Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(12): 6797–6839.
DOI:10.3934/jimo.2026227
Received: 08 September 2026
Revised: 09 November 2026
Accepted: 16 November 2026
Published: 23 November 2026

Research article

Platform traffic support, quality improvement, and carbon emission reduction under agency and reselling modes

Xueli Xin[1], **Tianwei Chen**[1] **and Rong Xin**[2,*]

[1] School of Management, Jiangsu University, Zhenjiang, P. R. China
[2] Business School of Ludong University, Yantai, P. R. China

* **Correspondence:** Email: 3237@ldu.edu.cn.

Appendix A. Proof of Proposition 1

Proof. We consider a Stackelberg differential game, in which the manufacturer acts as the leader and the platform acts as the follower. Accordingly, the equilibrium solution is derived using the backward induction method. First, let J_e^{TR} denote the value function of the platform. The platform's Hamilton–Jacobi–Bellman (HJB) equation is given by

$$\rho J_e^{TR} = \max_{P,I}\left\{(P-W)(\alpha+\beta G-\gamma P)-\frac{\mu}{2}I^2+\frac{dJ_e^{TR}}{dG}(\tau Q+\lambda H+\varepsilon I-\delta G)\right\}. \tag{TR.1}$$

Taking the first-order derivatives of equation (TR.1) with respect to the traffic support effort strategy I and the pricing strategy P, respectively, and setting them equal to zero, we obtain the platform's reaction functions as follows:

$$P^{TR}=\frac{\alpha+\beta G+\gamma W}{2\gamma}, \tag{TR.2}$$

$$I^{TR}=\frac{\varepsilon}{\mu}\frac{dJ_e^{TR}}{dG}. \tag{TR.3}$$

Now let J_m^{TR} denote the manufacturer's value function. The manufacturer's HJB equation is given by

$$\rho J_m^{TR} = \max_{W,Q,H}\left\{(W-c)(\alpha+\beta G-\gamma P)-\frac{\xi}{2}Q^2-\frac{\eta}{2}H^2+\frac{dJ_m^{TR}}{dG}(\tau Q+\lambda H+\varepsilon I-\delta G)\right\} \tag{TR.4}$$

Substituting equations (TR.2) and (TR.3) into the HJB equation (TR.4), we obtain

$$\rho J_m^{TR} = \max_{W,Q,H}\left\{\frac{1}{2}(W-c)(\alpha+\beta G-\gamma W)-\frac{\eta}{2}H^2-\frac{\xi}{2}Q^2+\frac{dJ_m^{TR}}{dG}\left(\tau Q+\lambda H+\frac{dJ_e^{TR}}{dG}\frac{\varepsilon^2}{\mu}-\delta G\right)\right\}. \tag{TR.5}$$

By taking the first-order derivatives of equation (TR.5) with respect to the wholesale price W, the product quality improvement effort Q, and the carbon emission reduction effort H, respectively, and setting them equal to zero, we obtain the following equilibrium solutions:

$$W^{TR} = \frac{\alpha + \beta G + \gamma c}{2\gamma}, \tag{TR.6}$$

$$Q^{TR} = \frac{\tau}{\xi}\frac{dJ_m^{TR}}{dG}, \tag{TR.7}$$

$$H^{TR} = \frac{\lambda}{\eta}\frac{dJ_m^{TR}}{dG}, \tag{TR.8}$$

$$P^{TR} = \frac{3\alpha + 3\beta G + \gamma c}{4\gamma}. \tag{TR.9}$$

Subsequently, substituting equations (TR.3) and (TR.6)–(TR.9) into the HJB equations (TR.1) and (TR.4), we obtain

$$\rho J_m^{TR} = \frac{(\alpha + \beta G - \gamma c)^2}{8\gamma} + \left(\frac{\tau^2}{2\xi} + \frac{\lambda^2}{2\eta}\right)\left(\frac{dJ_m^{TR}}{dG}\right)^2 + \frac{\varepsilon^2}{\mu}\frac{dJ_m^{TR}}{dG}\frac{dJ_e^{TR}}{dG} - \delta G\frac{dJ_m^{TR}}{dG}, \tag{TR.10}$$

$$\rho J_e^{TR} = \frac{(\alpha + \beta G - \gamma c)^2}{16\gamma} + \frac{\varepsilon^2}{2\mu}\left(\frac{dJ_e^{TR}}{dG}\right)^2 + \left(\frac{\tau^2}{\xi} + \frac{\lambda^2}{\eta}\right)\frac{dJ_e^{TR}}{dG}\frac{dJ_m^{TR}}{dG} - \delta G\frac{dJ_e^{TR}}{dG}. \tag{TR.11}$$

According to equations (TR.10) and (TR.11), the value functions J_m^{TR} and J_e^{TR} can be conjectured to be quadratic functions of the state variable G, given by $J_m^{TR} = \frac{1}{2}D_1^{TR}G^2 + D_2^{TR}G + D_3^{TR}$, and $J_e^{TR} = \frac{1}{2}F_1^{TR}G^2 + F_2^{TR}G + F_3^{TR}$, where D_1^{TR}, D_2^{TR}, D_3^{TR}, F_1^{TR}, F_2^{TR}, and F_3^{TR} are undetermined constants. Subsequently, substituting J_m^{TR} and J_e^{TR} as well as their first-order derivatives $\frac{dJ_m^{TR}}{dG} = D_1^{TR}G + D_2^{TR}$ and $\frac{dJ_e^{TR}}{dG} = F_1^{TR}G + F_2^{TR}$ into equations (TR.10) and (TR.11), and equating the coefficients of like powers of G on both sides, we obtain

$$\frac{D_1^{TR}\rho}{2} = \frac{1}{8}\left(\frac{\beta^2}{\gamma} + 4D_1^{TR}\left(-2\delta + \frac{D_1^{TR}\lambda^2}{\eta} + \frac{2F_1^{TR}\varepsilon^2}{\mu} + \frac{D_1^{TR}\tau^2}{\xi}\right)\right),$$

$$D_2^{TR}\rho = \frac{\beta(\alpha - c\gamma)}{4\gamma} + \frac{D_1^{TR}F_2^{TR}\varepsilon^2}{\mu} + D_2^{TR}\left(-\delta + \frac{D_1^{TR}\lambda^2}{\eta} + \frac{F_1^{TR}\varepsilon^2}{\mu} + \frac{D_1^{TR}\tau^2}{\xi}\right),$$

$$D_3^{TR}\rho = \frac{1}{8}\left(\frac{(\alpha - c\gamma)^2}{\gamma} + 4D_2^{TR}\left(\frac{D_2^{TR}\lambda^2}{\eta} + \frac{2F_2^{TR}\varepsilon^2}{\mu} + \frac{D_2^{TR}\tau^2}{\xi}\right)\right),$$

$$\frac{F_1^{TR}\rho}{2} = \frac{1}{16}\left(\frac{\beta^2}{\gamma} + 8F_1^{TR}\left(-2\delta + \frac{2D_1^{TR}\lambda^2}{\eta} + \frac{F_1^{TR}\varepsilon^2}{\mu} + \frac{2D_1^{TR}\tau^2}{\xi}\right)\right),$$

$$F_2^{TR}\rho = \frac{\beta(\alpha - c\gamma)}{8\gamma} - F_2^{TR}\delta + \frac{(D_2^{TR}F_1^{TR} + D_1^{TR}F_2^{TR})\lambda^2}{\eta} + \frac{F_1^{TR}F_2^{TR}\varepsilon^2}{\mu} + \frac{(D_2^{TR}F_1^{TR} + D_1^{TR}F_2^{TR})\tau^2}{\xi},$$

$$F_3^{TR}\rho = \frac{1}{16}\left(\frac{(\alpha - c\gamma)^2}{\gamma} + 8F_2^{TR}\left(\frac{F_2^{TR}\varepsilon^2}{\mu} + 2D_2^{TR}\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right)\right)\right).$$

It suffices to solve the first and the fourth equations simultaneously to obtain the coefficients D_1^{TR} and F_1^{TR}. Substituting these results into the remaining four equations then allows us to further determine D_2^{TR}, D_3^{TR}, F_2^{TR}, F_3^{TR}, thereby yielding the complete solution for all the coefficients. The resulting coupled coefficient system can be solved numerically. Admissible solutions are selected to satisfy the strategy feasibility conditions and the required stability condition.

Substituting the derivatives $\frac{dJ_m^{TR}}{dG} = D_1^{TR}G + D_2^{TR}$ and $\frac{dJ_e^{TR}}{dG} = F_1^{TR}G + F_2^{TR}$ as well as the decision variables $Q^{TR} = \frac{\tau}{\xi}\frac{dJ_m^{TR}}{dG}$, $H^{TR} = \frac{\lambda}{\eta}\frac{dJ_m^{TR}}{dG}$, and $I^{TR} = \frac{\varepsilon}{\mu}\frac{dJ_e^{TR}}{dG}$, which are determined by the optimality conditions, into the dynamic equation of the manufacturer's green reputation $G(t)$, the evolution of $G(t)$ is characterized by the following first-order linear differential equation:

$$\dot{G}(t) = \Lambda_2^{TR} + \Lambda_1^{TR}G(t). \tag{TR.12}$$

where

$$\Lambda_1^{TR} = \left(-\delta + \frac{D_1^{TR}\lambda^2}{\eta} + \frac{F_1^{TR}\varepsilon^2}{\mu} + \frac{D_1^{TR}\tau^2}{\xi}\right),$$

$$\Lambda_2^{TR} = \frac{D_2^{TR}\lambda^2}{\eta} + \frac{F_2^{TR}\varepsilon^2}{\mu} + \frac{D_2^{TR}\tau^2}{\xi},$$

subject to the initial condition $G(0) = G^0$.

For $\Lambda_1^{TR} \neq 0$, solving the nonhomogeneous differential equation (TR.12), we obtain

$$G(t) = \left(G^0 + \frac{\Lambda_2^{TR}}{\Lambda_1^{TR}}\right)e^{\Lambda_1^{TR}t} - \frac{\Lambda_2^{TR}}{\Lambda_1^{TR}}.$$

When $\Lambda_1^{TR} < 0$, the equilibrium trajectory converges to the asymptotically stable steady-state level $-\Lambda_2^{TR}/\Lambda_1^{TR}$. If $\Lambda_1^{TR} = 0$, the solution is $G(t) = G^0 + \Lambda_2^{TR}t$. This completes the proof.

□

Appendix B. Proof of Proposition 2

Proof. In case TA, the feedback Stackelberg equilibrium solution is derived following the same procedure as that adopted in the proof of Proposition 1. First, the platform's value function is denoted by J_e^{TA}. The HJB equation for the platform is then given by

$$\rho J_e^{TA} = \max_I\left\{\varphi P(\alpha + \beta G - \gamma P) - \frac{\mu}{2}I^2 + \frac{dJ_e^{TA}}{dG}(\tau Q + \lambda H + \varepsilon I - \delta G)\right\}. \tag{TA.13}$$

Taking the first-order derivative of equation (TA.13) with respect to the traffic-support strategy I and setting it equal to zero, we obtain the platform's reaction function as

$$I = \frac{\varepsilon}{\mu}\frac{dJ_e^{TA}}{dG}. \tag{TA.14}$$

Next, let J_m^{TA} denote the manufacturer's value function. The HJB equation for the manufacturer is given by

$$\rho J_m^{TA} = \max_{P,Q,H}\left\{[(1-\varphi)P - c](\alpha + \beta G - \gamma P) - \frac{\xi}{2}Q^2 - \frac{\eta}{2}H^2 + \frac{dJ_m^{TA}}{dG}(\tau Q + \lambda H + \varepsilon I - \delta G)\right\}. \quad \text{(TA.15)}$$

Substituting equation (TA.14) into the HJB equation (TA.15), we obtain

$$\rho J_m^{TA} = \max_{P,Q,H}\left\{[(1-\varphi)P - c](\alpha + \beta G - \gamma P) - \frac{\eta}{2}H^2 - \frac{\xi}{2}Q^2 + \frac{dJ_m^{TA}}{dG}\left(\tau Q + \lambda H + \frac{\varepsilon^2}{\mu}\frac{dJ_e^{TA}}{dG} - \delta G\right)\right\}. \quad \text{(TA.16)}$$

