Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(1): 554–580.
DOI:10.3934/jimo.2026021
Received: 27 June 2025
Revised: 24 October 2025
Accepted: 06 November 2025
Published: 16 December 2025

Research article

Pricing when customers have loss aversion and limited attention

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A. Appendix

A.1. Proof of Equation (3.10)

Proof. Assume S_N is the set of all signals that lead to customers not purchasing, i.e.,

$$1 - \pi_i = \int_{s \in S_N} F(ds|i), \quad i \in \Omega.$$

According to Lemma 1 of Matějka and McKay [38], each action corresponds uniquely to a specific signal. This implies that for any $s \in S_P$, $F(i|s)$ remains identical. Therefore, for any $i \in \Omega$ and any $s \in S_P$, we have $W(F(\cdot|s)) = U_i$. Similarly, for any $s \in S_N$, $F(i|s)$ is also identical, and we have $W(F(\cdot|s)) = U_0$.

Therefore, the first term in Equation (3.5) is

$$\begin{aligned}\sum_{i\in\Omega}\int_s W(F(\cdot|s))F(ds,i) &= \sum_{i\in\Omega}\int_s W(F(\cdot|s))F(ds|i)g_i \\ &= \sum_{i\in\Omega}\int_{s\in S_P S_N} W(F(\cdot|s))F(ds|i)g_i \\ &= \sum_{i\in\Omega}\left(\int_{s\in S_P} W(F(\cdot|s))F(ds|i)g_i + \int_{s\in S_N} W(F(\cdot|s))F(ds|i)g_i\right) \\ &= \sum_{i\in\Omega}\left(g_i U_i \int_{s\in S_P} F(ds|i) + g_i U_0 \int_{s\in S_N} F(ds|i)\right) \\ &= \sum_{i\in\Omega}(g_i U_i \pi_i + g_i U_0(1-\pi_i)) = \sum_{i\in\Omega} g_i(\pi_i U_i + (1-\pi_i)U_0).\end{aligned} \tag{A.1}$$

This exactly equals the first term in Equation (3.10). ■

A.2. Proof of Equation (4.3)

Proof. From Equations (3.9) and (3.11), we obtain

$$\pi_0^1 = \tau\pi_H^1 + (1-\tau)\pi_L^1 \tag{A.2}$$

$$\pi_H^1 = \frac{\pi_0^1 e^{(q_H^1-p)/\lambda}}{\pi_0^1 e^{(q_H^1-p)/\lambda} + (1-\pi_0^1)} \tag{A.3}$$

$$\pi_L^1 = \frac{\pi_0^1 e^{(q_L^1-p)/\lambda}}{\pi_0^1 e^{(q_L^1-p)/\lambda} + (1-\pi_0^1)}. \tag{A.4}$$

Substituting equations (A.3) and (A.4) into equation (A.2), we obtain

$$\pi_0^1 = \frac{\tau\pi_0^1 e^{(q_H^1-p)/\lambda}}{\pi_0^1 e^{(q_H^1-p)/\lambda} + (1-\pi_0^1)} + \frac{(1-\tau)\pi_0^1 e^{(q_L^1-p)/\lambda}}{\pi_0^1 e^{(q_L^1-p)/\lambda} + (1-\pi_0^1)}. \tag{A.5}$$

Multiplying both sides by

$$\frac{[\pi_0^1 e^{(q_H^1-p)/\lambda} + (1-\pi_0^1)][\pi_0^1 e^{(q_L^1-p)/\lambda} + (1-\pi_0^1)]}{\pi_0^1},$$

and simplifying, we have

$$\pi_0^1(e^{(q_H^1-p)/\lambda} - 1)(e^{(q_L^1-p)/\lambda} - 1) = 1 - \tau e^{(q_H^1-p)/\lambda} - (1-\tau)e^{(q_L^1-p)/\lambda}.$$

Therefore, we obtain the solution

$$\pi_0^1 = \frac{1 - \tau e^{(q_H^1-p)/\lambda} - (1-\tau)e^{(q_L^1-p)/\lambda}}{(e^{(q_H^1-p)/\lambda} - 1)(e^{(q_L^1-p)/\lambda} - 1)}. \tag{A.6}$$

Multiplying both the numerator and denominator of the right-hand side of Equation (A.6) by e^{2p}, we obtain

$$\pi_0^1(p) = \frac{e^{p/\lambda}(e^{p/\lambda} - \tau e^{q_H^1/\lambda} - (1-\tau)e^{q_L^1/\lambda})}{(e^{q_H^1/\lambda} - e^{p/\lambda})(e^{q_L^1/\lambda} - e^{p/\lambda})}. \tag{A.7}$$

