
Research article

Incorporating systematic carbon risk into global business portfolios for carbon-intensive industries

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APPENDIX: Derivation of Extended Criteria

The extended ranking device is derived in this Appendix. Searching for the optimal portfolio compatible with the double-index model is the same as the single-index model.¹ Suppose that short sales are not allowed, a riskless lending and borrowing rate exists, and the double-index model adequately reflects the correlation structure between securities. In addition to the symbols defined in the main text, the following are used:

R_f = the riskless lending-borrowing rate

X_i = the relative weights we place on each security ($X_i > 0$ for a long position)

R_P = return on the portfolio (a random variable)

σ_P = the standard deviation of the return on the portfolio

Then the problem is to find a set of X_i 's to maximize:

$$\theta = \frac{\bar{R}_P - R_f}{\sigma_P}$$

where the bar over a variable denotes its expected value. Now:

$$\bar{R}_P - R_f = \sum_{i=1}^N X_i (\bar{R}_i - R_f)$$

¹ Elton et al. (2014) discuss the portfolio optimization for the single-index model in Chapter 7, and for multi-index models in Chapter 8.

and

$$\sigma_p^2 = E \left(\sum_{i=1}^N X_i R_i - \sum_{i=1}^N X_i \bar{R}_i \right)^2 = \sum_{i=1}^N X_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N X_i X_j \sigma_{ij}$$

Then:

$$\theta = \frac{\sum_{i=1}^N X_i (\bar{R}_i - R_f)}{\left[\sum_{i=1}^N X_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N X_i X_j \sigma_{ij} \right]^{1/2}}$$

As noted in Elton, Gruber, and Padberg (1976), θ is homogeneous of degree zero. Thus, maximizing θ without the constraint on the sum of the X_i 's yields the same optimum as maximizing θ with the constraint. To find that set of X_i 's which maximize θ , I take the derivative of θ with respect to each X_i and set it equal to zero.

$$\frac{d\theta}{dX_i} = (\bar{R}_i - R_f) - (\lambda X_1 \sigma_{1i} + \lambda X_2 \sigma_{2i} + \dots + \lambda X_N \sigma_{Ni}) = 0 \quad \text{for } i=1, \dots, N.$$

Here, λ is a constant. Defining a new variable $Z_i = \lambda X_i$, substituting Z_i for λX_i , and moving the variance-covariance terms to the right-hand side of the equality yields:

$$\bar{R}_i - R_f = Z_i \sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^N Z_j \sigma_{ij} \quad i=1, \dots, N \quad (\text{A.1})$$

where $\bar{R}_i$ is the expected return of security i , R_f is the return on the riskless asset, σ_i^2 is the variance of security i , σ_{ij} is the covariance between securities i and j , and Z_i is proportional to the amount invested in security i .

An optimum portfolio can be obtained by solving the following system of simultaneous equations (A.1). If the double-index model in equations (1 ~ 7) is used to describe the structure of security returns, then the covariance between securities i and j is:

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 + \gamma_i \gamma_j \sigma_s^2$$

and the variance of security i is:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \gamma_i^2 \sigma_s^2 + \sigma_{ei}^2$$

Substituting these relationships that hold for the double-index model into the system of simultaneous equations (A.1) yields:

$$\bar{R}_i - R_f = Z_i (\beta_i^2 \sigma_m^2 + \gamma_i^2 \sigma_s^2 + \sigma_{ei}^2) + \sum_{\substack{j=1 \\ j \neq i}}^N Z_j (\beta_i \beta_j \sigma_m^2 + \gamma_i \gamma_j \sigma_s^2) \quad i=1, \dots, N$$

Incorporating the term $Z_i (\beta_i^2 \sigma_m^2 + \gamma_i^2 \sigma_s^2)$ within the summation term yields:

$$\bar{R}_i - R_f = Z_i \sigma_{ei}^2 + \sum_{j=1}^N Z_j (\beta_i \beta_j \sigma_m^2 + \gamma_i \gamma_j \sigma_s^2) \quad i=1, \dots, N$$

Solving for Z_i and taking the constants outside the summation yields:

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_{ei}^2} - \frac{\beta_i \sigma_m^2}{\sigma_{ei}^2} \sum_{j=1}^N Z_j \beta_j - \frac{\gamma_i \sigma_s^2}{\sigma_{ei}^2} \sum_{j=1}^N Z_j \gamma_j \quad i=1, \dots, N \quad (\text{A.2})$$

Since short sales are not allowed, I introduce the constraints that all $Z_i \geq 0$. This requires

employing the Kuhn-Tucker conditions. Since the variance-covariance matrix is positive definite, the Kuhn-Tucker conditions are necessary and sufficient for an optimum, as shown in (Elton, Gruber, and Padberg 1976; 1978). The equivalent of equation (A.2) using the Kuhn-Tucker conditions is:

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_{ei}^2} - \frac{\beta_i \sigma_m^2}{\sigma_{ei}^2} \sum_{j=1}^N Z_j \beta_j - \frac{\gamma_i \sigma_s^2}{\sigma_{ei}^2} \sum_{j=1}^N Z_j \gamma_j + \mu_i \quad (\text{A.3}),$$

where $Z_i \geq 0$, $\mu_i \geq 0$, and $\mu_i Z_i = 0$ for all i .

