Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(12): 6908–6935.
DOI:10.3934/jimo.2026230
Received: 27 August 2026
Revised: 12 November 2026
Accepted: 19 November 2026
Published: 26 November 2026

Research article

Consensus multiplier methods for heterogeneous reverse-convex programs

Zhonghui Xue* **and Yu Chen**

Shanghai Publishing and Printing College, Shanghai, China

* **Correspondence:** Email: hnlgxzh@163.com.

Appendix A. The two delayed-contact maps

The listings below specify the ideal edge-wise templates with exact local minimization. The archived edge-wise implementation uses the approximations and denominator regularization discussed in Section 5.2; it is not the corrected main benchmark. The reduced-position illustration instead uses an analytic minimizer. These listings are not the proximal copy map in Theorem 6.6. They are structural reductions of [1,19], not official clones. We write $p_i = P_i z_i$.

Algorithm A.1 Feasible-point template (edge-wise normal placement)

Require: z^0 as in Assumption 5.1, (ρ_i, γ_i), τ

copies $y_i^0 \leftarrow P_i z_i^0$; normals by (5.3)
for $k = 0, 1, \ldots$ **do**
  awake set $\mathcal{A}^k$; delayed stack $\hat{y}^k = y^{k-d(k)}$
  for $i \in \mathcal{A}^k$ **do**
    $z_i^{k+1} \leftarrow \arg\min_{z_i \in \hat{\mathcal{Z}}_i^k} f_i(z_i) + \frac{\rho_i}{2}\|r_i(z_i, \hat{y}_i^k) + \lambda_i^k/\rho_i\|^2 + \frac{\gamma_i}{2}\|z_i - z_i^k\|^2$
  end for
  $z_j^{k+1} \leftarrow z_j^k$ for $j \notin \mathcal{A}^k$
  y^{k+1} by (5.5), then average incident copies. This is not the proximal map of Theorem 6.6.
  $\lambda_i^{k+1} \leftarrow \lambda_i^k + \rho_i r_i(z_i^{k+1}, y_i^{k+1})$ for $i \in \mathcal{A}^k$
end for

Algorithm A.2 Infeasible-point template (residual linearization)

Require: z^0, (ρ_i, γ_i), $\tau, \beta \in (0,1)$

copies $y_i^0 \leftarrow P_i z_i^0$

for $k = 0, 1, \ldots$ **do**

  awake set $\mathcal{A}^k$; delayed stack $\hat{y}^k = y^{k-d(k)}$

  for $i \in \mathcal{A}^k$ **do**

    $z_i^{k+1} \leftarrow \arg\min_{z_i \in \mathcal{Z}_i} f_i(z_i) + \frac{\rho_i}{2}\|r_i(z_i, \hat{y}_i^k) + \lambda_i^k/\rho_i\|^2 + \frac{\gamma_i}{2}\|z_i - z_i^k\|^2$

  end for

  $z_j^{k+1} \leftarrow z_j^k$ for $j \notin \mathcal{A}^k$

  y^{k+1} by (5.6) with step $\beta \in (0, 1/2)$, then average incident copies. An overlap of a pair at distance $s \in (0, d)$ persists. The coincident case $s = 0$ is undefined. This is not the proximal map of Theorem 6.6.

  $\lambda_i^{k+1} \leftarrow \lambda_i^k + \rho_i r_i(z_i^{k+1}, y_i^{k+1})$ for $i \in \mathcal{A}^k$

end for

Appendix B. Proofs

Throughout, $\|\cdot\|$ is Euclidean. Full proofs with explicit constants are in this appendix; the proof roadmap is in Section 6. We write $r_i(z_i, y_i) := P_i z_i - y_i$ for the consensus residual of agent i, and $r(z, y)$ for the stacked residual. The inner product of stacked multipliers is $\langle \lambda, r\rangle = \sum_i \langle \lambda_i, r_i\rangle$. The copy blocks satisfy $\|y_i - y_i'\| \le C_W \|y - y'\|$, with $C_W = 1$ in (4.8).

Appendix B.1. Lifting and constraint qualification

Proof of Lemma 4.1. If (z, y) is feasible for (4.8) with the reverse-convex indicators, then $P_i z_i = y_i$ and $\|y_i^t - y_j^t\| \ge d_{ij}$, hence z satisfies (4.4) and $z_i \in \mathcal{Z}_i$. Conversely, setting $y_i = P_i z_i$ makes the consensus equalities identities and places y_e in the reverse-convex pair set.

If $\psi_{e,t} = \iota_{H_e^t(n_e^t)}$ and $\|n_e^t\| = 1$, the Cauchy–Schwarz inequality gives $(n_e^t)^\top(y_i^t - y_j^t) \le \|y_i^t - y_j^t\|$, so the halfspace forces the disk inequality. If $n_e^t = (y_i^t - y_j^t)/\|y_i^t - y_j^t\|$, then $(n_e^t)^\top(y_i^t - y_j^t) = \|y_i^t - y_j^t\|$, so disk feasibility implies this halfspace inequality. Alignment is sufficient, not necessary, when the separation is larger than d_{ij}. At contact, $\|y_i^t - y_j^t\| = d_{ij}$, and equality in the Cauchy–Schwarz inequality shows that alignment is necessary as well. □