Equation (A.7) represents the unconditional purchase probability when the customers choose to obtain information, which is equivalent to Equation (4.3). By combining this with the unconditional purchase probability when no information is obtained, we arrive at Equation (4.4). The derivation processes for Equations (4.6) and (4.7) follow the same methodology as described above and are therefore omitted here. ■

A.3. Proof of Remark 1

Proof. (i) The first derivative of $\overline{p}^1$ with respect to λ is

$$\frac{\partial \overline{p}^1}{\partial \lambda} = \ln[\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}] - \frac{\tau q_H^1 e^{q_H^1/\lambda} + (1-\tau)q_L^1 e^{q_L^1/\lambda}}{\lambda[\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}]}.$$

$\partial \overline{p}^1/\partial\lambda \le 0$ is equivalent to

$$\left[\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}\right] \ln[\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}] \le \tau \frac{q_H^1}{\lambda} e^{q_H^1/\lambda} + (1-\tau)\frac{q_L^1}{\lambda} e^{q_L^1/\lambda}. \tag{A.8}$$

We know that $f(x) = x\ln x$ is convex with respect to x. So we have $f(\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}) \le \tau f(e^{q_H^1/\lambda}) + (1-\tau)f(e^{q_L^1/\lambda})$, which means that the inequality (A.8) holds. Therefore, $\partial\overline{p}^1/\partial\lambda \le 0$.

The first derivative of $\underline{p}^1$ with respect to λ is

$$\frac{\partial \underline{p}^1}{\partial \lambda} = \frac{(1-\tau)q_H^1 e^{q_H^1/\lambda} + \tau q_L^1 e^{q_L^1/\lambda}}{\lambda[(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}]} - \ln[(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}].$$

$\partial \underline{p}^1/\partial\lambda \ge 0$ is equivalent to

$$(1-\tau)\frac{q_H^1}{\lambda} e^{q_H^1/\lambda} + \tau \frac{q_L^1}{\lambda} e^{q_L^1/\lambda} \ge [(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}] \ln[\tau e^{q_L^1/\lambda} + (1-\tau)e^{q_H^1/\lambda}].$$

It is obviously true according to the convexity of $f(x) = x\ln x$. So we have $\partial\underline{p}^1/\partial\lambda \ge 0$.

(ii) The first derivatives of $\overline{p}^1$ and $\underline{p}^1$ with respect to β, α, and τ are as follows:

$$\frac{\partial \overline{p}^1}{\partial \beta} = \frac{(q_L - q_H)(\tau e^{q_H^1/\lambda} + \alpha(1-\tau)e^{q_L^1/\lambda})}{\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}} < 0,$$

$$\frac{\partial \underline{p}^1}{\partial \beta} = (q_H - q_L)\left[\frac{(1-\tau)e^{q_H^1/\lambda} + \alpha\tau e^{q_L^1/\lambda}}{(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}} - 1 - \alpha\right] < q_L - q_H < 0,$$

$$\frac{\partial \overline{p}^1}{\partial \alpha} = \frac{(q_L - q_H)\beta(1-\tau)e^{q_L^1/\lambda}}{\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}} < 0.$$

$$\frac{\partial \underline{p}^1}{\partial \alpha} = \frac{(q_L - q_H)\beta(1-\tau)e^{q_H^1/\lambda}}{(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}} < 0,$$

$$\frac{\partial \overline{p}^1}{\partial \tau} = \frac{\lambda(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})}{\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_L^1/\lambda}} > 0,$$

$$\frac{\partial \underline{p}^1}{\partial \tau} = \frac{\lambda(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})}{(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}} > 0.$$

These prove that $\overline{p}^1$ and $\underline{p}^1$ both decrease with β and α, and increase with τ.

■

A.4. Proof of Proposition 2

Proof. Since the cases for $k = 1$ and $k = 2$ are similar, it suffices to prove the case for $k = 1$. First, it should be noted that $\pi_0^1(p)$ decreases with respect to p, since

$$\frac{\partial \pi_0^1(p)}{\partial p} = -\frac{\tau[\text{csch}((p + U_0 - q_L^1)/2\lambda)]^2 - (1-\tau)[\text{csch}((p + U_0 - q_H^1)/2\lambda)]^2}{4\lambda} \leq 0,$$

where

$$\text{csch}(x) = \frac{2}{e^x - e^{-x}}.$$

According to $\pi_0^1(p)$, when $p \leq \underline{p}^1$, we have $\pi_0^1(p) = 1$, and so $\pi_H^1(p) = \pi_L^1(p) = 1$. At this point, $R_i^1(p) = p * \pi_i^1(p) = p$ increases with respect to p, and therefore, $p_{i1}^* \geq \underline{p}^1$. When $p \geq \overline{p}^1$, we have $\pi_0^1(p) = 0$, and so $\pi_H^1(p) = \pi_L^1(p) = 0$. At this point, $R_i^1(p) = p * \pi_i^1(p) = 0$, and therefore, $p_{i1}^* \leq \overline{p}^1$. ■

A.5. Proof of Proposition 3

Proof.

