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Research article

Optimal control of persuasive communication for emergency management

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Appendix

A.1. Proof of Proposition 1

When $R_0 < 1$, the Jacobian matrix at the equilibrium point Q_0, denoted as $J(Q_0)$, is given by:

$$J(Q_0) = \begin{pmatrix} -v & 0 & 0 & -\frac{k\varepsilon}{v} & -\frac{k\varepsilon}{v} \\ 0 & -1 & 0 & \frac{\alpha k\varepsilon}{v} & \frac{\alpha k\varepsilon}{v} \\ 0 & 0 & -1 & \frac{(1-\alpha)k\varepsilon}{v} & \frac{(1-\alpha)k\varepsilon}{v} \\ 0 & \beta_1 & \beta_2 & -v & 0 \\ 0 & 1-\beta_1-v & 1-\beta_2-v & 0 & -v \end{pmatrix} \quad \text{(A1)}$$

Due to the complexity of $J(Q_0)$, it is difficult to determine its definiteness by directly computing its eigenvalues. Therefore, this study uses the principal minor criterion to determine the sign-definiteness of $J(Q_0)$. The determinants of the principal minors are as follows:

$$det(L_1) = -v < 0 \quad \text{(A2)}$$

$$det(L_2) = v > 0 \quad \text{(A3)}$$

$$det(L_3) = -v < 0 \quad \text{(A4)}$$

$$det(L_4) = -k\varepsilon[\beta_1\beta_2 + (1-\alpha)(\beta_2(1-\beta_1-v) + \beta_2 v) + \alpha(\beta_1(1-\beta_2-v) + \beta_1 v)] + v^2 = -k\varepsilon[\alpha\beta_1 + (1-\alpha)\beta_2] + v^2 \quad \text{(A5)}$$

$$det(L_5) = k\varepsilon v[1-v] - v^3 \quad \text{(A6)}$$

Next, we examine the sign of (A5) and (A6). Given that $R_0 < 1$, we have:

$$\mathrm{k}\varepsilon[1-v] < v^2 \quad \text{(A7)}$$

Since $v > 0$, it follows that $det(L_5) < 0$. Moreover, we have the following equation:

$$\alpha\beta_1 + (1-\alpha)\beta_2 < \alpha(1-v) + (1-\alpha)(1-v) = 1-v \quad \text{(A8)}$$

Therefore, $det(L_4) > -k\varepsilon[1-v] + v^2 > 0$.

In summary, since $det(L_1), det(L_3), det(L_5) < 0$, and $det(L_2), det(L_4) > 0$, according to the principal minor criterion, the matrix $J(Q_0)$ is negative definite. Thus, by the Routh-Hurwitz criterion, when $R_0 < 1$, the equilibrium point Q_0 of system (1) is locally asymptotically stable.

The proof is finished.

A.2. Proof of Proposition 2

When $R_0 > 1$, the Jacobian matrix at the equilibrium point Q^*, denoted as $J(Q^*)$, is given by:

$$J(Q^*) = \begin{pmatrix} -\frac{\varepsilon}{U^*} & 0 & 0 & -kU^* & -kU^* \\ \frac{\alpha(\varepsilon - vU^*)}{U^*} & -1 & 0 & \alpha kU^* & \alpha kU^* \\ \frac{(1-\alpha)(\varepsilon - vU^*)}{U^*} & 0 & -1 & (1-\alpha)kU^* & (1-\alpha)kU^* \\ 0 & \beta_1 & \beta_2 & -v & 0 \\ 0 & 1-\beta_1-v & 1-\beta_2-v & 0 & -v \end{pmatrix} \quad \text{(A9)}$$

Due to the complexity of $J(Q^*)$, the principal minor criterion is again used. The determinants of the principal minors are as follows:

$$det(L_1) = -\frac{\varepsilon}{U^*} < 0 \quad \text{(A10)}$$

$$det(L_2) = \frac{\varepsilon}{U^*} > 0 \quad \text{(A11)}$$

$$det(L_3) = -\frac{\varepsilon}{U^*} < 0 \quad \text{(A12)}$$

$$det(L_4) = \frac{\varepsilon v}{U^*} - kvU^*[(1-\alpha)(\beta_2(1-\beta_1-v) + \beta_2 v) + \alpha(\beta_1(1-\beta_2-v) + \beta_1 v) + \beta_1\beta_2] \quad \text{(A13)}$$

$$det(L_5) = kv^2U^*[1-v] - \frac{\varepsilon v^2}{U^*} \tag{A14}$$

For (A13), substituting $U^* = \frac{v}{k(1-v)}$ and using equation $\alpha\beta_1 + (1-\alpha)\beta_2 < 1-v$, we obtain:

$$det(L_4) = k\varepsilon[1-v] - \frac{v^2}{(1-v)}[\alpha\beta_1 + (1-\alpha)\beta_2] > k\varepsilon[1-v] - v^2 \tag{A15}$$

