Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(1): 282–309.
DOI:10.3934/jimo.2026011
Received: 23 June 2025
Revised: 12 October 2025
Accepted: 22 October 2025
Published: 25 November 2025

Research article

When engagement performs better: Revenue management on User-Generated Content platforms

Run Tang[1]**, Zhengyang Wang**[2]**, Yangyang Peng**[3,*]**, Ziyuan Zeng**[4]**, Tong Zhang**[5] **and Yingjun Shen**[6]

[1] School of Management Science and Engineering, Nanjing University of Finance & Economics, Nanjing 210023, China

[2] School of Government, Nanjing University, Nanjing 210023, China

[3] Shenzhen Research Institute of Big Data, Shenzhen 518172, China

[4] School of Public Policy, The Chinese University of Hong Kong, Shenzhen, Shenzhen 518172, China

[5] College of Art & Science, Boston University, Boston, Massachusetts 02215, USA

[6] Shenzhen Finance Institute, School of Management and Economics, The Chinese University of Hong Kong, Shenzhen 518172, China

* **Correspondence:** Email: pengyangyang@cuhk.edu.cn.

Supplementary

A: Proofs for major results

Proof. Proof of Proposition 1. For completeness, users who do not meet all of the constraints in analysis can be defined as not watching content and not interacting, known as Not View. Since very few users interact without watching a video at all, we exclude such users. As shown in Figure 2, we let υ_1 be the user type who is indifferent between View & Engage and View & Not Engage while letting υ_2 be the user type who is indifferent between view & Not Engage and Not view. Combined with the above analysis, we conclude the following:

- Users who choose to view the content and engage need to satisfy their rationality constraints ($U_A > 0$) and incentive-compatible constraints ($U_A > U_B$).
- Similarly, users who choose to view but not engage need to meet IR constraints ($U_B > 0$) and IC constraints ($U_B > U_A$).

$$\upsilon_1 = \frac{c_2 - q - \zeta q}{1-\zeta}, \upsilon_2 = \frac{c_1 - \zeta q}{\zeta}, \upsilon_3 = c_1 + c_2 - q,$$

$$\mathcal{X} = 1 - \frac{c_1 - \zeta q}{\zeta}, \mathcal{Y} = 1 - \frac{c_2 - q + \zeta q}{1-\zeta}.$$

□

Proof. Proof of Lemma 1:

The equilibrium solution of the linear model are: $q_1^* = \frac{2\kappa(m-\kappa)(\zeta-c_1)}{(\alpha\beta+2\kappa^2-2\kappa m)\zeta}$ and $\eta_1^* = \frac{b(m-\kappa)(\zeta-c_1)}{(\alpha\beta+2\kappa^2-2\kappa m)\zeta}$. According to the analysis in 3.1, we obtain $c_1 < \frac{\zeta}{1-\zeta}c_2$. According to Assumption 1, there is $c_2 < 1-\zeta$, so we get $\mathrm{c}_1 < \zeta$. Moreover, because the external revenue m must be greater than the commission κ, $q_1^* > 0$ and $n_1^* > 0$ are equivalent to $2\kappa^2 - 2m\kappa + \alpha\beta > 0$. When $m \leq \sqrt{2\alpha\beta}$, $\alpha\beta + 2\kappa^2 - 2\kappa m > 0$ is always true. When $m > \sqrt{2\alpha\beta}$, $\kappa < \underline{\kappa}$ or $\kappa > \bar{\kappa}$ should be satisfied, where $\underline{\kappa} = \frac{m}{2} - \frac{\sqrt{m^2-2\alpha\beta}}{2}$, $\bar{\kappa} = \frac{m}{2} + \frac{\sqrt{m^2-2\alpha\beta}}{2}$.