Similarly, it is sufficient to prove only that $k = 1$ here. (i) Let

$$\epsilon(\pi_i^1) = \frac{\pi_0^1}{1 - \pi_0^1} * \frac{1 - \pi_i^1}{\pi_i^1},$$

then rewrite the revenue function as

$$R_i^1(\pi_i^1) = (q_i - U_0 + (q_i - r)^+ + \alpha(q_i - r)^-)\pi_i^1 + \lambda\pi_i^1 \ln[\epsilon(\pi_i^1)].$$

Using Equation (3.11), π_H^1 and π_L^1 are given by

$$\pi_H^1 = \frac{e^{(q_H^1 - p)/\lambda}}{e^{(q_H^1 - p)/\lambda} + (1 - \pi_0^1)/\pi_0^1}$$

and

$$\pi_L^1 = \frac{e^{(q_L^1 - p)/\lambda}}{e^{(q_L^1 - p)/\lambda} + (1 - \pi_0^1)/\pi_0^1}.$$

We then represent π_L^1 as a function of π_H^1 as

$$\pi_L^1 = \frac{\pi_H^1 e^{q_L^1/\lambda}}{e^{q_H^1/\lambda} - \pi_H^1(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})},$$

and substitute it into equation (3.9) to obtain

$$\pi_0^1 = \tau\pi_H^1 + \frac{(1-\tau)\pi_H^1 e^{q_L^1/\lambda}}{e^{q_H^1/\lambda} - \pi_H^1(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})}.$$

Consequently, $\epsilon(\pi_H^1)$ is represented as

$$\epsilon(\pi_H^1) = \frac{\tau(1-\pi_H^1)e^{q_H^1/\lambda} + (1 - \tau(1-\pi_H^1))e^{q_L^1/\lambda}}{(1-\tau\pi_H^1)e^{q_H^1/\lambda} + \tau\pi_H^1 e^{q_L^1/\lambda}}.$$

The first and second derivatives of $\epsilon(\pi_H^1)$ with respect to π_H^1 are given by

$$\frac{\partial\epsilon(\pi_H^1)}{\partial\pi_H^1} = -\frac{\tau(1-\tau)(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})^2}{((1-\tau\pi_H^1)e^{q_H^1/\lambda} + \tau\pi_H^1 e^{q_L^1/\lambda})^2} \le 0$$

and

$$\frac{\partial^2\epsilon(\pi_H^1)}{\partial(\pi_H^1)^2} = -\frac{2\tau^2(1-\tau)(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})^3}{((1-\tau\pi_H^1)e^{q_H^1/\lambda} + \tau\pi_H^1 e^{q_L^1/\lambda})^3} \le 0.$$

Similarly, we represent π_H^1 as a function of π_L^1 as

$$\pi_H^1 = \frac{\pi_L^1 e^{q_H^1/\lambda}}{e^{q_L^1/\lambda} + \pi_L(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})}.$$

Then $\epsilon(\pi_L^1)$ is given by

$$\epsilon(\pi_L^1) = \frac{(\pi_L^1 + \tau(1-\pi_L^1))e^{q_H^1/\lambda} + (1-\tau)(1-\pi_L^1)e^{q_L^1/\lambda}}{(1-\tau)\pi_L^1 e^{q_H^1/\lambda} + (1-(1-\tau)\pi_L^1)e^{q_L^1/\lambda}}.$$

Therefore, the first and second derivatives of $\epsilon(\pi_L^1)$ with respect to π_L^1 are given by

$$\frac{\partial\epsilon(\pi_L^1)}{\partial\pi_L^1} = -\frac{\tau(1-\tau)(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})^2}{((1-(1-\tau)\pi_L^1)e^{q_L^1/\lambda} + (1-\tau)\pi_L^1 e^{q_H^1/\lambda})^2} \leq 0$$

and

$$\frac{\partial^2\epsilon(\pi_L^1)}{\partial(\pi_L^1)^2} = -\frac{2\tau(1-\tau)^2(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})^3}{((1-(1-\tau)\pi_L^1)e^{q_L^1/\lambda} + (1-\tau)\pi_L^1 e^{q_H^1/\lambda})^3} \leq 0.$$

Taking derivatives, the second derivative of $R_i^1(\pi_i^1)$ with respect to π_i^1 is given by

$$\frac{\partial^2 R_i^1(\pi_i^1)}{\partial(\pi_i^1)^2} = \frac{2\lambda\partial\epsilon(\pi_i^1)/\partial\pi_i^1}{\epsilon(\pi_i^1)} + \lambda\pi_i^1\left[-\left(\frac{\partial\epsilon(\pi_i^1)/\partial\pi_i^1}{\epsilon(\pi_i^1)}\right)^2 + \frac{\partial^2\epsilon(\pi_i^1)/\partial(\pi_i^1)^2}{\epsilon(\pi_i^1)}\right].$$

