, V(x-1,i)) &= V(x,i+1) - \min(V(x+1,i)+c_k, V(x,i)) \\ &\quad -\min(V(x,i+1)+c_k, V(x-1,i+1)) \\ &\quad +V(x,i)+c_k \end{aligned}$$

and

$$\begin{aligned} V(x,i+1) - \min(V(x+1,i)+c_k, V(x,i)) &\geq V(x,i+1) - V(x,i) \\ &= V(x,i+1) + c_k - V(x,i) - c_k \\ &\geq \min(V(x,i+1)+c_k, V(x-1,i+1)) \\ &\quad -V(x,i) - c_k \end{aligned}$$

Hence, it can be proved that I_2 in an increasing function about x.

The proof is complete. □

.3. *Proof of Proposition 3.4*

Proof. Similar to the proof of Theorem 3.2, the truthfulness of $D_1TV(x,i) \geq -c_1$ hinges on the validity of the following two statements:

$$D_1T_kV(x,i) \geq -c_1 \quad (.4)$$
$$D_1T^iV(x,i) \geq -c_1 \quad (.5)$$

To prove that the statement (.4) holds true, three cases should be considered. When $x \neq 0$,if $D_1Vx, i \geq -c_i$, then

$$D_1T^iV(x,i) \geq V(x,i) - V(x,i) - c_i \geq -c_i$$

If $D_1Vx, i \leq -c_i$, then

$$D_1T^iV(x,i) \geq [V(x+1,i) + c_i] - [V(x,i) + c_i] \geq V(x+1,k) - v(x,k) \geq -c_i$$

When $x = 0$, the inequality is obviously true.

Statement (.5) can be considered as a special case where $c_i = q$, which is also true.

The proof is complete. □

.4. *Proof of Theorem 3.2*

Proof. According to Theorem 3.1, it can be know that $V(x,i) \in V$. Meanwhile, the convexity (**P1**) of $V(x,i)$ implies that the optimal production rate strategy and inventory allocation strategy are threshold strategies that depend on the state. These two can derive to Properties 1 and 2 directly. Based on the definition of $S^*(i)$ and the upper bounding property (**P2**) of $V(x,i)$, we have $V(S^*(i)+1, i+1) - V(S^*(i), i+1) \geq V(S^*(i)+1, i) - V(S^*(i), i) \geq -q$, so we have $S^*(i+1) \leq S^*(i)$, which yields Property 3. There is $V(R_k^*(i), i) - V(R(i)-1, i) \geq -c_i \geq -c_{i-1}$, which implies $R_{k-1}^*(x,i) \leq R_k^*(x,i)$, yielding Property 4. Similarly, by the definition of $R^*(k)$ and Property 1, we have $V(R(i), i+1) - V(R^*(i), i+1) \geq V(R^*(i), k) - V(R^*(i)-1, i) \geq -c$, so we have $R^*(i+1) \leq R(i)$, yielding Property 5. Property 6 can be obtained directly from Property 3 and Theorem 3.1. □

.5. *Proof of Theorem 3.3*

Proof. We begin by observing that if $D_1(0,i,s_1) \leq -c_1$, then $\forall x : 0 \leq x \leq x_0, D_1(x,i,s_2) \leq -q, D_1(x+1,i,s_2) > -q$ holds true. Therefore, to prove the theorem, it suffices to demonstrate that if $D_1V(x,i,s_2) - D_1V(x,i,s_1) \geq 0$, then $D_1TV(x,i,s_2) - D_1V(x,i,s_1) \geq 0$. This is equivalent to proving the following inequalities:

$$D_1 \min_x \{V(x+1,i,s_2) + q, V(x,i,s_2)\} - D_1 \min_x \{V(x+1,i,s_1) + q, V(x,i,s_1)\} \geq 0 \quad (.6)$$
$$D_1 \min_x \{V(x,i,s_2) + c_k, V(x-1,i,s_2)\} - D_1 \min_x \{V(x,i,s_1) + c_k, V(x-1,i,s_1)\} \geq 0 \quad (.7)$$