□

Proof. Proof of Lemma 2: The equilibrium solutions of EOM are:

$$q_2^* = \frac{\kappa m\zeta\,(2-c_2-2\zeta) - 2\kappa^2\zeta\,(1-c_2-\zeta) - \kappa m c_1(1-\zeta)}{(\alpha\beta+2\kappa^2-2\kappa m)\,(1-\zeta)\zeta},$$

$$\eta_2^* = \frac{\alpha\beta\left[m\,(\zeta-c_1)\,(1-\zeta) - \zeta\kappa\,(1-c_2-\zeta)\right] - \kappa m(m-\kappa)\left[\zeta c_2 - c_1(1-\zeta)\right]}{\alpha\,(\alpha\beta+2\kappa^2-2\kappa m)\,(1-\zeta)\zeta}.$$

- (1) $q_2^* > 0$:
  The numerator of q^* is $\kappa m\zeta\,(2-c_2-2\zeta) - 2\kappa^2\zeta\,(1-c_2-\zeta) - \kappa m c_1(1-\zeta)$. According to Lemma 1, we obtain the inequality $c_1(1-\zeta) < \zeta c_2$, so the numerator of q^* is greater than $\kappa m\zeta\,(2-c_2-2\zeta) - 2\kappa^2\zeta\,(1-c_2-\zeta) - \kappa m\zeta c_2$, which can be rewritten as $2\kappa(m-\kappa)\zeta\,(1-c_2-\zeta)$. Because of $m > \kappa$ and $c_2 \leq 1-\zeta$, $2\kappa(m-\kappa)\zeta\,(1-c_2-\zeta)$ is positive, so the numerator of q^* is positive. So $q_2^* > 0$ is equivalent to $2\kappa^2 - 2m\kappa + \alpha\beta > 0$, which has been proved in the proof of lemma 1 .
- (2) $\eta_2^* > 0$:
  The numerator of η^* is $\alpha\beta\left[m\,(\zeta-c_1)\,(1-\zeta) - \zeta\kappa\,(1-c_2-\zeta)\right] - \kappa m(m-\kappa)\left[\zeta c_2 - c_1(1-\zeta)\right]$. We assume that $2\kappa^2 - 2m\kappa + \alpha\beta > 0$, the numerator of η^* is greater than $2\kappa(m-\kappa)\left[m\,(\zeta-c_1)\,(1-\zeta) - \zeta\kappa\,(1-c_2-\zeta)\right] - \kappa m(m-\kappa)\left[\zeta c_2 - c_1(1-\zeta)\right]$, which can be rewritten as $= \kappa(m-\kappa)\tilde{A}$. Where $\tilde{A} = 2\zeta m(1-\zeta) - mc_1(1-\zeta) - 2\zeta\kappa\,(1-c_2-\zeta) - \zeta m c_2$. Similarly, we use the inequality $c_1(1-\zeta) < \zeta c_2$, so we can obtain: $\tilde{A} > 2\zeta m(1-\zeta) - 2\zeta\kappa\,(1-c_2-\zeta) - 2\zeta m c_2$, which can be rewritten as $2\zeta(m-\kappa)\,(1-c_2-\zeta)$.It is obvious to tell that $2\zeta(m-\kappa)\,(1-c_2-\zeta) > 0$, which means when $2\kappa^2 - 2m\kappa + \alpha\beta > 0$, $n_2^* > 0$ holds. The rest of the proof process is the same as Lemma 1.

□

Proof. Proof of Proposition 2:

Under the assumption of $m \leq \sqrt{2\alpha\beta}$, we obtain $\frac{\partial q_1^*}{\partial\kappa} = \frac{2\alpha\beta(2\kappa-m)(\mathrm{c}_1-\zeta)}{(\alpha\beta+2\kappa(\kappa-m))^2\zeta}$ and $\frac{\partial\Pi_{11}^*}{\partial\kappa} = \frac{\alpha\beta^2(m-2\kappa)(\mathrm{c}_1-\zeta)^2}{(\alpha\beta+2\kappa(\kappa-m))^2\zeta^2}$. Obviously, when $\kappa < \frac{m}{2}$, $\frac{\partial q^*}{\partial\kappa} > 0$ and $\frac{\partial\Pi_{u1}^*}{\partial\kappa} > 0$ hold. When $\kappa > \frac{m}{2}$, $\frac{\partial q^*}{\partial\kappa} < 0$ and $\frac{\partial\Pi_{u1}^*}{\partial\kappa} < 0$. Similarly,

