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Research article

Fixed point theorems in $R-$metric spaces with applications

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Abstract: The purpose of this paper is to introduce the notion of R-metric spaces and give a real generalization of Banach fixed point theorem. Also, we give some conditions to construct the Brouwer fixed point. As an application, we find the existence of solution for a fractional integral equation.

Keywords: R-metric spaces; fixed point; strong R-compact metric spaces; fractional integral equations
Mathematics Subject Classification: 54H25, 47H10

1. Introduction

Fixed point theory is a powerful and important tool in the study of nonlinear phenomena. It is an interdisciplinary subject which can be applied in several areas of mathematics and other fields, like game theory, mathematical economics, optimization theory, approximation theory, variational inequality in [25], biology, chemistry, engineering, physics and etc. In 1886, Poincare was the first to work in this field.Then Brouwer in [7] in 1912, proved fixed point theorem for the $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimentional counterparts which was further extended by Kakutani in [17]. In 1922, Banach in [6] proved that a contraction mapping which its domain is complete posses a unique fixed point. The fixed point theory as well as Banach contraction principle has been studied and generalized in different spaces for example: In 1969, Nadler in [22] extended the Banach's principle to set valued mappings in complete metric spaces. In 1990 fixed point theory in modular function spaces was initiated by Khamsi, Kozlowski and Reich in [18]. Modular metric spaces were introduced in [8, 9]. Fixed point theory in modular metric spaces was studied by Abdou and Khamsi in [3]. In 2007, Huang and Zhang in [15] introduced cone metric spaces which are generalizations of metric spaces and they extended Banach's contraction principle to such spaces, whereafter many authors (for examples in [1, 2, 4, 10, 12, 16, 34] and references therein) studied fixed point theorems in cone metric spaces. Moreover, in the case when the underlying cone is normal led Khamsi in [19] to introduce a new type of spaces which he called metric-type spaces, satisfying basic properties of the associated space. Some fixed point results were obtained in metric-type spaces in the papers in [19]. The readers who are interested in hyperbolic type metrics defined on planar and multidimensional domains refer to [33]. In 2012, the notion of ordered spaces and normed ordered spaces were introduced by Al-Rawashdeh et al. in [31] and get a fixed point theorem. The authors for example in [21, 26, 27] considered fixed point theorems in E-metric spaces. In 2017, M. Eshaghi et al. in [14], introduced a new class of generalization metric spaces which are called orthogonal metric spaces. Subsequently they and many authors (for examples, in [5, 13, 28, 29]) gave an extension of Banach fixed point theorem. It is still going on. At the end it is advised to those who want to research the theory of fixed point read the valuable references mentioned in [11, 20].

2. R-metric spaces

Banach had been proved the following theorem in complete metric space X.

Theorem 2.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that, for some $\lambda \in (0, 1)$,*

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x,\ y \in X$. Then f has a unique fixed point in X.

Ran et al. in [30] gave an extension of Banach fixed point Theorem by weakened the contractivity condition on elements that are comparable in the partial order:

Theorem 2.2. *Let T be a partially ordered set such that every pair $x, y \in T$ has a lower bound and an upper bound. Furthermore, let d be a metric on T such that (T, d) is a complete metric space. If F is a continuous, monotone (i.e., either order-preserving or order-reversing) map from T into T such that*
$\exists 0 < c < 1 : d(F(x), F(y)) \leq cd(x, y), \forall x \geq y$,
$\exists x_0 \in T : x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$,
then F has a unique fixed point x. Moreover, for every $x \in T, \lim_{n\to\infty} f^n(x) = \overline{x}$.

Afterward Nieto et al. in [24] presented a new extension of Banach contractive mapping to partially ordered sets, where some valuable applications and examples are given:

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping such that there exists $k \in [0, 1)$ with*

$$d(f(x), f(y)) \leq kd(x, y),\ \forall x \geq y.$$

If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Recently Eshaghi et al. in [14] Proved Banach contraction principle in orthogonal metric spaces $(X, d, \perp)$, where $\perp$ is a relation on X.

In this paper among other things, we try to present a real generalization of the mentioned Banach's contraction principle by introducing R-metric spaces, where R is an arbitrary relation on X. We note that in especial case R can be considered as $R :=\leq$ [24, 30], $R :=\perp$ [14], or etc. Furthermore, in

Brouwer and Kakutani theorems the existence of fixed point is stated [17], and the Nash equilibrium is proved only based on the existence of fixed point [23]. Obtaining the exact value of the equilibrium point is usually difficult and only approximate value is obtained. If one can find a suitable replacement of Brouwer theorem which is determine the value of fixed point then many problems on game theory and economics can be solved. In this paper, we try to provide a structural method for finding a value of fixed point.

