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Research article

Advanced results in enumeration of hyperfields

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Abstract: The purpose of this paper is to introducing a computational method to construction and classification of finite hyperfields (in the sense of Krasner). In this regards first we introduce a mathematical method to produce hyperfields from a family of a non-empty subsets of a given multiplicative group under specific conditions and then we apply this method to enumerate all finite hyperfields of order less that 7, up to isomorphism by a computer programming. Of course this program can be used to produce hyperfields of finite higher orders, but it's commotional complexity is of high order and it must be used of very high-speed computers system.

1. Introduction

As it is well known theory of hyperstructures (also known as multialgebras) was introduced in 1934 at the "8th congress of Scandinavian Mathematicians", where a French mathematician Marty [1] presented some definitions and results on the hypergroup theory, which is a generalization of groups. He gave applications of hyperstructures to rational functions, algebraic functions and non-commutative groups. Hyperstructure theory is an extension of the classical algebraic structure. In algebraic hyperstructures, the composition of two elements is a set, while in a classical algebraic structure, the composition of two elements is an element. Algebraic hyperstructure theory has many applications in other disciplines. Over the years hypercompositional structures have been used in algebra, geometry, convexity, automata theory and even in some applied sciences (for more details see [2–11]). Hyperring and hyperfield theory first was introduced and studied by M. Krasner in [12, 13]. He used a field and a subgroup of its multiplicative group for making a hyperstructure named hyperfield. Then this hyperstructure was studied and extended by other researchers (e.g. [14–19]). As in a hyperstructure the operation result for two elements could be multi valued Then associated a

projective space to every finite commutative extension of **K** hyperfield. Hyperfield structure at first was introduced as a quotient structure but based on its definition Ch. G. Massouros [14] proved that there is non-quotient structure of a hyperfield. Thus if we desire to have hyperfield structure to be as introduced by Krasner we need to reintroduce Krasner hyperfield whit more conditions. By field theory studies and achievements one can find many results about this structure(quotient hyperfield). In [20] A. Connes and C. Consani introduced **K**-extension hyperfield and **K**-vector space and stated some properties for Krasner hyperfields. In their research they find isomorphism relation between Krasner hyperfields extension of **K** and projective space with at least 4 on each line which some of them are isomorphic to non-quotient Krasner hyperfield extensions of **K**.

As well more hyperfields introduced like infinite hyperfields and tropical hyperfields [16] which some of them are non-quotient types too. So this structure is applicable as is defined. In this way hyperfield theory was introduced as a field of study in hyperstructure theory. Basic Studies and results about this structure as there exist non-quotient types of it, is not in a way that one can describe and classify all hyperfields. Throughout the studies of hyperfield theory various manners of constructing a hyperfield have been issued [21]. It is not clear all hyperfields can be constructed by these manners. So we were interested in to find all hyperfields for a given number of elements. As a hyperfields a hyperstructure and the out put of an sum operation of two members is a set, so for verify some properties or relations one has to study it by primitive approaches. For instance for a given hypergroupoid if we want to check associativity property we have to verify for every triple (a, b, c) of elements of the hypergroupoid.

Also, in researches [22–27] one can looking for a certain ways to come across, algorithms, mathematical softwares and computer programming.

The purpose of this paper is to introduce a method to construct finite hyperfields. In this regards, we briefly introduce some new themes in hyperfield theory and then we bring some results to show that new results achieved from them. At the end we introduce a mathematical method and a computer programming to construct finite hyperfields and classify them up to isomorphism. In particular, we use this program and present all finite hyperfields of order less than 7, up to isomorphism. Of course, our computer program has capacity to compute and classify all finite hyperfields, but for the complexity of computation, it need to use very high-speed computers system.

2. Preliminaries

In this section, we review some definitions and results, which we need to development our paper. For more details see e.g. [28–31].

Let H be a nonempty set and $P^*(H)$ denotes the family of all nonempty subsets of H. A *hyperoperation(or hypercomposition)* on H is a function

$$\circ \; : H \times H \longrightarrow P^*(H).$$

For all x, y of H, $x \circ y$) is called a *hyperproduct* or a *hypercomposition* of x and y, sometimes we write xy instead $x \circ y$). A *hyperstructure* is a nonempty set together a family of finatary hyperoperation. A hyperstructure $(H, \circ)$ endowed with only a (binary) hyperoperation $\circ$ is called a *hypergroupoid*. The hyperoperation is extended to subsets of H in a natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B \}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Also, a *hypergroupoid* $(H, \cdot)$ is called a *semihypergroup* if for any $x, y, z \in H, (x \cdot y) \cdot z = x \cdot (y \cdot z)$, and a semihypergroup $(H, \cdot)$ is called a *hypergroup* if satisfies in the *reproduction axiom*, i.e. for any $x \in H, x \cdot H = H \cdot x = H$. Also, we identify every singleton $\{x\}$ with it's element x.

Definition 2.1. [32, Def. 3.1.1] A hyperring is an algebraic hyperstructure $(R, +, .)$ which satisfies:

(*i*) $(R, +)$ is a canonical hypergroup(or abelian hypergroup), i.e.:

- for every $x, y, z \in R, x + (y + z) = (x + y) + z$;
- for every $x, y \in R, x + y = y + x$;
- there exists an element $0 \in R$, such that $0 + x = x, \forall x \in R$;
- for every $x \in R$ there exist a unique element $x' \in R$, such that $0 \in x + x'$ (we write $-x$ for x');
- $z \in x + y$ implies $y \in z - x$ and $x \in z - y$.

(*ii*) $(R, .)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x.0 = 0.x = 0$.

(*iii*) the multiplication is distributive with respect to the hyperoperation $+$, i.e. $x.(y + z) = x.y + x.z$ and $(y + z).x = y.x + z.x$, for all $x, y, z \in R$.

Definition 2.2. [31, 32] A hyperfield is a hyperring which $(R \setminus \{0\}, .)$ is a group.

3. Krasner hyperfields and theirs extensions

The next result give a characterization of hyperfield extensions of **K**.

Example 3.1 (Krasner hyperfield)**.** The Krasner hyperfield **K** is the hyperfield

$$(\{0, 1\}, +, .),$$

with additive neutral element 0, usual multiplication with identity 1, and satisfying the hyperoperation

$$1 + 1 = \{0, 1\}.$$

In the category of hyperrings, **K** can be seen as the natural extension of the commutative pointed monoid $\mathbf{F}_1$, that is $(\mathbf{K}, .) = \mathbf{F}_1$.

Remark 3.2. (Krasner and $\mathbf{F}_1$). In the category of hyperrings, **K** can be seen as the natural extension of the commutative pointed monoid $\mathbf{F}_1$, that is, $(\mathbf{K}, .) = \mathbf{F}_1$. As remarked in [33], the Krasner hyperfield encodes the arithmetic of zero and nonzero numbers, just as $\mathbf{F}_2$ does for even and odd numbers.

Definition 3.3. [20] Let $(H, +)$ be a (canonical) hypergroup and $x \in H$. The set

$$O(x) = \{r \in \mathbb{Z} | \exists n \in \mathbb{Z} : 0 \in rx + n(x - x)\}$$

is a subgroup of $\mathbb{Z}$. We say that the order of x is infinite (i.e. $o(x) = \infty$) if $O(x) = \{0\}$. If $o(x) \neq \infty$, the smallest positive generator h of $O(x)$ is called the *principal order* of x (cf. [[7], Definition 57]). Let $q = min\{s \in \mathbb{N} | \exists m \neq 0, 0 \in mhx + s(x - x)\}$. The couple (h, q) is then called *the order* of x.