$\frac{\partial \eta_1^*}{\partial \kappa} = \frac{\beta(\alpha\beta - 2(\kappa-m)^2)(c_1-\zeta)}{(\alpha\beta+2\kappa(\kappa-m))^2\zeta}$ and $\frac{\partial \Pi_{p1}^*}{\partial \kappa} = \frac{\alpha\beta^2(\alpha\beta-2(\kappa-m)^2)(\kappa-m)(c_1-\zeta)^2}{(\alpha\beta+2\kappa(\kappa-m))^3\zeta^2}$. When $\kappa < m - \frac{\sqrt{2\alpha\beta}}{2}$, $\frac{\partial \eta_1^*}{\partial \kappa} > 0$ and $\frac{\partial \Pi_{p1}^*}{\partial \kappa} > 0$. When $\kappa > m - \frac{\sqrt{2\alpha\beta}}{2}$, $\frac{\partial \eta_1^*}{\partial \kappa} < 0$ and $\frac{\partial \Pi_{p1}^*}{\partial \kappa} < 0$.

□

Proof. Proof of Proposition 3:

- (1). $m \leq \sqrt{2\alpha\beta}$, the first order condition of the balanced quality in the EOM is: $\frac{\partial q_2^*}{\partial \kappa} = \frac{\alpha\beta\tilde{A}_2 - 2\kappa^2 m(\zeta c_2 - c_1(1-\zeta))}{(\alpha\beta+2\kappa(\kappa-m))^2(1-\zeta)\zeta}$. Where $\tilde{A}_2 = (-4\kappa\zeta(1-c_2-\zeta) + m\zeta(2-c_2-2\zeta) - mc_1(1-\zeta))$. Because of $c_1(1-\zeta) < \zeta c_2$, it can be obtained that $\tilde{A}_2$ is larger than $(m-2\kappa)\zeta(1-c_2-\zeta)$, which can be represented by $\tilde{A}_3$. When $\kappa < \frac{m}{2}$, it is obvious that $\tilde{A}_3 > 0$, which means $\tilde{A}_2 > 0$. Because $m \leq \sqrt{2\alpha\beta}$, the numerator of this first order condition $\alpha\beta\tilde{A}_2 - 2\kappa^2 m(\zeta c_2 - c_1(1-\zeta)) >$ can be scaled as $\frac{m^2}{2}\tilde{A}_2 - 2\kappa^2 m(\zeta c_2 - c_1(1-\zeta))$, which can be represented by $\tilde{A}_4$. Because $c_1(1-\zeta) < \zeta c_2$, we can obtain $\tilde{A}_4$ is greater than $m(m-2\kappa)\zeta(1- c_2 - \zeta)$, which is positive. Therefore, when $\kappa < \frac{m}{2}$, $\frac{\partial q_2^*}{\partial \kappa} > 0$ is true.
- (2). In addition, under the EOM, the first-order conditions for exposure and profit of both participants on κ ,respectively, are:

$$\frac{\partial \eta_2^*}{\partial \kappa} = \frac{b\left(c_1(2\kappa-m)m(-1+\zeta) + \zeta\left(-\alpha\beta(-1+c_2+\zeta) + 2\kappa^2(-1+c_2+\zeta) + T\right.\right.}{(\alpha\beta+2\kappa(\kappa-m))^2(-1+\zeta)\zeta},$$
$$T = -2\kappa m(-2+c_2+2\zeta) + m^2(-2+c_2+2\zeta),$$

$$\frac{\partial \Pi_{u2}^*}{\partial \kappa} = \frac{\tilde{B}_1 + \tilde{B}_2 - \tilde{B}_3}{\alpha(\alpha\beta+2\kappa(\kappa-m))^2(-1+\zeta)^2\zeta^2},$$
$$\frac{\partial \Pi_{p2}^*}{\partial \kappa} = \frac{\tilde{B}_4\tilde{B}_5}{(\alpha\beta+2\kappa(\kappa-m))^3(-1+\zeta)^2\zeta^2}.$$

