



Research article

Mean-constrained portfolio optimization under lambda expected shortfall: A loss-level decomposition approach

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This document accompanies the main article. References to numbered sections, equations, and propositions in the main article are identified explicitly. The supplementary procedures and tables are presented below.

A. Supplementary materials

A.1. Counterexample for the regularized crossing form

Proof of Proposition 4 of the main article. Consider two assets and two equally likely return scenarios

$$\mathbf{r}_1 = (-1/2, 0)^\top, \quad \mathbf{r}_2 = (-1, -1/2)^\top.$$

Let $\mathbf{w} = (p, 1 - p)^\top$, $0 \leq p \leq 1$. With $\hat{\boldsymbol{\mu}} = (-3/4, -1/4)^\top$ and $\eta = -3/4$, $\mathbf{w}^\top \hat{\boldsymbol{\mu}} = -1/4 - p/2 \geq -3/4$; thus every long-only, fully invested portfolio is feasible. Moreover, $L(\mathbf{w}) = (p/2, (p + 1)/2)^\top$, giving the empirical loss-level interval $[0, 1]$.

Set $\gamma = 1/2$ and $\Lambda(x) = 99/(99 + 11^x)$, so that $\Lambda(0) = 0.99$ and $\Lambda(1) = 0.90$. Since $\Lambda(x) > 1/2$ on $[0, 1]$, $\text{ES}_{\Lambda(x)}(L(\mathbf{w})) = (p + 1)/2$ throughout this interval. The resulting outer value function is

$$V_{1/2}(x) = \min_{0 \leq p \leq 1} \left\{ \left(\frac{p + 1}{2} \vee x \right) + \frac{1}{4}(p^2 + (1 - p)^2) \right\}.$$

For $x \leq 1/2$, the maximum term is $(p + 1)/2$. For $x \geq 1/2$, split the minimization at $p = 2x - 1$. The

resulting minimizer and value are

$$\begin{array}{c|ccc} x & [0, \frac{1}{2}] & [\frac{1}{2}, \frac{3}{4}] & [\frac{3}{4}, 1] \\ p^*(x) & 0 & 2x - 1 & \frac{1}{2} \\ V_{1/2}(x) & \frac{3}{4} & \frac{3}{4} + \frac{1}{2}(2x - 1)^2 & x + \frac{1}{8}. \end{array}$$

Thus, $V_{1/2}(x) > x$ on $[0, 1]$, while $V_{1/2}$ is strictly increasing on $[1/2, 1]$. If $V_{1/2}(x) = \widetilde{G}(x) \vee x$ for some nonincreasing $\widetilde{G}$, then $V_{1/2}(x) > x$ would imply $\widetilde{G}(x) = V_{1/2}(x)$ throughout $[0, 1]$, contradicting this strict increase.

For comparison, augmenting the fixed-confidence value function by the ℓ_2 term gives

$$G_{1/2}^{\text{reg}}(x) = \min_{0 \leq p \leq 1} \left\{ \text{ES}_{\Lambda(x)}(L(\mathbf{w})) + \frac{1}{4}(p^2 + (1-p)^2) \right\} = \frac{3}{4}, \quad x \in [0, 1].$$

Its crossing with x occurs at $x = 3/4$, whereas the true outer minimum $3/4$ is attained for every $x \in [0, 1/2]$, and $V_{1/2}(3/4) = 7/8$. Therefore, replacing the unregularized fixed-confidence value function by $G_{1/2}^{\text{reg}}$ in the crossing rule does not recover an outer minimizer in this instance. $\square$

A.2. Monotone crossing bisection procedure for MAES

For MAES, the scalar outer problem is

$$\min_{x \in [x_L, x_U]} H_\eta(x), \quad H_\eta(x) = G_\eta(x) \vee x, \quad G_\eta(x) = \min_{\mathbf{w} \in \mathcal{W}_\eta} \text{ES}_{\Lambda(x)}(L(\mathbf{w})). \quad (\text{A.1})$$

For fixed x , $G_\eta(x)$ is obtained from the corresponding fixed-confidence ES linear program over $\mathcal{W}_\eta$. Proposition 2 of the main article implies that $D_\eta(x) = G_\eta(x) - x$ is strictly decreasing; its sign therefore justifies both the endpoint tests and the bracket updates in Algorithm 1.

Algorithm 1 Monotone crossing bisection procedure for MAES.