Definition 2.1. Suppose (X, d) is a metric space and R is a relation on X. Then the triple (X, d, R) or in brief X is called R-metric space.

Example 2.1. Let $(\mathbb{R}, |\ \ |)$ be given and let $R :=\leq$, $R :=\geq$ or $R :=\perp, ...$, then with each R, $(\mathbb{R}, |\ \ |, R)$ is an R-metric space.

Example 2.2. Let $X = [0, \infty)$ equipped with Euclidean metric. Define xRy if $xy \leq (x \vee y)$ where $x \vee y = x$ or y. Then (X, d, R) is an R-metric space.

We note that for a given specified metric space (X, d) and any relation R on X we can consider an R-metric space (X, d, R) .

Definition 2.2. A sequence $\{x_n\}$ in an R-metric space X is called an R-sequence if $x_n R x_{n+k}$ for each $n, k \in \mathbb{N}$.

Definition 2.3. An R-sequence $\{x_n\}$ is said to converge to $x \in X$ if for every $\varepsilon > 0$ there is an integer N such that $d(x_n, x) < \varepsilon$ if $n \geq N$.

In this case, we write $x_n \overset{R}{\to} x$.

Example 2.3. Suppose $X = [1, 2]$ with Euclidean metric be given. Let $R :=\geq$ and $x_n = 1 + \frac{1}{n}$ for each $n \in \mathbb{N}$. Clearly $\{x_n\}$ is an R-sequence and $x_n \overset{R}{\to} 1$. Note that $x_n = 2 - \frac{1}{n}$ is not an R-sequence.

Example 2.4. Let $X = \mathbb{R}$. Consider $P(\mathbb{R})$ with metric $dist(A, B) = inf\{|a - b| :\ a \in A \ and \ b \in B\}$, let $R :=\subseteq$. Define $A_n = [1 + \frac{1}{n}, 4 - \frac{1}{n}]$, clearly $\{A_n\}$ is an R-sequence and $A_n \overset{R}{\to} (1, 4)$.

Remark 2.1. *Every subsequence of an R-sequence is an R-sequence too.*

In the followings it is supposed that X is an R-metric space.

Definition 2.4. Let $E \subseteq X$. $x \in X$ is called an R-limit point of E if there exists an R-sequence $\{x_n\}$ in E such that $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \overset{R}{\to} x$.

Definition 2.5. The set of all R-limit points of E is denoted by E'^R.

Definition 2.6. $E \subseteq X$ is called R-closed, if $E'^R \subseteq E$.

Definition 2.7. $E \subseteq X$ is called R-open, if E^C is R-closed.

Theorem 2.4. *$E \subseteq X$ is R-open if and only if for any $x \in E$ and any R-sequence $\{x_n\}$ which $x_n \overset{R}{\to} x$, there exists $N \in \mathbb{N}$ such that $x_n \in E$ for all $n \geq N$.*

Proof. Let $x \in E$. Suppose there exists an R-sequence $\{x_n\}$ which $x_n \xrightarrow{R} x$ but for every $N \in \mathbb{N}$, there exists a natural number $n \geq N$ such that $x_n \notin E$. Hence, we obtain an R-subsequence $\{x_{n_N}\}$ of $\{x_n\}$, which $x_{n_N} \xrightarrow{R} x$, and $x_{n_N} \notin E$. Therefore $x \in E^C$. This contradicts $x \in E$.

Conversely, if $x \in (E^C)'^R$, then $x \notin E$. Hence $(E^C)'^R \subset E^C$, so E^C is R-closed. Thus E is R-open. □

Definition 2.8. Let $E \subseteq X$. The R-closure of E is the set $\overline{E}^R = E \cup E'^R$.

Theorem 2.5. *If $E \subset X$, then*
(a) $\overline{E}^R$ is R-closed.
(b) $E = \overline{E}^R$ if and only if E is R-closed.