Here, $\tilde{B}_1 = -\kappa^2 m^3(c_1(-1+\zeta) + c_2\zeta)^2$, $\tilde{B}_2 = \alpha\beta\kappa m(c_1(-1+\zeta) + c_2\zeta)(c_1 m(-1+\zeta) + c_2 m\zeta + 2\kappa\zeta(-1+c_2+\zeta))$, $\tilde{B}_3 = a^2b^2\zeta(-1+c_2+\zeta)(c_1 m(-1+\zeta) + \zeta(m - m\zeta + 2\kappa(-1+c_2+\zeta)))$,

$$B_{41} = b(-c_1(2\kappa-m)m(-1+\zeta) + \zeta(\alpha\beta(-1+c_2+\zeta),$$

$$B_{42} = -2\kappa^2(-1+ c_2+\zeta) + 2\kappa m(-2+c_2+2\zeta) - m^2(-2+c_2+2\zeta),$$

$\tilde{B}_4 = B_{41} + B_{42}$, $\tilde{B}_5 = (\kappa(\kappa-m)m(c_1(-1+\zeta)+c_2\zeta) + \alpha\beta(c_1 m(-1+\zeta) + \zeta(m - m\zeta + \kappa(-1+c_2+\zeta))))$
The idea of the proof is the same as (1): Use the inequality $m \leq \sqrt{2\alpha\beta}$ to scale the first-order condition and replace all terms containing $\alpha\beta$. Then, the result can be scaled again using the inequality $c_1(1-\zeta) < \zeta c_2$, replacing the term containing ζc_2. Finally, the same kinds of items can be merged. It can be proved that at $\kappa < \frac{m}{2}$, $\frac{\partial \Pi_{u2}^*}{\partial \kappa} > 0$ holds. At $\kappa > \frac{m}{2}$, $\frac{\partial \Pi_{p2}^*}{\partial \kappa} < 0$ and $\frac{\partial \eta_2^*}{\partial \kappa} < 0$ hold.

□

Proof. Proof of Proposition 4 :

$$\Pi_{p2}^* - \Pi_{p1}^* = \frac{\kappa(c_2\zeta - c_1(1-\zeta))\widetilde{D}_1\widetilde{D}_2}{2a(\alpha\beta+2\kappa(\kappa-m))^2(-1+\zeta)^2\zeta^2}.$$

Here, $\widetilde{D}_1 = (\alpha\beta + (\kappa - m)m)$, $\widetilde{D}_2 = \kappa(\kappa - m)m\,(c_2\zeta - c_1(1-\zeta)) + \alpha\beta\widetilde{D}_3$, $\widetilde{D}_3 = c_1(\kappa - 2m)(1-\zeta) + 2m(1-\zeta)\zeta - \kappa\zeta\,(2 - c_2 - 2\zeta)$.

Since $m \leq \sqrt{2\alpha\beta}$, then it is obvious that $\widetilde{D}_2$ is greater than $\kappa(\kappa - m)m\,(c_2\zeta - c_1(1-\zeta)) + \frac{m^2}{2}\widetilde{D}_3$. Referring to the proof idea in proposition 2, it is easy to prove that $\kappa(\kappa - m)m\,(c_2\zeta - c_1(1-\zeta)) + \frac{m^2}{2}\widetilde{D}_3$ is positive, which means when $\kappa > \frac{m}{2}$, $\widetilde{D}_2 > 0$. Because of $\widetilde{D}_1 > 0$, therefore, when $\kappa > \frac{m}{2}$, $\Pi^*_{p2} > \Pi^*_{p1}$ is true. When $0 < \kappa < \frac{m}{2}$, only $\widetilde{D}_2 > 0$ holds. For $\Pi^*_{p2} > \Pi^*_{p1}$ to be true, $\widetilde{D}_1 > 0$ must be satisfied, so $\kappa > \frac{m^2-\alpha\beta}{m}$. In summary, when $\kappa > \frac{m^2-\alpha\beta}{m}$, $\Pi^*_{p2} > \Pi^*_{p1}$. When $\kappa < \frac{m^2-\alpha\beta}{m}$, $\Pi^*_{p2} < \Pi^*_{p1}$.

