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Research Article

Joint Distribution of Overtaking and Being Overtaken in the M/M/c Queue

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Supplementary

A. Proof of Proposition 1

Let

$$\sigma_k = \inf\{t > 0 : N(t) = k\}, \quad k = 0, 1, \ldots.$$

Thus, given $N(0) = n$, we have $\sigma_{n-1} = \sigma$. Let t_1 denote the first transition time of $N(t)$ after time 0, that is,

$$t_1 = \inf\{t > 0 : N(t) \neq N(0)\}.$$

Given $N(0) = n \geq c + 1$ and $J(0) = i$, the first transition occurs due to one of the following events:

- an arrival of a new customer from outside, with probability $\frac{\lambda}{\lambda+c\mu}$,
- service completion of a customer who arrived after the tagged customer, with probability $\frac{i\mu}{\lambda+c\mu}$,
- service completion of a customer who arrived before the tagged customer, with probability $\frac{(c-i-1)\mu}{\lambda+c\mu}$, and
- service completion of the tagged customer, with probability $\frac{\mu}{\lambda+c\mu}$.

Hence, for $|z| \leq 1$, $n \geq c + 1$ and $i \in \{0, 1, \ldots, c - 1\}$,

$$G_{ij}(z) = \mathbb{E}\left[z^{X(\sigma_{n-1})} \mathbb{1}_{\{\sigma_{n-1}<\tau,\ J(\sigma_{n-1})=j\}} \,\middle|\, N(0) = n,\ J(0) = i \right]$$

$$= \frac{\lambda}{\lambda + c\mu} \mathbb{E}\left[z^{X(\sigma_{n-1})} \mathbb{1}_{\{\sigma_{n-1}<\tau,\ J(\sigma_{n-1})=j\}} \,\middle|\, N(0) = n+1,\ J(0) = i \right]$$
$$+ \frac{i\mu}{\lambda + c\mu} z\, \delta_{i,j} + \frac{(c-i-1)\mu}{\lambda + c\mu} \delta_{i,j-1} \mathbb{1}_{\{j \ge 1\}}, \tag{A.1}$$

where $\delta_{i,j}$ is the Kronecker delta, i.e., $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. Since

$$\begin{aligned}
&\mathbb{E}\left[z^{X(\sigma_{n-1})} \mathbb{1}_{\{\sigma_{n-1}<\tau,\ J(\sigma_{n-1})=j\}} \,\middle|\, N(0) = n+1,\ J(0) = i \right] \\
&= \sum_{k=0}^{c-1} \mathbb{E}\left[z^{X(\sigma_{n-1})} \mathbb{1}_{\{\sigma_n<\tau,\ J(\sigma_n)=k, \sigma_{n-1}<\tau,\ J(\sigma_{n-1})=j\}} \,\middle|\, N(0) = n+1,\ J(0) = i \right] \\
&= \sum_{k=0}^{c-1} \mathbb{E}\left[z^{X(\sigma_n)} \mathbb{1}_{\{\sigma_n<\tau,\ J(\sigma_n)=k\}} z^{X(\sigma_{n-1})-X(\sigma_n)} \mathbb{1}_{\{\sigma_{n-1}<\tau,\ J(\sigma_{n-1})=j\}} \,\middle|\, N(0) = n+1,\ J(0) = i \right] \\
&= \sum_{k=0}^{c-1} \mathbb{E}\left[z^{X(\sigma_n)} \mathbb{1}_{\{\sigma_n<\tau,\ J(\sigma_n)=k\}} \,\middle|\, N(0) = n+1,\ J(0) = i \right] \\
&\qquad \times \mathbb{E}\left[z^{X(\sigma_{n-1})} \mathbb{1}_{\{\sigma_{n-1}<\tau,\ J(\sigma_{n-1})=j\}} \,\middle|\, N(0) = n,\ J(0) = k \right] \\
&= \sum_{k=0}^{c-1} G_{ik}(z) G_{kj}(z),
\end{aligned}$$