Proof. (a) If $x \in (\overline{E}^R)^C$, then x is neither a point of E nor an R-limit point of E. Let $\{x_n\}$ be an R-sequence converging to x, then there exists $N \in \mathbb{N}$ such that $x_n \in (\overline{E}^R)^C$ for $n \geq N$. Thus $(\overline{E}^R)^C$ is R-open so that $\overline{E}^R$ is R-closed.
(b) If $E = \overline{E}^R$, (a) implies that E is R-close. If E is R-closed, then $E'^R \subset E$ [by definitions 2.6 and 2.8], hence $\overline{E}^R = E$. □

Example 2.5. Suppose $X = \mathbb{R}$ equipped with standard topology. Let $R :=\leq$, let $E = (0, 1]$, then $\overline{E}^R = (0, 1]$. Hence E is R-closed but it is not closed.

Theorem 2.6. *The set $\{G \subseteq X | G$ is R-open$\}$ is a topology on X which is called an R-topology and is denoted by τ_R*

Proof. Trivially φ and X are R-open. Let $\{U_j\}_{j\in J}$ be a family of R-open sets. Put $U = \bigcup_{j\in J} U_j$, let x be an R-limit point of $U^C = \bigcap_{j\in J} U_j^C$, hence there exists an R-sequence $\{x_n\}$ in $U^C \setminus \{x\}$ such that $x_n \xrightarrow{R} x$. Therefore for each $j \in J$, x is an R-limit point of U_j^C. Since U_j^C is R-closed $x \in U_j^C$, so that $x \in U^C$, hence U is R-open.

Let $U_1, \ldots, U_n$ be R-open sets. Put $W = \bigcap_{j=1}^n U_j$ and let x be an R-limit point of $W^C = \bigcup_{j=1}^n U_j^C$, hence there exists an R-sequence $\{x_n\}$ in $W^C \setminus \{x\}$ such that $x_n \xrightarrow{R} x$. It is easy to show that there exists $1 \leq j \leq n$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in $U_j^C \setminus \{x\}$ which $x_{n_k} \xrightarrow{R} x$. Since U_j^C is R-closed $x \in U_j^C$, thus $x \in W^C$. It follows that W is R-open. □

Corollary 2.1. *Let $X = \mathbb{R}$ equipped with standard topology. Let $R :=\leq$, then:*
(a) The open intervals (a, b) ($a = -\infty$ or $b = +\infty$) form a basis for the standard topology on $\mathbb{R}$,
(b) The intervals (a, b) ($a = -\infty$ or $b = +\infty$) and $(a, b]$ form a basis for topology τ_R.

Theorem 2.7. *Let τ_d be a metric topology on X, then $\tau_d \subseteq \tau_R$.*

Proof. Let $G \in \tau_d$. Let x be an R-limit point of G^C. There exists an R-sequence $\{x_n\}$ in $G^C \setminus \{x\}$ such that $x_n \xrightarrow{R} x$. Since each R-sequence is a sequence hence x is a limit point of G^C. G^C is closed, thus $x \in G^C$. Therefore $G \in \tau_R$. □

Lemma 2.1. *Let $R := X \times X$, then $\tau_R = \tau_d$.*

Proof. By theorem 2.7, $\tau_d \subseteq \tau_R$. Let G is an R-open set and x is a limit point of G^C. There exists a sequence $\{x_n\}$ in $G^C \setminus \{x\}$ such that $x_n \to x$. Clearly $\{x_n\}$ is an R-sequence, hence $x_n \xrightarrow{R} x$. It follows that $x \in G^C$. Thus $\tau_R \subseteq \tau_d$. □

Remark 2.2. *Suppose (X, d) be a metric space and $R := X \times X$, then R-metric space (X, d, R) is equivalent to metric space (X, d).*

Definition 2.9. Let R be a relation on $\mathbb{R}^k$. $E \subseteq \mathbb{R}^k$ is called R-convex if $\lambda x + (1 - \lambda)y \in E$, whenever $x \in E$, $y \in E$, xRy, and $0 < \lambda < 1$.

Lemma 2.2. *Let E is a convex set, then E is R-convex.*

Proof. Let $x \in E,\ y \in E$, xRy, and $0 < \lambda < 1$. By the definition of convex set $\lambda x + (1 - \lambda)y \in E$. □

The converse is not true.

Example 2.6. Let $E = (0, 1] \cup (2, 4]$. Define xRy if $x,\ y \in (0, 1]$ or $x,\ y \in (2, 4]$. Clearly E is R-convex but it is not convex.

Definition 2.10. $K \subseteq X$ is called R-compact if every R-sequence $\{x_n\}$ in K has a convergent subsequence in K.

Example 2.7. Suppose the Euclidean metric space $X = \mathbb{R}$ be given. Let $K = [0, 1)$ and let $R :=\geq$. Then K is R-compact.