□

Proof. Proof of Proposition 5:

$$\Pi^*_{p2} - \Pi^*_{p1} = \frac{\kappa\,(c_2\zeta - c_1(1-\zeta))\,\tilde{F}_1}{2\alpha(\alpha\beta + 2\kappa(\kappa - m))(-1+\zeta)^2\zeta^2}$$

.

Here, $\tilde{F}_1 = \tilde{F}_2 + \tilde{F}_3$, $\tilde{F}_2 = \left[\kappa m^2 + 2\alpha\beta(\kappa - m)\right](c_2\zeta - c_1(1-\zeta))$, $\tilde{F}_3 = (2\kappa - m)\zeta(1 - c_2 - \zeta)$. $\tilde{F}_3 \geq 0$ is equivalent to $\kappa \geq \frac{m}{2}$. To ensure $\Pi^*_{p2} \geq \Pi^*_{p1}, \tilde{F}_2 \geq 0$ must be satisfied, i.e. $\kappa \geq \frac{2\alpha\beta m}{2\alpha\beta + m^2}$. If $\frac{m}{2} < \kappa < \frac{2\alpha\beta m}{2\alpha\beta + m^2}$, then $\tilde{F}_3 \geq 0$ and $\tilde{F}_2 < 0$. To ensure $\Pi^*_{p2} \geq \Pi^*_{p1}, \tilde{F}_1 \geq 0$ must be satisfied, i.e., $c_1 > \overline{c_1}$. Moreover, when $c_2 \leq \overline{c_2}$ and $\overline{c_1} < 0$, then $c_1 > \overline{c_1}$ constantly holds. In the case of $c_2 > \overline{c_2}, c_1 > \overline{c_1}$ must be satisfied. Therefore, it can be obtained that at $\frac{m}{2} < \kappa < \frac{2\alpha\beta m}{2\alpha\beta + m^2}$, it is necessary to meet the $c_2 \leq \overline{c_2}$ or $c_2 > \overline{c_2}, c_1 > \overline{c_1}$, and the proof is completed, where $\overline{c_1} = \frac{\zeta c_2\kappa m^2 - 2\alpha\beta\zeta\left[m(1-\zeta) - \kappa(2 - c_2 - \zeta)\right]}{\left[2\alpha\beta(\kappa - m) + \kappa m^2\right](1-\zeta)}$ and $\overline{c_2} = \frac{2\alpha\beta(m - 2\kappa)(1-\zeta)}{\kappa\left(m^2 - 2\alpha\beta\right)}$.

□

Proof. Proof of Corollary 1:

Comparing the conditions of Proposition 3 and Proposition 4, it can be concluded that when proposition 3 is true, proposition 4 must be true, and the proof is completed.

□

Proof. Proof of Corollary 2:

Let the equilibrium solution be $\left(q^*_1, n^*_1\right)$ in the linear sharing mode, and the equilibrium solution in the sharing mode based on the interaction rate is $\left(q^*_2, n^*_2\right)$.$q^*_2 - q^*_1 = \frac{\kappa(2\kappa - m)(c_2\zeta - c_1(1-\zeta))}{(\alpha\beta + 2\kappa(\kappa - m))(1-\zeta)\zeta}$. When $\kappa > \frac{m}{2}, q^*_2 > q^*_1$ holds. $n^*_2 - n^*_1 = \frac{\kappa(\alpha\beta + (\kappa - m)m)(c_2\zeta - c_1(1-\zeta))}{\alpha(\alpha\beta + 2\kappa(\kappa - m))(1-\zeta)\zeta}$. Additionally, $n^*_2 - n^*_1 > 0$ is equivalent to $\kappa > \frac{m^2 - \alpha\beta}{m}$. Thus, when the conditions of proposition 4 are satisfied, $q^*_2 > q^*_1$ and $n^*_2 > n^*_1$ hold.