Taking the first-order conditions of equation (TA.16) with respect to the product price P, the quality improvement effort Q, and the carbon emission reduction effort H, and setting them equal to zero, we derive the following equilibrium solutions:

$$P^{TA} = \frac{1}{2}\left(\frac{\alpha + \beta G}{\gamma} + \frac{c}{1-\varphi}\right), \quad \text{(TA.17)}$$

$$Q^{TA} = \frac{\tau}{\xi}\frac{dJ_m^{TA}}{dG}, \quad \text{(TA.18)}$$

$$H^{TA} = \frac{\lambda}{\eta}\frac{dJ_m^{TA}}{dG}. \quad \text{(TA.19)}$$

Finally, substituting equations (TA.14) and (TA.17)–(TA.19) into the HJB equations (TA.13) and (TA.15), we obtain

$$\rho J_e^{TA} = \frac{\varphi(\alpha + \beta G)^2}{4\gamma} - \frac{c^2\gamma\varphi}{4(1-\varphi)^2} + \frac{\varepsilon^2}{2\mu}\left(\frac{dJ_e^{TA}}{dG}\right)^2 + \left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right)\frac{dJ_e^{TA}}{dG}\frac{dJ_m^{TA}}{dG} - \delta G\frac{dJ_e^{TA}}{dG}, \quad \text{(TA.20)}$$

$$\rho J_m^{TA} = \frac{1-\varphi}{4\gamma}\left(\alpha + \beta G - \frac{c\gamma}{1-\varphi}\right)^2 + \frac{\varepsilon^2}{\mu}\frac{dJ_m^{TA}}{dG}\frac{dJ_e^{TA}}{dG} + \frac{1}{2}\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right)\left(\frac{dJ_m^{TA}}{dG}\right)^2 - \delta G\frac{dJ_m^{TA}}{dG}. \quad \text{(TA.21)}$$

According to equations (TA.20) and (TA.21), the value functions J_m^{TA} and J_e^{TA} can be conjectured to be quadratic functions of the state variable G, namely, $J_m^{TA} = \frac{1}{2}K_1^{TA}G^2 + K_2^{TA}G + K_3^{TA}$ and $J_e^{TA} = \frac{1}{2}L_1^{TA}G^2 + L_2^{TA}G + L_3^{TA}$, where K_1^{TA}, K_2^{TA}, K_3^{TA}, L_1^{TA}, L_2^{TA}, and L_3^{TA} are undetermined constants. Subsequently, substituting J_m^{TA} and J_e^{TA} as well as their first-order derivatives $\frac{dJ_m^{TA}}{dG} = K_1^{TA}G + K_2^{TA}$ and $\frac{dJ_e^{TA}}{dG} = L_1^{TA}G + L_2^{TA}$ into equations (TA.20) and (TA.21), and then equating the coefficients of like powers of G on both sides, we obtain

$$\frac{\rho K_1^{TA}}{2} = \frac{1}{4}\left(2K_1^{TA}\left(-2\delta + \frac{K_1^{TA}\lambda^2}{\eta} + \frac{2L_1^{TA}\varepsilon^2}{\mu} + \frac{K_1^{TA}\tau^2}{\xi}\right) - \frac{\beta^2(-1+\varphi)}{\gamma}\right),$$

$$\rho K_2^{TA} = \frac{K_1^{TA}L_2^{TA}\varepsilon^2}{\mu} + K_2^{TA}\left(-\delta + \frac{K_1^{TA}\lambda^2}{\eta} + \frac{L_1^{TA}\varepsilon^2}{\mu} + \frac{K_1^{TA}\tau^2}{\xi}\right) - \frac{\beta(c\gamma + \alpha(-1+\varphi))}{2\gamma},$$

$$\rho K_3^{TA} = \frac{1}{4}\left(2K_2^{TA}\left(\frac{K_2^{TA}\lambda^2}{\eta} + \frac{2L_2^{TA}\varepsilon^2}{\mu} + \frac{K_2^{TA}\tau^2}{\xi}\right) + \frac{(c\gamma + \alpha(-1+\varphi))^2}{\gamma(1-\varphi)}\right),$$

$$\frac{\rho L_1^{TA}}{2} = L_1^{TA}\left(-\delta + \frac{L_1^{TA}\varepsilon^2}{2\mu} + \frac{K_1^{TA}\lambda^2}{\eta} + \frac{K_1^{TA}\tau^2}{\xi}\right) + \frac{\beta^2\varphi}{4\gamma},$$

$$\rho L_2^{TA} = K_2^{TA}L_1^{TA}\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right) + L_2^{TA}\left(-\delta + \frac{K_1^{TA}\lambda^2}{\eta} + \frac{L_1^{TA}\varepsilon^2}{\mu} + \frac{K_1^{TA}\tau^2}{\xi}\right) + \frac{\alpha\beta\varphi}{2\gamma},$$

$$\rho L_3^{TA} = \frac{1}{4}\left[L_2^{TA}\left(\frac{2L_2^{TA}\varepsilon^2}{\mu} + 4K_2^{TA}\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right)\right) + \frac{\alpha^2\varphi}{\gamma} - \frac{c^2\gamma\varphi}{(-1+\varphi)^2}\right].$$

By jointly solving the first and the fourth equations, the coefficients K_1^{TA} and L_1^{TA} can be obtained. Substituting these results into the remaining four equations then yields K_2^{TA}, K_3^{TA}, L_2^{TA}, and L_3^{TA}, thereby providing the complete solution for all coefficients. The resulting coupled coefficient system can be solved numerically, with admissible solutions selected to satisfy strategy feasibility and the required stability condition.

Substituting the derivatives $\frac{dJ_m^{TA}}{dG} = K_1^{TA}G + K_2^{TA}$, $\frac{dJ_e^{TA}}{dG} = L_1^{TA}G + L_2^{TA}$, as well as the decision variables $Q^{TA} = \frac{\tau}{\xi}\frac{dJ_m^{TA}}{dG}$, $H^{TA} = \frac{\lambda}{\eta}\frac{dJ_m^{TA}}{dG}$, and $I^{TA} = \frac{\varepsilon}{\mu}\frac{dJ_e^{TA}}{dG}$, which are determined by the optimality conditions, into the dynamic equation of the manufacturer's green reputation $G(t)$, the evolution of $G(t)$ is characterized by the following first-order linear differential equation:

$$\dot{G}(t) = \Gamma_2^{TA} + \Gamma_1^{TA}G(t), \tag{TA.22}$$

where

$$\Gamma_1^{TA} = -\delta + \frac{K_1^{TA}\lambda^2}{\eta} + \frac{L_1^{TA}\varepsilon^2}{\mu} + \frac{K_1^{TA}\tau^2}{\xi},$$

$$\Gamma_2^{TA} = \frac{K_2^{TA}\lambda^2}{\eta} + \frac{L_2^{TA}\varepsilon^2}{\mu} + \frac{K_2^{TA}\tau^2}{\xi}.$$

subject to the initial condition $G(0) = G^0$.

For $\Gamma_1^{TA} \neq 0$, solving the nonhomogeneous differential equation (TA.22), we obtain

$$G(t) = \left(G^0 + \frac{\Gamma_2^{TA}}{\Gamma_1^{TA}}\right)e^{\Gamma_1^{TA}t} - \frac{\Gamma_2}{\Gamma_1^{TA}}.$$

When $\Gamma_1^{TA} < 0$, the equilibrium trajectory converges to the asymptotically stable steady-state level $-\Gamma_2^{TA}/\Gamma_1^{TA}$. If $\Gamma_1^{TA} = 0$, the solution is $G(t) = G^0 + \Gamma_2^{TA}t$. This completes the proof.

□

Appendix C. Proof of Proposition 3

Proof. We consider a Stackelberg differential game, in which the manufacturer acts as the leader and the platform acts as the follower. Accordingly, the equilibrium solution is derived using the backward induction method. The manufacturer's effort decisions consist only of product quality improvement

and carbon emission reduction. First, let J_e^{LR} denote the value function of the platform. The platform's Hamilton–Jacobi–Bellman (HJB) equation is given by

$$\rho J_e^{LR} = \max_P \left\{ (P - W)(\alpha + \beta G - \gamma P) + \frac{dJ_e^{LR}}{dG}(\tau Q + \lambda H - \delta G) \right\}. \tag{LR.1}$$

The first-order condition of equation (LR.1) with respect to the pricing strategy P yields the platform's reaction function as follows:

$$P^{LR} = \frac{\alpha + G\beta + W\gamma}{2\gamma}, \tag{LR.2}$$

Now let J_m^{LR} denote the manufacturer's value function. The manufacturer's HJB equation is given by

$$\rho J_m^{LR} = \max_{W,Q,H} \left\{ (W - c)(\alpha + \beta G - \gamma P) - \frac{\xi}{2}Q^2 - \frac{\eta}{2}H^2 + \frac{dJ_m^{LR}}{dG}(\tau Q + \lambda H - \delta G) \right\} \tag{LR.3}$$

Substituting equation (LR.2) into the HJB equation (LR.3), we obtain

$$\rho J_m^{LR} = \max_{W,Q,H} \left\{ \frac{1}{2}(W - c)(\alpha + \beta G - \gamma W) - \frac{\eta H^2}{2} - \frac{\xi Q^2}{2} + \frac{dJ_m^{LR}}{dG}(\tau Q + \lambda H - \delta G) \right\}. \tag{LR.4}$$

Taking the first-order derivatives of equation (LR.4) with respect to the wholesale price W, the product quality improvement effort Q, and the carbon emission reduction effort H, respectively, and setting them equal to zero, we obtain the following equilibrium solutions:

$$W^{LR} = \frac{\alpha + \beta G + \gamma c}{2\gamma}, \tag{LR.5}$$

$$Q^{LR} = \frac{\tau}{\xi}\frac{dJ_m^{LR}}{dG}, \tag{LR.6}$$

$$H^{LR} = \frac{\lambda}{\eta}\frac{dJ_m^{LR}}{dG}, \tag{LR.7}$$

$$P^{LR} = \frac{3\alpha + 3G\beta + \gamma c}{4\gamma}. \tag{LR.8}$$

Subsequently, substituting equations (LR.5)–(LR.8) into the HJB equations (LR.1) and (LR.3), we obtain

$$\rho J_m^{LR} = \frac{(\alpha + \beta G - \gamma c)^2}{8\gamma} + \left(\frac{\tau^2}{2\xi} + \frac{\lambda^2}{2\eta}\right)\left(\frac{dJ_m^{LR}}{dG}\right)^2 - \delta G\frac{dJ_m^{LR}}{dG} \tag{LR.9}$$

$$\rho J_e^{LR} = \frac{(\alpha + \beta G - \gamma c)^2}{16\gamma} + \left(\frac{\tau^2}{\xi} + \frac{\lambda^2}{\eta}\right)\frac{dJ_e^{LR}}{dG}\frac{dJ_m^{LR}}{dG} - \delta G\frac{dJ_e^{LR}}{dG} \tag{LR.10}$$

According to equations (LR.9) and (LR.10), the value functions J_m^{LR} and J_e^{LR} can be conjectured to be quadratic functions of the state variable G, given by $J_m^{LR} = \frac{1}{2}S_1^{LR}G^2 + S_2^{LR}G + S_3^{LR}$, $J_e^{LR} = \frac{1}{2}N_1^{LR}G^2 +$

$N_2^{LR}G + N_3^{LR}$, where S_1^{LR}, S_2^{LR}, S_3^{LR}, N_1^{LR}, N_2^{LR}, and N_3^{LR} are undetermined constants. Subsequently, substituting J_m^{LR} and J_e^{LR} as well as their first-order derivatives $\frac{dJ_m^{LR}}{dG} = S_1^{LR}G+S_2^{LR}$ and $\frac{dJ_e^{LR}}{dG} = N_1^{LR}G+N_2^{LR}$ into equations (LR.9) and (LR.10), and equating the coefficients of like powers on both sides, we obtain

$$\frac{S_1^{LR}\rho}{2} = \frac{1}{8}\left(\frac{\beta^2}{\gamma} + 4S_1^{LR}\left(-2\delta + \frac{S_1^{LR}\lambda^2}{\eta} + \frac{S_1^{LR}\tau^2}{\xi}\right)\right),$$

$$S_2^{LR}\rho = \frac{\beta(\alpha - c\gamma)}{4\gamma} + S_2^{LR}\left(-\delta + \frac{S_1^{LR}\lambda^2}{\eta} + \frac{S_1^{LR}\tau^2}{\xi}\right),$$

$$S_3^{LR}\rho = \frac{1}{8}\left(\frac{(\alpha - c\gamma)^2}{\gamma} + 4S_2^{LR}\left(\frac{S_2^{LR}\lambda^2}{\eta} + \frac{S_2^{LR}\tau^2}{\xi}\right)\right),$$

$$\frac{N_1^{LR}\rho}{2} = \frac{1}{16}\left(\frac{\beta^2}{\gamma} + 8N_1^{LR}\left(-2\delta + \frac{2S_1^{LR}\lambda^2}{\eta} + \frac{2S_1^{LR}\tau^2}{\xi}\right)\right),$$

$$N_2^{LR}\rho = \frac{\beta(\alpha - c\gamma)}{8\gamma} - N_2^{LR}\delta + \frac{(S_2^{LR}N_1^{LR} + S_1^{LR}N_2^{LR})\lambda^2}{\eta} + \frac{(S_2^{LR}N_1^{LR} + S_1^{LR}N_2^{LR})\tau^2}{\xi},$$

$$N_3^{LR}\rho = \frac{1}{16}\left(\frac{(\alpha - c\gamma)^2}{\gamma} + 16N_2^{LR}S_2^{LR}\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right)\right).$$

Solving the six algebraic equations above allows us to successively determine the manufacturer's coefficients S_1^{LR}, S_2^{LR}, and S_3^{LR} and the platform's coefficients N_1^{LR}, N_2^{LR}, and N_3^{LR}, thereby yielding the complete solutions for both value functions. The resulting nonlinear system can be solved numerically.