Since $\partial\epsilon(\pi_i^1)/\partial\pi_i^1 \leq 0$, $\partial^2\epsilon(\pi_i^1)/\partial(\pi_i^1)^2 \leq 0$, and $\epsilon(\pi_i^1) \geq 0$, we have $\partial^2 R_i^1(\pi_i^1)/\partial(\pi_i^1)^2 \leq 0$, which means that $R_i^1(\pi_i^1)$ is concave with respect to π_i^1.

(ii) For high quality, $\pi_H^1(p)$ is given by

$$\pi_H^1(p) = \frac{\pi_0^1(p)e^{(q_H^1-p)/\lambda}}{\pi_0^1(p)e^{(q_H^1-p)/\lambda} + (1-\pi_0^1(p))e^{U_0/\lambda}}.$$

Using Equation (9), we can represent $\pi_H^1(p)$ as

$$\pi_H^1(p) = \frac{e^{q_H^1/\lambda}}{\tau}\left[\frac{1}{e^{q_H^1/\lambda} - e^{q_L^1/\lambda}} - \frac{1-\tau}{e^{q_H^1/\lambda} - e^{(p+U_0)/\lambda}}\right].$$

Therefore, $R_H^1(p) = p * \pi_H^1(p)$, and the second derivative of $R_H^1(p) = p * \pi_H^1(p)$ with respect to p is

$$\frac{\partial^2 R_H^1(p)}{\partial p^2} = -\frac{(1-\tau)\left[\text{csch}((p+U_0-q_H^1)/2\lambda)\right]^2}{4\tau\lambda^2} * \left[2\lambda - p\text{coth}\left(\frac{p+U_0-q_H^1}{2\lambda}\right)\right],$$

where

$$\text{coth}(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Since $p < \overline{p} < q_H^1 - U_0$, we have $p + U_0 < q_H^1$, then $\text{coth}((p+U_0-q_H^1)/2\lambda) < 0$. Together with $-(1-\tau)\left[\text{csch}((p+U_0-q_H^1)/\lambda)\right]^2/4\tau\lambda^2 < 0$, consequently, $R_H^1(p)$ is concave with respect to p for $p \in [\underline{p}^1, \overline{p}^1]$.

Similarly, for low quality, we can represent $\pi_L^1(p)$ as

$$\pi_L^1(p) = \frac{e^{q_L^1/\lambda}}{1-\tau}\left[\frac{\tau}{e^{(p+U_0)/\lambda} - e^{q_L^1/\lambda}} - \frac{1}{e^{q_H^1/\lambda} - e^{(q_L^1)/\lambda}}\right].$$

So we obtain $R_L^1(p) = p * \pi_L^1(p)$ and

$$\frac{\partial R_L^1(p)}{\partial p} = -\frac{e^{q_L^1/\lambda}}{1-\lambda}\left[\frac{1}{e^{q_H^1/\lambda} - e^{q_L^1/\lambda}} + \tau\frac{\lambda e^{q_L^1/\lambda} + (p-\lambda)e^{(p+U_0)/\lambda}}{\lambda(e^{(p+U_0)/\lambda} - e^{q_L^1/\lambda})^2}\right].$$

If $\lambda \leq p$, then $\lambda e^{q_L^1/\lambda} + (p-\lambda)e^{(p+U_0)/\lambda} > 0$. If $\lambda > p$, to have $\lambda e^{q_L^1/\lambda} + (p-\lambda)e^{(p+U_0)/\lambda} \geq 0$, we must demonstrate that $(\lambda - p)e^{(p+U_0)/\lambda} \leq \lambda e^{q_L^1/\lambda}$, which, in turn, suggests that $e^{(p+U_0-q_L^1)/\lambda} \leq \lambda/(\lambda - p)$. As $p \to q_L^1 - U_0 (p > q_L^1 - U_0)$, $e^{(p+U_0-q_L^1)/\lambda} \leq \lambda/(\lambda-p)$ is satisfied. We can also see that $e^{(p+U_0-q_L^1)/\lambda}$ and $\lambda/(\lambda-p)$ both increase with respect to p. However, there is no value of p that can make the aforementioned inequality hold as an equality. Given the continuity of both sides, $e^{(p+U_0-q_L^1)/\lambda} \leq \lambda/(\lambda-p)$ is invariably satisfied; consequently, $\tau(p-\lambda)e^{(p+U_0)/\lambda} + \tau\lambda e^{q_L^1/\lambda} > 0$. Thus, we obtain $\partial R_L^1(p)/\partial p < 0$, and $R_L^1(p)$ decreases with respect to p. The second derivative of $R_L^1(p) = p * \pi_L^1(p)$ with respect to p is

$$\frac{\partial^2 R_L^1(p)}{\partial p^2} = -\frac{(1-\tau)\left[\operatorname{csch}((p+U_0-q_L^1)/2\lambda)\right]^2}{4\tau\lambda^2} * \left[2\lambda - p\coth\left(\frac{p+U_0-q_L^1}{2\lambda}\right)\right].$$