Proof of Lemma 4.2. Fix a feasible z at which every local inequality is inactive, and suppose the affine dynamics equalities have full row rank. Position controllability means that for each i, there exists a tangent vector w_i of the affine hull $\{G_i z_i = b_i(x_i(0))\}$ with $P_i^t w_i$ equal to any prescribed planar increment $\delta_i(t)$, $t = 1, \ldots, T$. Let $\bar{p}(t)$ be the centroid of $\{P_i^t z_i\}_{i \in \mathcal{V}}$ and set

$$\delta_i(t) := P_i^t z_i - \bar{p}(t).$$

The corresponding $w = (w_i)$ lies in $\prod_i T_{\mathcal{Z}_i}(z_i)$ because local inequalities are inactive. For an active disk $g_{ij}^t(z) = 0$, one has $\|P_i^t z_i - P_j^t z_j\| = d_{ij} > 0$, and (4.7) yields

$$\langle \nabla g_{ij}^t(z), w\rangle = -2(P_i^t z_i - P_j^t z_j)^\top(\delta_i(t) - \delta_j(t))$$

$$= -2\left\|P_i^t z_i - P_j^t z_j\right\|^2 = -2d_{ij}^2 < 0.$$

This is the Mangasarian–Fromovitz direction [21, Thm. 12.2]. □

Proof of Proposition 5.2. The set $\hat{\mathcal{Z}}_i^k$ is defined by unit-normal halfspaces. If it is nonempty, the exact minimizer lies in $\hat{\mathcal{Z}}_i^k$. The copy step (5.5) produces a per-time edge pair on the boundary or in the interior of $H_e^t(n_e^{t,k})$, before any averaging. Lemma 4.1 then gives $\|y_{e,i}^{t,+} - y_{e,j}^{t,+}\| \geq d_{ij}$ for that pair. Averaging incident copies can move the pair out of the halfspace, so the inequality is not claimed after averaging. The argument uses $\|n_e^{t,k}\| = 1$ only, not the age of $\hat{y}^k$. □

Appendix B.2. Delay

Proof of Lemma 6.5. Let $d := d(k) \leq \tau$. Then $\hat{y}^k - y^k = \sum_{s=1}^{d}(y^{k-s+1} - y^{k-s})$. The Cauchy–Schwarz inequality applied to at most τ terms yields $\|\hat{y}^k - y^k\|^2 \leq \tau \sum_{s=1}^{\tau} \|y^{k-s+1} - y^{k-s}\|^2$. □

We write $e^{k,s} := y^{k-s+1} - y^{k-s}$ for $s = 1, \ldots, \tau$ and

$$\Delta^k := c_\Delta \sum_{s=1}^{\tau}(\tau - s + 1)\|e^{k,s}\|^2, \qquad c_\Delta := 4\Big(\max_i \rho_i\Big)C_W^2\tau. \tag{B.1}$$

Theorem 6.6 instead uses the weighted saddle distance D^k as its Lyapunov function. The one-step identity

$$\Delta^k - \Delta^{k+1} = c_\Delta \sum_{s=1}^{\tau} \|e^{k,s}\|^2 - c_\Delta\tau\|y^{k+1} - y^k\|^2 \tag{B.2}$$

follows by expanding Δ^{k+1} and cancelling consecutive weights: the coefficient of $\|e^{k,s}\|^2$ drops by one for each $s < \tau$, the oldest term $\|e^{k,\tau}\|^2$ disappears, and the new increment $y^{k+1} - y^k$ enters with weight τ.

Appendix B.3. A local proximal inequality

Lemma B.1. *Let $i \in \mathcal{A}^k$, let f_i be convex, and let $\rho_i > 0$, $\gamma_i > 0$. Suppose the exact z_i-update is taken over a nonempty closed convex set C_i^k containing z_i^k. Then it satisfies*

$$\begin{aligned} f_i(z_i^{k+1}) + \frac{\rho_i}{2}\|r_i(z_i^{k+1}, \hat{y}_i^k)\|^2 + \langle \lambda_i^k, r_i(z_i^{k+1}, \hat{y}_i^k)\rangle + \frac{\gamma_i}{2}\|z_i^{k+1} - z_i^k\|^2 \\ \leq f_i(z_i^k) + \frac{\rho_i}{2}\|r_i(z_i^k, \hat{y}_i^k)\|^2 + \langle \lambda_i^k, r_i(z_i^k, \hat{y}_i^k)\rangle - \frac{\gamma_i}{2}\|z_i^{k+1} - z_i^k\|^2 - \frac{\rho_i}{2}\|P_i(z_i^{k+1} - z_i^k)\|^2. \end{aligned} \tag{B.3}$$

Proof. We write $\Delta z := z_i^{k+1} - z_i^k$ and

$$\phi_i(z_i) = f_i(z_i) + \frac{\rho_i}{2}\left\|P_i z_i - \hat{y}_i^k + \frac{\lambda_i^k}{\rho_i}\right\|^2 + \frac{\gamma_i}{2}\|z_i - z_i^k\|^2.$$

The map ϕ_i is $C^{1,1}$ and strongly convex with modulus γ_i. The update is the unique minimizer of ϕ_i on the convex set C_i^k. The variational inequality $\langle \nabla\phi_i(z_i^{k+1}), z_i^k - z_i^{k+1}\rangle \geq 0$ holds only when $z_i^k \in C_i^k$. Stale halfspaces can exclude z_i^k, and the lemma is not applied on those events. When the comparison point is feasible,

$$\langle \nabla f_i(z_i^{k+1}), -\Delta z\rangle + \rho_i\Big\langle r_i(z_i^{k+1}, \hat{y}_i^k) + \frac{\lambda_i^k}{\rho_i}, -P_i\Delta z\Big\rangle - \gamma_i\|\Delta z\|^2 \geq 0. \tag{B.4}$$