Substituting the derivative $\frac{dJ_m^{LR}}{dG} = S_1^{LR}G + S_2^{LR}$, together with the quality improvement and carbon emission reduction efforts determined by the optimality conditions, $Q^{LR} = \frac{\tau}{\xi}\frac{dJ_m^{LR}}{dG}$ and $H^{LR} = \frac{\lambda}{\eta}\frac{dJ_m^{LR}}{dG}$, into the dynamic equation of green reputation $\dot{G} = \tau Q(t) + \lambda H(t) - \delta G(t)$, with $G(0) = G^0$, we obtain the following first-order linear differential equation for $G(t)$ with respect to time t:

$$\dot{G} = \Lambda_2^{LR} + \Lambda_1^{LR}G(t) \tag{LR.11}$$

where

$$\Lambda_1^{LR} = \left(-\delta + \frac{S_1^{\mathrm{LR}}\lambda^2}{\eta} + \frac{S_1^{\mathrm{LR}}\tau^2}{\xi}\right),$$

$$\Lambda_2^{LR} = \frac{S_2^{\mathrm{LR}}\lambda^2}{\eta} + \frac{S_2^{\mathrm{LR}}\tau^2}{\xi}$$

subject to the initial condition $G(0) = G^0$.

For $\Lambda_1^{LR} \neq 0$, solving the nonhomogeneous differential equation (LR.11), we obtain

$$G(t) = (G^0 + \frac{\Lambda_2^{LR}}{\Lambda_1^{LR}})e^{\Lambda_1^{LR}t} - \frac{\Lambda_2^{LR}}{\Lambda_1^{LR}}$$

When $\Lambda_1^{LR} < 0$, the reputation trajectory converges to the asymptotically stable steady-state level $G_\infty^{LR} = -\Lambda_2^{LR}/\Lambda_1^{LR}$. If $\Lambda_1^{LR} = 0$, the trajectory is $G(t) = G^0 + \Lambda_2^{LR}t$. This completes the proof. □

Appendix D. Proof of Proposition 4

Proof. Let J_e^{LA} denote the value function of the platform. The platform's HJB equation is given by

$$\rho J_e^{LA} = \varphi P(\alpha + \beta G - \gamma P) + \frac{dJ_e^{LA}}{dG}(\tau Q + \lambda H - \delta G). \tag{LA.1}$$

Now let J_m^{LA} denote the manufacturer's value function. The manufacturer's HJB equation is given by

$$\rho J_m^{LA} = \max_{P,Q,H}\left\{[(1-\varphi)P - c](\alpha + \beta G - \gamma P) - \frac{\xi}{2}Q^2 - \frac{\eta}{2}H^2 + \frac{dJ_m^{LA}}{dG}(\tau Q + \lambda H - \delta G)\right\} \tag{LA.2}$$

Taking the first-order derivatives of equation (LA.2) with respect to the product price P, the product quality improvement effort Q, and the carbon emission reduction effort H, respectively, and setting them equal to zero, we obtain the following equilibrium solutions:

$$P^{LA} = \frac{1}{2}\left(\frac{\alpha + G\beta}{\gamma} + \frac{c}{1-\varphi}\right), \tag{LA.3}$$

$$Q^{LA} = \frac{\tau}{\xi}\frac{dJ_m^{LA}}{dG}, \tag{LA.4}$$

$$H^{LA} = \frac{\lambda}{\eta}\frac{dJ_m^{LA}}{dG}. \tag{LA.5}$$

Substituting equations (LA.3)–(LA.5) into the platform's value function equation (LA.1) and the manufacturer's HJB equation (LA.2), we obtain

$$\begin{aligned}\rho J_e^{LA} = {} & \frac{\varphi(\alpha + \beta G)^2}{4\gamma} - \frac{c^2\gamma\varphi}{4(1-\varphi)^2} \\ & + \left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right)\frac{dJ_e^{LA}}{dG}\frac{dJ_m^{LA}}{dG} - \delta G\frac{dJ_e^{LA}}{dG}.\end{aligned} \tag{LA.6}$$

$$\rho J_m^{LA} = \frac{1-\varphi}{4\gamma}\left(\alpha + G\beta - \frac{c\gamma}{1-\varphi}\right)^2 + \frac{1}{2}\left(\frac{dJ_m^{LA}}{dG}\right)^2\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right) - \delta G\frac{dJ_m^{LA}}{dG}. \tag{LA.7}$$

According to equations (LA.6) and (LA.7), the value functions J_m^{LA} and J_e^{LA} can be conjectured to be quadratic functions of the state variable G, given by $J_m^{LA} = \frac{1}{2}O_1^{LA}G^2 + O_2^{LA}G + O_3^{LA}$, $J_e^{LA} = \frac{1}{2}B_1^{LA}G^2 + B_2^{LA}G + B_3^{LA}$, where O_1^{LA}, O_2^{LA}, O_3^{LA}, B_1^{LA}, B_2^{LA}, and B_3^{LA} are undetermined constants. Subsequently, substituting J_m^{LA} and J_e^{LA} as well as their first-order derivatives $\frac{dJ_m^{LA}}{dG} = O_1^{LA}G + O_2^{LA}$ and $\frac{dJ_e^{LA}}{dG} = B_1^{LA}G + B_2^{LA}$ into the corresponding HJB equations, and equating the coefficients of like powers on both sides, we obtain

$$\frac{\rho O_1^{LA}}{2} = \frac{1}{4}\left(2O_1^{LA}\left(-2\delta + \frac{O_1^{LA}\lambda^2}{\eta} + \frac{O_1^{LA}\tau^2}{\xi}\right) - \frac{\beta^2(-1+\varphi)}{\gamma}\right),$$

$$\rho O_2^{LA} = O_2^{LA}\left(-\delta + \frac{O_1^{LA}\lambda^2}{\eta} + \frac{O_1^{LA}\tau^2}{\xi}\right) - \frac{\beta(c\gamma + \alpha(-1+\varphi))}{2\gamma},$$

$$\rho O_3^{LA} = \frac{1}{4}\left(2O_2^{LA}\left(\frac{O_2^{LA}\lambda^2}{\eta} + \frac{O_2^{LA}\tau^2}{\xi}\right) + \frac{(c\gamma + \alpha(-1+\varphi))^2}{\gamma(1-\varphi)}\right),$$

$$\frac{\rho B_1^{LA}}{2} = B_1^{LA}\left(-\delta + \frac{O_1^{LA}\lambda^2}{\eta} + \frac{O_1^{LA}\tau^2}{\xi}\right) + \frac{\beta^2\varphi}{4\gamma},$$

$$\rho B_2^{LA} = O_2^{LA}B_1^{LA}\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right) + B_2^{LA}\left(-\delta + \frac{O_1^{LA}\lambda^2}{\eta} + \frac{O_1^{LA}\tau^2}{\xi}\right) + \frac{\alpha\beta\varphi}{2\gamma},$$

$$\rho B_3^{LA} = \frac{1}{4}\left[B_2^{LA}\left(4O_2^{LA}\left(\frac{\lambda^2}{\eta} + \frac{\tau^2}{\xi}\right)\right) + \frac{\alpha^2\varphi}{\gamma} - \frac{c^2\gamma\varphi}{(-1+\varphi)^2}\right].$$

It suffices to solve the first and the fourth equations simultaneously to obtain the coefficients O_1^{LA} and B_1^{LA}. Substituting these results into the remaining four equations then yields O_2^{LA}, O_3^{LA}, B_2^{LA}, and B_3^{LA}, thereby providing the complete solution for all coefficients. The resulting nonlinear system in six unknowns can be solved numerically.

Substituting the derivative $\frac{dJ_m^{LA}}{dG} = O_1^{LA}G + O_2^{LA}$, together with the quality improvement and carbon emission reduction efforts determined by the optimality conditions, $Q^{LA} = \frac{\tau}{\xi}\frac{dJ_m^{LA}}{dG}$ and $H^{LA} = \frac{\lambda}{\eta}\frac{dJ_m^{LA}}{dG}$, into the dynamic equation of green reputation $\dot{G} = \tau Q(t) + \lambda H(t) - \delta G(t)$, with $G(0) = G^0$, we obtain the following first-order linear differential equation for $G(t)$:

$$\dot{G} = \Gamma_2^{LA} + \Gamma_1^{LA}G(t), \tag{LA.8}$$

where

$$\Gamma_1^{LA} = \left(-\delta + \frac{O_1^{\mathrm{LA}}\lambda^2}{\eta} + \frac{O_1^{\mathrm{LA}}\tau^2}{\xi}\right),$$

$$\Gamma_2^{LA} = \frac{O_2^{\mathrm{LA}}\lambda^2}{\eta} + \frac{O_2^{\mathrm{LA}}\tau^2}{\xi},$$

subject to the initial condition $G(0) = G^0$.

For $\Gamma_1^{LA} \neq 0$, solving the nonhomogeneous differential equation (LA.8), we obtain

$$G^{LA}(t) = \left(G^0 + \frac{\Gamma_2^{LA}}{\Gamma_1^{LA}}\right)e^{\Gamma_1^{LA}t} - \frac{\Gamma_2^{LA}}{\Gamma_1^{LA}}.$$

When $\Gamma_1^{LA} < 0$, the reputation trajectory converges to the asymptotically stable steady-state level

$$G_\infty^{LA} = -\frac{\Gamma_2^{LA}}{\Gamma_1^{LA}}.$$

If $\Gamma_1^{LA} = 0$, the reputation trajectory is $G^{LA}(t) = G^0 + \Gamma_2^{LA}t$. This completes the derivation of the platform's value function, the manufacturer's strategies and value function, and the reputation trajectory.

□

Appendix E. Proof of Proposition 5

Proof. In case BR, we follow the proof of Proposition 1 for case TR in exactly the same order, setting the carbon emission reduction and traffic support terms involving $\lambda, \eta, \varepsilon, \mu$ to zero while retaining the original form of all other formulas. The ratios involving these four parameters are removed together with the associated terms and are not evaluated as $0/0$. We consider a Stackelberg differential game, in which the manufacturer acts as the leader and the platform acts as the follower, and derive the equilibrium solution using the backward induction method. First, let J_e^{BR} denote the value function of the platform. The platform's Hamilton–Jacobi–Bellman (HJB) equation is given by

$$\rho J_e^{BR} = \max_P \left\{ (P - W)(\alpha + \beta G - \gamma P) + \frac{dJ_e^{BR}}{dG}(\tau Q - \delta G) \right\}. \tag{BR.1}$$

Taking the first-order derivative of equation (BR.1) with respect to the pricing strategy P and setting it equal to zero, we obtain the platform's reaction function as follows:

$$P^{BR} = \frac{\alpha + \beta G + \gamma W}{2\gamma}, \tag{BR.2}$$

Now let J_m^{BR} denote the manufacturer's value function. The manufacturer's HJB equation is given by

$$\rho J_m^{BR} = \max_{W,Q} \left\{ (W - c)(\alpha + \beta G - \gamma P) - \frac{\xi}{2} Q^2 + \frac{dJ_m^{BR}}{dG}(\tau Q - \delta G) \right\} \tag{BR.3}$$

Substituting equation (BR.2) into the HJB equation (BR.3), we obtain

$$\rho J_m^{BR} = \max_{W,Q} \left\{ \frac{1}{2}(W - c)(\alpha + \beta G - \gamma W) - \frac{\xi}{2} Q^2 + \frac{dJ_m^{BR}}{dG}(\tau Q - \delta G) \right\}. \tag{BR.4}$$

Taking the first-order derivatives of equation (BR.4) with respect to the wholesale price W and the product quality improvement effort Q, respectively, and setting them equal to zero, we obtain

$$W^{BR} = \frac{\alpha + \beta G + \gamma c}{2\gamma}, \tag{BR.5}$$

$$Q^{BR} = \frac{\tau}{\xi} \frac{dJ_m^{BR}}{dG}, \tag{BR.6}$$

$$P^{BR} = \frac{3\alpha + 3\beta G + \gamma c}{4\gamma}. \tag{BR.7}$$

Subsequently, substituting equations (BR.5), (BR.6), and (BR.7) into the HJB equations (BR.1) and (BR.3), we obtain

$$\rho J_m^{BR} = \frac{(\alpha + \beta G - \gamma c)^2}{8\gamma} + \left(\frac{\tau^2}{2\xi} \right) \left(\frac{dJ_m^{BR}}{dG} \right)^2 - \delta G \frac{dJ_m^{BR}}{dG}, \tag{BR.8}$$

$$\rho J_e^{BR} = \frac{(\alpha + \beta G - \gamma c)^2}{16\gamma} + \left(\frac{\tau^2}{\xi} \right) \frac{dJ_e^{BR}}{dG} \frac{dJ_m^{BR}}{dG} - \delta G \frac{dJ_e^{BR}}{dG}. \tag{BR.9}$$