The engagement-based sharing model leads to higher exposure $\left(n^*_2 > n^*_1\right)$, which increases the size of the total number of users. When calculating consumer surplus, the additional surplus brought about by the expansion of the total number of users is not considered, and only the change in consumer surplus of users who have remained in the market is calculated. Thus, the user's consumer surplus can be defined as:

$$CS = \int_{\upsilon_2}^{\upsilon_1} \zeta\upsilon + \zeta q^* - c_1 d\upsilon + \int_{\upsilon_1}^{1} \upsilon + q^* - c_1 - c_2\upsilon.$$

Using CS_1 to represent the consumer surplus in the linear mode and CS_2 to represent the consumer surplus in the interaction-rate-based sharing model yields:

$$CS_2 - CS_1 = \frac{\kappa(2\kappa - m)\left(c_2\zeta - c_1(1-\zeta)\right)\tilde{G}_1}{2(\alpha\beta + 2\kappa(\kappa - m))^2(-1+\zeta)^2\zeta^2}.$$

where $\tilde{G}_1 = \tilde{G}_2 + 2\alpha\beta\tilde{G}_3$, $\tilde{G}_2 = \kappa(c_1(-1+\zeta) + c_2\zeta)(m(3-4\zeta) + \kappa(-2+4\zeta))$, $\tilde{G}_3 = (-1 + c_1 + c_2)(-1+\zeta)\zeta$. Using the condition $2\alpha\beta \geq m^2$, $\tilde{G}_1$ is scaled down to obtain: $\tilde{G}_1 \geq \tilde{G}_2 + m^2\tilde{G}_3 = \tilde{G}_4 + \zeta c_2\tilde{G}_5$. Where $\tilde{G}_4 = (-1+\zeta)\left(-m^2\zeta + \mathrm{c}_1\left(\kappa m(3-4\zeta) + m^2\zeta + \kappa^2(-2+4\zeta)\right)\right)$ and $\tilde{G}_5 = m^2(-1+\zeta)\zeta + \kappa\zeta(m(3-4\zeta) + \kappa(-2+4\zeta))$. Using the condition $\zeta c_2 > c_1(1-\zeta)$, $\tilde{G}_4 + \zeta c_2\tilde{G}_5$ was scaled down to obtain: $\tilde{G}_4 + \zeta c_2\tilde{G}_5 > m^2(c_1 - \zeta)(-1+\zeta) > 0$. Therefore, $\tilde{G}_1 > 0$ holds. $CS_2 < CS_1$ holds when $\kappa < \frac{m}{2}$. When $\kappa > \frac{m}{2}$, $CS_2 > CS_1$ holds. In conclusion, $q_2^* > q_1^*$, $n_2^* > n_1^*$ and $CS_2 > CS_1$ all hold when the conditions of proposition 4 are satisfied.

□

Proof. Proof of Proposition 6:

(i)&(ii): Since $m > \sqrt{2\alpha\beta}, \underline{\kappa} < \frac{m}{2} < m - \frac{\sqrt{2\alpha\beta}}{2} < \bar{\kappa}$. where $\underline{\kappa} = \frac{m}{2} - \frac{\sqrt{m^2-2\alpha\beta}}{2}, \bar{\kappa} = \frac{m}{2} + \frac{\sqrt{m^2-2\alpha\beta}}{2}$. According to the conclusion of Proposition 1, it can be obtained: $\kappa < \underline{\kappa}$, $\frac{\partial q_1^*}{\partial\kappa} > 0$, $\frac{\partial\Pi_{u1}^*}{\partial\kappa} > 0$, $\frac{\partial n_1^*}{\partial\kappa} > 0$, $\frac{\partial\Pi_{p1}^*}{\partial\kappa} > 0$. At $\kappa > \bar{\kappa}$, $\frac{\partial q_1^*}{\partial\kappa} < 0$, $\frac{\partial\Pi_{u1}^*}{\partial\kappa} < 0$, $\frac{\partial n_1^*}{\partial\kappa} < 0$, $\frac{\partial\Pi_{p1}^*}{\partial\kappa} < 0$. Similarly, according to the conclusion of Proposition 2, it can be proved that $\kappa\left\langle\underline{\kappa}\right.$, $\frac{\partial q_2^*}{\partial\kappa} > 0$, $\frac{\partial\Pi_{u2}^*}{\partial\kappa} > 0$, $\frac{\partial n_2^*}{\partial\kappa} > 0$, $\frac{\partial\Pi_{p2}^*}{\partial\kappa} > 0$. At $\kappa > \bar{\kappa}$, $\frac{\partial q_2^*}{\partial\kappa} < 0$, $\frac{\partial\Pi_{u2}^*}{\partial\kappa} < 0$, $\frac{\partial n_2^*}{\partial\kappa} < 0$, $\frac{\partial\Pi_{p2}^*}{\partial\kappa} < 0$