Require: Return scenarios $\{\mathbf{r}_t\}_{t=1}^T$, sample mean $\hat{\mu}$, feasible set $\mathcal{W}_\eta$, confidence function $\Lambda(\cdot)$, loss-level interval $[x_L, x_U]$, and tolerance $\varepsilon_x > 0$.

Ensure: Loss level $\hat{x}$, portfolio $\hat{\mathbf{w}}$, and value $\hat{H}$.

- 1: Let $\text{SolveFixedES}(x)$ return $(G_\eta(x), \mathbf{w}(x), D_\eta(x))$, where $D_\eta(x) = G_\eta(x) - x$.
 - 2: Evaluate $(G_L, \mathbf{w}_L, D_L) \leftarrow \text{SolveFixedES}(x_L)$ and $(G_U, \mathbf{w}_U, D_U) \leftarrow \text{SolveFixedES}(x_U)$.
 - 3: **if** $D_L \leq 0$ **then**
 - 4: **return** $(x_L, \mathbf{w}_L, G_L \vee x_L)$.
 - 5: **end if**
 - 6: **if** $D_U \geq 0$ **then**
 - 7: **return** $(x_U, \mathbf{w}_U, G_U \vee x_U)$.
 - 8: **end if**
 - 9: Set $\underline{x} \leftarrow x_L$ and $\bar{x} \leftarrow x_U$.
 - 10: **while** $\bar{x} - \underline{x} > \varepsilon_x$ **do**
 - 11: Set $m \leftarrow (\underline{x} + \bar{x})/2$ and evaluate $(G_m, \mathbf{w}_m, D_m) \leftarrow \text{SolveFixedES}(m)$.
 - 12: **if** $D_m > 0$ **then**
 - 13: $\underline{x} \leftarrow m$.
 - 14: **else**
 - 15: $\bar{x} \leftarrow m$.
 - 16: **end if**
 - 17: **end while**
 - 18: Set $C \leftarrow \{\underline{x}, (\underline{x} + \bar{x})/2, \bar{x}\}$, and evaluate any previously unevaluated $x \in C$.
 - 19: Select $\hat{x} \in \arg \min_{x \in C} \{G_\eta(x) \vee x\}$.
 - 20: Set $\hat{\mathbf{w}} \leftarrow \mathbf{w}(\hat{x})$ and $\hat{H} \leftarrow G_\eta(\hat{x}) \vee \hat{x}$.
 - 21: **return** $(\hat{x}, \hat{\mathbf{w}}, \hat{H})$.
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In the interior case, each iteration halves the bracket. BIS therefore requires

$$O\left(\log \frac{x_U - x_L}{\varepsilon_x}\right)$$

fixed-loss-level ES evaluations, apart from the endpoint and final candidate evaluations. The BIS stopping diagnostic is the final bracket width, which measures loss-level localization accuracy rather than an objective-value gap.

The numerical implementation solves the equivalent fixed-level epigraph linear program (LP) for $H_\eta(x)$, with $z \geq x$ and $z \geq \text{ES}_{\Lambda(x)}(L(\mathbf{w}))$. If $z > x$, the returned ES value equals $G_\eta(x)$. If $z = x$, it need not equal $G_\eta(x)$, but both values are no greater than x . The LP therefore provides the sign classification needed for the bracket update. In the numerical experiments, this comparison uses a tolerance for values near zero and an iteration cap, as specified in Section 5.2 of the main article.

A.3. Adaptive loss-level discretization benchmark

Adaptive loss-level discretization provides a finite-grid benchmark for $\min_{x \in [x_L, x_U]} V_\gamma(x)$, where $V_\gamma(x)$ is defined in (3.5) of the main article. It applies to both MAES and MAES- ℓ_2 : SOLVEATLEVEL(x) records a numerical fixed-level objective and its portfolio at each trial level, and successive rounds refine the grid around the best evaluation. A feasible evaluated portfolio gives an upper bound on the continuous outer minimum. Thus, unlike BB, GRID solves no interval relaxation and provides no lower bound or certified gap over the continuous loss-level interval. Section 5.2 of the main article specifies the solver acceptance and feasibility checks used in the numerical comparison.

Algorithm 2 Adaptive loss-level discretization benchmark.

Require: Return scenarios $\{\mathbf{r}_t\}_{t=1}^T$, sample mean $\hat{\mu}$, confidence function $\Lambda(\cdot)$, regularization parameter γ , return target η , loss-level interval $[x_L, x_U]$, grid size $M \geq 6$, maximum rounds $K \geq 1$, and loss-level tolerance $\varepsilon_x > 0$.