Since $R_0 > 1$, it follows that $det(L_4) > 0$.

For (A14), substituting U^* as above, we obtain:

$$det(L_5) = v^3 - vk\varepsilon[1-v] \tag{A16}$$

Given $R_0 > 1$ and $v > 0$, it follows that $det(L_5) < 0$.

In summary, since $det(L_1), det(L_3), det(L_5) < 0$ and $det(L_2), det(L_4) > 0$, according to the principal minor criterion, the matrix $J(Q^*)$ is negative definite. Thus, by the Routh–Hurwitz criterion, when $R_0 > 1$, the equilibrium point Q^* of system (1) is locally asymptotically stable.

The proof is finished.

A.3. Proof of Proposition 3

From Equation (4), we have $\Delta = P^* - NP^* = \frac{(\varepsilon - vU^*)}{v}[2\alpha\beta_1 + 2(1-\alpha)\beta_2 + v - 1]$

Let $\Delta_{part} = 2\alpha\beta_1 + 2(1-\alpha)\beta_2 + v - 1$, then the original equation can be expressed as (A17).

$$\Delta = \frac{(\varepsilon - vU^*)}{v}\Delta_{part} \tag{A17}$$

Given that $R_0 = \frac{k\varepsilon(1-v)}{v^2} > 1$, the term can be derived as $\frac{(\varepsilon - vU^*)}{v} = \frac{k\varepsilon(1-v)-v^2}{k(1-v)v} > 0$; thus, the sign of Δ depends solely on the term Δ_{part}.

Because β_1, $\beta_2 \in [0,1-v]$ and $\alpha \in [0,1]$, so $\Delta_{part} \in [v-1, 1-v]$, indicating that the interval is symmetric about zero.

The partial derivatives of Δ_{part} with respect to α, β_1, β_2 are obtained as (A18).

$$\frac{\partial\Delta_{part}}{\partial\alpha} = 2(\beta_1 - \beta_2),\ \frac{\partial\Delta_{part}}{\partial\beta_1} = 2\alpha,\ \frac{\partial\Delta_{part}}{\partial\beta_2} = 2(1-\alpha) \tag{A18}$$

It is found that when $\beta_1 > \beta_2$, $\frac{\partial\Delta_{part}}{\partial\alpha} > 0$, indicating that an increase in α enhances Δ_{part}; conversely, when $\beta_1 \le \beta_2$, $\frac{\partial\Delta_{part}}{\partial\alpha} \le 0$, indicating that an decrease in α enhances Δ_{part}.

Moreover, the marginal impacts of β_1 and β_2 are moderated by α: higher α amplifies the effect of β_1, whereas lower α elevates the influence of β_2.

Since the sign of Δ depends on Δ_{part}, and Δ_{part} is governed by the values of α, β_1 and β_2, the size relationship between P^* and NP^* depends on the values of the parameters α, β_1, β_2.

The proof is finished.

A.4. The equilibrium sensitivity analysis of ε, k, v

There are different combinations of α, β_1, β_2, corresponding to the cognitions of the netizens and persuasion strategies. We choose eight combinations to test whether the changes in ε, k, v do not affect the relative sizes of equilibrium P^* and NP^*.

To represent different netizens' cognitions and persuasion strategies, we consider eight representative combinations of α, β_1, β_2. For each combination, we vary ε, k, and v over plausible ranges and examine whether such changes alter the relative sizes of the persuaded and not-persuaded groups. In numerical simulation, we use long-run values $P(\infty)$ and $NP(\infty)$ as proxies for the equilibrium values P^* and NP^*. The results are shown in Figures A1–A8.