According to equations (BR.8) and (BR.9), the two value functions can be conjectured to be quadratic functions of G, given by $J_m^{BR} = \frac{1}{2}M_1^{BR}G^2 + M_2^{BR}G + M_3^{BR}$, and $J_e^{BR} = \frac{1}{2}N_1^{BR}G^2 + N_2^{BR}G + N_3^{BR}$, where M_1^{BR}, M_2^{BR}, M_3^{BR}, N_1^{BR}, N_2^{BR}, and N_3^{BR} are undetermined constants. We retain the coefficient notation M and N used for case BR in the main text, which corresponds to D and F, respectively, in the proof for case TR. Subsequently, substituting the two value functions and their first-order derivatives $\frac{dJ_m^{BR}}{dG} = M_1^{BR}G + M_2^{BR}$ and $\frac{dJ_e^{BR}}{dG} = N_1^{BR}G + N_2^{BR}$ into equations (BR.8) and (BR.9), and equating the coefficients of like powers on both sides, we obtain

$$\frac{M_1^{BR}\rho}{2} = \frac{1}{8}\left(\frac{\beta^2}{\gamma} + 4M_1^{BR}\left(-2\delta + \frac{M_1^{BR}\tau^2}{\xi}\right)\right),$$

$$M_2^{BR}\rho = \frac{\beta(\alpha - c\gamma)}{4\gamma} + M_2^{BR}\left(-\delta + \frac{M_1^{BR}\tau^2}{\xi}\right),$$

$$M_3^{BR}\rho = \frac{1}{8}\left(\frac{(\alpha - c\gamma)^2}{\gamma} + 4M_2^{BR}\left(\frac{M_2^{BR}\tau^2}{\xi}\right)\right),$$

$$\frac{N_1^{BR}\rho}{2} = \frac{1}{16}\left(\frac{\beta^2}{\gamma} + 8N_1^{BR}\left(-2\delta + \frac{2M_1^{BR}\tau^2}{\xi}\right)\right),$$

$$N_2^{BR}\rho = \frac{\beta(\alpha - c\gamma)}{8\gamma} - N_2^{BR}\delta + \frac{(M_2^{BR}N_1^{BR} + M_1^{BR}N_2^{BR})\tau^2}{\xi},$$

$$N_3^{BR}\rho = \frac{1}{16}\left(\frac{(\alpha - c\gamma)^2}{\gamma} + 8N_2^{BR}\left(2M_2^{BR}\left(\frac{\tau^2}{\xi}\right)\right)\right).$$

By jointly solving the first and the fourth equations, the coefficients M_1^{BR} and N_1^{BR} can be obtained. Substituting these results into the remaining four equations then allows us to determine M_2^{BR}, M_3^{BR}, N_2^{BR}, and N_3^{BR}, thereby yielding all coefficients. The resulting coefficient system can be solved numerically, with admissible solutions selected to satisfy strategy feasibility and the required stability condition.

Substituting the derivatives $\frac{dJ_m^{BR}}{dG} = M_1^{BR}G + M_2^{BR}$ and $\frac{dJ_e^{BR}}{dG} = N_1^{BR}G + N_2^{BR}$ as well as the decision variable determined by the optimality conditions, $Q^{BR} = \frac{\tau}{\xi}\frac{dJ_m^{BR}}{dG}$ into the dynamic equation of the manufacturer's green reputation $G(t)$, we obtain the following first-order linear differential equation:

$$\dot{G}(t) = \Lambda_2^{BR} + \Lambda_1^{BR}G^{BR}(t). \tag{BR.10}$$

where

$$\Lambda_1^{BR} = \left(-\delta + \frac{M_1^{BR}\tau^2}{\xi}\right),$$

$$\Lambda_2^{BR} = \frac{M_2^{BR}\tau^2}{\xi},$$

subject to the initial condition $G(0) = G^0$.

For $\Lambda_1^{BR} \neq 0$, solving the nonhomogeneous differential equation (BR.10), we obtain

$$G(t) = \left(G^0 + \frac{\Lambda_2^{BR}}{\Lambda_1^{BR}}\right)e^{\Lambda_1^{BR}t} - \frac{\Lambda_2^{BR}}{\Lambda_1^{BR}}.$$

When $\Lambda_1^{BR} < 0$, the reputation trajectory converges to the asymptotically stable steady-state level $-\Lambda_2^{BR}/\Lambda_1^{BR}$. If $\Lambda_1^{BR} = 0$, the solution is $G(t) = G^0 + \Lambda_2^{BR} t$. This completes the proof. □

Appendix F. Proof of Proposition 6

Proof of Proposition 6. First, let J_e^{BA} denote the value function of the platform. The platform's HJB equation is given by

$$\rho J_e^{BA} = \left\{\varphi P(\alpha + \beta G - \gamma P) + \frac{dJ_e^{BA}}{dG}(\tau Q - \delta G)\right\}. \tag{BA.1}$$

Next, let J_m^{BA} denote the manufacturer's value function. The manufacturer's HJB equation is given by

$$\rho J_m^{BA} = \max_{P,Q}\left\{[(1-\varphi)P - c](\alpha + \beta G - \gamma P) - \frac{\xi}{2}Q^2 + \frac{dJ_m^{BA}}{dG}(\tau Q - \delta G)\right\}. \tag{BA.2}$$

The first-order conditions of equation (BA.2) with respect to the product price P and the quality improvement effort Q yield

$$P^{BA} = \frac{1}{2}\left(\frac{\alpha + \beta G}{\gamma} + \frac{c}{1-\varphi}\right), \tag{BA.3}$$

$$Q^{BA} = \frac{\tau}{\xi}\frac{dJ_m^{BA}}{dG}, \tag{BA.4}$$

Finally, substituting equations (BA.3) and (BA.4) into equations (BA.1) and (BA.2), we obtain

$$\rho J_e^{BA} = \frac{\varphi(\alpha + \beta G)^2}{4\gamma} - \frac{c^2\gamma\varphi}{4(1-\varphi)^2} + \left(\frac{\tau^2}{\xi}\right)\frac{dJ_e^{BA}}{dG}\frac{dJ_m^{BA}}{dG} - \delta G\frac{dJ_e^{BA}}{dG}, \tag{BA.5}$$

$$\rho J_m^{BA} = \frac{1-\varphi}{4\gamma}\left(\alpha + \beta G - \frac{c\gamma}{1-\varphi}\right)^2 + \frac{1}{2}\left(\frac{\tau^2}{\xi}\right)\left(\frac{dJ_m^{BA}}{dG}\right)^2 - \delta G\frac{dJ_m^{BA}}{dG}. \tag{BA.6}$$

According to equations (BA.5) and (BA.6), the value functions J_m^{BA} and J_e^{BA} can be conjectured to be quadratic functions of the state variable G, given by $J_m^{BA} = \frac{1}{2}X_1^{BA}G^2 + X_2^{BA}G + X_3^{BA}$ and $J_e^{BA} = \frac{1}{2}Y_1^{BA}G^2 + Y_2^{BA}G + Y_3^{BA}$, where X_1^{BA}, X_2^{BA}, X_3^{BA}, Y_1^{BA}, Y_2^{BA}, and Y_3^{BA} are undetermined constants. We retain the coefficient notation X and Y used for case BA in the main text, which corresponds to K and L, respectively, in the proof for case TA. The coefficient K_j^{BA} in the quality strategy in the main text is identical to X_j^{BA} here, that is, $K_j^{BA} \equiv X_j^{BA}$ $(j = 1, 2)$. Subsequently, substituting the two value functions and their first-order derivatives $\frac{dJ_m^{BA}}{dG} = X_1^{BA}G + X_2^{BA}$ and $\frac{dJ_e^{BA}}{dG} = Y_1^{BA}G + Y_2^{BA}$ into equations (BA.5) and (BA.6), and equating the coefficients of like powers on both sides, we obtain

$$\frac{\rho X_1^{BA}}{2} = \frac{1}{4}\left(2X_1^{BA}\left(-2\delta + \frac{X_1^{BA}\tau^2}{\xi}\right) - \frac{\beta^2(-1+\varphi)}{\gamma}\right),$$

$$\rho X_2^{BA} = X_2^{BA}\left(-\delta + \frac{X_1^{BA}\tau^2}{\xi}\right) - \frac{\beta(c\gamma + \alpha(-1+\varphi))}{2\gamma},$$

$$\rho X_3^{BA} = \frac{1}{4}\left(2X_2^{BA}\left(\frac{X_2^{BA}\tau^2}{\xi}\right) + \frac{(c\gamma + \alpha(-1+\varphi))^2}{\gamma(1-\varphi)}\right),$$

$$\frac{\rho Y_1^{BA}}{2} = Y_1^{BA}\left(-\delta + \frac{X_1^{BA}\tau^2}{\xi}\right) + \frac{\beta^2\varphi}{4\gamma},$$

$$\rho Y_2^{BA} = X_2^{BA}Y_1^{BA}\left(\frac{\tau^2}{\xi}\right) + Y_2^{BA}\left(-\delta + \frac{X_1^{BA}\tau^2}{\xi}\right) + \frac{\alpha\beta\varphi}{2\gamma},$$

$$\rho Y_3^{BA} = \frac{1}{4}\left[Y_2^{BA}\left(4X_2^{BA}\left(\frac{\tau^2}{\xi}\right)\right) + \frac{\alpha^2\varphi}{\gamma} - \frac{c^2\gamma\varphi}{(-1+\varphi)^2}\right].$$

By jointly solving the first and the fourth equations, the coefficients X_1^{BA} and Y_1^{BA} can be obtained. Substituting these results into the remaining four equations then allows us to determine X_2^{BA}, X_3^{BA}, Y_2^{BA}, and Y_3^{BA}, thereby yielding all coefficients. The resulting coefficient system can be solved numerically, with admissible solutions selected to satisfy strategy feasibility and the required stability condition.

Substituting the derivatives $\frac{dJ_m^{BA}}{dG} = X_1^{BA}G + X_2^{BA}$, $\frac{dJ_e^{BA}}{dG} = Y_1^{BA}G + Y_2^{BA}$, as well as the decision variable determined by the optimality conditions, $Q^{BA} = \frac{\tau}{\xi}\frac{dJ_m^{BA}}{dG}$ into the dynamic equation of the manufacturer's green reputation $G(t)$, we obtain the following first-order linear differential equation:

$$\dot{G}(t) = \Gamma_2^{BA} + \Gamma_1^{BA}G^{BA}(t), \tag{BA.7}$$

where

$$\Gamma_1^{BA} = -\delta + \frac{X_1^{BA}\tau^2}{\xi},$$

$$\Gamma_2^{BA} = \frac{X_2^{BA}\tau^2}{\xi}.$$

subject to the initial condition $G(0) = G^0$.

For $\Gamma_1^{BA} \neq 0$, solving the nonhomogeneous differential equation (BA.7), we obtain

$$G(t) = \left(G^0 + \frac{\Gamma_2^{BA}}{\Gamma_1^{BA}}\right)e^{\Gamma_1^{BA}t} - \frac{\Gamma_2^{BA}}{\Gamma_1^{BA}}.$$

When $\Gamma_1^{BA} < 0$, the reputation trajectory converges to the asymptotically stable steady-state level $-\Gamma_2^{BA}/\Gamma_1^{BA}$. If $\Gamma_1^{BA} = 0$, the solution is $G(t) = G^0 + \Gamma_2^{BA}t$. This completes the proof. □

Appendix G. Based on the derivations of Propositions 1–6, Appendix G provides a unified representation and simplification of the equilibrium equations for the models as a basis for the subsequent comparative analysis. The notation corresponds as follows:

Case j	(a_j, d_j, k_j)	(b_j, e_j, l_j)
TR	$(D_1^{TR}, D_2^{TR}, D_3^{TR})$	$(F_1^{TR}, F_2^{TR}, F_3^{TR})$
TA	$(K_1^{TA}, K_2^{TA}, K_3^{TA})$	$(L_1^{TA}, L_2^{TA}, L_3^{TA})$
LR	$(S_1^{LR}, S_2^{LR}, S_3^{LR})$	$(N_1^{LR}, N_2^{LR}, N_3^{LR})$
LA	$(O_1^{LA}, O_2^{LA}, O_3^{LA})$	$(B_1^{LA}, B_2^{LA}, B_3^{LA})$
BR	$(M_1^{BR}, M_2^{BR}, M_3^{BR})$	$(N_1^{BR}, N_2^{BR}, N_3^{BR})$
BA	$(X_1^{BA}, X_2^{BA}, X_3^{BA})$	$(Y_1^{BA}, Y_2^{BA}, Y_3^{BA})$

To simplify the expressions below, let $\kappa_B = \frac{\tau^2}{\xi}$, $\kappa_L = \kappa_T = \frac{\tau^2}{\xi} + \frac{\lambda^2}{\eta}$, $\chi_B = \chi_L = 0$, $\chi_T = \frac{\varepsilon^2}{\mu}$, $\sigma = \rho + 2\delta$, $\bar{\kappa} = \kappa_T$, $\bar{\chi} = \chi_T$, and $(q_m^R, q_e^R) = \left(\frac{\beta^2}{4\gamma}, \frac{\beta^2}{8\gamma}\right)$, $(q_m^A, q_e^A) = \left(\frac{(1-\varphi)\beta^2}{2\gamma}, \frac{\varphi\beta^2}{2\gamma}\right)$. Substituting the original equilibrium retail and wholesale prices into the profit functions yields the returns before effort costs:

$$r_m^R(g) = \frac{(\alpha + \beta g - \gamma c)^2}{8\gamma}, \qquad r_e^R(g) = \frac{(\alpha + \beta g - \gamma c)^2}{16\gamma}, \tag{G.1}$$

$$r_m^A(g) = \frac{u}{4\gamma}\left(\alpha + \beta g - \frac{\gamma c}{u}\right)^2, \qquad r_e^A(g) = \frac{1-u}{4\gamma}\left[(\alpha + \beta g)^2 - \frac{\gamma^2 c^2}{u^2}\right], \tag{G.2}$$

Here $u = 1 - \varphi$. For a fixed s, let $r_i^s(g) = \frac{1}{2}q_i^s g^2 + \ell_i^s g + n_i^s$. These returns are unchanged across T, L, B. By the optimality conditions for effort in Appendices A–F, the reduced HJB equations can be written uniformly as

$$\rho J_m = r_m^s + \tfrac{1}{2}\kappa (J_m')^2 + \chi J_m' J_e' - \delta g J_m', \tag{G.3}$$

$$\rho J_e = r_e^s + \tfrac{1}{2}\chi (J_e')^2 + \kappa J_m' J_e' - \delta g J_e'. \tag{G.4}$$

Here $\kappa = \kappa_k$ and $\chi = \chi_k$. Moving from model T to L or B removes the corresponding effort term by setting the effective coefficient χ or λ^2/η, respectively, to zero; the cost denominators themselves need not be set to zero.