(A.1) leads to

$$G_{ij}(z) = \frac{\lambda}{\lambda + c\mu} \sum_{k=0}^{c-1} G_{ik}(z) G_{kj}(z) + \frac{i\mu}{\lambda + c\mu} z\, \delta_{i,j} + \frac{(c-i-1)\mu}{\lambda + c\mu} \delta_{i,j-1} \mathbb{1}_{\{j \ge 1\}}.$$

This is written in matrix form as equation (2.2). Since $J(t)$ is nondecreasing in t over the interval $[0, \tau]$, we have

$$G_{ij}(z) = 0, \quad \text{if } i > j. \tag{A.2}$$

Equation (2.1) then follows from (2.2), (A.2), and the inequalities $|G_{ij}(z)| \le 1$.

By applying an argument similar to the one used to derive equation (2.2), we can derive the following equations:

$$G_c(z) = \frac{\lambda}{\lambda + c\mu} G(z)\, G_c(z) + \frac{c\mu}{\lambda + c\mu} B_c(z), \tag{A.3}$$

$$G_n(z) = \frac{\lambda}{\lambda + n\mu} A_n\, G_{n+1}(z)\, G_n(z) + \frac{c\mu}{\lambda + n\mu} B_n(z), \quad n = 2, 3, \ldots, c-1. \tag{A.4}$$

Equations (2.3) and (2.4) are obtained from (A.3) and (A.4), respectively. This completes the proof. □

B. Proof of Proposition 2

Define σ_k, $k = 0, 1, \ldots$, and t_1 as in Section A. Given $N(0) = n \ge c$ and $J(0) = i$, the first transition occurs due to one of the four events as described in Section A. For $|z| \le 1$, $|w| \le 1$, $n \ge c$ and $i \in \{0, 1, \ldots, c-1\}$,

$$h_i(z, w) = \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)} \mathbb{1}_{\{\tau \le \sigma_{n-1}\}} \,\middle|\, N(0) = n,\ J(0) = i \right]$$

$$= \frac{\lambda}{\lambda + c\mu}\mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right] + \frac{\mu}{\lambda+c\mu}w^{c-i-1}. \tag{B.1}$$

Note that

$$\begin{aligned}
&\mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right]\\
&\quad= \mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\tau\le\sigma_{n}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right]\\
&\qquad+ \mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\sigma_n<\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right].
\end{aligned} \tag{B.2}$$

Since

$$\mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\tau\le\sigma_{n}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right] = h_i(z,w)$$

and

$$\begin{aligned}
&\mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\sigma_n<\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right]\\
&= \sum_{k=0}^{c-1}\mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\sigma_n<\tau,\,J(\sigma_n)=k,\,\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right]\\
&= \sum_{k=0}^{c-1}\mathbb{E}\left[z^{X(\sigma_n)}\,\mathbb{1}_{\{\sigma_n<\tau,\ J(\sigma_n)=k\}}\,z^{X(\tau)-X(\sigma_n)}w^{Y(\tau)}\mathbb{1}_{\{\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right]\\
&= \sum_{k=0}^{c-1}\mathbb{E}\left[z^{X(\sigma_n)}\,\mathbb{1}_{\{\sigma_n<\tau,\ J(\sigma_n)=k\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right]\\
&\qquad\times\mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n,\ J(0)=k\right]\\
&= \sum_{k=0}^{c-1}G_{ik}(z)h_k(z,w),
\end{aligned}$$

(B.2) leads to

$$\mathbb{E}\left[z^{X(\tau)}w^{Y(\tau)}\,\mathbb{1}_{\{\tau\le\sigma_{n-1}\}}\,\middle|\,N(0)=n+1,\ J(0)=i\right] = h_i(z,w) + \sum_{k=0}^{c-1}G_{ik}(z)h_k(z,w). \tag{B.3}$$