Example 3.4 (Sign hyperfield). We let **S** be the hyperfield $\mathbf{S} = \{-1, 0, 1\}$ with the hyperaddition given by the *rule of signs* $1 + 1 = 1, -1 - 1 = -1, 1 - 1 = -1 + 1 = \{-1, 0, 1\}$ and the usual multiplication also given by the rule of multiplication of signs.

One can sees that the monoid underlying **S** is $\mathbf{F}_{1^2}$, i.e. $(\mathbf{S}, .) = \mathbf{F}_{1^2}$, where the order of the element $1 \in \mathbf{S}$ is the pair $(1, 1)$. The homomorphism absolute value $\pi : \mathbf{S} \longrightarrow \mathbf{K}, f(x) = |x|$ is an epimorphism of hyperrings. Given a hyperring R and a subset $S \subseteq R$ containing 0 and 1 and such that for any $x, y \in S$ one has $xy \in S$, $x - y \subseteq S$, the subset S is a hyperring with the induced operations. This suggests the following definition of an extension.

Definition 3.5. [20] (Homomorphism of hyperrings) Let R and S be two hyperrings. A map $f : R \longrightarrow S$ is called a *(strict) homomorphism* if

1. $\forall a, b \in R, f(a + b) \subseteq f(a) + f(b)(f(a + b) = f(a) + f(b))$
2. $\forall a, b \in R, f(ab) = f(a)f(b)$

Definition 3.6. [20] A map f is said to be an *isomorphism* if it is a bijective strict homomorphism.

Definition 3.7. [20, Def. 2.3] Let $R_1 \subseteq R_2$ be hyperrings, we say that R_2 is an *extension* of R_1 when the inclusion $R_1 \subset R_2$ is a strict homomorphism.

Note that this is a stronger requirement than simply asking the inclusion $R_1 \subseteq R_2$ to be a homomorphism. It implies that for $x, y \in R_1$ the sum $x + y$ is the same when computed in R_1 or in R_2. To become familiar with the operations in hyperstructures, see the following simple results.

Proposition 3.8. [20] *In a hyperring extension R of the Krasner hyperfield* **K** *one has $x + x = \{0, x\}$ for any $x \in R$ and moreover*

$$a \in a + b \quad \Leftrightarrow \quad b \in \{0, a\}.$$

In particular, there is no hyperfield extension of K of cardinality 3 *or* 4.

Remark 3.9. The same proof shows that in a hyperring extension R of the hyperfield S one has

$$a \in a + b \quad \Leftrightarrow \quad b \in \{0, a, -a\}.$$

Krasner gave in [12] a construction of a hyperring as the quotient of a ring R by a multiplicative subgroup G of the group $R^\times$ of the invertible elements of R. This result states as follows:

Example 3.10. Consider $(R, +, \cdot)$ a ring with identity, G a normal subgroup of multiplicative semigroup $(R^\times, \cdot)$ and take $\bar{R} = R/G = \{aG | a \in R\}$ with the hyperaddition and multiplication given by:
$$\begin{cases} aG \oplus bG = \{cG | c \in aG + bG\} \\ aG \odot bG = abG. \end{cases}$$
Then $(R, \oplus, \odot)$ is a hyperring, which is called a quotient hyperring.

Remark 3.11. Note that in above example the normal condition for G is not necessary, since Massouros in [34] generalized this construction using for no normal multiplicative subgroups, since he proved that in a ring there exist multiplicative subgroups G of multiplicative semigroup $(R, .)$ which satisfy the property $xGyG = xyG$, even though they are not normal.

Remark 3.12. In the above definition we replace R by any field F, then $(\bar{F}, \oplus, \odot)$ is a hyperfield, which is a quotient hyperfield. Note that a hyperring(resp. hyperfield) S is said to be quotientable hyperring(resp. hyperfield) if S isomorphic to $\bar{R}$ as above. In [34] Massouros prove that there exists non-quotientable hyperring.

One of important classes of quotient hyperfield is introduce in the following example.

Example 3.13. Let R be the finite field $\mathbf{F}_{q^m}$, where q is a prime power and m is a positive integer, and let G be the multiplicative group $\mathbf{F}_{q^m}$, and hence $\mathbf{F}_q^{\times} \leq \mathbf{F}_{q^m}$. Then we can see R naturally as an m-dimensional $\mathbf{F}_q$-vector space, or better: as an $(m-1)$-dimensional $\mathbf{F}_q$-projective space. In the latter case, projective points are the cosets xG with $x \neq 0$. And lines, for instance, are of the form $(xG + yG)/G$. Once one lets m go to 1, one naturally constructs the Krasner hyperfield $\mathbf{K}$. These examples will be very important in what is to come.

Example 3.14. This simple example is an application of the above results and it shows that there exists a hyperfield extension of $\mathbf{K}$ of cardinality 5. Let H be the union of 0 with the powers of α, $\alpha^4 = 1$. It is a set with 5 elements and the table of hyper-addition in H is given by the following matrix

$$\begin{bmatrix} 0 & 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & \{0,1\} & \{\alpha^2, \alpha^3\} & \{\alpha, \alpha^3\} & \{\alpha, \alpha^2\} \\ \alpha^2 & \{\alpha, \alpha^3\} & \{1, \alpha^3\} & \{0, \alpha^2\} & \{1, \alpha\} \\ \alpha^3 & \{\alpha, \alpha^2\} & \{1, \alpha^2\} & \{1, \alpha\} & \{0, \alpha^3\} \end{bmatrix}$$

This hyperfield structure is obtained, with $\alpha = 1 + \sqrt{-1}$, as the quotient of the finite field $\mathbf{F}_9 = \mathbf{F}_3(\sqrt{-1})$ by the multiplicative group $\mathbf{F}_3^{\times} = \{-1, 1\}$. It follows from Proposition 3.16 that $F = \mathbf{F}_9/\mathbf{F}_9^{\times}$ is a hyperfield extension of $\mathbf{F}$. Notice that the addition has a very easy description since for any two distinct non-zero elements x, y the sum $x + y$ is the complement of $\{x, y, 0\}$ (cf. [5] and Proposition 3.16 below for a more general construction).

Proposition 3.15. [36]. *Let $\mathbf{K}$ be a field with at least three elements. Then the hyperring $\mathbf{K}/\mathbf{K}^{\times}$ is isomorphic to the Krasner hyperfield. If, in general, R is a commutative ring and $G \subseteq \mathbf{K}^{\times}$ is a proper subgroup of the group of units of R, then the hyperring R/G defined as above contains $\mathbf{K}$ as a subhyperfield if and only if $\{0\} \cup G$ is a subfield of R.*

One of the important example of Krasner hyperfield, was constructed by above proposition is the next example.

Example 3.16. (Important Example) (Adéle class space and Krasner). Consider a global field $\mathbf{K}$. Its adéle class space $\mathbf{H} = \mathbf{A_K}/\mathbf{K}^{\times}$ is the quotient of a commutative ring $\mathbf{A_K}$ by $G = \mathbf{K}^{\times}$, and $\{0\} \cup G = K$, so it is a hyperring extension of $\mathbf{K}$.

Remark 3.17. Remark that the adéle class space plays a very important role in the non-commutative program of solving the Riemann Hypothesis. (See for instance [37].)

Proposition 3.18. [36] *Let $\mathbf{H} \supseteq \mathbf{K}$ be a finite commutative hyperfield extension of $\mathbf{K}$. Then one of the following cases occurs:*

(i) $\mathbf{H} = \mathbf{K}[G]$ *for a finite abelian group G.*

(ii) There exists a finite field extension $\mathbf{F}_q \subseteq \mathbf{F}_{q^m}$ such that $\mathbf{H} = \mathbf{F}_{q^m}/\mathbf{F}_q$.

(iii) There exists a finite non-Desarguesian projective plane admitting a sharply point-transitive automorphism group G, and G is the abelian incidence group associated to $\mathbf{H}$.