It is intuitive that $-(1-\tau)\left[\operatorname{csch}((p+U_0-q_L^1)/2\lambda)\right]^2/4\tau\lambda^2 \leq 0$. In addition, according to the property of $\coth(x)$, we obtain $2\lambda/(p+U_0-q_L^1) < \coth((p+U_0-q_L^1)/2\lambda)$, which implies $2\lambda - (p+U_0-q_L^1)\coth((p+U_0-q_L^1)/2\lambda) < 0$. Since $p > q_L^1 - U_0$, we also have $2\lambda - p\coth((p+U_0-q_L^1)/2\lambda) < 0$. Therefore, $\partial^2 R_L^1(p)/\partial p^2 > 0$ and $R_L^1(p)$ is convex with respect to p. ■

A.6. Proof of Corollary 1

Proof.

This result can be easily obtained via Proposition 2 and Proposition 3. ■

A.7. Proof of Proposition 4

Proof.

Similarly, it suffices to prove the case for $k = 1$.

(i) To simplify the proof, we rewrite the unconditional purchase probability as

$$\pi_0^1(p) = \frac{1 - \tau e^{U_H^1/\lambda} - (1-\tau)e^{U_L^1/\lambda}}{(e^{U_H^1/\lambda} - 1)(e^{U_L^1/\lambda} - 1)},$$

where $U_H^1 = 2q_H - p - r$, $U_L^1 = (1+\alpha)q_L - p - \alpha r$, and $r = \beta q_H + (1-\beta)q_L$. The first derivative of $\pi_0^1(p)$ with respect to λ is

$$\frac{\partial \pi_0^1(p)}{\partial \lambda} = \frac{M}{\lambda^2(e^{U_H^1/\lambda} - 1)^2(e^{U_L^1/\lambda} - 1)^2},$$

where

$$\begin{aligned} M =& [(\tau U_H e^{U_H^1/\lambda} + (1-\tau)U_L e^{U_L^1/\lambda})(e^{U_H^1/\lambda} - 1)(e^{U_L^1/\lambda} - 1) \\ &+ (1 - \tau e^{U_H^1/\lambda} - (1-\tau)e^{U_L^1/\lambda})(U_H e^{U_H/\lambda}(e^{U_L/\lambda} - 1) + U_L e^{U_L/\lambda}(e^{U_H/\lambda} - 1))]. \end{aligned}$$

Since $\lambda^2(e^{U_H^1/\lambda}-1)^2(e^{U_L^1/\lambda}-1)^2 \geq 0$, we only need to focus on the positive and negative signs of the numerator M. After calculation, M is ultimately simplified to

$$M = \tau U_L e^{U_L^1/\lambda}(2e^{U_H^1/\lambda} - e^{2U_H^1/\lambda} - 1) + (1-\tau)U_H e^{U_H^1/\lambda}(2e^{U_L^1/\lambda} - e^{2U_L^1/\lambda} - 1).$$

Note that $2e^x - e^{2x} - 1 \leq 0$ and $U_H > 0, U_L < 0$, so $\tau U_L(2e^{U_H^1/\lambda} - e^{2U_H^1/\lambda} - 1) \geq 0$ and $(1-\tau)U_H(2e^{U_L^1/\lambda} - e^{2U_L^1/\lambda} - 1) \leq 0$. Since $\partial\pi_0^1(p)/\partial\lambda$ decreases with τ, when τ approaches 0, $\partial\pi_0^1(p)/\partial\lambda \leq 0$, and when τ approaches 1, $\partial\pi_0^1(p)/\partial\lambda \geq 0$. Therefore there is a unique τ_c^1 such that when τ is lower than τ_c^1, $\partial\pi_0^1(p)/\partial\lambda \leq 0$, and $\partial\pi_0^1(p)/\partial\lambda \geq 0$ otherwise.

(*ii*) For low quality, we know that $R_L^{1*} = p_L^{1*} = \underline{p}^1 = q_H^1 + q_L^1 - \lambda\ln[\tau e^{q_H^1/\lambda} + (1-\tau)e^{q_H^1/\lambda}]$. According to the proof of Remark 1(i), we have $\partial p_L^{1*}/\partial\lambda \geq 0$. Therefore, R_L^{1*} and p_L^{1*} both increase with respect to λ.