Each f_i is convex, hence $f_i(z_i^k) \geq f_i(z_i^{k+1}) + \langle \nabla f_i(z_i^{k+1}), -\Delta z \rangle$, and (B.4) yields

$$f_i(z_i^{k+1}) + \rho_i \Big\langle r_i(z_i^{k+1}, \hat{y}_i^k) + \frac{\lambda_i^k}{\rho_i}, P_i \Delta z \Big\rangle + \gamma_i \|\Delta z\|^2 \leq f_i(z_i^k).$$

The identity

$$\begin{aligned} &\frac{\rho_i}{2}\|r_i(z_i^{k+1}, \hat{y}_i^k)\|^2 - \frac{\rho_i}{2}\|r_i(z_i^k, \hat{y}_i^k)\|^2 + \langle \lambda_i^k, P_i \Delta z \rangle \\ &\qquad = \rho_i \Big\langle r_i(z_i^{k+1}, \hat{y}_i^k) + \frac{\lambda_i^k}{\rho_i}, P_i \Delta z \Big\rangle - \frac{\rho_i}{2}\|P_i \Delta z\|^2 \end{aligned} \tag{B.5}$$

then implies (B.3) after moving $\gamma_i \|\Delta z\|^2/2$ to the right-hand side. The comparison requires $z_i^k \in C_i^k$; it is not available merely because a new stale-contact set is nonempty. □

Appendix B.4. Dual identity

For $i \in \mathcal{A}^k$, the dual update is $\lambda_i^{k+1} = \lambda_i^k + \rho_i r_i(z_i^{k+1}, y_i^{k+1})$, so

$$\frac{1}{2\rho_i}(\|\lambda_i^{k+1}\|^2 - \|\lambda_i^k\|^2) = \langle \lambda_i^k, r_i(z_i^{k+1}, y_i^{k+1}) \rangle + \frac{\rho_i}{2}\|r_i(z_i^{k+1}, y_i^{k+1})\|^2. \tag{B.6}$$

This identity concerns the awake agent's multiplier increment. If the copy step changes y_i for a sleeping agent, then r_i and $\mathcal{L}$ can change while λ_i does not; the identity alone does not control that change. In Theorem 6.6, every agent is awake, so this difficulty does not arise. No asynchronous descent conclusion is drawn from this identity.

Appendix B.5. Lyapunov function

Proof of Theorem 6.6. In independent input coordinates, the position map has the affine form $a_i + P_i z_i$. In the formulas below, $P_i z_i$ denotes this full position map, whereas P_i applied to a difference or transposed in a gradient denotes its linear part. The fixed offsets cancel in every displayed difference and in the feasibility identity, so the same proof applies to both coordinate choices. Write $z_i^+ = z_i^{k+1}$, $y_i^+ = y_i^{k+1}$, $\lambda_i^+ = \lambda_i^{k+1}$, and $r_i^+ = P_i z_i^+ - y_i^+$. Saddle optimality gives, for every $z_i \in \mathcal{Z}_i$ and every $y \in \mathcal{Y}$,

$$\begin{aligned} \langle \nabla f_i(z_i^\star), z_i - z_i^\star \rangle + \langle \lambda_i^\star, P_i(z_i - z_i^\star) \rangle &\geq 0, \\ \langle \lambda^\star, y^\star - y \rangle &\geq 0. \end{aligned}$$

Exact minimization on $\mathcal{Z}_i$ gives

$$\langle \nabla f_i(z_i^+) + \gamma_i(z_i^+ - z_i^k) + P_i^\top(\lambda_i^k + \rho_i(P_i z_i^+ - y_i^k)), z_i^\star - z_i^+ \rangle \geq 0.$$

The dual step rearranges the penalty gradient:

$$\lambda_i^k + \rho_i(P_i z_i^+ - y_i^k) = \lambda_i^+ + \rho_i(y_i^+ - y_i^k).$$

Strong convexity is the inner-product form $\langle \nabla f_i(z_i^+) - \nabla f_i(z_i^\star), z_i^+ - z_i^\star \rangle \geq \mu_i \|z_i^+ - z_i^\star\|^2$. Add the z-step inequality to the saddle inequality at z_i^+ and cancel $\nabla f_i(z_i^\star)$ against this gap:

$$\gamma_i \langle z_i^+ - z_i^k, z_i^\star - z_i^+ \rangle + \langle \lambda_i^+ - \lambda_i^\star + \rho_i(y_i^+ - y_i^k), P_i(z_i^\star - z_i^+) \rangle \geq \mu_i \|z_i^+ - z_i^\star\|^2. \tag{B.7}$$

The identity $P_i(z_i^\star - z_i^+) = (y_i^\star - y_i^+) - r_i^+$ converts the summed form of (B.7) into

$$\sum_i \gamma_i \langle z_i^+ - z_i^k, z_i^\star - z_i^+ \rangle + \langle \lambda^+ - \lambda^\star, y^\star - y^+ \rangle - \langle \lambda^+ - \lambda^\star, r^+ \rangle + \sum_i \rho_i \langle y_i^+ - y_i^k, y_i^\star - y_i^+ \rangle - \sum_i \rho_i \langle y_i^+ - y_i^k, r_i^+ \rangle \geq \sum_i \mu_i \|z_i^+ - z_i^\star\|^2.$$