Lemma 2.3. *Suppose $K \subseteq X$ is compact, then K is R-compact.*

Proof. Let $\{x_n\}$ be an R-sequence in K. It is clear that $\{x_n\}$ is a sequence too. So by theorem 3.6 (a) [32], there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in K. The theorem follows. □

The converse is not true.

Example 2.8. Suppose the Euclidean metric space $X = \mathbb{R}$ be given. Let $k = [0, 1)$, and let R defined on X by

$$xRy \Longleftrightarrow \begin{cases} x \leq y \leq \frac{1}{3} \\ or \\ x = 0 \qquad if \quad y > \frac{1}{3} \end{cases}.$$

Since K is not closed so it is not compact. Let $\{x_n\}$ be an R-sequence in K. Then:
i) For all $n \in \mathbb{N}$, $x_n = 0$, hence $\{x_n\}$ converges to 0.
ii) $\{x_n\}$ is increasing and bounded above to $\frac{1}{3}$, therefore it is convergent.
It follows that K is R-compact.

Example 2.9. Suppose $X = \mathbb{R}$ equipped with the Euclidean metric. Let $K = (0, 1]$. Let $R :=\leq$, and let $\{x_n\}$ be an R-sequence in K. It is clear that $\{x_n\}$ is increasing and bounded above, hence it is convergent. Therefore K is R-compact but it is not compact.

Theorem 2.8. *R-compact subsets of R-metric spaces are R-closed.*

Proof. Let K be an R-compact subset of X. Let $x \in K'^R$. Then there exists an R-sequence $\{x_n\}$ in K such that $x_n \xrightarrow{R} x$. By definition 2.10, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in K, hence $x_{n_k} \xrightarrow{R} x$, so that $x \in K$. The theorem follows. □

Theorem 2.9. *An R-closed subset of an R-compact set, is R-compact.*

Proof. Suppose $F \subseteq K \subseteq X$. F is R-closed (relative to X), and K is R-compact. Let $\{x_n\}$ be an R-sequence in F, hence $\{x_n\}$ is an R-sequence in K too. Since K is R-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \stackrel{R}{\to} x$. Since F is R-closed $x \in F$.The theorem follows. □

Corollary 2.2. *Let F be R-closed and K be R-compact. Then $F \cap K$ is R-compact.*

Proof. By theorems 2.6 and 2.8, $F \cap K$ is R-closed; since $F \cap K \subset K$, theorem 2.9 shows that $F \cap K$ is R-compact. □

Definition 2.11. A set $K \subseteq X$ is called strong R-compact, if each R-sequence $\{x_n\}$ in K, that has a subsequence $\{x_{n_k}\}$ which converges to $x^* \in K$, i.e, $x_{n_k} \stackrel{R}{\to} x^*$, then $x_{n_k+1} \stackrel{R}{\to} x^*$.

Example 2.10. Suppose $X = \mathbb{R}$ with standard topology be given, let $R :=\leq$ and let $K = (0, 10]$. Suppose $\{x_n\}$ is an R-sequence, then each subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is increasing and bounded above, so there exists $x^* \in K$, such that $x_{n_k} \stackrel{R}{\to} x^*$ and $x_{n_k+1} \stackrel{R}{\to} x^*$. Therefore K is a strong R-compact set.

Definition 2.12. An R-sequence $\{x_n\}$ in X is said to be an R-Cauchy sequence, if for every $\varepsilon > 0$ there exists an integer N such that $d(x_n, x_m) < \varepsilon$ if $n \geq N$ and $m \geq N$. It is clear that $x_n R x_m$ or $x_m R x_n$.

Lemma 2.4. *(a) Every convergent R-sequence in X is an R-Cauchy sequence.*
(b) Suppose K be an R-compact set and $\{x_n\}$ be a R-Cauchy sequence in K. Then $\{x_n\}$ converges to $x \in K$.

Definition 2.13. X is said to be R-complete if every R-Cauchy sequence in X converges to a point in X.

Corollary 2.3. *Every R-compact space is R-complete, but the converse is not true.*

Example 2.11. Suppose $X = \mathbb{R}$ with standard topology be given and let $R :=\leq$. It is easy to show that X is R-complete but it is not R-compact. Since the R-sequence $\{n\}$ has no convergent subsequence.