Ensure: Selected loss level $\hat{x}_{\text{GRID}}$, portfolio $\hat{\mathbf{w}}_{\text{GRID}}$, and evaluated objective $\hat{V}_{\text{GRID}}$.

- 1: Let SOLVEATLEVEL(x) solve the fixed-level LP with HiGHS when $\gamma = 0$, and the fixed-level convex QP with OSQP when $\gamma > 0$; for the latter, use SLSQP only as the documented fallback.
 - 2: Set $\underline{x} \leftarrow x_L$, $\bar{x} \leftarrow x_U$, and $\mathcal{S} \leftarrow \emptyset$.
 - 3: **for** $k = 1, \dots, K$ **do**
 - 4: Set $\Delta_k \leftarrow (\bar{x} - \underline{x})/(M - 1)$ and construct M equally spaced points $\mathcal{X}_k$ on $[\underline{x}, \bar{x}]$.
 - 5: **for all** $x_j \in \mathcal{X}_k$ not previously evaluated **do**
 - 6: Compute $(V_j, \mathbf{w}_j) \leftarrow \text{SOLVEATLEVEL}(x_j)$ and add $(x_j, V_j, \mathbf{w}_j)$ to $\mathcal{S}$.
 - 7: **end for**
 - 8: Retain $\mathcal{S}$ across all rounds, so that each trial loss level is solved at most once.
 - 9: Select the cumulative incumbent $(\bar{x}, \bar{V}, \bar{\mathbf{w}}) \in \arg \min_{(x, V, \mathbf{w}) \in \mathcal{S}} V$, breaking an exact tie by the smallest x .
 - 10: **if** $\Delta_k \leq \varepsilon_x$ **then**
 - 11: **break**
 - 12: **end if**
 - 13: Update $\underline{x} \leftarrow \max\{x_L, \bar{x} - 2\Delta_k\}$ and $\bar{x} \leftarrow \min\{x_U, \bar{x} + 2\Delta_k\}$.
 - 14: **end for**
 - 15: Select $(\hat{x}_{\text{GRID}}, \hat{V}_{\text{GRID}}, \hat{\mathbf{w}}_{\text{GRID}}) \in \arg \min_{(x, V, \mathbf{w}) \in \mathcal{S}} V$, using the same tie-breaking rule.
 - 16: **return** $(\hat{x}_{\text{GRID}}, \hat{\mathbf{w}}_{\text{GRID}}, \hat{V}_{\text{GRID}})$.
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A.4. Interval branch-and-bound with verified bounds

Algorithm 3 applies to both MAES and MAES- ℓ_2 . It uses the fixed-loss-level problem $P(x)$ and interval relaxation $R(I)$ established in Section 4.2 of the main article. Incumbent upper bounds are recomputed from portfolios repaired and verified exactly as described in Section 4.2 of the main article.

For an interval $I = [l, b]$, the safe lower bound combines the analytic term $l + \gamma/(2N)$, the inherited parent bound, and any verified dual bound for $R(I)$. The analytic term follows from $z \geq l$ and $\|\mathbf{w}\|_2^2 \geq 1/N$ on the long-only unit simplex.

Algorithm 3 Interval branch-and-bound with verified bounds.

Require: $P(x)$, $R(I)$, asset count N , regularization $\gamma \geq 0$, $[x_L, x_U]$, gap tolerances $\varepsilon_{\text{abs}}, \varepsilon_{\text{rel}} > 0$, minimum width $\varepsilon_x > 0$, node limit N_{max} , and time limit t_{max} .

Ensure: Feasible incumbent $(\hat{x}, \hat{\mathbf{w}}, UB)$, safe LB , bound gap, and termination status.