We now turn our attention to proving Equation 3.7. We have

$$\begin{aligned}
& D_1 \min_x \{V(x+1,i,s_2)+q, V(x,i,s_2)\} - D_1 \min_x \{V(x+1,i,s_1)+q, V(x,i,s_1)\} \\
= & \min_x \{V(x+2,i,s_2)+q, V(x+1,i,s_2)\} - \min_x \{V(x+1,i,s_2)+q, V(x,i,s_2)\} \\
& - \min_x \{V(x+2,i,s_1)+q, V(x+1,i,s_1)\} + \min_x \{V(x+1,i,s_1)+q, V(x,i,s_1)\} \\
= & V(x+1,i,s_2) - V(x,i,s_2) - \min_x \{V(x+2,i,s_1)+q, V(x+1,i,s_1)\} \\
& + \min_x \{V(x+1,i,s_1)+q, V(x,i,s_1)\}
\end{aligned}$$

Using the fact that $\min_x\{V(x+2,i,s_1)+q, V(x+1,i,s_1)\} \le V(x+1,i,s_1)$, we can derive

$$\begin{aligned}
& V(x+1,i,s_2) - V(x,i,s_2) - \min_x \{V(x+2,i,s_1)+q, V(x+1,i,s_1)\} \\
& + \min_x \{V(x+1,i,s_1)+q, V(x,i,s_1)\} \\
\ge & V(x+1,i,s_2) - V(x,i,s_2) - V(x+1,i,s_1) + V(x,i,s_1) \\
= & D_1 V(x,i,s_2) - D_2 V(x,i,s_1) \ge 0
\end{aligned}$$

Hence, Equation 3.7 holds true.

Next, we proceed to prove Equation 3.8. By expanding the expression, we obtain

$$\begin{aligned}
& D_1 \min_x \{V(x,i,s_2)+c_k, V(x-1,i,s_2)\} - D_1 \min_x \{V(x,i,s_1)+c_k, V(x-1,i,s_1)\} \\
= & \min_x \{V(x+1,i,s_2)+c_k, V(x,i,s_2)\} - \min_x \{V(x,i,s_2)+c_k, V(x-1,i,s_2)\} \\
& - \min_x \{V(x+1,i,s_1)+c_k, V(x,i,s_1)\} + \min_x \{V(x,i,s_1)+c_k, V(x,i,s_1)\} \\
= & V(x+1,i,s_2) - V(x,i,s_2) - min_x\{V(x+2,i,s_1)+c_k, V(x+1,i,s_1)\} \\
& + \min_x \{V(x+1,i,s_1)+c_k, V(x,i,s_1)\}
\end{aligned}$$

If $D_1V(x-1,i,s_2) \ge D_1V(x-1,i,s_1) \ge -c_k$, then

$$\begin{aligned}
& D_1 \min_x \{V(x,i,s_2)+c_k, V(x-1,i,s_2)\} - D_1 \min_x \{V(x,i,s_1)+c_k, V(x-1,i,s_1)\} \\
= & D_1V(x-1,i,s_1) - D_2V(x-1,i,s_1) \ge 0
\end{aligned}$$

If $D_1V(x-1,i,s_2) \ge -c_k \ge D_1V(x-1,i,s_1)$, then

$$\begin{aligned}
& D_1 \min_x \{V(x,i,s_2)+c_k, V(x-1,i,s_2)\} - D_1 \min_x \{V(x,i,s_1)+c_k, V(x-1,i,s_1)\} \\
\ge & D_1V(x-1,i,s_2) - V(x,i,s_1) + V(x,i,s_1) + c_k \\
= & D_1V(x-1,i,s_1) + c_k \ge 0
\end{aligned}$$

If $-c_k \ge D_1V(x-1,i,s_2) \ge D_1V(x-1,i,s_1)$, then

$$\begin{aligned}
& D_1 \min_x \{V(x,i,s_2)+c_k, V(x-1,i,s_2)\} - D_1 \min_x \{V(x,i,s_1)+c_k, V(x-1,i,s_1)\} \\
\ge & V(x+1,i,s_2) + c_k - V(x,i,s_2) + c_k - V(x+1,i,s_1) + V(x,i,s_1) + c_k \\
= & D_1V(x,i,s_2) - D_1V(x,i,s_1) \ge 0
\end{aligned}$$

Hence, Equation 3.8 holds true. And the proof is complete. □

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