Proposition 3.19. *In every hyperfield* **H**. *Then the following statements are satisfied:*

(*i*) *the hyperaddition completely is determined by* $\{1 + x\}$ *for* $x \in$ **H**;

(*ii*) *If* $0 \in 1 + x$ *for some* $x \in \mathbb{H}$, *then* $x^2 = 1$.

(*iii*) *Let* G *be a subgroup of multiplicative group* $(\mathbf{H}^\times, .)$ *of odd index. Then* $G = -G$.

Proof. (*i*) Let $x, y \in$ **H**. Then $x + y = x(1 + x^{-1}y)$.

(*ii*) Let $0 \in 1 + x$, then $0 \in x(1 + x) = x + x^2$. Then $x^2 = 1$.

(*iii*) By definition $\mathbf{F}/G$ is a hyperfield and by hypothesis it's multiplicative group is of odd number. By (*ii*) previous 0 belongs to $G + G$. Thus there exist $a, b \in G$, such that $a + b = 0$. Thus $1 + a^{-1}b = 0$, and hence $a^{-1}b = -1$. So, $-1 \in G$ and $\forall c \in G, -c \in G$. Therefore $G = -G$. □

Let $(G, .)$ be a group with identity 1. Letting $R = G \cup \{0\}, 0 \notin G$ and $\{A_x\}_{x \in R}$ be a family of non-empty subsets of R such that $\bigcup_{x \in R} A_x = R$. Define the hyperaddition + by $1 + x = x + 1 = A_x, A_0 = \{0\}, x + y = x.A_{x^{-1}y}$. Moreover, suppose that the family $\{A_x\}_{x \in R}$ are satisfied the following properties:

1. $0 \in A_0$;
2. $x.A_{x^{-1}y} = A_{xy^{-1}}.y, \forall x, y$;
3. $A_x + y = x + A_y$, where $A_x + y = \bigcup_{t \in A_x}(t + y)$;
4. $y \in A_x \Rightarrow -x \in A_{-y}$.

The next result immediate consequence of Proposition 3.19.

Theorem 3.20. $(R, +, .)$ *is a hyperfield.*

Proof. For non-zero elements x, y and z in R, one has

$$(x + y) + z = y[(y^{-1}x + 1) + y^{-1}z] = y[A_{y^{-1}x} + y^{-1}z] =$$
$$y[y^{-1}x + A_{y^{-1}z}] = y[y^{-1}x + (1 + y^{-1}z)] = x + (y + z),$$

$(R, +)$ is associative. $(R, +, .)$ is distributive since one has,

$$x(y + z) = xyA_{y^{-1}z} = xyA_{y^{-1}x^{-1}xz} = xy + xz.$$

On the other hands, we have

$$(y + z)x = yA_{y^{-1}z}x = A_{yz^{-1}}zx = A_{yxx^{-1}z}zx = yxA_{x^{-1}y^{-1}zx} = yxA_{(yx)^{-1}zx} = yx + zx.$$

Clearly, 0 is identity element of $(R, +)$. Also, $\forall x \in R$, consider $-x = -1.x$, that $0 \in x + (-x)$. If $x \in y + z$, then $y^{-1}x \in 1 + y^{-1}z = A_{y^{-1}z} \Rightarrow -y^{-1}z \in A_{-y^{-1}x} = 1 - y^{-1}x \Rightarrow z \in x - y$. Similarly, we have $y \in x - z$. Therefore, $(R, +)$ is a canonical hypergroup. Thus $(R, +, .)$ is a hyperfield. □

By the next examples we show that there exist, such a family of subsets for infinite case.

Example 3.21. Let $R = \{0, \mathbb{Q}^\times\} \bigcup_{x \in \mathbb{Q}^c} x.\mathbb{Q}^\times$. Letting

$$A_0 = \{1 \equiv \mathbb{Q}^\times\}, A_1 = \{0, 1\},$$

and for $x \in R \setminus \{0, 1\}, A_x = 1 \oplus x = \bigcup_{t,u \in \mathbb{Q}^\times}(t + ux)$. It is easy to verify that the family $\{A_x\}_{x \in R}$ satisfies the conditions of the last theorem. So, $(R, \oplus, .)$ is an infinite hyperfield extension of **K**.

Example 3.22. Let $\mathbb{R}_{\geq 0}$ be the set of non-zero real numbers. Consider the family

$$\{A_x = [|1-x|, 1+x]\}_{x \in \mathbb{R}_{\geq 0}}, A_x = 1 \oplus x.$$

This family of subsets also satisfies the theorem conditions. So, $(\mathbb{R}_{\geq 0}, \oplus, .)$ is a hyperfield (triangle hyperfield [16]).

Proposition 3.23. *Let $(H, +, .)$ and $(R, \oplus, \odot)$ be Krasner hyperfields and φ be a group isomorphism between $(H^\times, .)$ and $(R^\times, \odot)$. If for every $x \in H, \varphi(1 + x) = 1 \oplus \varphi(x)$, then one can extend φ to a hyperfield isomorphism between H and R by defining $\varphi(0) = 0$.*

Proof. It is adequate to prove that for every $x, y \in H, \varphi(x+y) = \varphi(x) \oplus \varphi(y)$. One can write $x + y, x \neq 0$ as $x(1 + x^{-1}y)$. So, we have

$$
\begin{aligned}
\varphi(x+y) &= \varphi(x) \odot \varphi(1 + x^{-1}y) \\
&= \varphi(x) \odot (1 \oplus \varphi(x^{-1}y)) \\
&= \varphi(x) \odot (1 \oplus \varphi(x)^{-1} \odot \varphi(y)) \\
&= \varphi(x) \oplus \varphi(y). \quad \square
\end{aligned}
$$

4. C^{++} **Package**

In this paper classification of finite hyperfields of order less than 7 is performed through three steps. At first step by means of the theory of classification of finite groups all groups G of order $m = n - 1$ is obtained. In other words, multiplication action for hyperfield is given by the group action. In the second step hyperoperation $\oplus$ is defined in a way that for $H = G \cup \{0\}, (H, \oplus, .)$ is a Krasner hyperfield. Referring to Theorem 3.18 it is enough to define hyperoperation $+$ only for the element of the form $1 + x$ for every $x \in H$. By the computer programming power set $P(H)$ of H is constructed. Then a family of non-empty members of $P(H)$, $\{A_x\}_{x \in H}$ is chosen(is not necessary unique). So for every $x \in H, A_x$ is devoted to each set $1 \oplus x$ as set value meaning $A_x = 1 \oplus x$. Then it is checked that the conditions of theorem 1 is satisfied or not. Third step is about classifying these hyperfields up to isomorphism. After computing all hyperfields of order n by referring to theorem 3.20, the program finds automorphisms of multiplication group G as needed. Then it is checked the when the conditions of theorem 3.20 is satisfied or not. Note that for each isomorphism the program simultaneously is finding all isomorphisms between hyperfields. At last part of the program, isomorphism class of each hyperfield is computed which by can find number of up to isomorphism hyperfields of order n. Main part of the program with some comments is presented the appendix. At the following we will present some results of running the program to construct all finite hyperfield of order less than or equal to 6 up to isomorphism. Because the computational complexity of the program is exponentially high, we just are able to compute hyperfields of order less than or equal to 6.

In row 4, columns 2 and 3 of the Table 1, the numbers $11, 16$ represent the number of hyperfields with $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ as their multiplicative groups respectively. So totally there exist $11 + 16 = 27$ hyperfield of order 5. Similarly discussed, there exist $1 + 1 = 2$ hyperfield of order 5, extension of **K**. In [39] hyperfields of order less than 6 is presented by a different approach. For the hyperfields of order 5 it is claimed that there exist 33 of them up to isomorphism. But for cases that do not presented here, the addition fails to be associative.