Definition 2.14. Let $f : X \to X$ be a mapping. f is said to be R-continuous at $x \in X$ if for every R-sequence $\{x_n\}$ in X with $x_n \stackrel{R}{\to} x$, we have $f(x_n) \longrightarrow f(x)$. Also, f is said to be R-continuous on X if f is R-continuous in each $x \in X$.

Lemma 2.5. *Every continuous mapping $f : X \to X$, is R-continuous.*

Proof. Since each R-sequence is a sequence. □

The converse is not true.

Example 2.12. Suppose $X = [0, 1]$ equipped with standard topology and let $f : X \to X$ be a Dirichlet mapping, i.e,

$$f(x) = \begin{cases} 1 & if \quad x \in Q \cap [0, 1] \\ 0 & if \quad x \in Q^C \cap [0, 1] \end{cases} .$$

Let R be an equality relation on X, then f is discontinuous at each point of X but f is R-continuous on X.

Example 2.13. Suppose the Euclidean metric space $X = \mathbb{R}$ be given. Define xRy if $x,\ y \in (n+\frac{2}{3}, n+\frac{4}{5})$ for some $n \in \mathbb{Z}$ or $x = 0$. Define $f : X \to X$ by $f(x) = [x]$. Let $x \in X$ and $\{x_n\}$ be an arbitrary R-sequence in X such that converges to x, then the following cases are satisfied:

Case 1: If $x_n = 0$ for all n, then $x = 0$, and $f(x_n) = 0 = f(x)$.

Case 2: If $x_n \neq 0$ for some n, then there exists $m \in \mathbb{Z}$ such that $x \in [m+\frac{2}{3}, m+\frac{4}{5}]$, and $f(x_n) = m = f(x)$.

Therefore f is R-continuous on X, but it is not continuous on X.

Definition 2.15. A mapping $f : X \to X$ is said to be an R-contraction with Lipschutz constant $0 < \lambda < 1$ if for all $x, y \in X$ such that xRy, we have

$$d(f(x), f(y)) \leq \lambda d(x, y)\ .$$

It is easy to show that every contraction is an R-contraction but the converse is not true. See the next example.

Example 2.14. Let $X = [0, 0.99)$, and X equipped with Euclidean metric. Let xRy if $xy \in \{x, y\}$ for all $x, y \in X$. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} x^2 & if \quad x \in Q \cap X \\ 0 & if \quad x \in Q^C \cap X \end{cases} .$$

Suppose $x = 0.9$, $y = \frac{\sqrt{2}}{2}$ and $0 < \lambda < 1$, then

$$|f(x) - f(y)| = 0.81 \nleq \lambda |0.9 - \frac{\sqrt{2}}{2}|\ .$$

Hence f is not a contraction. Now, let xRy, therefore $x = 0$ or $y = 0$. Suppose $x = 0$, thus

$$|f(x) - f(y)| = \begin{cases} y^2 & if \quad y \in Q \cap X \\ 0 & if \quad y \in Q^C \cap X \end{cases} .$$

Hence by choosing $\lambda = 0.99$ it follows that

$$|f(x) - f(y)| \leq y^2 \leq \lambda |0 - y| = \lambda y\ .$$

So that f is R-contraction.

Definition 2.16. Let $f : X \to X$ be a mapping f is called R-preserving if xRy, then $f(x)Rf(y)$ for all $x, y \in X$.

Example 2.15. Suppose $X = \mathbb{R}$ with standard topology be given and let $R :=\geq$. Let $f : X \to X$ be a mapping defined by $f(x) = x^3$. Let $x_1 \geq x_2$, then $f(x_1) = x_1^3 \geq x_2^3 = f(x_2)$. Hence f is R-preserving.

3. The main results

In this section, it is proved two main theorems. The first one is the real extension of one of the most important theorem in mathematics which is named Banach contraction principle theorem 2.1, and the second one is the version of Brouwer fixed point theorem.

Theorem 3.1. *Let X be an R-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be R-continuous, R-contraction with Lipschutz constant λ and R-preserving. Suppose there exists $x_0 \in X$ such that $x_0 R y$ for all $y \in f(X)$. Then f has a unique fixed point x^*. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.*

Proof. Let $x_1 = f(x_0)$, $x_2 = f(x_1) = f^2(x_0), \dots,\ x_n = f(x_{n-1}) = f^n(x_0), \dots$, for all $n \in \mathbb{N}$. let $n, m \in \mathbb{N}$, and $n < m$, put $k = m - n$. We have $x_0 R f^k(x_0)$ since f is R-preserving $(x_n = f^n(x_0))R(f^{n+k}(x_0) = x_m)$. Hence $\{x_n\}$ is an R-sequence. On the other hand, f is R-contraction, thus we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \\ \lambda d(x_{n-1}, x_n) \leq \cdots \leq \lambda^n d(x_1, x_0)\ , \end{aligned}$$

for all $n \in \mathbb{N}$. If $m, n \in \mathbb{N}$, and $n \leq m$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda}\, d(x_1, x_0)\ . \end{aligned}$$