The y-step with $\eta = 0$ has smooth gradient $-\lambda^+$, so $\langle \lambda^+, y^+ - y^\star \rangle \geq 0$. Together with the saddle inequality, this is $\langle \lambda^+ - \lambda^\star, y^+ - y^\star \rangle \geq 0$. The three-point identity $\langle u - v, w - u \rangle = \frac{1}{2}(\|v - w\|^2 - \|u - w\|^2 - \|u - v\|^2)$ with $(u, v, w) = (z_i^+, z_i^k, z_i^\star)$, and the same identity for y_i and for the multiplier update $\lambda_i^+ - \lambda_i^k = \rho_i r_i^+$ gives

$$\begin{aligned} D^k - D^{k+1} = {} & \sum_i \gamma_i \langle z_i^+ - z_i^k, z_i^\star - z_i^+ \rangle + \sum_i \frac{\gamma_i}{2} \|z_i^+ - z_i^k\|^2 \\ & + \sum_i \rho_i \langle y_i^+ - y_i^k, y_i^\star - y_i^+ \rangle + \sum_i \frac{\rho_i}{2} \|y_i^+ - y_i^k\|^2 \\ & - \langle \lambda^+ - \lambda^\star, r^+ \rangle + \sum_i \frac{\rho_i}{2} \|r_i^+\|^2. \end{aligned}$$

Insert the variational lower bound and use $\langle \lambda^+ - \lambda^\star, y^+ - y^\star \rangle \geq 0$:

$$\begin{aligned} D^k - D^{k+1} \geq {} & \sum_i \mu_i \|z_i^+ - z_i^\star\|^2 + \sum_i \frac{\gamma_i}{2} \|z_i^+ - z_i^k\|^2 + \langle \lambda^+ - \lambda^\star, y^+ - y^\star \rangle \\ & + \sum_i \left(\frac{\rho_i}{2} \|r_i^+\|^2 + \rho_i \langle y_i^+ - y_i^k, r_i^+ \rangle + \frac{\rho_i}{2} \|y_i^+ - y_i^k\|^2 \right). \end{aligned}$$

The expression in parentheses is $\frac{\rho_i}{2} \|P_i z_i^+ - y_i^k\|^2$. This is (6.3). Nonnegativity of the right-hand side yields $D^{k+1} \leq D^k$. Since $D^k \geq 0$, the sum of the decreases is at most D^0. Hence $\sum_k \|z^k - z^\star\|^2 < \infty$, so $z^k \to z^\star$ and $\sum_k \|P z^{k+1} - y^k\|^2 < \infty$. Therefore, $y^k \to P z^\star = y^\star$ and $r^k = P z^k - y^k \to 0$. The dual increment is $\rho_i r_i^{k+1}$, so $\lambda^{k+1} - \lambda^k \to 0$ and $y^{k+1} - y^k \to 0$. The multiplier bound is the corresponding piece of $D^k \leq D^0$. Let $\bar{\lambda}$ be a limit point of λ^k. Pass to the limit in the two variational inequalities, using vanishing increments, to obtain the saddle system at $(z^\star, y^\star, \bar{\lambda})$. The same argument gives Fejér monotonicity of the distance to $(z^\star, y^\star, \bar{\lambda})$. A subsequence of that distance tends to zero, and a nonincreasing nonnegative sequence with a zero subsequence tends to zero. Thus $\lambda^k \to \bar{\lambda}$. □

Proof of Corollary 6.12. Summing (6.3) and using $D^k \geq 0$ gives the two series in Corollary 6.12. The residual tends to zero because (z^k, y^k) converges to a feasible saddle. Lemma 6.8 remains a separate fact: a square-summable residual sequence need not bound λ. □

Proof of Corollary 6.11. Under the hypotheses of Theorem 6.6, every agent updates with the current copy, the normals are fixed, and $\eta = 0$. Theorems 6.6 and 6.10 then apply with $C_W = 1$ in (4.8). This conclusion does not follow from setting the delay parameter to zero while retaining a different copy map or refreshing the normals at every iteration. □

Proof of Proposition 6.23. Section 6.3 and Appendix B prove Theorems 6.19 and 6.21, along with Lemma 6.20, using planar geometry and (5.5), (5.6) only. No step invokes V^k, (6.1), or Lemma B.1. □

Proof of Lemma 6.8. The update rule gives $\|\lambda_i^{k+1} - \lambda_i^k\| = \rho_i \|r_i(z_i^{k+1}, y_i^{k+1})\|$ for $i \in \mathcal{A}^k$, and $\lambda_i^{k+1} = \lambda_i^k$ otherwise. A residual $r^k = k^{-1}e$ along one fixed awake coordinate, with a unit vector e and a fixed positive penalty ρ, satisfies $\sum_k \|r^k\|^2 < \infty$, whereas $\lambda^k = \lambda^0 + \rho \sum_{s=1}^k s^{-1} e$ is unbounded. Thus square-summability alone does not bound the multipliers. This counterexample concerns the proposed inference; it need not be an orbit of the synchronous iteration in Theorem 6.6. Absolute convergence of the multiplier increments under $\sum_k \sum_{i \in \mathcal{A}^k} \|r_i^{k+1}\| < \infty$ follows from $\|\lambda^{k+1} - \lambda^k\| \le (\max_i \rho_i) \sum_{i \in \mathcal{A}^k} \|r_i^{k+1}\|$. □

Proof of Lemma 6.9. The separation lower bound makes normalization continuous, so the normals and the offsets of each local contact halfspace converge. If $q^{k_j} \in \hat{\mathcal{Z}}_i^{k_j}$ and $q^{k_j} \to q$, closedness of $\mathcal{Z}_i$ and passage to the limit in the finite affine inequalities give $q \in \hat{\mathcal{Z}}_i^\star$. This proves the outer-limit inclusion without an additional regularity assumption.