For any given case j, to simplify the notation, the coefficients (a_j, d_j, k_j) and (b_j, e_j, l_j) in the table are abbreviated below as (a, d, k) and (b, e, l), respectively. Comparing the coefficients of g^2, g, and the constant terms in the value-function equations gives Comparing the coefficients of $g^2, g, 1$, the original six sets of equations are equivalent to

$$\sigma a = q_m^s + \kappa a^2 + 2\chi ab, \qquad \sigma b = q_e^s + \chi b^2 + 2\kappa ab, \tag{G.5}$$

$$\begin{pmatrix} h - z & -\chi a \\ -\kappa b & h - z \end{pmatrix}\begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \ell_m^s \\ \ell_e^s \end{pmatrix}, \qquad z = \kappa a + \chi b, \quad h = \rho + \delta, \tag{G.6}$$

$$\rho k = n_m^s + \tfrac{1}{2}\kappa d^2 + \chi de, \qquad \rho l = n_e^s + \tfrac{1}{2}\chi e^2 + \kappa de. \tag{G.7}$$

If $\omega_j = \delta - \kappa_k a_j - \chi_k b_j > 0$, the state trajectory for each case is

$$\bar{G}^j = \frac{\kappa_k d_j + \chi_k e_j}{\omega_j}, \qquad G^j(t) = \bar{G}^j + (G^0 - \bar{G}^j)e^{-\omega_j t}. \tag{G.8}$$

Here ω_j and $\bar{G}^j$ vary across cases. The drift term is zero at the steady state, and (G.3) and (G.4) give $\rho J_i^j(\bar{G}^j) = \bar{\pi}_i^j$.

Appendix G.1 Proof of Corollary 1

The model assumes that all cost parameters and effect coefficients, as well as $\rho, \delta, \alpha, \beta, \gamma, c$, are positive, and that $0 < \varphi < 1$. In addition to these assumptions, to establish the comparison in Proposition 7, we further require

$$\alpha > \frac{\gamma c}{1 - \varphi}, \tag{G.9}$$

$$\bar{\kappa} q_m^s + \bar{\chi} q_e^s < \sigma r_* - \frac{3}{2} r_*^2, \qquad r_* = \min\{\delta, \sigma/4\}. \tag{G.10}$$

Condition (G.9) ensures that, at any nonnegative level of green goodwill, equilibrium demand is positive under both reselling and agency selling. In addition, this condition ensures that the equations for the quadratic coefficients have a unique solution in the set of nonnegative coefficients $\mathcal{D}_r$ constructed below, that the remaining coefficient equations uniquely determine the other value-function coefficients, and that the corresponding green goodwill trajectories converge to their respective steady states.

Proof. When (G.9) and (G.10) hold and the three models share the same initial goodwill $G^0 \geq 0$, let the sales mode be $s \in \{R, A\}$. We compare the manufacturer's green goodwill, quality improvement effort, and carbon emission reduction effort, as well as the manufacturer's and platform's value functions, along each model's own equilibrium trajectory, and then derive the corresponding rankings at their respective steady states.

(1) Comparison of value-function coefficients.

Under condition (G.9), q_i^s, ℓ_i^s, n_i^s are all positive. By condition (G.10), we can choose $0 < r < \min\{\delta, \sigma/4\}$, such that $\bar{\kappa} q_m^s + \bar{\chi} q_e^s \leq \sigma r - \frac{3}{2} r^2$. Consider the closed set $\mathcal{D}_r = \{(a,b) : a, b \geq 0,\ \bar{\kappa} a + \bar{\chi} b \leq r\}$ and the mapping defined by the equations for the quadratic coefficients, $\mathcal{T}_{\kappa,\chi}(a,b) = \frac{1}{\sigma}\begin{pmatrix} q_m^s + \kappa a^2 + 2\chi ab \\ q_e^s + \chi b^2 + 2\kappa ab \end{pmatrix}$. For $0 < \kappa \leq \bar{\kappa}$, $0 \leq \chi \leq \bar{\chi}$, let $x = \bar{\kappa} a$ and $y = \bar{\chi} b$. When $(a,b) \in \mathcal{D}_r$, $x, y \geq 0$ and $x + y \leq r$, so

$$\bar{\kappa}\mathcal{T}_1 + \bar{\chi}\mathcal{T}_2 \leq \frac{\bar{\kappa} q_m^s + \bar{\chi} q_e^s + x^2 + y^2 + 4xy}{\sigma} \leq \frac{\bar{\kappa} q_m^s + \bar{\chi} q_e^s + \frac{3}{2} r^2}{\sigma} \leq r.$$

Here we used $x^2 + y^2 + 4xy \leq \frac{3}{2}(x+y)^2$. Thus, $\mathcal{T}_{\kappa,\chi}$ maps $\mathcal{D}_r$ into itself.

Under the weighted norm $\|(v_1, v_2)\|_w = \bar{\kappa}|v_1| + \bar{\chi}|v_2|$ the two weighted column sums of the Jacobian matrix are bounded above by $(2x + 4y)/\sigma$ and $(4x + 2y)/\sigma$, respectively. Hence its induced norm is at most $4r/\sigma < 1$. By the contraction mapping theorem, there is a unique fixed point in $\mathcal{D}_r$.

Starting the iteration from $(a^{(0)}, b^{(0)}) = (0, 0)$, because the mapping is componentwise nondecreasing in both a, b, the iterates are nonnegative and increasing, and converge to this fixed point. The mapping is also componentwise nondecreasing in κ, χ; therefore, comparing iterations from the same initial point and taking limits shows that a, b on this branch are nondecreasing in κ, χ. From $\kappa_T = \kappa_L \geq \kappa_B$, $\chi_T \geq \chi_L = \chi_B = 0$, we obtain $a_{Ts} \geq a_{Ls} \geq a_{Bs} \geq 0$, $b_{Ts} \geq b_{Ls} \geq b_{Bs} \geq 0$. At the same time, $z = \kappa a + \chi b \leq \bar{\kappa} a + \bar{\chi} b \leq r < \delta$, Thus, this branch satisfies the stability condition $\omega = \delta - z > 0$. The uniqueness established above applies only to the quadratic branch within $\mathcal{D}_r$ and does not exclude other algebraic roots or other forms of feedback solutions outside this set.

We next consider the linear coefficients. Let $C_0 = \begin{pmatrix} z & \chi a \\ \kappa b & z \end{pmatrix}$. The equations for the linear coefficients in (G.6) give $\begin{pmatrix} d \\ e \end{pmatrix} = (hI - C_0)^{-1} \begin{pmatrix} \ell_m^s \\ \ell_e^s \end{pmatrix}$. The two weighted column sums of matrix C_0 are bounded above by $z + y$ and $z + x$, respectively; hence $\|C_0\|_w \leq 2r < \sigma/2 < h$. Therefore, $(hI - C_0)^{-1} = \frac{1}{h}\sum_{n=0}^{\infty}\left(\frac{C_0}{h}\right)^n \geq 0$. Because C_0 is elementwise nondecreasing in κ, χ, the above series of nonnegative matrices is also elementwise nondecreasing in these parameters. Consequently, d, e are positive and satisfy $d_{Ts} \geq d_{Ls} \geq d_{Bs}$, $e_{Ts} \geq e_{Ls} \geq e_{Bs}$. The equations for the constant coefficients in the preceding subsection then give $k_{Ts} \geq k_{Ls} \geq k_{Bs} > 0$, $l_{Ts} \geq l_{Ls} \geq l_{Bs} > 0$.

In summary, for any given $g \geq 0$,

$$J_i^{Ts}(g) \geq J_i^{Ls}(g) \geq J_i^{Bs}(g), \qquad i \in \{m, e\}.$$

Moreover, because $a_j, b_j \geq 0$ and $d_j, e_j > 0$, each value function is strictly increasing in goodwill over the nonnegative state interval.

(2) Comparison of the manufacturer's green goodwill G.

Substituting the equilibrium quality improvement effort, carbon emission reduction effort, and traffic support for each case into the dynamics of the manufacturer's goodwill, and denoting the right-hand side of the resulting equation uniformly by $f_j(g) = \kappa_k(a_j g + d_j) + \chi_k(b_j g + e_j) - \delta g$, $j = ks$. the goodwill trajectory for each case then satisfies $\dot{G}^j(t) = f_j(G^j(t))$, $G^j(0) = G^0$.

By the ordering of the value-function coefficients established above and $\kappa_T = \kappa_L \geq \kappa_B$, $\chi_T \geq \chi_L = \chi_B = 0$, for any $g \geq 0$, we have $f_{Ts}(g) \geq f_{Ls}(g) \geq f_{Bs}(g)$. The rate-of-change function $f_j(g)$ for each case is affine in g, and $f_j(0) = \kappa_k d_j + \chi_k e_j > 0$. thus the goodwill trajectories remain nonnegative when starting from nonnegative initial goodwill. The scalar differential-equation comparison theorem and the common initial condition yield $G^{Ts}(t) \geq G^{Ls}(t) \geq G^{Bs}(t)$, $t \geq 0$.

(3) Comparison of quality effort Q and carbon emission reduction effort H.

From the above ordering of the value-function coefficients $a_{Ts} \geq a_{Ls} \geq 0$ $d_{Ts} \geq d_{Ls}$, and the ordering of the goodwill trajectories, $G^{Ts}(t) \geq G^{Ls}(t) \geq 0$, we obtain $a_{Ts}G^{Ts}(t) + d_{Ts} - a_{Ls}G^{Ls}(t) - d_{Ls} = a_{Ts}[G^{Ts}(t) - G^{Ls}(t)] + (a_{Ts} - a_{Ls})G^{Ls}(t) + (d_{Ts} - d_{Ls}) \geq 0$. Similarly, $a_{Ls}G^{Ls}(t) + d_{Ls} \geq a_{Bs}G^{Bs}(t) + d_{Bs}$. Therefore, $a_{Ts}G^{Ts}(t) + d_{Ts} \geq a_{Ls}G^{Ls}(t) + d_{Ls} \geq a_{Bs}G^{Bs}(t) + d_{Bs}$.

The equilibrium quality improvement effort is $Q^j(g) = \tau(a_j g + d_j)/\xi$, and $\tau/\xi > 0$; thus the above inequalities directly give $Q^{Ts}(G^{Ts}(t)) \geq Q^{Ls}(G^{Ls}(t)) \geq Q^{Bs}(G^{Bs}(t))$.

In the traffic support and low-carbon models, the manufacturer's equilibrium carbon emission reduction effort can be written as $H^j(g) = \frac{\lambda}{\eta}(a_j g + d_j)$, $j \in \{Ts, Ls\}$. As established above, $a_{Ts}G^{Ts}(t) + d_{Ts} \geq a_{Ls}G^{Ls}(t) + d_{Ls}$, together with $\lambda/\eta > 0$, this gives $H^{Ts}(G^{Ts}(t)) \geq H^{Ls}(G^{Ls}(t))$. This completes the proof of the rankings of quality improvement effort and carbon emission reduction effort.

(4) Comparison of manufacturer and platform value functions J.