In the sequel, we present all hyperfields of order less than or equal 6 up to isomorphism.

Table 1. Number of finite hyperfields up to isomorphism for number of elements less than 7.

Num. of elements	Num. of hyperfields	Num. of hyperfields extension of **K**
2	2	1
3	5	0
4	7	0
5	11+16=27	1+1=2
6	16	1

4.1. Hyperfields of order 3

There are 5 hyperfields of order 3 up to isomorphism which all of them are quotient hyperfields.

Remark 4.1. Note that the underling multiplicative in all cases is isomorphic to $\mathbb{Z}_2$. At the following by HF_{mn}, we mean n^{th} hyperfield of order m.

1. $HF_{31} \cong (\mathbf{S}, \oplus, \odot)$

+	0	1	−1
0	0	1	−1
1	1	1	**S**
−1	−1	**S**	−1

2. $HF_{32} = \mathbb{Z}_3$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

3. $HF_{33} \cong \mathbb{Z}_5/<4>$

+	0	1	a
0	0	1	a
1	1	$\{0,a\}$	$\{1,a\}$
a	a	$\{1,a\}$	$\{0,1\}$

4. $HF_{34} \cong \mathbb{Z}_7/<4>$

+	0	1	a
0	0	1	a
1	1	$\{1,a\}$	$\{0,1,a\}$
a	a	$\{0,1,a\}$	$\{1,a\}$

5. HF_{35} quotient (see Proposition 3.17 in [35] for hyperfield HF_{33})

+	0	1	a
0	0	1	a
1	1	$\{0,1,a\}$	$\{1,a\}$
a	a	$\{1,a\}$	$\{0,1,a\}$

4.2. Hyperfields of order 4 up to isomorphism

There are 7 hyperfields of order 4. There exists only one non-quotient hyperfield of order 4, HF_{44} which is the smallest non-quotient hyperfield.

Remark 4.2. Multiplication group in all cases is isomorphic to $\mathbb{Z}_3$

1. $HF_{41} = \mathbb{F}_4$

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

2. $HF_{42} \cong \mathbb{Z}_7/<6>$

+	0	1	a	b
0	0	1	a	b
1	1	$\{0,a\}$	$\{1,b\}$	$\{a,b\}$
a	a	$\{1,b\}$	$\{0,b\}$	$\{1,a\}$
b	b	$\{a,b\}$	$\{1,a\}$	$\{0,1\}$

3. $HF_{43} \cong \mathbf{F}_{16}/<\alpha>, \alpha^5 = 1$

+	0	1	a	b
0	0	1	a	b
1	1	$\{0,1,a\}$	$\{1,b\}$	$\{a,b\}$
a	a	$\{1,b\}$	$\{0,a,b\}$	$\{1,a\}$
b	b	$\{a,b\}$	$\{1,a\}$	$\{0,1,b\}$

4. HF_{44} non-quotient (Theorem 3.1 in [34])

+	0	1	a	b
0	0	1	a	b
1	1	$\{0,a,b\}$	$\{1,a\}$	$\{1,b\}$
a	a	$\{1,a\}$	$\{0,1,b\}$	$\{a,b\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a\}$

5. $HF_{45} \cong \mathbb{Z}_{13}/ <5>$

+	0	1	a	b
0	0	1	a	b
1	1	$\{0,a,b\}$	$\{1,a,b\}$	$\{1,a,b\}$
a	a	$\{1,a,b\}$	$\{0,1,b\}$	$\{1,a,b\}$
b	b	$\{1,a,b\}$	$\{1,a,b\}$	$\{0,1,a\}$

6. HF_{46}

+	0	1	a	b
0	0	1	a	b
1	1	$\{0,1,a,b\}$	$\{1,a\}$	$\{1,b\}$
a	a	$\{1,a\}$	$\{0,1,a,b\}$	$\{a,b\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a,b\}$

7. $HF_{47} \cong \mathbb{Z}_{19}/ <8>$

+	0	1	a	b
0	0	1	a	b
1	1	$\{0,1,a,b\}$	$\{1,a,b\}$	$\{1,a,b\}$
a	a	$\{1,a,b\}$	$\{0,1,a,b\}$	$\{1,a,b\}$
b	b	$\{1,a,b\}$	$\{1,a,b\}$	$\{0,1,a,b\}$

4.3. Hyperfields of order 5

There are 27 hyperfields of order 5 which the multiplication group of 11 of them is $\mathbb{Z}_2 \times \mathbb{Z}_2$ and 16 of them is $\mathbb{Z}_4$. For every multiplicative group there exist a hyperfiled extension of **K**.

4.3.1. Hyperfields of order 5 up to isomorphism with multiplication group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$

1. HF_{51}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	1	$\{1,a\}$	$\{1,b\}$	$\{0,1,a,b,c\}$
a	a	$\{1,a\}$	a	$\{0,1,a,b,c\}$	$\{a,c\}$
b	b	$\{1,b\}$	$\{0,1,a,b,c\}$	b	$\{b,c\}$
c	c	$\{0,1,a,b,c\}$	$\{a,c\}$	$\{b,c\}$	c

2. HF_{52}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	{0, 1}	{b, c}	{a, c}	{a, b}
a	a	{b, c}	{0, a}	{1, c}	{1, b}
b	b	{a, c}	{1, c}	{0, b}	{1, a}
c	c	{a, b}	{1, b}	{1, a}	{0, c}

3. HF_{53}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	{1, a}	{1, a}	{1, a, b, c}	{0, 1, a, b, c}
a	a	{1, a}	{1, a}	{0, 1, a, b, c}	{1, a, b, c}
b	b	{1, a, b, c}	{0, 1, a, b, c}	{1, c}	{b, c}
c	c	{0, 1, a, b, c}	{1, a, b, c}	{b, c}	{b, c}

4. HF_{54}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	{1, c}	{1, a}	{1, b}	{0, 1, a, b, c}
a	a	{1, a}	{a, b}	{0, 1, a, b, c}	{a, c}
b	b	{1, b}	{0, 1, a, b, c}	{a, b}	{b, c}
c	c	{0, 1, a, b, c}	{a, c}	{b, c}	{1, c}

5. HF_{55}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	{1, a, b}	{1, a, b, c}	{1, a, b, c}	{0, 1, a, b, c}
a	a	{1, a, b, c}	{1, a, c}	{0, 1, a, b, c}	{1, a, b, c}
b	b	{1, a, b, c}	{0, 1, a, b, c}	{1, b, c}	{1, a, b, c}
c	c	{0, 1, a, b, c}	{1, a, b, c}	{1, a, b, c}	{a, b, c}

6. HF_{56}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	{a, b, c}	{1, a, b, c}	{1, a, b, c}	{0, a, b}
a	a	{1, a, b, c}	{1, b, c}	{0, 1, c}	{1, a, b, c}
b	b	{1, a, b, c}	{0, 1, c}	{1, a, c}	{1, a, b, c}
c	c	{0, a, b}	{1, a, b, c}	{1, a, b, c}	{1, a, b}

7. HF_{57} Non-quotient (see Proposition 2.7. in [38])

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,a,b,c\}$	$\{1,a\}$	$\{1,b\}$	$\{1,c\}$
a	a	$\{1,a\}$	$\{0,1,b,c\}$	$\{a,b\}$	$\{a,c\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a,c\}$	$\{b,c\}$
c	c	$\{1,c\}$	$\{a,c\}$	$\{b,c\}$	$\{0,1,a,b\}$

8. HF_{58} Non-quotient (by Theorem 3.1 in [34] and Theorem 1.3 in [40])

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{0,1,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
b	b	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,c\}$	$\{1,a,b,c\}$
c	c	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b\}$