For the reverse inclusion, use the strict point required in Lemma 6.9. Let $w_i \in \mathcal{Z}_i$ satisfy every limiting contact inequality strictly, and let $q \in \hat{\mathcal{Z}}_i^\star$. For every fixed $\varepsilon \in (0, 1]$, the point $q_\varepsilon = (1 - \varepsilon)q + \varepsilon w_i$ lies in $\mathcal{Z}_i$ and satisfies every limiting contact inequality strictly. Since there are finitely many inequalities and all coefficients converge, $q_\varepsilon \in \hat{\mathcal{Z}}_i^{k_j}$ for all sufficiently large j. A diagonal choice $\varepsilon_j \downarrow 0$ gives $q_{\varepsilon_j} \in \hat{\mathcal{Z}}_i^{k_j}$ with $q_{\varepsilon_j} \to q$. This establishes the inner-limit inclusion and hence Painlevé–Kuratowski convergence. In particular, intersection convergence has been proved using the strict point; it is not inferred from intersection of arbitrary convergent closed sets.

To pass normal vectors to the limit, suppose $q_j \in \hat{\mathcal{Z}}_i^{k_j}$, $q_j \to q$, $v_j \in N_{\hat{\mathcal{Z}}_i^{k_j}}(q_j)$, and $v_j \to v$. For any $w \in \hat{\mathcal{Z}}_i^\star$, choose the approximating points $w_j \in \hat{\mathcal{Z}}_i^{k_j}$ just constructed. The normal inequality $\langle v_j, w_j - q_j \rangle \le 0$ passes to $\langle v, w - q \rangle \le 0$. Thus $v \in N_{\hat{\mathcal{Z}}_i^\star}(q)$. This proves the claimed outer convergence of the normal graphs for convergent normal vectors. It does not supply boundedness of such vectors or of algorithmic multipliers. □

Proof of Theorem 6.10. By Theorem 6.6, the entire synchronous sequence converges to $(z^\star, y^\star, \bar{\lambda})$, its primal and dual increments vanish, and $Pz^\star = y^\star$. The local-step variational inequality is taken on $\mathcal{Z}_i$, not on a contact cut of $\mathcal{Z}_i$. Passing to the limit there gives

$$\langle \nabla f_i(z_i^\star) + P_i^\top \bar{\lambda}_i,\ z_i - z_i^\star \rangle \ge 0 \qquad (z_i \in \mathcal{Z}_i).$$

The copy-step inequality gives $\langle -\bar{\lambda}, y - y^\star \rangle \ge 0$ for every $y \in \mathcal{Y}$. These are exactly (6.4) with $\psi = \iota_{\mathcal{Y}}$ and $C_i = \mathcal{Z}_i$. The sets and the normals are fixed in this argument, so Lemma 6.9 is not needed. No stationarity conclusion for a nonlinear disk-indicator step, a nonconvex penalty step, or an averaged edge-wise map follows from this synchronous convex theorem.

For the disk claim, we express the normal to the polyhedron $\mathcal{Y}$ as a nonnegative combination of its active halfspace normals. Write $\nu_e^t \ge 0$ for these coefficients, using the halfspace residual $d_{ij} - (n_e^t)^\top (y_i^t - y_j^t) \le 0$. If every active halfspace normal is aligned with the realized separation, its active equality implies $\|y_i^t - y_j^t\| = d_{ij}$, and

$$-\nu_e^t n_e^t = -2\mu_e^t (y_i^t - y_j^t), \qquad \mu_e^t = \frac{\nu_e^t}{2d_{ij}} \ge 0.$$

Take $\mu_e^t = 0$ on the other disk inequalities. The local normal $N_{\mathcal{Z}_i}(z_i^\star)$ is retained unchanged, so active local bounds pose no difficulty in this algebraic conversion. Feasibility, stationarity, and complementary slackness for the disk program follow. A constraint qualification is not required to convert

already-existing multipliers. Lemma 4.2 supplies a sufficient constraint qualification under its separate hypotheses, but it is neither asserted at every accumulation point nor used to infer convergence of a normal-refresh procedure. □

Proof of Proposition 6.14. If $\rho_i > 0$ and $\gamma_i > 0$, then $\max_i \rho_i > 0$ and $\max_i \gamma_i > 0$, which is Assumption 6.4 with one pair (ρ, γ). If γ_i is proportional to L_i, then $(\max_i \gamma_i)/(\min_i \gamma_i) = \max_i L_i / \min_i L_i$. □

Appendix B.6. Saddle geometry

Write $d = 2r$ for the common safety distance of the packing subgraph, $R_\star = d/(2\sin(\pi/M))$, and $p_i^\star = R_\star(\cos(2\pi(i-1)/M), \sin(2\pi(i-1)/M))$. Let J denote a 90° rotation matrix and let $v_i = Jp_i^\star$ be the rotational field. Adjacent differences satisfy $\|p_i^\star - p_j^\star\| = d$ and $(p_i^\star - p_j^\star)^\top (v_i - v_j) = 0$.