Substituting (B.3) into (B.1) yields

$$h_i(z,w) = \frac{\lambda}{\lambda+c\mu}\left(h_i(z,w) + \sum_{k=0}^{c-1}G_{ik}(z)h_k(z,w)\right) + \frac{\mu}{\lambda+c\mu}w^{c-i-1}.$$

Rearranging the above expression, we obtain the following:

$$h_i(z,w) = \rho\sum_{k=0}^{c-1}G_{ik}(z)h_k(z,w) + \frac{1}{c}w^{c-i-1}.$$

Furthermore, it can be expressed in matrix form as:

$$\boldsymbol{h}(z,w) = \rho G(z)\boldsymbol{h}(z,w) + \boldsymbol{b}(w), \tag{B.4}$$

where

$$\boldsymbol{b}(w) = \frac{1}{c}(w^{c-1},\ w^{c-2},\ \cdots, w,\ 1)^{\top}.$$

which yields (2.5).

By applying an argument similar to the one used to derive (B.4), we can derive the following equations: For $n = 1, 2, \ldots, c-1$, $|z| \le 1$ and $|w| \le 1$,

$$\boldsymbol{h}_n(z,w) = \frac{\lambda}{\lambda + n\mu} A_n \left(\boldsymbol{h}_{n+1}(z,w) + G_{n+1}(z)\boldsymbol{h}_n(z,w)\right) + \frac{c\mu}{\lambda + n\mu}\boldsymbol{b}_n(w).$$

This yields (2.6). The proof is complete. □

C. Proof of Proposition 3

For $|z| \le 1$, $|w| \le 1$, $n = 1, 2, \ldots$, and $i = 0, 1, \ldots, \min\{n-1, c-1\}$,

$$\begin{aligned}\phi_{n,i}(z,w) &= \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)}\, \mathbb{1}_{\{\tau \le \sigma\}} \,\middle|\, N(0) = n,\ J(0) = i \right] \\ &\quad + \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)}\, \mathbb{1}_{\{\sigma < \tau\}} \,\middle|\, N(0) = n,\ J(0) = i \right]. \end{aligned} \tag{C.1}$$

Note

$$\mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)}\, \mathbb{1}_{\{\tau \le \sigma\}} \,\middle|\, N(0) = n,\ J(0) = i \right] = h_{n,i}(z,w),$$

and, if $n \ge 2$, then

$$\begin{aligned} &\mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)}\, \mathbb{1}_{\{\sigma < \tau\}} \,\middle|\, N(0) = n,\ J(0) = i \right] \\ &= \sum_{k=0}^{\min\{n-2,c-1\}} \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)}\, \mathbb{1}_{\{\sigma < \tau,\, J(\sigma) = k\}} \,\middle|\, N(0) = n,\ J(0) = i \right] \\ &= \sum_{k=0}^{\min\{n-2,c-1\}} \mathbb{E}\left[z^{X(\sigma)} \mathbb{1}_{\{\sigma < \tau,\, J(\sigma) = k\}}\, z^{X(\tau) - X(\sigma)} w^{Y(\tau)} \,\middle|\, N(0) = n,\ J(0) = i \right] \\ &= \sum_{k=0}^{\min\{n-2,c-1\}} \mathbb{E}\left[z^{X(\sigma)} \mathbb{1}_{\{\sigma < \tau,\, J(\sigma) = k\}} \,\middle|\, N(0) = n,\ J(0) = i \right] \\ &\qquad\qquad \times \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)} \,\middle|\, N(0) = n-1,\ J(0) = k \right] \\ &= \sum_{k=0}^{\min\{n-2,c-1\}} G_{n,ik}(z)\, \phi_{n-1,k}(z,w). \end{aligned}$$

Hence, (C.1) is written as

$$\begin{aligned} \phi_{1,i}(z,w) &= h_{1,i}(z,w), \\ \phi_{n,i}(z,w) &= h_{n,i}(z,w) + \sum_{k=0}^{\min\{n-2,c-1\}} G_{n,ik}(z)\, \phi_{n-1,k}(z,w), \quad n = 2, 3, \ldots. \end{aligned}$$