- 1: Let $\text{tol}(u) = \varepsilon_{\text{abs}} + \varepsilon_{\text{rel}} \max\{1, |u|\}$. Let $\text{EVAL}(x)$ update the incumbent $(\hat{x}, \hat{\mathbf{w}}, UB)$ only with a checked feasible solution of $P(x)$.
- 2: For $I = [l, b]$, let $\text{BOUND}(I, L_p)$ return $L_s(I) = \max\{l + \gamma/(2N), L_p, L_d(I)\}$, where $L_d(I)$ is a verified dual bound from $R(I)$, or $-\infty$ if unavailable.
- 3: Evaluate EVAL at $x_L, (x_L + x_U)/2, x_U$; retain the best incumbent.
- 4: Set $I_0 = [x_L, x_U]$, $L_s(I_0) \leftarrow \text{BOUND}(I_0, -\infty)$, and $\mathcal{A} = \mathcal{L} = \{I_0\}$, where $\mathcal{L}$ contains every leaf.
- 5: **while** true **do**
- 6: Set $\tau \leftarrow \text{tol}(UB)$, $LB \leftarrow \min_{I \in \mathcal{L}} L_s(I)$, and remove from $\mathcal{A}$ nodes with $L_s(I) \geq UB - \tau$.
- 7: **if** $LB > UB$ **then** status $\leftarrow$ bound-inconsistency; **break**.
- 8: **if** $UB - LB \leq \tau$ **then** status $\leftarrow$ certified-gap; **break**.
- 9: **if** $\mathcal{A} = \emptyset$ or a budget is exhausted **then** record the status; **break**.
- 10: Remove $I^* \in \arg \min_{I \in \mathcal{A}} L_s(I)$ from $\mathcal{A}$.
- 11: **if** $\text{diam}(I^*) \leq \varepsilon_x$ **then continue**.
- 12: Bisect I^* into I_1, I_2 and replace it in $\mathcal{L}$.
- 13: **for** each child I_j **do**
- 14: Set $L_s(I_j) \leftarrow \text{BOUND}(I_j, L_s(I^*))$.
- 15: **if** $L_s(I_j) < UB - \text{tol}(UB)$ **then** call EVAL at the midpoint of I_j .
- 16: **if** $L_s(I_j) < UB - \text{tol}(UB)$ and $\text{diam}(I_j) > \varepsilon_x$ **then** add I_j to $\mathcal{A}$.
- 17: **end for**
- 18: **end while**
- 19: Set $g \leftarrow UB - LB$ and certified $\leftarrow$ (status = certified-gap).
- 20: **return** $(\hat{x}, \hat{\mathbf{w}}, UB, LB, g, g/\max\{1, |UB|\}, \text{certified}, \text{status})$.

The computed primal relaxation objective $\widehat{L}(I)$ is not used as a lower bound or to screen intervals. Numerical relaxation solves provide candidate dual witnesses. If a witness cannot be verified, the analytic and inherited bounds remain available. Intervals removed from the active set remain in the leaf cover used to compute LB , so $LB \leq V^* \leq UB$. Only a run satisfying $UB - LB \leq \text{tol}(UB)$ receives a certified-gap status.

A.5. Additional computational diagnostics

Tables A1, A2, and A3 supplement Section 5.2 of the main article. Table A1 reports objectives, induced effective confidence levels, maximum runtimes, and point and interval evaluations. Tables A2 and A3 report certificate and QP-solver diagnostics.

Table A1. Computational performance of the decomposition-based algorithms.

Model	Dataset	Algorithm	Mean objective	Mean Λ	Max runtime (s)	Max point evaluations	Max interval evaluations
MAES	FF49	GRID	0.029662	0.963037	0.55	58	–
MAES	FF49	BIS	0.029636	0.962966	0.23	22	–
MAES	FF49	BB	0.029637	0.962969	1.38	20	33
MAES	FTSE100	GRID	0.019903	0.968031	0.92	58	–
MAES	FTSE100	BIS	0.019893	0.967967	0.39	24	–
MAES	FTSE100	BB	0.019893	0.967973	1.81	19	33
MAES	SSE Industry	GRID	0.020566	0.958752	0.69	56	–
MAES	SSE Industry	BIS	0.020554	0.958670	0.25	21	–
MAES	SSE Industry	BB	0.020554	0.958673	1.68	18	31
MAES	100SBM	GRID	0.033254	0.952789	0.92	58	–
MAES	100SBM	BIS	0.033226	0.952691	0.39	23	–
MAES	100SBM	BB	0.033227	0.952693	1.95	20	33
MAES- ℓ_2	FF49	GRID	0.034831	0.962432	52.77	58	–
MAES- ℓ_2	FF49	BB	0.034822	0.962437	66.12	59	107
MAES- ℓ_2	FTSE100	GRID	0.023995	0.967839	5.95	58	–
MAES- ℓ_2	FTSE100	BB	0.023982	0.967795	54.12	28	49
MAES- ℓ_2	SSE Industry	GRID	0.024269	0.957961	44.82	58	–
MAES- ℓ_2	SSE Industry	BB	0.024266	0.957947	61.63	42	79
MAES- ℓ_2	100SBM	GRID	0.038148	0.952052	92.01	58	–
MAES- ℓ_2	100SBM	BB	0.038136	0.952011	126.64	53	101

Note. Times and evaluation counts are maxima across windows. BB time includes preprocessing, search, and a certificate audit. Point evaluations solve $P(x)$ at a fixed loss level. Interval evaluations construct verified dual bounds or analytic safe lower bounds.