(1) $\alpha = 0.2,\ \beta_1 = 0.2,\ \beta_2 = 0.8$

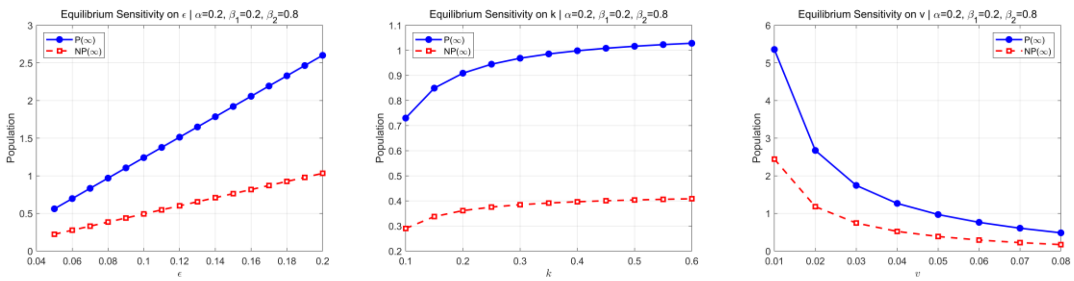

Figure A1. Equilibrium sensitivity analysis under $\alpha = 0.2,\ \beta_1 = 0.2,\ \beta_2 = 0.8$.

(2) $\alpha = 0.2,\ \beta_1 = 0.8,\ \beta_2 = 0.2$

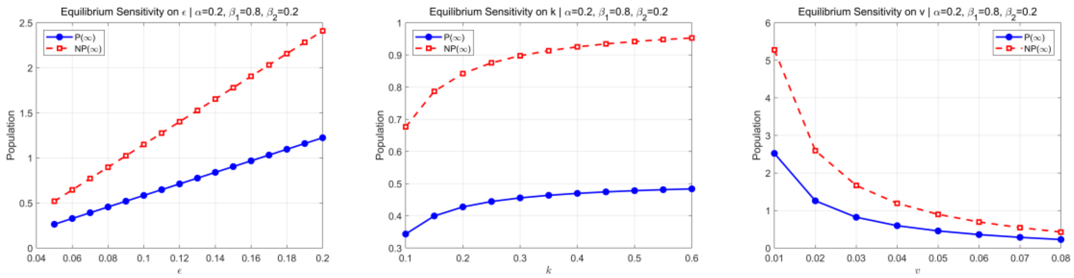

Figure A2. Equilibrium sensitivity analysis under $\alpha = 0.2,\ \beta_1 = 0.8,\ \beta_2 = 0.2$.

(3) $\alpha = 0.8,\ \beta_1 = 0.2,\ \beta_2 = 0.8$

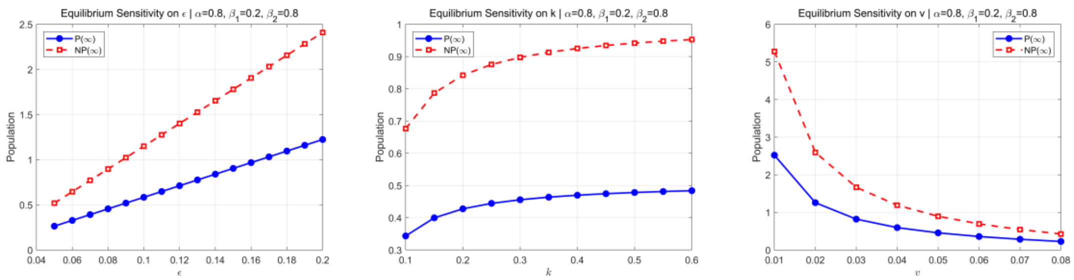

Figure A3. Equilibrium sensitivity analysis under $\alpha = 0.8,\ \beta_1 = 0.2,\ \beta_2 = 0.8$.

(4) $\alpha = 0.8,\ \beta_1 = 0.8,\ \beta_2 = 0.2$

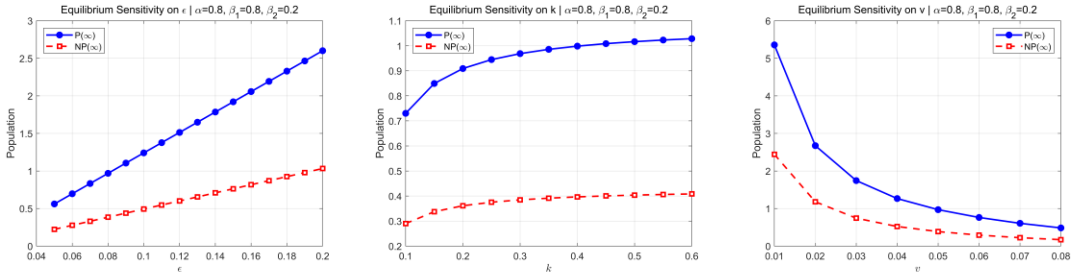

Figure A4. Equilibrium sensitivity analysis under $\alpha = 0.8,\ \beta_1 = 0.8,\ \beta_2 = 0.2$.