These are written in matrix form as

$$\begin{aligned}\phi_1(z,w) &= \boldsymbol{h}_1(z,w),\\ \phi_n(z,w) &= \boldsymbol{h}_n(z,w) + G_n(z)\,\phi_{n-1}(z,w), \quad n = 2,3,\ldots.\end{aligned}$$

Thus, (2.7) and (2.8) are proved. From the last equation,

$$\begin{aligned}\widetilde{\phi}(z,w) &= \sum_{n=c}^{\infty} \rho^{n-c} \left(\boldsymbol{h}_n(z,w) + G_n(z)\,\phi_{n-1}(z,w)\right)\\ &= \sum_{n=c}^{\infty} \rho^{n-c}\boldsymbol{h}(z,w) + G_c(z)\,\phi_{c-1}(z,w) + \sum_{n=c+1}^{\infty} \rho^{n-c} G(z)\,\phi_{n-1}(z,w)\\ &= \frac{1}{1-\rho}\boldsymbol{h}(z,w) + G_c(z)\,\phi_{c-1}(z,w) + \rho\, G(z)\,\widetilde{\phi}(z,w),\end{aligned}$$

which yields (2.9). The proof is complete. □

D. Proof of Lemma 1

Let $\mathcal{N}_0$ denote the total number of customers in the system, including the arbitrary customer, immediately after the arrival of the arbitrary customer. By the *Poisson Arrivals See Time Averages* (PASTA) property, $\mathcal{N}_0 - 1$ has the same distribution as the stationary distribution of the number of customers in the system. That is, $\mathbb{P}(\mathcal{N}_0 - 1 = n) = p_n$, $n = 0, 1, 2, \ldots$, where

$$p_n = \begin{cases} \dfrac{1}{D_c(c\rho,\rho)} \dfrac{(c\rho)^n}{n!}, & \text{if } 0 \le n \le c,\\ \dfrac{1}{D_c(c\rho,\rho)} \dfrac{(c\rho)^c}{c!}\rho^{n-c}, & \text{if } n \ge c+1. \end{cases}$$

Thus, the distribution of $\mathcal{N}_0$ is given by

$$\mathbb{P}(\mathcal{N}_0 = n) = p_{n-1}, \quad n = 1,2,\ldots. \tag{D.1}$$

To derive the distribution of $\mathcal{N}$, we make the following observations:

1. If $\mathcal{N}_0 = n$, for $1 \le n \le c$, the arbitrary customer begins service immediately upon arrival. Thus,

$$\mathbb{P}(\mathcal{N} = n) = \mathbb{P}(\mathcal{N}_0 = n) = p_{n-1}, \quad n = 1,2,\ldots,c. \tag{D.2}$$

2. If $\mathcal{N}_0 = n$, for $n > c$, the arbitrary customer begins service after $n-c$ service completions following its arrival. Note that each service completion occurs after an exponentially distributed time with mean $1/(c\mu)$. Therefore, the number of arrivals during one service time follows a geometric distribution with probability mass function $\frac{1}{1+\rho}(\frac{\rho}{1+\rho})^k$, $k = 0,1,2,\ldots$, and hence its probability generating function is given by $\frac{1}{1+\rho-\rho z}$. Consequently, we have

$$\mathbb{E}\left[z^{\mathcal{N}-c}\,\middle|\,\mathcal{N}_0 = n\right] = \frac{1}{(1+\rho-\rho z)^{n-c}}, \quad n > c. \tag{D.3}$$