Now, let me assume for a moment that I can find all securities, which would be in an optimal portfolio (call the set of such security k) and arrange these securities as $i=1, 2, \dots, k$. For the sub-population of securities that make up the optimal portfolio, equation (A.3) can be written as:

$$Z_i = \frac{\bar{R}_i - R_f}{\sigma_{ei}^2} - \frac{\beta_i \sigma_m^2}{\sigma_{ei}^2} \sum_{j=1}^k Z_j \beta_j - \frac{\gamma_i \sigma_s^2}{\sigma_{ei}^2} \sum_{j=1}^k Z_j \gamma_j \quad (\text{A.4}),$$

where $\mu_i = 0$. Note that $\sum_{j=1}^k Z_j \beta_j = \sum_{j=1}^N Z_j \beta_j$ and $\sum_{j=1}^k Z_j \gamma_j = \sum_{j=1}^N Z_j \gamma_j$ since the set k contains all securities with positive Z_i 's. I express Z_i in terms that do not invoke: $\sum_{j=1}^k Z_j \beta_j$ and $\sum_{j=1}^k Z_j \gamma_j$. First, multiplying both sides of equation (A.4) by β_i , summing over all securities in k , and rearranging yields:

$$\sum_{j=1}^k Z_j \beta_j = \frac{\sum_{j=1}^k [(\bar{R}_j - R_f) \beta_j / \sigma_{ej}^2]}{1 + \sum_{j=1}^k (\beta_j^2 \sigma_m^2 / \sigma_{ej}^2)} - \frac{\sum_{j=1}^k (\beta_j \gamma_j \sigma_s^2 / \sigma_{ej}^2)}{1 + \sum_{j=1}^k (\beta_j^2 \sigma_m^2 / \sigma_{ej}^2)} \sum_{j=1}^k Z_j \gamma_j \quad (\text{A.5})$$

Next, multiplying both sides of equation (A.4) by γ_i , summing over all securities in k , and rearranging yields:

$$\sum_{j=1}^k Z_j \gamma_j = \frac{\sum_{j=1}^k [(\bar{R}_j - R_f) \gamma_j / \sigma_{ej}^2]}{1 + \sum_{j=1}^k (\gamma_j^2 \sigma_s^2 / \sigma_{ej}^2)} - \frac{\sum_{j=1}^k (\beta_j \gamma_j \sigma_m^2 / \sigma_{ej}^2)}{1 + \sum_{j=1}^k (\gamma_j^2 \sigma_s^2 / \sigma_{ej}^2)} \sum_{j=1}^k Z_j \beta_j \quad (\text{A.6})$$

Next, the solution involves solving the system of simultaneous equations consisting of equations (A.5) and equation (A.6). The system can be written as:

$$\sum_{j=1}^k Z_j \beta_j = \phi_{1,k} - \phi_{2,k} \sum_{j=1}^k Z_j \gamma_j \quad (\text{A.7})$$

$$\sum_{j=1}^k Z_j \gamma_j = \phi_{3,k} - \phi_{4,k} \sum_{j=1}^k Z_j \beta_j \quad (\text{A.8})$$

where

$$\begin{aligned} \phi_{1,k} &= \frac{\sum_{j=1}^k [(\bar{R}_j - R_f) \beta_j / \sigma_{ej}^2]}{1 + \sum_{j=1}^k (\beta_j^2 \sigma_m^2 / \sigma_{ej}^2)} \\ \phi_{2,k} &= \frac{\sum_{j=1}^k (\beta_j \gamma_j \sigma_s^2 / \sigma_{ej}^2)}{1 + \sum_{j=1}^k (\beta_j^2 \sigma_m^2 / \sigma_{ej}^2)} \\ \phi_{3,k} &= \frac{\sum_{j=1}^k [(\bar{R}_j - R_f) \gamma_j / \sigma_{ej}^2]}{1 + \sum_{j=1}^k (\gamma_j^2 \sigma_s^2 / \sigma_{ej}^2)} \\ \phi_{4,k} &= \frac{\sum_{j=1}^k (\beta_j \gamma_j \sigma_m^2 / \sigma_{ej}^2)}{1 + \sum_{j=1}^k (\gamma_j^2 \sigma_s^2 / \sigma_{ej}^2)} \end{aligned}$$

Substituting equation (A.8) into equation (A.7) and solving for $\sum_{j=1}^k Z_j \beta_j$ yields:

$$\sum_{j=1}^k Z_j \beta_j = (\phi_{1,k} - \phi_{2,k} \phi_{3,k}) / (1 - \phi_{2,k} \phi_{4,k}) \quad (\text{A.9})$$

Substituting equation (A.9) into equation (A.8) and solving for $\sum_{j=1}^k Z_j \gamma_j$ yields:

$$\sum_{j=1}^k Z_j \gamma_j = (\phi_{3,k} - \phi_{1,k} \phi_{4,k}) / (1 - \phi_{2,k} \phi_{4,k}) \quad (\text{A.10})$$

Substituting equation (A.9) and equation (A.10) into equation (A.4) and rearranging, I obtain the following expression for Z_i :

$$Z_i = \frac{\omega_i}{\sigma_{ei}^2} \left[\frac{\bar{R}_i - R_f}{\omega_i} - \Phi_k \right]$$

, where $\mu_i = 0$, $\omega_i = \sqrt{\beta_i^2 \sigma_m^2 + \gamma_i^2 \sigma_s^2}$, and:

$$\Phi_k = \frac{\beta_i \sigma_m^2}{\omega_i} \left(\frac{\phi_{1,k} - \phi_{2,k} \phi_{3,k}}{1 - \phi_{2,k} \phi_{4,k}} \right) + \frac{\gamma_i \sigma_s^2}{\omega_i} \left(\frac{\phi_{3,k} - \phi_{1,k} \phi_{4,k}}{1 - \phi_{2,k} \phi_{4,k}} \right)$$

Q.E.D.