The calculations above show that each member's value function increases with goodwill, and that, for any $g \geq 0$, $J_i^{Ts}(g) \geq J_i^{Ls}(g)$. Combining this with the ordering of goodwill trajectories, $G^{Ts}(t) \geq G^{Ls}(t)$, for $i \in \{m, e\}$ we obtain $J_i^{Ts}(G^{Ts}(t)) \geq J_i^{Ts}(G^{Ls}(t))geqJ_i^{Ls}(G^{Ls}(t))$. The first inequality uses $G^{Ts}(t) \geq G^{Ls}(t)$ and the monotonicity of J_i^{Ts} in goodwill; the second uses the ordering of value functions at a given state. Similarly, $J_i^{Ls}(G^{Ls}(t)) \geq J_i^{Ls}(G^{Bs}(t)) \geq J_i^{Bs}(G^{Bs}(t))$. Therefore, $J_i^{Ts}(G^{Ts}(t)) \geq J_i^{Ls}(G^{Ls}(t)) \geq J_i^{Bs}(G^{Bs}(t))$, $i \in \{m, e\}$.

(5) Verification of the value functions and steady-state comparisons.

The maximization problems in the HJB equations in Appendices A–F are strictly concave in each party's respective decision variables. Furthermore, $\alpha > \gamma c/(1 - \varphi)$, and the coefficient positivity established above ensures that the optimal responses are feasible.

Because each case satisfies $\omega_j > 0$, (G.8) shows that $G^j(t)$ converges to its own steady state $\bar{G}^j$ and always lies between G^0 and $\bar{G}^j$. Thus, $J_i^j(G^j(t))$ is bounded along the corresponding equilibrium trajectory. Together with $\rho > 0$, this implies $\lim_{t\to\infty} e^{-\rho t} J_i^j(G^j(t)) = 0$. Let $\pi_i^j(t)$ denote the instantaneous net profit earned by member i in case j along its equilibrium trajectory. Rewrite (G.3)–(G.4) as

$\rho J_i^j(G^j(t)) = \pi_i^j(t) + (J_i^j)'(G^j(t))\dot{G}^j(t)$. Hence, $\frac{d}{dt}\left[e^{-\rho t}J_i^j(G^j(t))\right] = -e^{-\rho t}\pi_i^j(t)$. Integrating over time and using the above limiting condition gives $J_i^j(G^0) = \int_0^\infty e^{-\rho t}\pi_i^j(t)\,dt$. Thus, the constructed value function equals the infinite-horizon discounted profit under the corresponding equilibrium strategies.

The maximization conditions in the corresponding HJB equations imply that, under the decision structure of the game, the discounted payoff from any other feasible strategy adopted by either party does not exceed $J_i^j(G^0)$. Because the derived equilibrium strategies attain this value function, the constructed value function is consistent with the corresponding equilibrium strategies.

In the above comparison inequalities for the manufacturer's green goodwill, quality improvement effort, and carbon emission reduction effort, and for both parties' value functions, let $t \to \infty$. Because the goodwill trajectory in each case converges to its own steady state, and the manufacturer's quality improvement effort, carbon emission reduction effort, and value functions are continuous in goodwill, the above ordering continues to hold at the respective steady states of the manufacturer's green goodwill in the three models. In particular, $J_i^{Ts}(\bar{G}^{Ts}) \geq J_i^{Ls}(\bar{G}^{Ls}) \geq J_i^{Bs}(\bar{G}^{Bs})$, $i \in \{m, e\}$. At the respective steady states of the manufacturer's green goodwill in the three models, $\dot{G}^j = 0$. From (G.3)–(G.4) and the definition of steady-state instantaneous net profit, we obtain $\bar{\pi}_i^j = \rho J_i^j(\bar{G}^j)$. Because the three models have the same positive discount rate ρ, the above ordering of value functions further yields $\bar{\pi}_i^{Ts} \geq \bar{\pi}_i^{Ls} \geq \bar{\pi}_i^{Bs}$, $i \in \{m, e\}$. This completes the proof of Proposition 7. □

Appendix G.2 Proof of Corollary 2

Proof. This section considers the traditional model ($k = B$) and the low-carbon model ($k = L$). In both models, the platform provides no traffic support, so $\chi_k = 0$. Substituting $\chi_k = 0$ into the coefficient equations in Appendix G, the manufacturer's quadratic coefficient a and linear coefficient d satisfy, respectively, $\kappa_k a^2 - \sigma a + q_m^s = 0$, $(h - \kappa_k a)d = \ell_m^s$. The stable root for the manufacturer's quadratic coefficient satisfies $a = \frac{\sigma - \sqrt{\sigma^2 - 4\kappa q_m^s}}{2\kappa}$, $d = \frac{\ell_m^s}{h - \kappa a}$. The assumption $B_k < v$ ensures $\kappa q_m^s < v$, so this root has $\kappa a < \delta$; the other quadratic root satisfies $\kappa a > \delta$ and does not belong to the stable branch. Using $\kappa a^2 - \sigma a + q_m^s = 0$, we have $(h - \kappa a)(\delta - \kappa a) = \delta h - \sigma\kappa a + \kappa^2 a^2 = \delta h - \kappa q_m^s$. Therefore, $\bar{G} = \frac{\kappa d}{\delta - \kappa a} = \frac{\kappa \ell_m^s}{\delta h - \kappa q_m^s}$. Substituting $q_m^R = \beta^2/(4\gamma)$, $\ell_m^R = \beta(\alpha - \gamma c)/(4\gamma)$, $q_m^A = u\beta^2/(2\gamma)$ and $\ell_m^A = \beta(u\alpha - \gamma c)/(2\gamma)$ gives (7.1). Both denominators are positive; after cross-multiplication, the sign of $\bar{G}^{kA} - \bar{G}^{kR}$ equals $2\delta h(\alpha - \gamma c) - \varphi(4\delta h\alpha - \kappa_k \beta^2 c)$ the sign of this expression, giving (7.2). Because $B_k < v$ and $C < \alpha$, we have $B_k C < v\alpha$, so the denominator of the threshold is positive, and $\varphi_{G,k}/\varphi_{\max} = v\alpha/(2v\alpha - B_k C) \in (0, 1)$. Steady-state flow balance gives $a\bar{G} + d = \delta\bar{G}/\kappa_k$; hence $\bar{Q}^{ks} = \frac{\tau\delta}{\xi\kappa_k}\bar{G}^{ks}$, $\bar{H}^{Ls} = \frac{\lambda\delta}{\eta\kappa_L}\bar{G}^{Ls}$. This proves (i).

At the steady state, the manufacturer's total effort cost is $\kappa_k(a\bar{G} + d)^2/2 = \delta^2\bar{G}^2/(2\kappa_k)$, and the platform incurs no traffic cost. Therefore, $\bar{\pi}_m^{ks} = r_m^s(\bar{G}^{ks}) - \frac{\delta^2(\bar{G}^{ks})^2}{2\kappa_k}$, $\bar{\pi}_e^{ks} = r_e^s(\bar{G}^{ks})$. Substituting the steady-state green goodwill in (7.2) into the above profit expression and simplifying gives the manufacturer's steady-state instantaneous profits under agency selling and reselling:

$$M_k(u) = \frac{\delta^2(u\alpha - C)^2(h^2 - B_k u)}{4\gamma u(v - B_k u)^2}, \quad M_k^R = \frac{\delta^2(\alpha - C)^2(h^2 - B_k/2)}{8\gamma(v - B_k/2)^2}.$$

Similarly, the platform's steady-state instantaneous profits under agency selling and reselling are

$$E_k(u) = \frac{1 - u}{4\gamma}\left[\frac{(\alpha v - B_k C)^2}{(v - B_k u)^2} - \frac{C^2}{u^2}\right], \quad E_k^R = \frac{(\alpha - C)^2 v^2}{16\gamma(v - B_k/2)^2}.$$

To prove uniqueness of the manufacturer's threshold, consider agency selling for $\frac{C}{\alpha} < u < 1$. The quantities q_m^A, ℓ_m^A, n_m^A all increase strictly with u, where $\frac{\partial n_m^A}{\partial u} = \frac{\alpha^2 - C^2/u^2}{4\gamma} > 0$. For $\frac{C}{\alpha} < u < 1$, q_m^A, ℓ_m^A, and n_m^A all increase strictly with u. Because the stable root satisfies $\kappa_k a < \delta$, we have $\frac{\partial a}{\partial q_m^A} = \frac{1}{\sigma - 2\kappa_k a} > 0$. Thus, a increases strictly with q_m^A and hence strictly with u. From $d = \frac{\ell_m^A}{h - \kappa_k a}$, $k_{kA} = \frac{n_m^A + \kappa_k d^2/2}{\rho}$, $\bar{G}^{kA} = \frac{\kappa_k d}{\delta - \kappa_k a}$, we see that d, k_{kA}, and $\bar{G}^{kA}$ also increase strictly with u. Because the value-function coefficients and steady-state green goodwill are nonnegative, $J_m^{kA}(\bar{G}^{kA})$ increases strictly with u. Using $u = 1 - \varphi$ and $M_k(1 - \varphi) = \rho J_m^{kA}(\bar{G}^{kA})$, we find that the manufacturer's steady-state instantaneous profit under agency selling decreases strictly with the commission rate.

Take $\varphi_0 = \varphi_{\max}/4$ and substitute it into the above steady-state profit formula; this gives $M_k(1 - \varphi_0) > M_k^R$. As the commission rate approaches the upper bound $\varphi_{\max}$ from below, u approaches C/α, and both ℓ_m^A and n_m^A approach zero. Because $\kappa_k q_m^A = B_k u \le B_k < v$, both $h - \kappa_k a$ and $\delta - \kappa_k a$ remain positive at this endpoint. Therefore, d, k_{kA}, and $\bar{G}^{kA}$ all approach zero, as does the manufacturer's steady-state instantaneous profit under agency selling, whereas its steady-state instantaneous profit under reselling remains positive.

By continuity and strict monotonicity of the profit difference, the commission-rate threshold in conclusion (ii) exists and is unique in $(0, \varphi_{\max})$.

Substituting $u = 1 - \varphi$ into the platform's steady-state profit expression yields

$$\begin{aligned} 4\gamma\rho[J_e^{kA}(\bar{G}^{kA}) - J_e^{kR}(\bar{G}^{kR})] &= 4\gamma[E_k(1 - \varphi) - E_k^R] \\ &= \varphi\left[\frac{(\alpha v - B_k C)^2}{[v - B_k(1 - \varphi)]^2} - \frac{C^2}{(1 - \varphi)^2}\right] - \frac{(\alpha - C)^2 v^2}{4(v - B_k/2)^2}. \end{aligned}$$

Because $4\gamma > 0$, the platform's steady-state profit under agency selling exceeds that under reselling if and only if the right-hand side above is positive, that is, if (7.4) holds.

As the commission rate approaches the lower bound 0 from within the feasible interval, in $E_k(1 - \varphi)$, the factor φ approaches zero, while the expression in square brackets remains bounded; therefore, the platform's steady-state profit under agency selling approaches zero. As the commission rate approaches the upper bound $\varphi_{\max}$, u approaches C/α, and both terms in square brackets in $E_k(u)$ approach α^2; therefore, the platform's steady-state profit under agency selling also approaches zero. Because the steady-state profit E_k^R under reselling is positive and independent of the commission rate, continuity implies that the platform prefers reselling near both ends of the feasible interval. This proves conclusion (iii). □

Appendix G.3 Proof of Corollary 3

Proof. Because $\gamma > 0$, (7.4) directly gives conclusion (i). At the initial time, substitute $G^{kR}(0) = G^{kA}(0) = G^0$ into this expression and rearrange to obtain conclusion (ii). If green goodwill under both modes converges to the respective steady states, by the continuity of equilibrium prices in green goodwill, substitute into (7.4) the respective values $\bar{G}^{kR}$ and $\bar{G}^{kA}$ to obtain conclusion (iii). □

Appendix H.1. Proof of Proposition 7

Proof. We consider a Stackelberg differential game in which the manufacturer is the leader and the platform is the follower. Accordingly, we derive the equilibrium solution by backward induction.