Hence $d(x_n, x_m) \longrightarrow 0$ as $m, n \to \infty$. Therefore $\{x_n\}$ is an R-Cauchy sequence. Since X is R-complete, there exists $x^* \in X$ such that $x_n \xrightarrow{R} x^*$. On the other hand, f is R-continuous so $f(x_n) \xrightarrow{R} f(x^*)$, therefore $f(x^*) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$. Thus x^* is a fixed point of f.

To prove the uniqueness, let $y^* \in X$ be a fixed point of f, then $x_0 R f(y^*) = y^*$. Hence $(x_n = f^n(x_0))R y^*$ for all $n \in \mathbb{N}$. Therefore, by triangle inequality, we have

$$\begin{aligned} d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) &\leq d(f^n(x^*), f^n(x_0)) + d(f^n(x_0), f^n(y^*)) \\ &\leq \lambda^n d(x^*, x_0) + \lambda^n d(x_0, y^*) \longrightarrow 0\ , \end{aligned}$$

as $n \to \infty$. Thus it follows that $x^* = y^*$.

Finally, let x be an arbitrary element of X. We have $x_0 R f(x)$, hence $f^n(x_0) R f^{n+1}(x)$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} d(x^*, f^n(x)) = d(f^n(x^*), f^n(x)) &\leq d(f^{n-1}(x^*), f^{n-1}(x_0)) + d(f^{n-1}(x_0), f^{n-1}(f(x))) \\ &\leq \lambda^{n-1} d(x^*, x_0) + \lambda^{n-1} d(x_0, f(x)) \longrightarrow 0\ , \end{aligned}$$

as $n \to \infty$. Hence $\lim_{n\to\infty} f^n(x) = x^*$. □

We now show that our theorem is an extension of Banach contraction principle.

Corollary 3.1. *(Banach contraction principle) Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that for some $\lambda \in (0, 1)$,*

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x,\ y \in X$. Then f has a unique fixed point in X.

Proof. Suppose that $R := X \times X$. Fix $x_0 \in X$. Clearly, $x_0 R y$ for all $y \in f(X)$. Since X is complete, it is R-complete. It is clear that f is R-preserving, R-contraction and R-continuous. Using Theorem 3.1, f has a unique fixed point in X. □

The following example, shows that our theorem is a real extension of Banach contraction principle. Moreover, the next example is not satisfied [24, Theorem 2.1] and satisfy Theorem 3.1 in this paper.

Example 3.1. Let $X = [0, 1)$ and let the metric on X be the Euclidean metric. Define xRy if $xy \in \{x, y\}$, that is, $x = 0$ or $y = 0$. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} \frac{x^2}{3} & : \ x \leq \frac{1}{3} \\ 0 & : \ x > \frac{1}{3} \end{cases} .$$

Let $\{x_n\}$ be an R-sequence in X. It is obvious that there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N$, $n \in \mathbb{N}$. Hence $x_n \overset{R}{\to} 0$. Therefore X is R-complete but not complete. f is R-continuous but not continuous. Suppose xRy, thus $x = 0$ or $y = 0$. Let $x = 0$, then

$$|f(x) - f(y)| = \begin{cases} \frac{y^2}{3} & if \ \ y \leq \frac{1}{3} \\ 0 & if \ \ y > \frac{1}{3} \end{cases} .$$

Hence by choosing $\lambda = 0.99$ it follows that

$$|f(x) - f(y)| \leq \frac{y^2}{3} \leq \lambda |0 - y| = \lambda y .$$

So f is R-contraction. Let xRy, then $x = 0$ or $y = 0$, hence $f(x) = 0$ or $f(y) = 0$, so $f(x)Rf(y)$, that is, f is R-preserving. Let $y \in f(X)$, then $0Ry$. Therefore using Theorem 3.1, f has a unique fixed point in X. However, by Banach contraction principle and [24, Theorem 2.1], we can not find any fixed point of f in X.

As, we know in Brouwer theorem a continuous mapping $f : E \to E$ from a convex and compact, set E $(\subseteq \mathbb{R}^n)$ into E has a fixed point without mentioning how to find the fixed point.