Proof of Theorem 6.19. The Lagrangian of the position program is $L(p, \mu) = \sum_i \frac{1}{2}\|p_i + \alpha p_i^\star\|^2 + \sum_{e \in \mathcal{E}_\circ} \mu_e (d^2 - \|p_i - p_j\|^2)$. Stationarity at $p^\star$ reads, for each i,

$$(1+\alpha)p_i^\star - 2 \sum_{j:\{i,j\} \in \mathcal{E}_\circ} \mu_{ij}(p_i^\star - p_j^\star) = 0.$$

On the regular cycle, the two adjacent differences sum to $2(1 - \cos(2\pi/M))p_i^\star$, so the choice $\mu_\circ = (1+\alpha)/(4(1-\cos(2\pi/M)))$ solves the system with $\mu_e = \mu_\circ > 0$ on $\mathcal{E}_\circ$. Primal feasibility is by construction, complementary slackness holds with equality on $\mathcal{E}_\circ$, and dual feasibility is $\mu \geq 0$. Hence $(p^\star, \mu)$ is KKT.

The M active cycle gradients are linearly independent in $\mathbb{R}^{2M}$. If $\sum_{i=1}^M \eta_i \nabla g_{i,i+1}(p^\star) = 0$ (indices modulo M), the block at agent i reads $\eta_i(p_i^\star - p_{i+1}^\star) + \eta_{i-1}(p_i^\star - p_{i-1}^\star) = 0$. The two chords issuing from $p_i^\star$ are linearly independent for $M \geq 3$ (the interior angle is $(M-2)\pi/M \in (0, \pi)$), hence every coefficient vanishes. Thus, the LICQ holds.

The rotational field satisfies $\langle \nabla g_e(p^\star), v \rangle = 0$ on $\mathcal{E}_\circ$ and $\mu_e > 0$ there, so $v \in C(p^\star, \mu)$. Moreover, $\|v\|^2 = MR_\star^2$ and $\|v_i - v_{i+1}\| = \|p_i^\star - p_{i+1}^\star\| = d$ on the ring, hence

$$\langle \nabla^2_{pp} L(p^\star, \mu)v, v \rangle = MR_\star^2 - 2\mu_\circ M d^2 = -M\alpha R_\star^2 < 0,$$

where the last step uses $d = 2R_\star \sin(\pi/M)$ and $1 - \cos(2\pi/M) = 2\sin^2(\pi/M)$. The second-order necessary condition [21,Thm. 12.5] therefore fails. The rigid rotation $p_i(\theta)$ through an angle θ about the origin is a feasible C^∞ curve in $\{g \leq 0\}$ along which $f(p(\theta)) = \frac{M}{2}R_\star^2(1 + \alpha^2 + 2\alpha\cos\theta)$, so the second derivative at $\theta = 0$ is $-M\alpha R_\star^2 < 0$ and $p^\star$ is not a local minimizer.

Finally, $n_e^{t,\star} = (p_i^\star - p_j^\star)/d$ is a unit normal, $p^\star$ lies on the boundary of $H_e^t(n_e^{t,\star})$, and $(n_e^{t,\star})^\top (v_i - v_j) = 0$, so the rotational field is tangent to every frozen plane. □

Proof of Lemma 6.20. At a rest equilibrium of the stated integrator dynamics, zero input keeps the planar block at $p_i^\star$, so $P_i z_i^0 = p_i^\star \mathbf{1}_T$. For a triple integrator, this requires zero initial acceleration as well as zero velocity. With $x_i(0)$ fixed, $z_i = z_i^0 + S_i u_i$ and $P_i z_i = p_i^\star \mathbf{1}_T + Q_i u_i$. The map $u_i \mapsto P_i z_i$ is affine. When every local inequality is inactive, every input perturbation induces a tangent vector $S_i \delta u$ to $\mathcal{Z}_i$, since $G_i S_i = 0$. Surjectivity of Q_i then provides, for the prescribed increments $\delta p_i(t) = Jp_i^\star$, a perturbation δu with $Q_i \delta u$ equal to those increments and with $\delta x_i(0) = 0$. This implication can fail at an active input or corridor bound, or when Q_i is not surjective. For the position cost in Theorem 6.19, the position gradient equals $(1+\alpha)p_i^\star$ at $p_i^\star$, which is not zero. For a general weighted tracking cost, its

actual gradient must instead be used; a cancellation in the reduced position model does not by itself prove stationarity of an arbitrary unrolled cost. Under the additional cost assumptions of Lemma 6.20, repeat the ring multiplier $\mu_\circ$ from Theorem 6.19 at each prediction time. The contact-gradient sum cancels the position-tracking gradient at each time before composition with the dynamics. Pulling that zero sum back by $Q_i^\top$ therefore gives zero in the independent input coordinates. The extra quadratic input cost has a zero gradient at zero input by hypothesis, and inactive local bounds contribute zero multipliers. This proves only the stated conditional first-order cancellation; it does not establish negative curvature or trap persistence in the unrolled program. □