(iii): Since $m > \sqrt{2\alpha\beta}$, so $\underline{\kappa} < \frac{m^2-\alpha\beta}{m} < \frac{2\alpha\beta m}{2\alpha\beta+m^2} < \bar{\kappa}$. So according to the conclusion of proposition 3 and proposition 4 , when $\kappa > \bar{\kappa}, \Pi_{p2}^* > \Pi_{p1}^*$ and $\Pi_{u2}^* > \Pi_{u1}^*$ hold. So when $\kappa < \underline{\kappa}$, $\Pi_{p2}^* < \Pi_{p1}^*$ and $\Pi_{u2}^* < \Pi_{u1}^*$ hold.

□

Proof. Proof of Proposition 7:

(1). $\Pi_{p1}^*\left(\kappa_1^*\right) = \frac{(c_1-\zeta)^2}{4(-2+\sqrt{2}m)^2\zeta^2}, \Pi_{p2}(0) = \frac{m^2(c_1-\zeta)^2}{2\zeta^2}, and \Pi_{p2}(m) = \frac{m^2(c_1(-1+\zeta)+c_2\zeta)^2}{2(-1+\zeta)^2\zeta^2}$, So there is $\Pi_{p2}(0) < \Pi_{p1}^*\left(\kappa_1^*\right) < \Pi_{p2}\left(\kappa_2^*\right)$, and by the intermediate value theorem, $\xi_1 \in \left(0, \kappa_2^*\right)$ exists, such that $\Pi_{p2}\left(\xi_1\right) = \Pi_{p1}^*\left(\kappa_1^*\right)$. Since the κ_2 is increasing on $\left(0, \kappa_2^*\right)$, ξ_1 is unique. In the same way, there is $\Pi_{p2}(m) < \Pi_{p1}^*\left(\kappa_1^*\right) < \Pi_{p2}\left(\kappa_2^*\right)$, which can be known from the intermediate value theorem and monotonicity, and there is only one unique $\xi_2 \in \left(\kappa_2^*, m\right)$. Thus we have $\Pi_{p_2}\left(\xi_2\right) = \Pi_{p1}^*\left(\kappa_1^*\right)$. Therefore, when $\xi_1 < \kappa_2 < \xi_2, \Pi_{p2}\left(\kappa_2\right) > \Pi_{p1}^*\left(\kappa_1^*\right)$ holds.

(2). $\Pi_{u1}^*\left(\kappa_1^*\right) = \frac{(\sqrt{2}-2m)(C1-\zeta)^2}{4(-\sqrt{2}+m)\zeta^2}$,

$$\Pi_{u2}(m) = \frac{m^2(c_1(-1+\zeta) + c_2\zeta)\left(c_1 m^2(-1+\zeta) + \left(2 + c_2\left(-2+m^2\right) - 2\zeta\right)\zeta\right)}{2(-1+\zeta)^2\zeta^2},$$

$\Pi_{u2}(0) = 0$. Similarly, the existence of $\eta_1 and \eta_2 \in (0, m)$ means $\Pi_{u2}\left(\eta_1\right) = \Pi_{u2}\left(\eta_2\right) = \Pi_{u1}^*\left(\kappa_1^*\right)$ is true and can be proved by the intermediate value theorem and monotonicity. Therefore, when $\eta_1 < \kappa_2 < \eta_2, \Pi_{u2}\left(\kappa_2\right) > \Pi_{u1}^*\left(\kappa_1^*\right)$. In other words, there is a maximum value of Π_{p2} in the interval (ξ_1, ξ_2) and a maximum value of Π_{u2} in the interval (η_1, η_2).