9. HF_{59}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$
b	b	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
c	c	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$

10. HF_{510}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,1,a,b,c\}$	$\{1,a\}$	$\{1,b\}$	$\{1,c\}$
a	a	$\{1,a\}$	$\{0,1,a,b,c\}$	$\{a,b\}$	$\{a,c\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a,b,c\}$	$\{b,c\}$
c	c	$\{1,c\}$	$\{a,c\}$	$\{b,c\}$	$\{0,1,a,b,c\}$

11. HF_{511}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
b	b	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$
c	c	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$

4.3.2. Hyperfields of order 5 up to isomorphism with multiplication group isomorphic to $\mathbb{Z}_4$

12. HF_{512}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	1	$\{1,a\}$	$\{0,1,a,b,c\}$	$\{1,c\}$
a	a	$\{1,a\}$	a	$\{a,b\}$	$\{0,1,a,b,c\}$
b	b	$\{0,1,a,b,c\}$	$\{a,b\}$	b	$\{b,c\}$
c	c	$\{1,c\}$	$\{0,1,a,b,c\}$	$\{b,c\}$	c

13. $HF_{513} \cong \mathbf{F}_9/\mathbf{F}_3^\times$

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,1\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b\}$
a	a	$\{b,c\}$	$\{0,a\}$	$\{1,c\}$	$\{1,b\}$
b	b	$\{a,c\}$	$\{1,c\}$	$\{0,b\}$	$\{1,a\}$
c	c	$\{a,b\}$	$\{1,b\}$	$\{1,a\}$	$\{0,c\}$

14. $HF_{514} = \mathbb{Z}_5$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

15. HF_{515} quotient (by Proposition 4 in [14] for hyperfield HF_{514})

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{1,a\}$	$\{1,a,c\}$	$\{0,1,a,b,c\}$	$\{1,b,c\}$
a	a	$\{1,a,c\}$	$\{a,b\}$	$\{1,a,b\}$	$\{0,1,a,b,c\}$
b	b	$\{0,1,a,b,c\}$	$\{1,a,b\}$	$\{b,c\}$	$\{a,b,c\}$
c	c	$\{1,b,c\}$	$\{0,1,a,b,c\}$	$\{a,b,c\}$	$\{1,c\}$

16. HF_{516}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{1,b\}$	$\{1,a\}$	$\{0,1,a,b,c\}$	$\{1,c\}$
a	a	$\{1,a\}$	$\{a,c\}$	$\{a,b\}$	$\{0,1,a,b,c\}$
b	b	$\{0,1,a,b,c\}$	$\{a,b\}$	$\{1,b\}$	$\{b,c\}$
c	c	$\{1,c\}$	$\{0,1,a,b,c\}$	$\{b,c\}$	$\{a,c\}$

17. $HF_{517} \cong \mathbb{Z}_{13}/<3>$

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{a,b\}$	$\{1,a,c\}$	$\{0,a,c\}$	$\{1,b,c\}$
a	a	$\{1,a,c\}$	$\{b,c\}$	$\{1,a,b\}$	$\{0,1,b\}$
b	b	$\{0,a,c\}$	$\{1,a,b\}$	$\{1,c\}$	$\{a,b,c\}$
c	c	$\{1,b,c\}$	$\{0,1,b\}$	$\{a,b,c\}$	$\{1,a\}$

18. $HF_{518} \cong \mathbb{Z}_{17}/<4>$

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,a,b\}$	$\{1,b,c\}$	$\{1,a,b,c\}$	$\{a,b,c\}$
a	a	$\{1,b,c\}$	$\{0,b,c\}$	$\{1,a,c\}$	$\{1,a,b,c\}$
b	b	$\{1,a,b,c\}$	$\{1,a,c\}$	$\{0,1,c\}$	$\{1,a,b\}$
c	c	$\{a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b\}$	$\{0,1,a\}$

19. HF_{519}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{1,a,b\}$	$\{1,a,c\}$	$\{0,1,a,b,c\}$	$\{1,b,c\}$
a	a	$\{1,a,c\}$	$\{a,b,c\}$	$\{1,a,b\}$	$\{0,1,a,b,c\}$
b	b	$\{0,1,a,b,c\}$	$\{1,a,b\}$	$\{1,b,c\}$	$\{a,b,c\}$
c	c	$\{1,b,c\}$	$\{0,1,a,b,c\}$	$\{a,b,c\}$	$\{1,a,c\}$

20. HF_{520}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,1,a,b\}$	$\{1,b,c\}$	$\{1,a,b,c\}$	$\{a,b,c\}$
a	a	$\{1,b,c\}$	$\{0,a,b,c\}$	$\{1,a,c\}$	$\{1,a,b,c\}$
b	b	$\{1,a,b,c\}$	$\{1,a,c\}$	$\{0,1,b,c\}$	$\{a,b,c\}$
c	c	$\{a,b,c\}$	$\{1,a,b,c\}$	$\{a,b,c\}$	$\{0,1,a,c\}$

21. $HF_{521} \cong \mathbb{Z}_{29}/<7>$

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{1,a,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{1,a,b\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$
b	b	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{a,b,c\}$	$\{1,a,b,c\}$
c	c	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,b,c\}$

22. HF_{522}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{a,b,c\}$	$\{1,a,b,c\}$	$\{0,a,c\}$	$\{1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{1,b,c\}$	$\{1,a,b,c\}$	$\{0,1,b\}$
b	b	$\{0,a,c\}$	$\{1,a,b,c\}$	$\{1,a,c\}$	$\{1,a,b,c\}$
c	c	$\{1,a,b,c\}$	$\{0,1,b\}$	$\{1,a,b,c\}$	$\{1,a,b\}$

23. HF_{523} Non-quotient (see Theorem3.1 in [35])

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,a,b,c\}$	$\{1,a\}$	$\{1,b\}$	$\{1,c\}$
a	a	$\{1,a\}$	$\{0,1,b,c\}$	$\{a,b\}$	$\{a,c\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a,c\}$	$\{b,c\}$
c	c	$\{1,c\}$	$\{a,c\}$	$\{b,c\}$	$\{0,1,a,b\}$

24. $HF_{524} \cong \mathbb{Z}_{41}/<4>$

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{0,1,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
b	b	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,c\}$	$\{1,a,b,c\}$
c	c	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b\}$

25. $HF_{525} \cong \mathbb{Z}_{37}/<7>$

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$
b	b	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
c	c	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$

26. HF_{526}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,1,a,b,c\}$	$\{1,a\}$	$\{1,b\}$	$\{1,c\}$
a	a	$\{1,a\}$	$\{0,1,a,b,c\}$	$\{a,b\}$	$\{a,c\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a,b,c\}$	$\{b,c\}$
c	c	$\{1,c\}$	$\{a,c\}$	$\{b,c\}$	$\{0,1,a,b,c\}$

27. HF_{527}

+	0	1	a	b	c
0	0	1	a	b	c
1	1	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
a	a	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$
b	b	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$	$\{1,a,b,c\}$
c	c	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,b,c\}$

4.4. Hyperfields of order 6 up to isomorphism

There are 16 hyperfields of order 6. Notice that as you can see against for fields the number of elements is not necessarily a power of a prime number. Really there are hyperfield of any finite order $n \neq 1$.