Proof of Theorem 6.21. (i) At k_0, the copies equal $p^\star$ and, by hypothesis, so do the delayed copies. The frozen planes are then the contact planes of Theorem 6.19. In the reduced position model, choose the consensus multiplier $\lambda_i^\star = -(1+\alpha)p_i^\star$. It is induced by the halfspace multipliers $\nu_e = 2d\mu_\circ$ in Theorem 6.19, since the sum of the two outward unit chord normals at vertex i equals $2(1-\cos(2\pi/M))p_i^\star/d$. At $p_i^\star$, the gradient of the local tracking cost plus the consensus linear term is zero, and both the consensus and proximal quadratic gradients are zero. The local objective is strongly convex, so $p_i^\star$ is its unique minimizer on its contact cut as well as on the whole reduced position space. The edge map (5.5) leaves every contact pair unchanged because its signed separation is already d. The average of the identical incident copies is $p_i^\star$; the residual is zero and the consensus multiplier stays fixed. Induction proves persistence at later events while the delayed history equals this same configuration. This proof concerns the reduced position model and does not infer persistence for arbitrary unrolled dynamics from Lemma 6.20.

(ii) The step is taken with $\lambda_i = 0$, so the linear term does not cancel $(1+\alpha)p_i^\star$. The reduced position step minimizes

$$\frac{1}{2}\|p_i + \alpha p_i^\star\|^2 + \frac{\rho_i+\gamma_i}{2}\|p_i - p_i^\star\|^2.$$

Its unique minimizer is $p_i^+ = \kappa_i p_i^\star$ with $\kappa_i = (\rho_i+\gamma_i-\alpha)/(1+\rho_i+\gamma_i) = 1-(1+\alpha)/(1+\rho_i+\gamma_i) < 1$. Set $p_i(\kappa) = \kappa p_i^\star$ and keep the neighbor at $p_j^\star$, with angle $\theta = 2\pi/M$. Then

$$\|p_i(\kappa) - p_j^\star\|^2 = R_\star^2(\kappa^2 + 1 - 2\kappa\cos\theta),$$

while $d^2 = R_\star^2 \cdot 2(1-\cos\theta)$. The strict inequality $\|p_i(\kappa) - p_j^\star\| < d$ rearranges to $2\cos\theta - 1 < \kappa < 1$. The inward displacement is $(1-\kappa)R_\star$, so the violation occurs precisely when that displacement is positive and smaller than $2(1-\cos\theta)R_\star$. For $M = 8$, $\cos\theta = \sqrt{2}/2$ and the bound equals $(2-\sqrt{2})R_\star$. For the displayed reduced minimizer, this is equivalent to $\rho_i + \gamma_i > (1+\alpha)/(2(1-\cos\theta)) - 1$. Increasing the combined quadratic weight shrinks the inward displacement. A smaller weight can jump past $\kappa = 2\cos\theta - 1$ and restore separation. This computation fixes the neighbor; simultaneous motion of both endpoints has a different distance formula. Equation (5.6) updates y and leaves P unchanged, so it neither creates nor removes the violation of $g_e(P)$. If $s = \|p_i - p_j\| \in (0, d)$ and both endpoints move as in (5.6), the updated separation equals $s + 2\beta(d-s)$. It remains strictly below d if and only if $\beta < 1/2$. The formula $(1-\beta)s + \beta d$ would be a one-sided step and is not (5.6). At $s = 0$, the unit direction is undefined.

(iii) Let $v(p) = p_i - p_j$. For nonzero vectors a, b, set $r = \|a\|$ and $s = \|b\|$. The identity $\|a-b\|^2 = (r-s)^2 + rs\|a/r - b/s\|^2$ shows that normalization is Lipschitz with constant $2/d$ when both norms are at least $d/2$. This argument applies directly to the nonconvex separation domain. For the stacked Euclidean

norm, $\|v(p) - v(q)\| \le \sqrt{2}\|p - q\|$ because $\|\Delta p_i\| + \|\Delta p_j\| \le \sqrt{2}\sqrt{\|\Delta p_i\|^2 + \|\Delta p_j\|^2}$. The composite map therefore has a Lipschitz constant of at most $2\sqrt{2}/d$. If $\|p^{k-d} - p^\star\| \le d/(4\sqrt{2})$, the stale separation is at least $d - \sqrt{2}\,d/(4\sqrt{2}) = 3d/4$, so the same Lipschitz bound applies. Thus $\|n^{k-d} - n_e^\star\| \le 1/2$ and

$$(n^{k-d})^\top n_e^\star = 1 - \tfrac{1}{2}\|n^{k-d} - n_e^\star\|^2 \ge 7/8.$$

The sign of the outward chord is unchanged inside this ball. This is a sufficient local estimate; no maximal radius is claimed. □

The identification of $p^\star$ as a KKT strict saddle is proved in Theorem 6.19, not imported. The negative curvature belongs to the constrained Lagrangian along the critical rotational field. The reduced objective itself is convex, with a Hessian equal to the identity, and its gradient at $p^\star$ is nonzero. Unconstrained strict-saddle escape results [23] therefore do not directly apply to this point.

Appendix B.7. Neighbor projection and normal refresh

Proof of Proposition 6.15. The matrix Q is positive semidefinite and its diagonal entry is $Q_{aa} = w_i^{-1} + w_j^{-1} > 0$. Holding the other coordinates fixed, (6.7) is the exact maximizer of q over $\nu_a \ge 0$. For its increment δ_a, the one-dimensional quadratic identity and optimality give

$$q(\nu + \delta_a e_a) - q(\nu) = -s_a\delta_a - \tfrac{1}{2}Q_{aa}\delta_a^2 \ge \tfrac{1}{2}Q_{aa}\delta_a^2.$$

The last inequality holds for an interior coordinate maximizer by equality and for a boundary maximizer by its one-sided optimality inequality.