□

Proof. Proof of Corollary 3:

(1)We need to prove that the set $\left(\underline{\kappa_2}, \overline{\kappa_2}\right)$ is a non-empty set. Here, $\underline{\kappa_2} = \max\{\xi_1, \eta_1\}, \overline{\kappa_2} = \min\{\xi_2, \eta_2\}$. The maximum point of $\Pi_{p2}(\kappa_2)$ is represented by κ_a, and the maximum point of $\Pi_{u2}(\kappa_2)$ is represented by κ_b, where $\kappa_a = \kappa_2^* = \frac{A-\sqrt{B}}{2\zeta(1-c_2-\zeta)}$, $\kappa_b = \frac{m(c_1-\zeta)(-1+\zeta)}{c_2m^2(-1+\zeta)+\left(2+c_2\left(-2+m^2\right)-2\zeta\right)\zeta}$, $A = m\left(\zeta\left(2-c_2-2\zeta\right)+c_1\zeta-c_1\right)$, $B = c_1^2m^2(1-\zeta)^2 - 2c_1m^2(1-\zeta)^2\zeta + \left(-c_2^2\left(m^2- 2\right) + 2c_2\left(m^2-2\right)(1-\zeta) + 2(1-\zeta)^2\right)\zeta^2$. According to the conclusion of proposition 7, $\xi_1 < \kappa_a < \xi_2$ and $\eta_1 < \kappa_b < \eta_2$ hold true. It can be proved that when $m < \sqrt{2\alpha\beta}, \kappa_a < \frac{m}{2} < \kappa_b$ holds, so $\xi_1 < \frac{m}{2} < \eta_2$ holds. On the interval (κ_a, ξ_2), there is $\Pi_{p2}(\kappa_a) > \Pi_{p2}\left(\frac{m}{2}\right) > \Pi_{p1}^*\left(\kappa_1^*\right) = \Pi_{p2}(\xi_2)$. Since $\kappa_a < \frac{m}{2}$, and given monotonicity and the Intermediate Value Theorem, we have that $\frac{m}{2} < \xi_2$. Because $\Pi_{u2}(\kappa_b) > \Pi_{u2}\left(\frac{m}{2}\right) > \Pi_{u1}^*\left(\kappa_1^*\right) = \Pi_{p2}(\eta_1)$ and $\frac{m}{2} < \kappa_b$, and given monotonicity and the Intermediate Value Theorem, it can be concluded that $\eta_1 < \frac{m}{2}$. To sum up, $\eta_1 < \frac{m}{2} < \eta_2$ and $\xi_1 < \frac{m}{2} < \xi_2$, then $\frac{m}{2}$ must be an element in $\left(\underline{\kappa_2}, \overline{\kappa_2}\right)$, and the set $\left(\underline{\kappa_2}, \overline{\kappa_2}\right)$ is a non-empty set. Figure 4 shows a set of feasible parameters. (2)According to (1), it is established that the set $\left(\underline{\kappa_2}, \overline{\kappa_2}\right)$ is non-empty. Assuming the validity of the proof for $q_1^* > q_2(m/2) > q_2^*$, it becomes evident that $\left(\underline{\kappa_2}, \overline{\kappa_2}\right)$ constitutes a subset of $\left(q_2^*, q_1^*\right)$. Consequently, our sole objective reduces to demonstrating that $q_1^* > q_2(m/2) > q_2^*$, which can be readily inferred from the monotonicity property. Figure 4 shows a set of feasible parameters. □