Table A1 shows closely matching mean objectives and mean effective confidence levels for BIS and BB under MAES. BIS uses at most 24 fixed-level evaluations, versus 56–58 for GRID, consistent with the computational advantage of monotone crossing. Under MAES- ℓ_2 , BB uses up to 107 interval evaluations and 126.64 seconds, both within the search limits. These counts quantify the work needed for a certified regularized solution.

Table A2 reports independently audited Algorithm 3 certificates and solver diagnostics. The certified gap is $UB - LB$, and the prescribed stopping tolerance is $\tau = 10^{-6} + 10^{-8} \max\{1, |UB|\}$. All feasible windows met this tolerance. The OSQP violation column concerns accepted numerical QP solutions. Table A3 checks 21 fixed-level MAES- ℓ_2 QPs from the 2019–2021 algorithm-verification sample with SLSQP and Clarabel: 12 certified BB-incumbent layers from early, middle, and late windows, and nine first, middle, and last recorded GRID fallback calls from datasets with such calls. The 12 BB-incumbent cases also have accepted OSQP solutions for comparison. GRID and BB share the fixed-level formulation; the GRID cases do not directly test BB fallback calls.

Table A2. Certified bounds and solver diagnostics across rolling windows.

Model	Dataset	Feasible windows	Audited windows	Max certified gap (10^{-6})	Verified bounds	QP fallbacks	Max OSQP violation
MAES	FF49	25	25	0.9806	717	0	–
MAES	FTSE100	25	25	0.9469	691	0	–
MAES	SSE Industry	23	23	0.9001	655	0	–
MAES	100SBM	24	24	1.0042	712	0	–
MAES- ℓ_2	FF49	25	25	1.0072	1533	100	1.40×10^{-8}
MAES- ℓ_2	FTSE100	25	25	0.9981	871	0	1.22×10^{-8}
MAES- ℓ_2	SSE Industry	23	23	1.0091	1143	44	8.70×10^{-9}
MAES- ℓ_2	100SBM	24	24	1.0066	1384	60	8.30×10^{-8}

Note. The stopping tolerance includes absolute and relative terms, $10^{-6} + 10^{-8} \max\{1, |UB|\}$. Certified gaps slightly above 10^{-6} can therefore satisfy the stopping criterion.

In Table A2, the audited count equals the feasible count for every model and dataset, and each maximum certified gap meets the prescribed tolerance. The regularized BB runs record 204 QP fallback calls, yet their certificates pass the same audit as runs without fallback. The largest violation among accepted OSQP solutions is 8.30×10^{-8} , below the 10^{-7} acceptance threshold. Solver fallback changes how a QP candidate is obtained. It does not replace the checks on the feasible upper bound and the safe interval lower bounds. In Table A3, f_O , f_S , and f_C denote the OSQP, SLSQP, and Clarabel objective values.

Table A3. Independent QP cross-checks on 2019–2021 algorithm-verification windows.

Dataset	Max $ f_O - f_C $	Max $ f_S - f_C $	Max SLSQP violation	Max OSQP duality gap	Max Clarabel KKT residual
FF49	1.44×10^{-10}	6.35×10^{-9}	2.41×10^{-14}	2.11×10^{-12}	2.32×10^{-9}
FTSE100	1.46×10^{-10}	3.18×10^{-9}	6.00×10^{-15}	1.36×10^{-12}	7.72×10^{-10}
SSE Industry	1.87×10^{-11}	4.38×10^{-9}	1.48×10^{-14}	6.14×10^{-14}	2.94×10^{-10}
100SBM	1.03×10^{-10}	1.88×10^{-9}	9.33×10^{-15}	6.68×10^{-14}	3.51×10^{-10}

Table A3 gives maximum OSQP–Clarabel and SLSQP–Clarabel objective differences of 1.46×10^{-10} and 6.35×10^{-9} . The maximum SLSQP violation and Clarabel KKT residual are 2.41×10^{-14} and 2.32×10^{-9} . These results support the sampled QP solves. Table A2 separately certifies the global BB gaps.



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