(*iii*) For high quality, we have

$$\lim_{\lambda\to 0} R_H^{1*} = q_H^1 > \tau q_H^1 + (1-\tau)q_L^1 = \lim_{\lambda\to\infty} R_H^{1*}.$$

The first derivative of $R_H^1(\pi_H^1)$ with respect to π_H^1 is

$$\frac{\partial R_H^1(\pi_H^1)}{\partial \pi_H^1} = q_H^1 - U_0 + \lambda\ln[\epsilon(\pi_H^1)] + \lambda\pi_H^1\frac{\epsilon^{(1)}(\pi_H^1)}{\epsilon(\pi_H^1)},$$

where $\epsilon^{(1)}(\pi_H^1)$ represents the first derivative of $\epsilon(\pi_H^1)$ with respect to π_H^1. $\partial R_H^1(\pi_H^1)/\partial\pi_H^1$ in the limit as $\pi_H^1 \to 1$ becomes

$$q_H^1 - U_0 + \left\{\lambda\ln\left[\frac{e^{q_L^1/\lambda}}{(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}}\right] - \lambda\tau(1-\tau)\frac{(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})^2}{e^{q_L^1/\lambda}((1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda})}\right\}.$$

We refer to the content enclosed within curly braces {} as A. Since $(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})^2/(e^{q_L^1/\lambda}((1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda})) \geq 0$ and $e^{q_L^1/\lambda}/((1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}) \leq 1$, we obtain $A \leq 0$. The first derivative of A with respect to λ is

$$\begin{aligned}\frac{\partial A}{\partial\lambda} = &\underbrace{\frac{(1-\tau)(q_H^1 - q_L^1)e^{q_H^1/\lambda}}{\lambda((1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda})} + \ln\left[\frac{e^{q_L^1/\lambda}}{(1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda}}\right]}_{B}\\ &+ \underbrace{\frac{\tau(1-\tau)e^{-q_L^1/\lambda}(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})}{\lambda((1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda})^2}}_{C}\\ &\times \underbrace{\begin{matrix}((1-\tau)(q_H^1 - q_L^1 - \lambda)e^{2q_H^1/\lambda} + \tau\lambda e^{2q_L^1/\lambda}\\ +((q_H^1 - q_L^1)(1+\tau) + \lambda(1-2\tau))e^{(q_H^1+q_L^1)/\lambda})\end{matrix}}_{D}.\end{aligned}$$

Firstly, B decreases with respect to λ since

$$\frac{\partial B}{\partial\lambda} = -\frac{\tau(1-\tau)(q_H^1 - q_L^1)^2 e^{(q_H^1+q_L^1)/\lambda}}{\lambda^3((1-\tau)e^{q_H^1/\lambda} + \tau e^{q_L^1/\lambda})^2} \leq 0.$$

Also note that with $\lim_{\lambda\to\infty} B = 0$, we have $B \geq 0$. It is also obvious that $C \geq 0$. Next, D is convex with respect to q_H^1 since

$$\frac{\partial^2 D}{\partial (q_H^1)^2} = \frac{e^{q_H^1/\lambda}(4e^{q_H^1/\lambda}(q_H^1 - q_L^1)(1-\tau) + e^{q_L^1/\lambda}((q_H^1 - q_L^1)(1+\tau) + 3\lambda))}{\lambda^2} \geq 0,$$

When $q_H^1 = q_L^1$, D achieves its minimum value, which is 0. Hence, $D \geq 0$. As a result, A increases with respect to λ. We also have $\lim_{\lambda\to 0} A = -\infty$ and $\lim_{\lambda\to\infty} A = -(1-\tau)(q_H^1 - q_L^1)$. Thus, we find that when $\pi_H^1 \to 1$, $\partial R_H^1(\pi_H^1)/\partial \pi_H^1 = q_H^1 - U_0 + A$ increases with respect to λ. Moreover, when $\lambda \to 0$, $\partial R_H^1(\pi_H^1)/\partial \pi_H^1$ approaches $-\infty$, and when $\lambda \to \infty$, $\partial R_H^1(\pi_H^1)/\partial \pi_H^1$ approaches $\tau q_H^1 + (1-\tau) q_L^1 - U_0$. Therefore, there is a unique threshold $\lambda = \lambda_c^1$ and $\pi_{H1}^* = 1$ for $\lambda > \lambda_c^1$. This means that for $\lambda > \lambda_c^1$, the revenue-optimal price and revenue of both sellers will be the same (R_L^{1*} and p_L^{1*} both increase with respect to λ as proved above).

■

A.8. Proof of Proposition 5

Proof.

Similarly, it also suffices to prove the case for $k = 1$.