(5) $\alpha = 0.2,\ \beta_1 = 0.2,\ \beta_2 = 0.2$

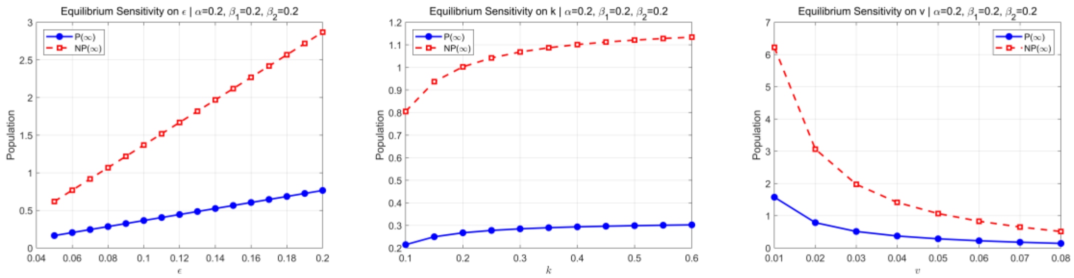

Figure A5. Equilibrium sensitivity analysis under $\alpha = 0.2,\ \beta_1 = 0.2,\ \beta_2 = 0.2$.

(6) $\alpha = 0.8,\ \beta_1 = 0.2,\ \beta_2 = 0.2$

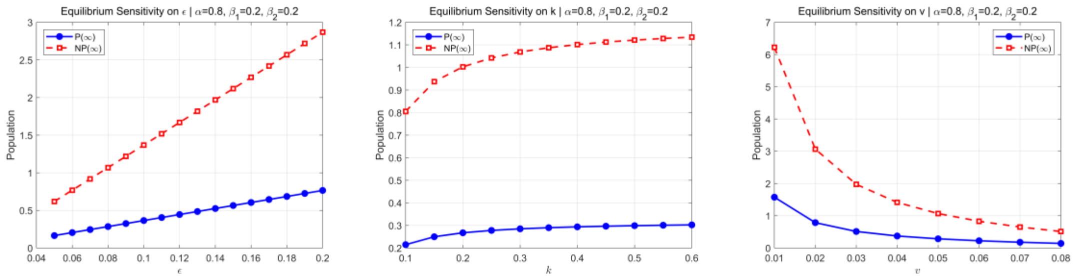

Figure A6. Equilibrium sensitivity analysis under $\alpha = 0.8,\ \beta_1 = 0.2,\ \beta_2 = 0.2$.

(7) $\alpha = 0.2,\ \beta_1 = 0.8,\ \beta_2 = 0.8$

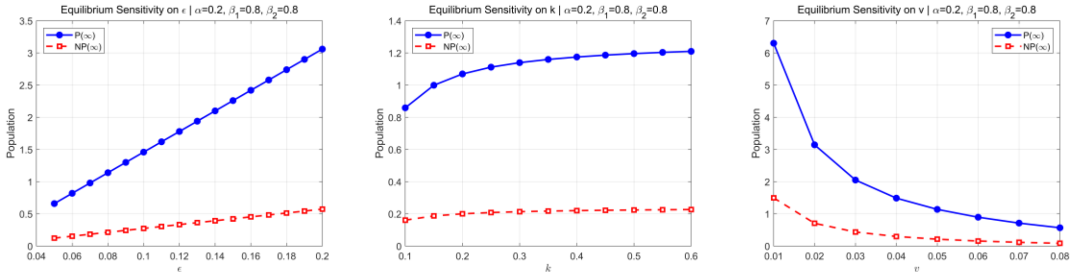

Figure A7. Equilibrium sensitivity analysis under $\alpha = 0.2,\ \beta_1 = 0.8,\ \beta_2 = 0.8$.