In our theorem, we omit the convexity and substitute the compactness with strong R-compactness to show how the fixed point can be constructed.

Theorem 3.2. *Suppose $X = \mathbb{R}^n$ equipped with standard topology and let $E \subsetneq X$ be a strong R-compact set. Let $f : E \to E$ is R-continuous and R-preserving. Suppose $x_0 \in E$ be given such that $x_0 R y$ for all $y \in f(E)$. Then f has a fixed point.*

Proof. Let $x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0)) = f^2(x_0), \ldots, x_n = f^n(x_0), \ldots$, for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ where $n < m$, put $k = m - n$. $x_0 R f^k(x_0)$, hence $f^n(x_0) R f^{n+k}(x_0) = f^m(x_0)$, i.e, $x_n R x_m$, therefore, the sequence $\{x_n\}$ is an R-sequence. Since E is R-compact, there exists a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \overset{R}{\to} x^*$. Thus $f(x_{n_k}) \overset{R}{\to} f(x^*)$, therefore $f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} x_{n_k+1} = x^*$. □

Example 3.2. Suppose $X = \mathbb{R}$ with standard topology be given. Let $R :=\leq$ and $E = (0, 1]$. Let $f : E \to E$ be a mapping defined by $f(x) = \frac{x+1}{2}$. Let $x_0 = \frac{1}{2}$, then $x_1 = f(x_0) = \frac{3}{4}$, $x_2 = f(x_1) = \frac{7}{8}$, $x_3 = f(x_2) = \frac{15}{16}$, $x_4 = f(x_3) = \frac{31}{32}, \ldots$. Hence we obtain an R-sequence $\{x_n\}$ such that $x_n \to 1 = x^*$. Thus $f(x^*) = f(1) = \frac{1+1}{2} = 1 = x^*$.

Example 3.3. Suppose $X = \mathbb{R}$ with standard topology be given. Let $R :=\leq$ and let $E = (-1, 2] \cup (3, 4]$. Let $f : E \to E$ be a mapping defined by $f(x) = \frac{x+1}{4}$. Put $x_0 = 0$, then $x_1 = f(x_0) = f(0) = \frac{1}{4}$, $x_2 = f(x_1) = f(\frac{1}{4}) = \frac{5}{4^2}$, $x_3 = f(x_2) = f(\frac{5}{4^2}) = \frac{21}{4^3}$, $x_4 = f(x_3) = f(\frac{21}{4^3}) = \frac{85}{4^4}, \ldots$. It is easy to show that $x_n \xrightarrow{R} \frac{1}{3} = x^*$. Hence the fixed point is $\frac{1}{3}$.

In the above example E is neither compact nor convex but f has a fixed point.

4. Application to fractional integral equations

Our aim here is to apply Theorem 3.1 to prove the existence and uniqueness of solution for the following fractional integral equation of the type

$$\begin{cases} x(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(u)\,(t-u)^{\alpha-1}\, x(u)du + f(t) \quad , \quad t \in I = [0, T] \\ f(t) \geq 1 \quad , \quad t \in I \end{cases} \tag{4.1}$$

where $0 < \alpha < 1$ and $a,\ b,\ f : [0, T] \to \mathbb{R}$ are continuous functions.

Theorem 4.1. *Under the above conditions, for all $T > 0$ the fractional integral equation* (4.1) *has a unique solution.*

Proof. Let $X = \Big\{u \in C(I, \mathbb{R}) :\ u(t) > 0,\ \forall t \in I\Big\}$. We define the following relation R in X:

$$xRy \iff x(t)\, y(t) \geq (x(t) \vee y(t)),\ \forall t \in I.$$

Define

$$\|x\|_r = \max_{t\in I} e^{-rt}|x(t)|,\ x \in X,\ for\ r > (\|a\| \cdot \|b\|)^{\frac{1}{\alpha}} \tag{4.2}$$

It is easy to show that (X, d) is an R-metric space. Suppose $x_0(t) \equiv 1$ for all $t \in I$, then x_0Rx for all $x \in X$.Let $\{x_n\} \subseteq X$ be a Cauchy R-sequence. It is easy to show that $\{x_n\}$ is converging to an element x uniformly in $C(I, \mathbb{R})$. Fix $t \in I$, the definition of R implies $x_n(t)\ x_{n+k}(t) \geq (x_n(t) \vee x_{n+k}(t))$ for each $n, k \in \mathbb{N}$. Since $x_n(t) > 0$ for all $n \in \mathbb{N}$, there exists an R-subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}(t) \geq 1$ for all $i \in \mathbb{N}$. Clearly $\{x_{n_i}\}$ also converges to x, therefore $x \geq 1$, hence $x \in X$.
Define $\mathcal{F} : X \to X$ by

$$\mathcal{F}x(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(u)\,(t-u)^{\alpha-1}\, x(u)du + f(t)\,.$$

Clearly the fixed points of $\mathcal{F}$ are the solutions of (4.1) .
It is enough to prove the following three steps:

Step (1) $\mathcal{F}$ is R-preserving.