Remark 4.3. Multiplication group in all cases is isomorphic to $\mathbb{Z}_5$

1. $HF_{61} \cong \mathbf{F}_{16}/\mathbf{F}_4^{\times}$

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1\}$	$\{b,c,d\}$	$\{a,c,d\}$	$\{a,b,d\}$	$\{a,b,c\}$
a	a	$\{b,c,d\}$	$\{0,a\}$	$\{1,c,d\}$	$\{1,b,d\}$	$\{1,b,c\}$
b	b	$\{a,c,d\}$	$\{1,c,d\}$	$\{0,b\}$	$\{1,a,d\}$	$\{1,a,c\}$
c	c	$\{a,b,d\}$	$\{1,b,d\}$	$\{1,a,d\}$	$\{0,c\}$	$\{1,a,b\}$
d	d	$\{a,b,c\}$	$\{1,b,c\}$	$\{1,a,c\}$	$\{1,a,b\}$	$\{0,d\}$

2. $HF_{62} \cong \mathbb{Z}_{11}/<10>$

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,a\}$	$\{1,c\}$	$\{c,d\}$	$\{a,b\}$	$\{b,d\}$
a	a	$\{1,c\}$	$\{0,b\}$	$\{a,d\}$	$\{1,d\}$	$\{b,c\}$
b	b	$\{c,d\}$	$\{a,d\}$	$\{0,c\}$	$\{1,b\}$	$\{1,a\}$
c	c	$\{a,b\}$	$\{1,d\}$	$\{1,b\}$	$\{0,d\}$	$\{a,c\}$
d	d	$\{b,d\}$	$\{b,c\}$	$\{1,a\}$	$\{a,c\}$	$\{0,1\}$

3. HF_{63} Non-quotient (by Theorem 3.1 in [34] and Theorem 1.3 in [40])

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,a\}$	$\{1,b,c,d\}$	$\{a,c,d\}$	$\{a,b,d\}$	$\{a,b,c,d\}$
a	a	$\{1,b,c,d\}$	$\{0,a,b\}$	$\{1,a,c,d\}$	$\{1,b,d\}$	$\{1,b,c\}$
b	b	$\{a,c,d\}$	$\{1,a,c,d\}$	$\{0,b,c\}$	$\{1,a,b,d\}$	$\{1,a,c\}$
c	c	$\{a,b,d\}$	$\{1,b,d\}$	$\{1,a,b,d\}$	$\{0,c,d\}$	$\{1,a,b,c\}$
d	d	$\{a,b,c,d\}$	$\{1,b,c\}$	$\{1,a,c\}$	$\{1,a,b,c\}$	$\{0,1,d\}$

4. HF_{64} Non-quotient (by Theorem 3.1 in [34] and Theorem 1.3 in [40])

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,a,b\}$	$\{1,b,c,d\}$	$\{1,a,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
a	a	$\{1,b,c,d\}$	$\{0,b,c\}$	$\{a,b,c,d\}$	$\{1,a,b,d\}$	$\{1,b,c,d\}$
b	b	$\{1,a,c,d\}$	$\{a,b,c,d\}$	$\{0,c,d\}$	$\{1,a,b,d\}$	$\{1,a,b,c\}$
c	c	$\{a,b,c,d\}$	$\{1,a,b,d\}$	$\{1,a,b,d\}$	$\{0,1,d\}$	$\{1,a,b,c\}$
d	d	$\{a,b,c,d\}$	$\{1,b,c,d\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a\}$

5. $HF_{65} \cong \mathbb{Z}_{31}/<6>$

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,a,b\}$	$\{1,b,c,d\}$	$\{1,a,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
a	a	$\{1,b,c,d\}$	$\{0,a,b,c\}$	$\{a,b,c,d\}$	$\{1,a,b,d\}$	$\{1,b,c,d\}$
b	b	$\{1,a,c,d\}$	$\{a,b,c,d\}$	$\{0,b,c,d\}$	$\{1,a,b,d\}$	$\{1,a,b,c\}$
c	c	$\{a,b,c,d\}$	$\{1,a,b,d\}$	$\{1,a,b,d\}$	$\{0,1,c,d\}$	$\{1,a,b,c\}$
d	d	$\{a,b,c,d\}$	$\{1,b,c,d\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{0,1,a,d\}$

6. HF_{66}non-quotient (by Theorem 3.1 in [34] and Theorem 1.3 in [40])

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,b,c\}$	$\{b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{a,b,c\}$
a	a	$\{b,c,d\}$	$\{0,1,c,d\}$	$\{1,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
b	b	$\{1,a,b,c,d\}$	$\{1,c,d\}$	$\{0,1,b,d\}$	$\{1,a,d\}$	$\{1,a,b,c,d\}$
c	c	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,d\}$	$\{0,1,a,c\}$	$\{1,a,b\}$
d	d	$\{a,b,c\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b\}$	$\{0,a,b,d\}$

7. HF_{67} non-quotient (by Theorem 3.1 in [34] and Theorem 1.3 in [40])

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,a,b,c\}$	$\{1,b,d\}$	$\{1,a,b\}$	$\{1,c,d\}$	$\{a,c,d\}$
a	a	$\{1,b,d\}$	$\{0,b,c,d\}$	$\{1,a,c\}$	$\{a,b,c\}$	$\{1,a,d\}$
b	b	$\{1,a,b\}$	$\{1,a,c\}$	$\{0,1,c,d\}$	$\{a,b,d\}$	$\{1,a,b\}$
c	c	$\{1,c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{0,1,a,d\}$	$\{1,b,c\}$
d	d	$\{a,c,d\}$	$\{1,a,d\}$	$\{1,a,b\}$	$\{1,b,c\}$	$\{0,1,a,b\}$

8. $HF_{68} \cong \mathbb{Z}_{41}/ < 3 >$

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,a,b,c\}$	$\{1,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{a,b,c,d\}$
a	a	$\{1,b,c,d\}$	$\{0,b,c,d\}$	$\{1,a,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
b	b	$\{1,a,b,c,d\}$	$\{1,a,c,d\}$	$\{0,1,c,d\}$	$\{1,a,b,d\}$	$\{1,a,b,c,d\}$
c	c	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,d\}$	$\{0,1,a,d\}$	$\{1,a,b,c\}$
d	d	$\{a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c\}$	$\{0,1,a,b\}$

9. HF_{69} non-quotient (by Theorem 3.1 in [34] and Theorem 1.3 in [40])

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,a,b,c\}$	$\{1,b,d\}$	$\{1,a,b\}$	$\{1,c,d\}$	$\{a,c,d\}$
a	a	$\{1,b,d\}$	$\{0,a,b,c,d\}$	$\{1,a,c\}$	$\{a,b,c\}$	$\{1,a,d\}$
b	b	$\{1,a,b\}$	$\{1,a,c\}$	$\{0,1,b,c,d\}$	$\{a,b,d\}$	$\{1,a,b\}$
c	c	$\{1,c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{0,1,a,c,d\}$	$\{1,b,c\}$
d	d	$\{a,c,d\}$	$\{1,a,d\}$	$\{1,a,b\}$	$\{1,b,c\}$	$\{0,1,a,b,d\}$

10. $HF_{610} \cong \mathbb{Z}_{61}/ < 21 >$

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,a,b,c\}$	$\{1,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{a,b,c,d\}$
a	a	$\{1,b,c,d\}$	$\{0,a,b,c,d\}$	$\{1,a,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
b	b	$\{1,a,b,c,d\}$	$\{1,a,c,d\}$	$\{0,1,b,c,d\}$	$\{1,a,b,d\}$	$\{1,a,b,c,d\}$
c	c	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,d\}$	$\{0,1,a,c,d\}$	$\{1,a,b,c\}$
d	d	$\{a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c\}$	$\{0,1,a,b,d\}$

11. HF_{611} non-quotient (see Theorem 3.1 in [35])

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,a,b,c,d\}$	$\{1,a\}$	$\{1,b\}$	$\{1,c\}$	$\{1,d\}$
a	a	$\{1,a\}$	$\{0,1,b,c,d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a,c,d\}$	$\{b,c\}$	$\{b,d\}$
c	c	$\{1,c\}$	$\{a,c\}$	$\{b,c\}$	$\{0,1,a,b,d\}$	$\{c,d\}$
d	d	$\{1,d\}$	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{0,1,a,b,c\}$