(8) $\alpha = 0.8,\ \beta_1 = 0.8,\ \beta_2 = 0.8$

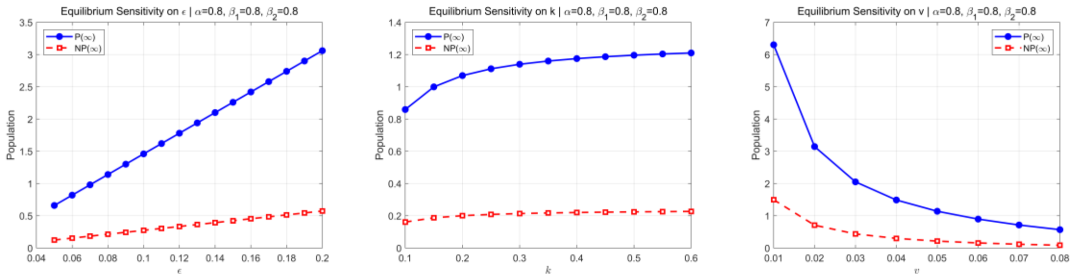

Figure A8. Equilibrium sensitivity analysis under $\alpha = 0.8,\ \beta_1 = 0.8,\ \beta_2 = 0.8$.

According to the simulation results, although the absolute levels of $P(\infty)$ and $NP(\infty)$ vary, the changes in ε, k, v do not affect the relative sizes of $P(\infty)$ and $NP(\infty)$. Therefore, the findings reported in the main text are robust.

A.5. Proof of Proposition 5

Within the interval $[0,1-v]$, the expressions of optimal persuasions are shown in (A19) and (A20).

$$\beta_1^*(t) = \frac{1}{C_1}\left(\lambda_{np}(t) - \lambda_p(t)\right) I_1(t) \tag{A19}$$

$$\beta_2^*(t) = \frac{1}{C_2}\left(\lambda_{np}(t) - \lambda_p(t)\right) I_2(t) \tag{A20}$$

Let us identify the optimal level of the steady-state by the superscript “s”. By the definitions, the candidate steady-state conditions are $\dot{U}(t) = \dot{I}_1(t) = \dot{I}_2(t) = \dot{P}(t) = \dot{NP}(t) = \dot{\lambda_p} = \dot{\lambda_{np}} = 0$. The following conditions are obtained as in (A21).

$$\begin{cases} \varepsilon - kU^s(P^s + NP^s) - vU^s = 0 \\ \alpha kU^s(P^s + NP^s) - I_1^s = 0 \\ (1-\alpha)kU^s(P^s + NP^s) - I_2^s = 0 \\ \beta_1 I_1^s + \beta_2 I_2^s - vP^s = 0 \\ (1-\beta_1 - v)I_1^s + (1-\beta_2 - v)I_2^s - vNP^s = 0 \\ \rho\lambda_p^s + 1 + \lambda_u^s kU^s - \lambda_{I_1}^s \alpha kU^s - \lambda_{I_2}^s(1-\alpha)kU^s + \lambda_p^s v = 0 \\ \rho\lambda_{np}^s - 1 + \lambda_u^s kU^s - \lambda_{I_1}^s \alpha kU^s - \lambda_{I_2}^s(1-\alpha)kU^s + \lambda_{np}^s v = 0 \end{cases} \quad \text{(A21)}$$

Some calculations yield the following steady-state in (A22), within the interval $[0,1-v]$.

$$\beta_1^s = \frac{2\alpha\left(k\varepsilon(1-v)-v^2\right)}{C_1(\rho+v)k(1-v)},\ \beta_2^s = \frac{2(1-\alpha)\left(k\varepsilon(1-v)-v^2\right)}{C_2(\rho+v)k(1-v)} \quad \text{(A22)}$$

In order to reflect the boundary $[0,1-v]$, the expressions are revised as in (A23).

$$\beta_1^s = min\left\{max\left\{\frac{2\alpha\left(k\varepsilon(1-v)-v^2\right)}{C_1(\rho+v)k(1-v)}, 0\right\}, 1-v\right\},\ \beta_2^s = min\left\{max\left\{\frac{2(1-\alpha)\left(k\varepsilon(1-v)-v^2\right)}{C_2(\rho+v)k(1-v)}, 0\right\}, 1-v\right\} \quad \text{(A23)}$$

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