First, let $J_e^{TR,\theta}$ denote the platform's value function. The platform's Hamilton–Jacobi–Bellman (HJB) equation is

$$\rho J_e^{TR,\theta} = \max_{P,I}\left\{(P-W)(\alpha+\beta G+\theta H-\gamma P)-\frac{\mu I^2}{2}+\frac{dJ_e^{TR,\theta}}{dG}(\tau Q+\lambda H+\varepsilon I-\delta G)\right\}. \tag{H.1}$$

Taking the first-order derivatives of equation (H.1) with respect to the traffic support effort strategy I and the pricing strategy P, respectively, and setting them equal to zero, we obtain the platform's reaction functions as follows:

$$P^{TR,\theta} = \frac{\alpha+\beta G+\theta H+\gamma W}{2\gamma}, \tag{H.2}$$

$$I^{TR,\theta} = \frac{\varepsilon}{\mu}\frac{dJ_e^{TR,\theta}}{dG}. \tag{H.3}$$

Now let $J_m^{TR,\theta}$ denote the manufacturer's value function. The manufacturer's HJB equation is given by

$$\rho J_m^{TR,\theta} = \max_{W,Q,H}\left\{(W-c)(\alpha+\beta G+\theta H-\gamma P)-\frac{\xi Q^2}{2}-\frac{\eta H^2}{2}+\frac{dJ_m^{TR,\theta}}{dG}(\tau Q+\lambda H+\varepsilon I-\delta G)\right\}. \tag{H.4}$$

Substituting (H.2) and (H.3) into the HJB equation (H.4), we obtain

$$\rho J_m^{TR,\theta} = \max_{W,Q,H}\left\{\frac{W-c}{2}(\alpha+\beta G+\theta H-\gamma W)-\frac{\xi Q^2}{2}-\frac{\eta H^2}{2}+\frac{dJ_m^{TR,\theta}}{dG}\left(\tau Q+\lambda H+\frac{\varepsilon^2}{\mu}\frac{dJ_e^{TR,\theta}}{dG}-\delta G\right)\right\}. \tag{H.5}$$

By taking the first-order derivatives of equation (H.5) with respect to the wholesale price W, the product quality improvement effort Q, and the carbon emission reduction effort H, respectively, and setting them equal to zero, we obtain the following equilibrium solutions:

$$\begin{aligned}
W^{TR,\theta} &= \frac{\alpha+\beta G+\theta H^{TR,\theta}+\gamma c}{2\gamma},\\
Q^{TR,\theta} &= \frac{\tau}{\xi}\frac{dJ_m^{TR,\theta}}{dG},\\
H^{TR,\theta} &= \frac{\lambda}{\Delta_R}\frac{dJ_m^{TR,\theta}}{dG}+\frac{\theta(\alpha+\beta G-\gamma c)}{4\gamma\Delta_R},\\
P^{TR,\theta} &= \frac{3\alpha+3\beta G+3\theta H^{TR,\theta}+\gamma c}{4\gamma}.
\end{aligned} \tag{H.6}$$

where $\Delta_R = \eta - \frac{\theta^2}{4\gamma} > 0$. Subsequently, substituting equations (H.3) and (H.6) into the HJB equations (H.1) and (H.4), we obtain

$$\rho J_m = \frac{\eta A_R^2}{8\gamma\Delta_R}+\frac{\theta\lambda A_R}{4\gamma\Delta_R}\frac{dJ_m^{TR,\theta}}{dG}+\frac{k_R}{2}\left(\frac{dJ_m^{TR,\theta}}{dG}\right)^2+\frac{\varepsilon^2}{\mu}\frac{dJ_m^{TR,\theta}}{dG}\frac{dJ_e^{TR,\theta}}{dG}-\delta G\frac{dJ_m^{TR,\theta}}{dG}, \tag{H.7}$$

$$\rho J_e = \frac{1}{16\gamma}\left(\frac{\eta A_R+\theta\lambda\frac{dJ_m^{TR,\theta}}{dG}}{\Delta_R}\right)^2+\frac{\varepsilon^2}{2\mu}\left(\frac{dJ_e^{TR,\theta}}{dG}\right)^2+\frac{dJ_e^{TR,\theta}}{dG}\left(k_R\frac{dJ_m^{TR,\theta}}{dG}+\frac{\theta\lambda}{4\gamma\Delta_R}A_R-\delta G\right). \tag{H.8}$$

where $A_R(G) = \alpha - \gamma c + \beta G$, $k_R = \frac{\tau^2}{\xi} + \frac{\lambda^2}{\Delta_R}$.

According to equations (H.6) and (H.7), the value functions $J_m^{TR,\theta}$ and $J_e^{TR,\theta}$ can be conjectured to be quadratic functions of the state variable G, given by $J_m^{TR} = \frac{1}{2}D_1^{TR,\theta}G^2 + D_2^{TR,\theta}G + D_3^{TR,\theta}$, and $J_e^{TR} = \frac{1}{2}F_1^{TR,\theta}G^2 + F_2^{TR,\theta}G + F_3^{TR,\theta}$, where $D_1^{TR,\theta}$, $D_2^{TR,\theta}$, $D_3^{TR,\theta}$, $F_1^{TR,\theta}$, $F_2^{TR,\theta}$, and $F_3^{TR,\theta}$ are undetermined constants. Subsequently, substituting $J_m^{TR,\theta}$ and $J_e^{TR,\theta}$ as well as their first-order derivatives $\frac{dJ_m^{TR,\theta}}{dG} = D_1^{TR,\theta}G + D_2^{TR,\theta}$ and $\frac{dJ_e^{TR,\theta}}{dG} = F_1^{TR,\theta}G + F_2^{TR,\theta}$ into equations (H.6) and (H.7), and equating the coefficients of like powers of G on both sides, we obtain

$$
\begin{aligned}
\frac{\rho D_1^{TR,\theta}}{2} &= \frac{\eta\beta^2}{8\gamma\Delta_R} + \frac{\theta\lambda\beta D_1^{TR,\theta}}{4\gamma\Delta_R} + \frac{k_R(D_1^{TR,\theta})^2}{2} + \frac{\varepsilon^2}{\mu}D_1^{TR,\theta}F_1^{TR,\theta} - \delta D_1^{TR,\theta},\\
\rho D_2^{TR,\theta} &= \frac{\eta\beta(\alpha-\gamma c)}{4\gamma\Delta_R} + \frac{\theta\lambda}{4\gamma\Delta_R}\left[\beta D_2^{TR,\theta} + (\alpha-\gamma c)D_1^{TR,\theta}\right] + k_R D_1^{TR,\theta}D_2^{TR,\theta}\\
&\quad + \frac{\varepsilon^2}{\mu}\left(D_1^{TR,\theta}F_2^{TR,\theta} + D_2^{TR,\theta}F_1^{TR,\theta}\right) - \delta D_2^{TR,\theta},\\
\rho D_3^{TR,\theta} &= \frac{\eta(\alpha-\gamma c)^2}{8\gamma\Delta_R} + \frac{\theta\lambda(\alpha-\gamma c)D_2^{TR,\theta}}{4\gamma\Delta_R} + \frac{k_R(D_2^{TR,\theta})^2}{2} + \frac{\varepsilon^2}{\mu}D_2^{TR,\theta}F_2^{TR,\theta},\\
\frac{\rho F_1^{TR,\theta}}{2} &= \frac{(\eta\beta + \theta\lambda D_1^{TR,\theta})^2}{16\gamma\Delta_R^2} + \frac{\varepsilon^2}{2\mu}(F_1^{TR,\theta})^2 + F_1^{TR,\theta}\left(k_R D_1^{TR,\theta} + \frac{\theta\lambda\beta}{4\gamma\Delta_R} - \delta\right),\\
\rho F_2^{TR,\theta} &= \frac{(\eta\beta + \theta\lambda D_1^{TR,\theta})[\eta(\alpha-\gamma c) + \theta\lambda D_2^{TR,\theta}]}{8\gamma\Delta_R^2} + \frac{\varepsilon^2}{\mu}F_1^{TR,\theta}F_2^{TR,\theta} + F_1^{TR,\theta}\left(k_R D_2^{TR,\theta} + \frac{\theta\lambda(\alpha-\gamma c)}{4\gamma\Delta_R}\right)\\
&\quad + F_2^{TR,\theta}\left(k_R D_1^{TR,\theta} + \frac{\theta\lambda\beta}{4\gamma\Delta_R} - \delta\right),\\
\rho F_3^{TR,\theta} &= \frac{[\eta(\alpha-\gamma c) + \theta\lambda D_2^{TR,\theta}]^2}{16\gamma\Delta_R^2} + \frac{\varepsilon^2}{2\mu}(F_2^{TR,\theta})^2 + F_2^{TR,\theta}\left(k_R D_2^{TR,\theta} + \frac{\theta\lambda(\alpha-\gamma c)}{4\gamma\Delta_R}\right).
\end{aligned}
$$

It suffices to solve the first and the fourth equations simultaneously to obtain the coefficients $D_1^{TR,\theta}$ and $F_1^{TR,\theta}$. Substituting these results into the remaining four equations then allows us to further determine $D_2^{TR,\theta}$, $D_3^{TR,\theta}$, $F_2^{TR,\theta}$, $F_3^{TR,\theta}$, thereby yielding the complete solution for all the coefficients. The resulting coupled coefficient system can be solved numerically. Admissible solutions are selected to satisfy the strategy feasibility conditions and the required stability condition.

Substituting the derivatives $\frac{dJ_m^{TR,\theta}}{dG} = D_1^{TR,\theta}G + D_2^{TR,\theta}$ and $\frac{dJ_e^{TR,\theta}}{dG} = F_1^{TR,\theta}G + F_2^{TR,\theta}$ as well as the decision variables $Q^{TR,\theta} = \frac{\tau}{\xi}\frac{dJ_m^{TR,\theta}}{dG}$, $H^{TR,\theta} = \frac{\lambda}{\Delta_R}\frac{dJ_m^{TR,\theta}}{dG} + \frac{\theta}{4\gamma\Delta_R}(\alpha + \beta G - \gamma c)$, and $I^{TR,\theta} = \frac{\varepsilon}{\mu}\frac{dJ_e^{TR,\theta}}{dG}$, which are determined by the optimality conditions, into the dynamic equation of the manufacturer's green reputation $G(t)$, the evolution of $G(t)$ is characterized by the following first-order linear differential equation:

$$\dot{G} = \Lambda_2^{TR,\theta} + \Lambda_1^{TR,\theta}G, \tag{H.9}$$

where

$$\Lambda_1^{TR,\theta} = k_R D_1 + \frac{\varepsilon^2}{\mu} F_1 + \frac{\theta\lambda\beta}{4\gamma\Delta_R} - \delta,$$
$$\Lambda_2^{TR,\theta} = k_R D_2 + \frac{\varepsilon^2}{\mu} F_2 + \frac{\theta\lambda(\alpha - \gamma c)}{4\gamma\Delta_R}.$$

subject to the initial condition $G(0) = G^0$.

For $\Lambda_1^{TR,\theta} \neq 0$, solving the nonhomogeneous differential equation (H.8), we obtain

$$G^{TR,\theta}(t) = \left(G^0 + \frac{\Lambda_2^{TR,\theta}}{\Lambda_1^{TR,\theta}}\right) e^{\Lambda_1^{TR,\theta} t} - \frac{\Lambda_2^{TR,\theta}}{\Lambda_1^{TR,\theta}}.$$

When $\Lambda_1^{TR,\theta} < 0$, the equilibrium trajectory converges to the asymptotically stable steady-state level $-\Lambda_2^{TR,\theta}/\Lambda_1^{TR,\theta}$. If $\Lambda_1^{TR,\theta} = 0$, the solution is $G(t) = G^0 + \Lambda_2^{TR,\theta} t$. This completes the proof. □

Appendix H.2. Proof of Proposition 8

Proof. First, let $J_e^{TA,\theta}$ denote the platform's value function. The platform's HJB equation is given by

$$\rho J_e^{TA,\theta} = \max_I \left\{\varphi P(\alpha + \beta G + \theta H - \gamma P) - \frac{\mu I^2}{2} + \frac{dJ_e^{TA,\theta}}{dG}(\tau Q + \lambda H + \varepsilon I - \delta G)\right\}. \tag{H.10}$$

Taking the first-order derivative of equation (H.9) with respect to the traffic-support strategy I and setting it equal to zero, we obtain the platform's reaction function as

$$I = \frac{\varepsilon}{\mu}\frac{dJ_e^{TA,\theta}}{dG}. \tag{H.11}$$

Next, let $J_m^{TA,\theta}$ denote the manufacturer's value function. The manufacturer's HJB equation is given by

$$\rho J_m^{TA,\theta} = \max_{P,Q,H} \left\{[(1-\varphi)P - c](\alpha + \beta G + \theta H - \gamma P) - \frac{\xi Q^2}{2} - \frac{\eta H^2}{2} + \frac{dJ_m^{TA,\theta}}{dG}(\tau Q + \lambda H + \varepsilon I - \delta G)\right\}. \tag{H.12}$$

Substituting (H.11) into the HJB equation (H.12), we obtain

$$\rho J_m^{TA,\theta} = \max_{P,Q,H} \left\{[(1-\varphi)P - c](\alpha + \beta G + \theta H - \gamma P) - \frac{\xi Q^2}{2} - \frac{\eta H^2}{2} + \frac{dJ_m^{TA,\theta}}{dG}\left(\tau Q + \lambda H + \frac{\varepsilon^2}{\mu}\frac{dJ_e^{TA,\theta}}{dG} - \delta G\right)\right\}. \tag{H.13}$$

Taking the first-order conditions of equation (H.12) with respect to the product price P, the quality improvement effort Q, and the carbon emission reduction effort H, and setting them equal to zero, we derive the following equilibrium solutions:

$$P^{TA,\theta} = \frac{\eta(\alpha + \beta G) + \theta\lambda\frac{dJ_m^{TA,\theta}}{dG} - \theta^2 c + \frac{\eta\gamma c}{1-\varphi}}{2\gamma\eta - \theta^2(1-\varphi)}, \tag{H.14}$$

$$Q^{TA,\theta} = \frac{\tau}{\xi}\frac{dJ_m^{TA,\theta}}{dG}, \tag{H.15}$$

$$H^{TA,\theta} = \frac{2\gamma\lambda\frac{dJ_m^{TA,\theta}}{dG} + \theta(1-\varphi)(\alpha+\beta G) - \gamma\theta c}{2\gamma\eta - \theta^2(1-\varphi)}. \tag{H.16}$$

Finally, substituting equations (H.11) and (H.14)–(H.16) into the HJB equations (H.10) and (H.12), we obtain

$$\rho J_e^{TA,\theta} = \frac{\varphi Z_A^2}{4\gamma} + \frac{\varphi c Z_A}{2(1-\varphi)} + \frac{E}{2}(\frac{dJ_e^{TA,\theta}}{dG})^2 + \frac{dJ_e^{TA,\theta}}{dG}\left(k_A\frac{dJ_m^{TA,\theta}}{dG} + \frac{(1-\varphi)\theta\lambda}{2\gamma\Delta_A}A_A - \delta G\right), \tag{H.17}$$

$$\rho J_m^{TA,\theta} = \frac{(1-\varphi)\eta A_A^2}{4\gamma\Delta_A} + \frac{(1-\varphi)\theta\lambda A_A}{2\gamma\Delta_A}\frac{dJ_m^{TA,\theta}}{dG} + \frac{k_A}{2}(\frac{dJ_m^{TA,\theta}}{dG})^2 + E\frac{dJ_m^{TA,\theta}}{dG}\frac{dJ_e^{TA,\theta}}{dG} - \delta G\frac{dJ_m^{TA,\theta}}{dG}. \tag{H.18}$$

where $\bar{A}_A = \alpha - \frac{\gamma c}{1-\varphi}$, $A_A(G) = \bar{A}_A + \beta G$, $k_A = \frac{\tau^2}{\xi} + \frac{\lambda^2}{\Delta_A}$, $E = \frac{\varepsilon^2}{\mu}$, $\Delta_A \equiv \eta - (1-\varphi)\theta^2/(2\gamma) > 0$.