12. HF_{612} non-quotient (by Theorem 3.1 in [34] and Theorem 1.3 in [40])

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,a,b,c,d\}$	$\{1,a,c\}$	$\{1,b,c,d\}$	$\{1,a,b,c\}$	$\{1,b,d\}$
a	a	$\{1,a,c\}$	$\{0,1,b,c,d\}$	$\{a,b,d\}$	$\{1,a,c,d\}$	$\{1,a,b,c\}$
b	b	$\{1,b,c,d\}$	$\{a,b,d\}$	$\{0,1,a,c,d\}$	$\{1,b,c\}$	$\{1,a,b,d\}$
c	c	$\{1,a,b,c\}$	$\{1,a,c,d\}$	$\{1,b,c\}$	$\{0,1,a,b,d\}$	$\{1,a,d\}$
d	d	$\{1,b,d\}$	$\{1,a,b,c\}$	$\{1,a,b,d\}$	$\{1,a,d\}$	$\{0,1,a,b,c\}$

13. $HF_{613} \cong \mathbb{Z}_{71}/ < 51 >$

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
a	a	$\{1,a,b,c,d\}$	$\{0,1,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
b	b	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{0,1,a,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
c	c	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{0,1,a,b,d\}$	$\{1,a,b,c,d\}$
d	d	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{0,1,a,b,c\}$

14. HF_{614}

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,a,b,c,d\}$	$\{1,a\}$	$\{1,b\}$	$\{1,c\}$	$\{1,d\}$
a	a	$\{1,a\}$	$\{0,1,a,b,c,d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	b	$\{1,b\}$	$\{a,b\}$	$\{0,1,a,b,c,d\}$	$\{b,c\}$	$\{b,d\}$
c	c	$\{1,c\}$	$\{a,c\}$	$\{b,c\}$	$\{0,1,a,b,c,d\}$	$\{c,d\}$
d	d	$\{1,d\}$	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{0,1,a,b,c,d\}$

15. HF_{615} quotient (see Proposition 4 in [35] for hyperfield HF_{62})

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,a,b,c,d\}$	$\{1,a,c\}$	$\{1,b,c,d\}$	$\{1,a,b,c\}$	$\{1,b,d\}$
a	a	$\{1,a,c\}$	$\{0,1,a,b,c,d\}$	$\{a,b,d\}$	$\{1,a,c,d\}$	$\{1,a,b,c\}$
b	b	$\{1,b,c,d\}$	$\{a,b,d\}$	$\{0,1,a,b,c,d\}$	$\{1,b,c\}$	$\{1,a,b,d\}$
c	c	$\{1,a,b,c\}$	$\{1,a,c,d\}$	$\{1,b,c\}$	$\{0,1,a,b,c,d\}$	$\{1,a,d\}$
d	d	$\{1,b,d\}$	$\{1,a,b,c\}$	$\{1,a,b,d\}$	$\{1,a,d\}$	$\{0,1,a,b,c,d\}$

16. HF_{616} quotient(see Proposition 4 in [35] for hyperfield HF_{613})

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	$\{0,1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
a	a	$\{1,a,b,c,d\}$	$\{0,1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
b	b	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{0,1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$
c	c	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{0,1,a,b,c,d\}$	$\{1,a,b,c,d\}$
d	d	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{1,a,b,c,d\}$	$\{0,1,a,b,c,d\}$

All presented hyperfields are commutative. For example we presented a finite non-commutative K-extension hyperfield of order 7:

+	0	1	a	b	c	d	e	$\cdot$	0	1	a	b	c	d	e
0	0	1	a	b	c	d	e	0	0	0	0	0	0	0	0
1	1	$\{0,1\}$	$\{b,c,d,e\}$	$\{a,c,d,e\}$	$\{a,b,d,e\}$	$\{a,b,c,e\}$	$\{a,b,c,d\}$	1	0	1	a	b	c	d	e
a	a	$\{b,c,d,e\}$	$\{0,a\}$	$\{1,c,d,e\}$	$\{1,b,d,e\}$	$\{1,b,c,e\}$	$\{1,b,c,d\}$	a	0	a	1	e	d	c	b
b	b	$\{a,c,d,e\}$	$\{1,c,d,e\}$	$\{0,b\}$	$\{1,a,d,e\}$	$\{1,a,c,e\}$	$\{1,a,c,d\}$	b	0	b	d	1	e	a	c
c	c	$\{a,b,d,e\}$	$\{1,b,d,e\}$	$\{1,a,d,e\}$	$\{0,c\}$	$\{1,a,b,e\}$	$\{1,a,b,d\}$	c	0	c	e	d	1	b	a
d	d	$\{a,b,c,e\}$	$\{1,b,c,e\}$	$\{1,a,c,e\}$	$\{1,a,b,e\}$	$\{0,d\}$	$\{1,a,b,c\}$	d	0	d	b	c	a	e	1
e	e	$\{a,b,c,d\}$	$\{1,b,c,d\}$	$\{1,a,c,d\}$	$\{1,a,b,d\}$	$\{1,a,b,c\}$	$\{0,e\}$	e	0	e	c	a	b	1	d

5. Conclusions

Here we presented all finite hyperfields of order less than 7 with their tables of operations which could be considered as a source of study on finite hyperfields and verify properties of fields as like for hyperfields. For instance by extending characteristic notion in field theory to hyperfield theory, we see that in HF_{515}, $0 \notin 1 + 1$ and $0 \notin 1 + 1 + 1$, but $0 \in 1 + 1 + 1 + 1$. Which means that the characteristic of a hyperfield is not necessarily a prime number as it is for fields. Watching these tables we find out that it is not possible to introduce number of construction methods for all finite hyperfields whereas there are hyperfields that could not be construct by the methods since known. Hyperfields $\mathbf{S}, HF_{51}$ and HF_{512} are characteristic 1 which by one can study the extension of characteristic 1 geometry on hyperfileds. Also hyperfields presented here, could be considered in coding and cryptography theory.

6. Appendix

Here we present main part of the program which produce all hyperfields of order 5 of the given group G with some comments:

At first after inputting elements of hyperfield and defining group action $M[i][j] = arr[i] * arr[j]$ an elements the program makes power set $\mathcal{P}(H)$ of hyperfield elements:

```
H={0,1,a,b,c}, a.a=1, b.b=1, c.c=1,a.b=c,a.c=b,b.c=a;
      for(l=0; l<n; l++){
    cin >> arr[l];}
```

```
for(i=0;i<n;i++){
    for(j=0;j<n;j++)
    cout<<"Enter "<<arr[i]<<"*"<<arr[j]<<"=";
        cin >> M[i][j];}
          for(i=0; i<(1<<n);i++){
          for(j=0; j<n; j++)}
           if(i & (1 << j)){
              A[i].insert(arr[j]);}
```

0 must belong to one and only one $1 + x, x \in H \setminus \{0\}$,

```
for(h=0;h<n-1;h++){
if(B[h].find('0')!=B[h].end())
        w+=1;}
    if(w==1){
     w=0;}
```

Every non zero element have to belong to at least one $1 + x, x \in H$,

```
for(q=2;q<(1<<n);q++){
 for(h=0;h<n-1;h++)}
   if(B[h].find(arr[q])!=B[h].end()){
      w+=1;}
       if(w!=0){
         z+=1;}
       if(z+2==(1<<n)){
      w=0;
      z=0;}
```