Choose $y_0 \in \mathcal{Y}$ and $\bar{y} = 2y_0$. Since $d_a > 0$, $B\bar{y} - d \ge d \ge \sigma\mathbf{1}$ with $\sigma = \min_a d_a > 0$. For every $\nu \ge 0$, minimizing the Lagrangian over y yields

$$q(\nu) \le \tfrac{1}{2}\|\bar{y} - c\|_R^2 - \sigma\|\nu\|_1.$$

Thus an ascent level set $q(\nu) \ge q(\nu^0)$ is bounded. The dual objective is nondecreasing and bounded above by the primal optimum, so the sum of the squared coordinate increments is finite. Every coordinate increment therefore tends to zero; since a cycle has finitely many coordinates, the total change within a cycle also tends to zero.

Take a cluster point of the dual iterates at cycle boundaries. The point at which any one coordinate is updated has the same cluster limit. Passing to the one-coordinate optimality conditions gives, for every a,

$$\nu_a \ge 0, \qquad (Q\nu + Bc - d)_a \ge 0, \qquad \nu_a(Q\nu + Bc - d)_a = 0.$$

These are the KKT conditions of the concave dual quadratic program. Primal recovery gives $By \ge d$ and $R(y - c) - B^\top\nu = 0$, so the primal and dual KKT systems hold. The primal objective is strictly convex, hence its minimizer $y^\star$ is unique. Every cluster point of the bounded recovered primal sequence is $y^\star$, which proves convergence of the full primal sequence. Redundant rows may change the limiting dual solution without changing $y^\star$. □

Proof of Theorem 6.17. At a feasible u^r, every separation is at least $d_a > 0$, so n_a^r is defined and $(n_a^r)^\top\Delta_a(u^r) = \|\Delta_a(u^r)\| \ge d_a$. Thus $u^r \in C_r$, the subproblem is nonempty, and its compact feasible set and strongly convex objective give a unique minimizer. For $u \in C_r$, the Cauchy–Schwarz inequality

implies $\|\Delta_a(u)\| \geq d_a$. Induction proves the feasibility of all outer iterates. Comparing the subproblem minimizer with u^r gives (6.9). The objective is bounded below on compact U, so summation proves $\sum_r \|u^{r+1} - u^r\|^2 < \infty$ and hence $u^{r+1} - u^r \to 0$.

Set $h_a^r(u) = d_a - (n_a^r)^\top \Delta_a(u)$ and $G_r = \nabla F(u^{r+1}) + \kappa(u^{r+1} - u^r)$. The convex subproblem has nonnegative contact multipliers ν_a^r and a normal $v^r \in N_U(u^{r+1})$ with

$$0 = G_r - \sum_a \nu_a^r A_a^\top n_a^r + v^r, \quad \nu_a^r h_a^r(u^{r+1}) = 0. \tag{B.8}$$

Existence follows from the affine contact constraints and the polyhedral local set; the strict point also supplies the usual convex constraint qualification. Take the inner product with $w^r - u^{r+1}$. The normal inequality is $(v^r)^\top(w^r - u^{r+1}) \leq 0$, and complementarity gives

$$\sum_a \nu_a^r (h_a^r(w^r) - h_a^r(u^{r+1})) \leq -\sigma \|v^r\|_1.$$

Consequently,

$$\sigma\|v^r\|_1 \leq G_r^\top(w^r - u^{r+1}) \leq \operatorname{diam}(U)\,\|G_r\|.$$

The right side is uniformly bounded by compactness and continuity. Thus the contact multipliers are uniformly bounded; (B.8) then bounds the aggregate normal v^r as well. This does not claim uniqueness or boundedness of every possible scalar representation of the local polyhedral normal.

Choose a subsequence $u^{r_j} \to u^\star$. Vanishing increments give $u^{r_j+1} \to u^\star$. Feasibility and $d_a > 0$ give $n_a^{r_j} \to n_a^\star = \Delta_a(u^\star)/\|\Delta_a(u^\star)\|$. Pass to a further subsequence with $v^{r_j} \to v^\star$ and $v^{r_j} \to v^\star$. The normal-cone graph of fixed closed convex U is closed, so $v^\star \in N_U(u^\star)$. The limit of (B.8) is

$$0 = \nabla F(u^\star) - \sum_a \nu_a^\star A_a^\top n_a^\star + v^\star, \qquad \nu_a^\star(d_a - \|\Delta_a(u^\star)\|) = 0.$$

If $\nu_a^\star > 0$, the last equality implies $\|\Delta_a(u^\star)\| = d_a$, and $\nabla g_a(u^\star) = -2d_a A_a^\top n_a^\star$. Set $\mu_a^\star = \nu_a^\star/(2d_a)$, including zero when $\nu_a^\star = 0$. This yields

$$0 \in \nabla F(u^\star) + \sum_a \mu_a^\star \nabla g_a(u^\star) + N_U(u^\star), \quad \mu_a^\star \geq 0, \quad \mu_a^\star g_a(u^\star) = 0.$$

These are the original-disk KKT conditions, with all local normals retained. No inference from finite approximate alignment to exact KKT is used: alignment follows from the continuity of the outer update at a cluster point. □



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)