(i) The first derivative of $\pi_0^1(p)$ with respect to r is

$$\frac{\partial \pi_0^1(p)}{\partial r} = \frac{K}{\lambda(e^{U_H^1/\lambda} - 1)^2 (e^{U_L^1/\lambda} - 1)^2},$$

where

$$\begin{aligned} K =& [(\tau e^{U_H^1/\lambda} + (1-\tau)\alpha e^{U_L^1/\lambda})(e^{U_H^1/\lambda} - 1)(e^{U_L^1/\lambda} - 1) \\ &+ (1 - \tau e^{U_H^1/\lambda} - (1-\tau) e^{U_L^1/\lambda})((1+\alpha) e^{(U_H^1 + U_L^1)/\lambda} - e^{U_H^1/\lambda} - \alpha e^{U_L^1/\lambda})]. \end{aligned}$$

Since $\lambda(e^{U_H^1/\lambda} - 1)^2 (e^{U_L^1/\lambda} - 1)^2 \geq 0$, we only need to focus on the positive and negative signs of the numerator K. After calculation, K is ultimately simplified to

$$K = \tau\alpha e^{U_L^1/\lambda}(2e^{U_H^1/\lambda} - e^{2U_H^1/\lambda} - 1) + (1-\tau) e^{U_H^1/\lambda}(2e^{U_L^1/\lambda} - e^{2U_L^1/\lambda} - 1) \leq 0.$$

Therefore, $\partial \pi_0^1(p)/\partial r \leq 0$. Moreover, because $\partial r/\partial \beta \geq 0$, we have $\partial \pi_0^1(p)/\partial \beta \leq 0$.

(ii) We rewrite q_H^1 and q_L^1 as $q_H^1 = 2q_H - r$ and $q_L^1 = (1+\alpha) q_L - \alpha r$. Since

$$\frac{\partial R_L^{1*}}{\partial r} = -1 - \alpha + \frac{\alpha\tau e^{q_L^1/\lambda} + (1+\tau) e^{q_H^1/\lambda}}{\tau e^{q_L^1/\lambda} + (1+\tau) e^{q_H^1/\lambda}} < -1 < 0$$

and $\partial r/\partial \beta > 0$, R_L^{1*} and p_L^{1*} are both decreasing with respect to β.

The first derivative of $R_H^1(p)$ with respect to r is

$$\frac{\partial R_H^1(p)}{\partial r} = \frac{p}{\tau\lambda}\left[\frac{(1-\alpha) e^{(q_L^1 - q_H^1)/\lambda}}{(1 - e^{(q_L^1 - q_H^1)/\lambda})^2} - \frac{(1-\tau) e^{(p - q_H^1)/\lambda}}{(1 - e^{(p - q_H^1)/\lambda})^2}\right] < 0.$$

We can see that

$$\frac{\partial^2 R_H^1(p)}{\partial p \partial r} = \frac{1}{\tau\lambda}\left[\frac{(1-\alpha)e^{(q_L^1-q_H^1)/\lambda}}{(1-e^{(q_L^1-q_H^1)/\lambda})^2} - \frac{(1-\tau)e^{(p-q_H^1)/\lambda}}{(1-e^{(p-q_H^1)/\lambda})^2}\right] + \frac{p}{\tau}\left[-\frac{2(1-\tau)e^{(p-q_H^1)/\lambda}}{\lambda^2(1-e^{(p-q_H^1)/\lambda})^3} - \frac{e^{(p-q_H^1)/\lambda}}{\lambda(1-e^{(p-q_H^1)/\lambda})^2}\right] < 0$$

Therefore,

$$\frac{\partial p_H^{1*}}{\partial r} = -\frac{\partial^2 R_H^1(p)/\partial p \partial r}{\partial^2 R_H^1(p)/\partial p^2} < 0.$$

Furthermore, because $\partial r/\partial \beta \geq 0$, we have $\partial p_H^{1*}/\partial \beta < 0$ and $\partial R_H^{1*}/\partial \beta < 0$. ■

A.9. Proof of Proposition 6

Proof.

(i) For $k = 1$, we have

$$\pi_0^1(p) = \frac{1 - \tau e^{U_H^1/\lambda} - (1-\tau)e^{U_L^1/\lambda}}{(e^{U_H^1/\lambda} - 1)(e^{U_L^1/\lambda} - 1)}.$$

The first derivative of $\pi_0^1(p)$ with respect to α is

$$\frac{\partial \pi_0^1(p)}{\partial \alpha} = \frac{\tau\beta(q_L - q_H)e^(U_L/\lambda)}{\lambda(e^{U_L^1/\lambda} - 1)^2} < 0,$$

So $\pi_0^1(p)$ is decreasing with respect to α.

For $k = 2$, because the customers will not feel the loss at this time, the unconditional purchase probability $\pi_0^2(p)$ is independent of α.