By (D.1), (D.2) and (D.3), we have

$$
\begin{aligned}
\mathbb{E}[z^{\mathcal{N}}] &= \sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}_0 = n)\, \mathbb{E}\left[z^{\mathcal{N}} \middle| \mathcal{N}_0 = n \right] \\
&= \sum_{n=1}^{c} p_{n-1} z^n + \sum_{n=c+1}^{\infty} p_{n-1} \frac{z^c}{(1+\rho-\rho z)^{n-c}} \\
&= \frac{1}{D_c(c\rho,\rho)} \left[\sum_{n=1}^{c} \frac{(c\rho)^{n-1}}{(n-1)!} z^n + \sum_{n=c+1}^{\infty} \frac{(c\rho)^c}{c!} \rho^{n-c-1} \frac{z^c}{(1+\rho-\rho z)^{n-c}} \right] \\
&= \frac{1}{D_c(c\rho,\rho)} \left[\sum_{n=1}^{c} \frac{(c\rho)^{n-1}}{(n-1)!} z^n + \frac{(c\rho)^c}{c!} \frac{z^c}{1-\rho z} \right] \\
&= \frac{1}{D_c(c\rho,\rho)} \left[\sum_{n=1}^{c-1} \frac{(c\rho)^{n-1}}{(n-1)!} z^n + \frac{(c\rho)^{c-1}}{(c-1)!}(1+\rho) z^c + \sum_{n=c+1}^{\infty} \frac{(c\rho)^{c-1}}{(c-1)!} \rho^{n-c+1} z^n \right].
\end{aligned}
$$

This completes the proof of the lemma. □

E. Glossary of notation

For convenience, Table 1 summarizes the key notation used throughout the paper.

Table 1. List of notations.

Symbol	Description
ρ	$\rho = \frac{\lambda}{c\mu}$
$N(t)$	the total number of customers in the system at time t, including the tagged customer
$J(t)$	the number of customers in service at time t who arrived after the tagged customer
$X(t)$	the number of customers who overtake the tagged customer during the time interval $[0, t]$
$Y(t)$	the number of customers who are overtaken by the tagged customer over the same interval
σ	$\sigma = \inf\{t > 0 : N(t) = N(0) - 1\}$
τ	the departure time of the tagged customer from the system
$G_n(z)$	the $(\min\{n-1, c-1\}+1) \times (\min\{n-2, c-1\}+1)$ matrix with entries $$G_{n,ij}(z) = \mathbb{E}\left[z^{X(\sigma)}\, \mathbb{1}_{\{\sigma<\tau,\ J(\sigma)=j\}} \,\middle|\, N(0) = n,\ J(0) = i\right],$$ where $0 \le i \le \min\{n-1, c-1\}$, $0 \le j \le \min\{n-2, c-1\}$
$G(z)$	$G(z) = G_n(z)$ for $n \ge c+1$
$\boldsymbol{h}_n(z, w)$	the $(\min\{n-1, c-1\}+1)$-dimensional column vector with components $$\mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)}\, \mathbb{1}_{\{\tau\le\sigma\}} \,\middle|\, N(0) = n,\ J(0) = i\right],$$ where $0 \le i \le \min\{n-1, c-1\}$
$\boldsymbol{h}(z, w)$	$h(z, w) = h_n(z, w)$ for $n \ge c$
$\phi_n(z, w)$	the $(\min\{n-1, c-1\}+1)$-dimensional column vector with components $$\phi_{n,i}(z, w) = \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)} \,\middle|\, N(0) = n,\ J(0) = i\right],$$ where $0 \le i \le \min\{n-1, c-1\}$
$\tilde{\phi}(z, w)$	$\widetilde{\phi}(z, w) = \sum_{n=c}^{\infty} \rho^{n-c} \phi_n(z, w)$
$\mathcal{X}$	the number of times an arbitrary customer is overtaken
$\mathcal{Y}$	the number of times an arbitrary customer overtakes others
$\mathcal{N}$	the total number of customers in the system, including the arbitrary customer, immediately after that customer begins service
$\Phi(z, w)$	$\Phi(z, w) = \mathbb{E}[z^{\mathcal{X}} w^{\mathcal{Y}}]$

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