According to equations (H.17) and (H.18), the value functions $J_m^{TA,\theta}$ and $J_e^{TA,\theta}$ can be conjectured to be quadratic functions of the state variable G, namely, $J_m^{TA,\theta} = \frac{1}{2}K_1^{TA,\theta}G^2 + K_2^{TA,\theta}G + K_3^{TA,\theta}$ and $J_e^{TA,\theta} = \frac{1}{2}L_1^{TA,\theta}G^2 + L_2^{TA,\theta}G + L_3^{TA,\theta}$, where $K_1^{TA,\theta}$, $K_2^{TA,\theta}$, $K_3^{TA,\theta}$, $L_1^{TA,\theta}$, $L_2^{TA,\theta}$, and $L_3^{TA,\theta}$ are undetermined constants. Subsequently, substituting $J_m^{TA,\theta}$ and $J_e^{TA,\theta}$ as well as their first-order derivatives $\frac{dJ_m^{TA,\theta}}{dG} = K_1^{TA,\theta}G + K_2^{TA,\theta}$ and $\frac{dJ_e^{TA,\theta}}{dG} = L_1^{TA,\theta}G + L_2^{TA,\theta}$ into equations (G.17) and (G.18), and then equating the coefficients of like powers of G on both sides, we obtain

$$\begin{aligned}
\frac{\rho K_1}{2} &= \frac{(1-\varphi)\eta\beta^2}{4\gamma\Delta_A} + \frac{(1-\varphi)\theta\lambda\beta K_1}{2\gamma\Delta_A} + \frac{k_A K_1^2}{2} + EK_1L_1 - \delta K_1,\\
\rho K_2 &= \frac{(1-\varphi)\eta\beta\bar{A}_A}{2\gamma\Delta_A} + \frac{(1-\varphi)\theta\lambda}{2\gamma\Delta_A}(\beta K_2 + \bar{A}_A K_1) + k_A K_1 K_2 + E(K_1L_2 + K_2L_1) - \delta K_2,\\
\rho K_3 &= \frac{(1-\varphi)\eta\bar{A}_A^2}{4\gamma\Delta_A} + \frac{(1-\varphi)\theta\lambda\bar{A}_A K_2}{2\gamma\Delta_A} + \frac{k_A K_2^2}{2} + EK_2L_2,\\
\frac{\rho L_1}{2} &= \frac{\varphi z_{1A}^2}{4\gamma} + \frac{EL_1^2}{2} + L_1\left(k_A K_1 + \frac{(1-\varphi)\theta\lambda\beta}{2\gamma\Delta_A} - \delta\right),\\
\rho L_2 &= \frac{\varphi z_{1A}z_{0A}}{2\gamma} + \frac{\varphi c z_{1A}}{2(1-\varphi)} + EL_1L_2 + L_1\left(k_A K_2 + \frac{(1-\varphi)\theta\lambda\bar{A}_A}{2\gamma\Delta_A}\right) + L_2\left(k_A K_1 + \frac{(1-\varphi)\theta\lambda\beta}{2\gamma\Delta_A} - \delta\right),\\
\rho L_3 &= \frac{\varphi z_{0A}^2}{4\gamma} + \frac{\varphi c z_{0A}}{2(1-\varphi)} + \frac{EL_2^2}{2} + L_2\left(k_A K_2 + \frac{(1-\varphi)\theta\lambda\bar{A}_A}{2\gamma\Delta_A}\right).
\end{aligned}$$

By jointly solving the first and the fourth equations, the coefficients $K_1^{TA,\theta}$ and $L_1^{TA,\theta}$ can be obtained. Substituting these results into the remaining four equations then yields $K_2^{TA,\theta}$, $K_3^{TA,\theta}$, $L_2^{TA,\theta}$, and $L_3^{TA,\theta}$, thereby providing the complete solution for all coefficients. The resulting coupled coefficient system can be solved numerically, with admissible solutions selected to satisfy strategy feasibility and the required stability condition.

Substituting the derivatives $\frac{dJ_m^{TA,\theta}}{dG} = K_1^{TA,\theta}G + K_2^{TA,\theta}$, $\frac{dJ_e^{TA,\theta}}{dG} = L_1^{TA,\theta}G + L_2^{TA,\theta}$, as well as the decision variables $Q^{TA,\theta} = \frac{\tau}{\xi}\frac{dJ_m^{TA,\theta}}{dG}$, $H^{TA,\theta} = \frac{\lambda}{\eta}\frac{dJ_m^{TA,\theta}}{dG}$, and $I^{TA,\theta} = \frac{\varepsilon}{\mu}\frac{dJ_e^{TA,\theta}}{dG}$, which are determined by the optimality

conditions, into the dynamic equation of the manufacturer's green reputation $G(t)$, the evolution of $G(t)$ is characterized by the following first-order linear differential equation:

$$\dot{G} = \Gamma_2^{TA,\theta} + \Gamma_1^{TA,\theta} G, \tag{H.19}$$

where

$$\Gamma_1^{TA,\theta} = k_A K_1 + EL_1 + \frac{(1-\varphi)\theta\lambda\beta}{2\gamma\Delta_A} - \delta,$$
$$\Gamma_2^{TA,\theta} = k_A K_2 + EL_2 + \frac{(1-\varphi)\theta\lambda\bar{A}_A}{2\gamma\Delta_A}.$$

subject to the initial condition $G(0) = G^0$.

For $\Gamma_1^{TA,\theta} \neq 0$, solving the nonhomogeneous differential equation (G.19), we obtain

$$G^{TA,\theta}(t) = \left(G^0 + \frac{\Gamma_2^{TA,\theta}}{\Gamma_1^{TA,\theta}}\right) e^{\Gamma_1^{TA,\theta} t} - \frac{\Gamma_2^{TA,\theta}}{\Gamma_1^{TA,\theta}}.$$

When $\Gamma_1^{TA,\theta} < 0$, the equilibrium trajectory converges to the asymptotically stable steady-state level $-\Gamma_2^{TA,\theta}/\Gamma_1^{TA,\theta}$. If $\Gamma_1^{TA,\theta} = 0$, the solution is $G(t) = G^0 + \Gamma_2^{TA,\theta} t$. This completes the proof. □

Appendix H.3. Proof of Proposition 9

Proof. In the traffic support model, the first-order optimality conditions for quality improvement effort and carbon emission reduction effort are, respectively, $Q^j = \frac{\tau}{\xi}\frac{dJ_m^j}{dG}$, $H^j = \frac{\lambda}{\eta}\frac{dJ_m^j}{dG}$.

It follows that the ratio of the two efforts is $\frac{H^j}{Q^j} = R^0 = \frac{\lambda\xi}{\eta\tau}$, $j \in \{TR, TA\}$.

In the extended model, define $v_j^\theta \equiv \frac{dJ_m^{j,\theta}}{dG}$, where $j \in \{TR, TA\}$.

Let $D^{TR,\theta}(G)$ and $D^{TA,\theta}(G)$ denote the equilibrium demand at state G in the two extended cases, respectively. From the equilibrium pricing formulas in Propositions 7 and 8, we obtain

$$M_{TR}^\theta(G) = \frac{W^{TR,\theta}(G) - c}{2} = \frac{D^{TR,\theta}(G)}{\gamma} > 0,$$
$$M_{TA}^\theta(G) = (1-\varphi)P^{TA,\theta}(G) - c = \frac{(1-\varphi)D^{TA,\theta}(G)}{\gamma} > 0.$$

where the strict inequality follows from the positive-demand conditions for the two extended cases and $\gamma > 0$ and $1 - \varphi > 0$.

Under reselling, the first-order optimality conditions for quality improvement effort and carbon emission reduction effort are $Q^{TR,\theta} = \frac{\tau}{\xi} v_{TR}^\theta$, and $H^{TR,\theta} = \frac{\lambda}{\eta} v_{TR}^\theta + \frac{\theta}{2\eta}(W^{TR,\theta} - c)$. It follows that the ratio of the two efforts is

$$\frac{H^{TR,\theta}}{Q^{TR,\theta}} - \frac{\lambda\xi}{\eta\tau} = \frac{\xi\theta(W^{TR,\theta} - c)}{2\eta\tau v_{TR}^\theta} > 0. \tag{H.19}$$

Similarly, under agency selling, the first-order optimality conditions for quality improvement effort and carbon emission reduction effort give $Q^{TA,\theta} = \frac{\tau}{\xi} v_{TA}^\theta$, and $H^{TA,\theta} = \frac{\lambda}{\eta} v_{TA}^\theta + \frac{\theta}{\eta}\left[(1-\varphi)P^{TA,\theta} - c\right]$. It follows that the ratio of the two efforts is

$$\frac{H^{TA,\theta}}{Q^{TA,\theta}} - \frac{\lambda\xi}{\eta\tau} = \frac{\xi\theta\left[(1-\varphi)P^{TA,\theta} - c\right]}{\eta\tau v_{TA}^\theta} > 0. \tag{H.20}$$

Equations (H.19) and (H.20) show that, under both reselling and agency selling, the ratio of carbon emission reduction effort to quality improvement effort in the extended model is higher than the corresponding ratio in the traffic support model, namely,

$$R_j^\theta \equiv \frac{H^{j,\theta}}{Q^{j,\theta}} > R_j^0 = \frac{\lambda\xi}{\eta\tau}, \qquad j \in \{TR, TA\}.$$

We further examine how this change in the effort ratio affects the structure of the manufacturer's effort costs. In the traffic support model, the costs of quality improvement effort and carbon emission reduction effort are, respectively, $C_Q = \frac{\xi Q^2}{2}$ and $C_H = \frac{\eta H^2}{2}$. Thus, the share of the manufacturer's total effort cost spent on carbon emission reduction effort is

$$S_H = \frac{C_H}{C_Q + C_H} = \frac{\eta H^2}{\xi Q^2 + \eta H^2}.$$

Let $R = H/Q > 0$. Dividing both the numerator and denominator by Q^2 gives

$$S_H(R) = \frac{\eta R^2}{\xi + \eta R^2}, \qquad \frac{dS_H(R)}{dR} = \frac{2\eta\xi R}{(\xi + \eta R^2)^2} > 0. \tag{H.21}$$

It follows that the cost share of carbon emission reduction effort increases strictly with the ratio of carbon emission reduction effort to quality improvement effort.

Because $R_j^\theta > R_j^0 > 0$ and $S_H(R)$ is strictly increasing in R, we have

$$S_H^{j,\theta}(G) = S_H(R_j^\theta) > S_H(R_j^0).$$

It should be noted that, in the traffic support model, the effort ratio under both reselling and agency selling is R_j^0; hence $S_H^{j,0}(G) = S_H(R_j^0)$.

Furthermore, substituting $R_j^0 = \frac{\lambda\xi}{\eta\tau}$ into the cost-share function gives

$$S_H^{j,\theta}(G) > S_H^{j,0}(G) = \frac{\lambda^2\xi}{\eta\tau^2 + \lambda^2\xi}, \qquad j \in \{TR, TA\}. \tag{H.22}$$

Therefore, the additional immediate demand effect increases the share of the manufacturer's total effort cost spent on carbon emission reduction effort under both reselling and agency selling. This completes the proof. □

AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)