(ii) For low quality, we have $R_L^{1*} = p_L^{1*} = q_H^1 + q_L^1 - \lambda \ln[\tau e^{q_L^1/\lambda} + (1-\tau)e^{q_H^1/\lambda}]$. The first derivative of R_L^{1*} with respect to α is

$$\frac{\partial R_L^{1*}}{\partial \alpha} = \frac{\beta(q_L - q_H)(1-\tau)e^{q_H^1/\lambda}}{\tau e^{q_L^1/\lambda} + (1-\tau)e^{q_H^1/\lambda}} < 0.$$

Thus R_L^{1*} and p_L^{1*} are decreasing with respect to α.

For high quality, we note that

$$R_H^1(p) = p * \pi^1(p) = \frac{p}{\tau}\left[\frac{1}{1 - e^{(q_L^1-q_H^1)/\lambda} - \frac{1-\tau}{1-e^{(p-q_H^1)/\lambda}}}\right].$$

The first derivative of $R_H^1(p)$ with respect to α is

$$\frac{\partial R_H^1(p)}{\partial \alpha} = \frac{p}{\tau}\left[\frac{\beta(q_L - q_H)e^{(q_L^1-q_H^1)/\lambda}}{\lambda(1-e^{(q_L^1-q_H^1)/\lambda})^2}\right] < 0.$$

Note that

$$\frac{\partial^2 R_H^1(p)}{\partial p \partial \alpha} = \frac{1}{\tau}\left[\frac{\beta(q_L - q_H)e^{(q_L^1 - q_H^1)/\lambda}}{\lambda(1 - e^{(q_L^1 - q_H^1)/\lambda})^2}\right] < 0.$$

We also have $\partial^2 R_H^1(p)/\partial^2 p < 0$, so $-(\partial^2 R_H^1(p)/\partial p \partial \alpha)/(\partial^2 R_H^1(p)/\partial^2 p) < 0$. In summary, the optimal revenue R_H^{1*} and price p_H^{1*} for the high-quality seller are also decreasing with respect to α.

For $k = 2$, because the customers will not feel the loss at this time, R_H^{2*} and the price p_H^{2*} are independent of α. ■

A.10. Proof of Proposition 7

Proof.

Similarly, it also suffices to prove the case for $k = 1$.

(i) The first derivative of $\pi_0^1(p)$ with respect to τ is

$$\frac{\partial \pi_0^1(p)}{\partial \tau} = \frac{e^{U_L^1/\lambda} - e^{U_H/\lambda}}{(e^{U_H/\lambda} - 1)(e^{U_L/\lambda} - 1)}.$$

Since $e^{U_H/\lambda} - 1 > 0$ and $e^{U_L/\lambda} - 1 < 0$, we have $\partial \pi_0^1(p)/\partial \tau > 0$.

(ii) For low quality, we have $R_L^{1*} = q_H^1 + q_L^1 - \lambda \ln[\tau e^{q_L^1/\lambda} + (1-\tau)e^{q_H^1/\lambda}]$. Since

$$\frac{\partial R_L^{1*}}{\partial \tau} = \frac{\lambda(e^{q_H^1/\lambda} - e^{q_L^1/\lambda})}{\tau e^{q_L^1/\lambda} + (1-\tau)e^{q_H^1/\lambda}} \geq 0,$$

R_L^{1*} is increasing with respect to τ.

For high quality, we have

$$R_H^1(p) = p * \pi_H^1(p) = \frac{p}{\tau}\left[\frac{1}{1 - e^{(q_L^1 - q_H^1)/\lambda}} - \frac{(1-\tau)}{1 - e^{(p - q_H^1)/\lambda}}\right].$$

The first derivative of $R_H^1(p)$ with respect to τ is

$$\frac{\partial R_H^1(p)}{\partial \tau} = -\frac{p}{\tau^2}\left[\frac{1}{1 - e^{(q_L^1 - q_H^1)/\lambda}} - \frac{1}{1 - e^{(p - q_H^1)/\lambda}}\right] > 0.$$

Note that

$$\frac{\partial^2 R_H^1(p)}{\partial p \partial \tau} = -\frac{1}{\tau^2}\left[\frac{1}{1 - e^{(q_L^1 - q_H^1)/\lambda}} - \frac{1}{1 - e^{(p - q_H^1)/\lambda}}\right] + \frac{p}{\tau^2}\left[\frac{e^{(p - q_H^1)/\lambda}}{\lambda(1 - e^{(p - q_H^1)/\lambda})^2}\right] > 0.$$

We also have $\partial^2 R_H^1(p)/\partial^2 p < 0$, so $-(\partial^2 R_H^1(p)/\partial p \partial \tau)/(\partial^2 R_H^1(p)/\partial^2 p) > 0$. Therefore, $R_H^{1*}(p)$ and p_H^{1*} are both increasing with respect to τ. ■

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