Distributivity of multiplication whit respect to addition from left,

$$F[h][q] = arr[h] + arr[q] = arr[h](1 + arr[y])(arr[h]arr[y] = arr[q]) \text{ and from right,}$$
$$G[h][q] = arr[h] + arr[q] = (1 + arr[y])arr[q](arr[y]arr[q] = arr[h]).$$

```
for(h=0;h<n;h++){
for(q=0;q<n;q++){
for(y=1;y<n;y++){
if(M[h][y]==arr[q]){
for(p=0;p<n;p++){
if(B[y-1].find(arr[p])!=B[y-1].end())
F[h][q].insert(M[h][p]);}}}}}
    for(h=0;h<n;h++){
    for(q=0;q<n;q++){
    for(y=1;y<n;y++){
    if(M[y][q]==arr[h]){
    for(p=0;p<n;p++){
    if(B[y-1].find(arr[p])!=B[y-1].end())
    G[h][q].insert(M[p][q]);}}}}}
```

```
for (h=0;h<n;h++){
for (q=0;q<n;q++){
if (F[h][q]!=G[h][q])
 z+=1;}
 if (z==0)}
```

This part of program checks associativity. It is enough to check
$C[h][q] = (x + 1) + y = x + (1 + y) = D[h][q]$. For every $x, y \in H$.

```
for (h=0;h<n-2;h++){
        if (w!=0)
        break;
        else
for (q=h+2;q<n;q++){
     if (B[h].find(arr[0])!=B[h].end())
     C[h][q].insert(arr[q]);
     if (B[h].find(arr[1])!=B[h].end()){
for (it=B[q-1].begin();it!=B[q-1].end();++it)
        C[h][q].insert(*it);}
for (t=2;t<n;t++){
   if (B[h].find(arr[t])!=B[h].end()){
for (it=F[t][q].begin();it!=F[t][q].end();++it)
       C[h][q].insert(*it);}}
   if (B[q-1].find(arr[0])!=B[q-1].end())
  D[h][q].insert(arr[h+1]);
   if (B[q-1].find(arr[1])!=B[q-1].end()){
   for (it=B[h].begin();it!=B[h].end();++it)
          D[h][q].insert(*it);}
     for (t=2;t<n;t++){
   if (B[q-1].find(arr[t])!=B[q-1].end()){
  for (it=F[h+1][t].begin();it!=F[h+1][t].end();++it)
       D[h][q].insert(*it);}}
     if (C[h][q]!=D[h][q]){
        w+=1;}
   if (w==0){\sum\sum
```

This part of program checks reversibility:

$arr[q] \in (1 + arr[h + 1]) = B[h] \Leftrightarrow arr[h + 1] \in arr[q] - 1 = arr[q] + arr[y + 1] = F[q][y + 1],$

```
for (y=0;y<n-1;y++){  if (B[y].find('0')!=B[y].end())}
    for (h=0;h<n-1;h++){
    for (q=1;q<n;q++)}
if (B[h].find(arr[q])!=B[h].end()){
```

```
&&F[q][y+1].find(arr[h+1])==F[q][y+1].end()
            x++;
}
if(x==0){
        m+=1;
```

This part of program is related to third step. Computation goes on for finding automorphism on the given group *G*. At first it finds an one to one mapping.

```
cout <<"state:"<<m<<"\n";
for(h=1;h<n;h++){
Setvalue[m][h-1]=B[h-1];
      cout <<"1+"<<arr[h]<<"={";
for(it=B[h-1].begin();it!=B[h-1].end();++it){
      cout <<*it;
          if(it!=--B[h-1].end())
              cout <<",";}
        cout <<"}"<<"\n";{
      for(h=0;h<n;h++){
      for(q=0;q<n;q++){
       C[h][q].clear();
       D[h][q].clear();
       F[h][q].clear();
       G[h][q].clear();}}}
        cout <<"m="<<m<<endl;
      Phi[0]=arr[0];
      Phi[1]=arr[1];
      x=0;
      for(h=0;h<n;h++){
      for(q=0;q<n;q++){
       Matx[h][q]=0;}}
      for(i=2;i<n;++i){
      Phi[2]=arr[i];
      Matx[2][i]=1;
        for(j=2;j<n;++j){
         Phi[3]=arr[j];
         Matx[3][j]=1;
         for(r=2;r<n;++r){
        Phi[4]=arr[r];
        Matx[4][r]=1;
    Matx[0][0]=1;
      Matx[1][1]=1;
for(h=0;h<n;h++)
      {w=0;
        for(q=0;q<n;q++){
```

```
        w+=Matx [ q ] [ h ] ; }
       if (w!=1){
       w=0;
        break ; } }
     if (w==1){
```

In this part the program checks that whether or not the mapping is an automorphism.

```
    for ( h=0;h<n ; h++){
    for ( q=0;q<n ; q++){
     C[ h ] [ q ] . c l e a r ( ) ;
     D[ h ] [ q ] . c l e a r ( ) ; } }
        for ( h=1;h<n ; h++){
        for ( q=1;q<n ; q++){
        for ( k=1;k<n ; k++){
        if (M[ h ] [ q]== a r r [ k ] ) {
        C[ h ] [ q ] . i n s e r t ( Phi [ k ] ) ; } } } }
        for ( h=1;h<n ; h++){
        for ( q=1;q<n ; q++){
        for ( k=1;k<n ; k++){
        for ( l=1; l <n ; l ++){
        if ( Phi [ h]== a r r [ k]&&Phi [ q]== a r r [ l ] )
        D[ h ] [ q ] . i n s e r t (M[ k ] [ l ] ) ; } } } }
    w=0;
     for ( h=1;h<n ; h++){
     for ( q=1;q<n ; q++){
     if (C[ h ] [ q]!=D[ h ] [ q ] ) {
        w+=1;} } }
        if (w==0){
        for ( h=0;h<n ; h++){
        cout <<" Phi ["<<h<<"]="<<Phi [ h]<< endl ; }
            x+=1;
cout <<" for automorphism "<<x<<" one has these
 isomorphisms between hyp
 e r f i e l d s :"<< endl ;
```

For the automorphism by proposition 10 it checks isomorphism between hyperfields.

```
    for ( h=1;h<m+1  ; h++){
    for ( q=1;q<n ; q++){
    for ( k=0;k<n ; k++){
     if ( Setvalue [ h ] [ q−1]. find ( arr [ k ])!= Setvalue [ h ] [ q−1]. end ( ) )
     F[ h ] [ q−1]. i n s e r t ( Phi [ k ] ) ; } } }
     for ( h=1;h<m+1;h++){
     for ( k=1;k<n ; k++){
     for ( q=1;q<n ; q++){
```

```
   if (Phi[k]==arr[q])
   G[h][k-1]=Setvalue[h][q-1];}}}
 for (h=1;h<m+1;h++){
 for (q=1;q<m+1;q++){
 for (k=0;k<n-1;k++){
    if (F[h][k]==G[q][k])
    w+=1;
    if (w==n-1){
cout<<"Hyperfield of state:"<< h <<", is isomorphic to
 hyperfield of state:" << q << endl;{
```

At the end the program computes the number of isomorphism class for each hyperfield which by one can state the number of hyperfields of order 5, up to isomorphism.

```
   Isonum[h].insert(q);}}}
   w=0;}}
 for (h=1;h<m+1;h++){
 for (q=0;q<n-1;q++){
    F[h][q].clear();
    G[h][q].clear();}}
 for (h=0;h<n;h++){
   Matx[4][h]=0;}}
 for (h=0;h<n;h++){
   Matx[3][h]=0;}
 for (h=0;h<n;h++){
   Matx[2][h]=0;}}
 for (h=1;h<m+1;h++){
 z=0;
for (itin=Isonum[h].begin();itin!=Isonum[h].end();++itin)
      z+=1;}
cout<<"Isomorphism class of state "<<h<<" contains "<<z<<"
elements."<<endl;}
```

Conflict of interest

Authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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