Proof. Let xRy and $t \in I$, $\mathcal{F}x(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(u)\,(t-u)^{\alpha-1}\, x(u)du + f(t) \geq 1$ which implies that $\mathcal{F}_x(t)\,\mathcal{F}_y(t) \geq \mathcal{F}_y(t)$. Thus $\mathcal{F}xR\mathcal{F}y$.

Step (2) $\mathcal{F}$ is R- contraction.
Proof. Let xRy and $t \in I$,

$$\begin{aligned}
e^{-rt}|\mathcal{F}x(t) - \mathcal{F}y(t)| &\leq e^{-rt}\,\frac{a(t)}{\Gamma(\alpha)}\int_0^t b(u)\,(t-u)^{\alpha-1}\,|x(u)-y(u)|\,du \\
&\leq e^{-rt}\,\frac{a(t)}{\Gamma(\alpha)}\int_0^t b(u)\,(t-u)^{\alpha-1}\,e^{ru}e^{-ru}\,|x(u)-y(u)|\,du \\
&\leq e^{-rt}\,\frac{a(t)}{\Gamma(\alpha)}\,\|x-y\|_r\int_0^t b(u)\,(t-u)^{\alpha-1}\,e^{ru}du \\
&\leq \frac{1}{\Gamma(\alpha)}\,\|a\|\,\|b\|\,\|x-y\|_r\,\frac{1}{r^{\alpha-1}}\int_0^t r^{\alpha-1}\,(t-u)^{\alpha-1}\,e^{ru}e^{-rt}du\,.
\end{aligned}$$

Put $w = r(t-u)$, then $0 \leq w \leq rt$, we get

$$\begin{aligned}
e^{-rt}|\mathcal{F}x(t) - \mathcal{F}y(t)| &\leq \frac{1}{\Gamma(\alpha)}\,\|a\|\,\|b\|\,\|x-y\|_r\,\frac{1}{r^{\alpha}}\int_0^{rt} w^{\alpha-1}\,e^{-w}dw \\
&\leq (\|a\|\,\|b\|\,\frac{1}{r^{\alpha}})\|x-y\|_r\,.
\end{aligned}$$

Since t is an arbitrary element of I, by (4.2) it follows that $0 \leq \lambda = (\|a\|\,\|b\|\,\frac{1}{r^{\alpha}}) < 1$, hence $\|\mathcal{F}x - \mathcal{F}y\|_r \leq \lambda\|x-y\|_r$.

Step (3) $\mathcal{F}$ is R-continuous.
Proof: Let $\{x_n\}$ be an R-sequence converging to $x \in X$. Using the first part of the proof $x(t) \geq 1$ for all $t \in I$, hence $x_n(t)x(t) \geq x_n(t)$ for all $n \in \mathbb{N}$ and all $t \in I$, therefore x_nRx. By step (2), we have

$$e^{-rt}|\mathcal{F}x_n(t) - \mathcal{F}x(t)| \leq \lambda\|x_n - x\|_r\,.$$

Thus

$$\|\mathcal{F}x_n - \mathcal{F}x\|_r \leq \lambda\|x_n - x\|_r\,.$$

Therefore $\mathcal{F}x_n \xrightarrow{R} \mathcal{F}x$.
Now applying theorem 3.1 it follows that the fractional integral equation (4.1) has a unique solution. □

5. Conclusion

In this paper, by introducing a new and applicable metric space which was called $R-$metric space, we have shown a real generalization of one of the most important theorems in mathematics which is named Banach fixed point Theorem. The results have immediately applied to show the existence and uniqueness of fractional integral equations and can be extended to ordinary differential equations and etc. Moreover, the results we have obtained suggest a constructive method to find the value of Brouwer fixed point which is very important in many fields like game theory, mathematical economics, physics, chemistry. engineering and etc.

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Conflict of interest

The authors declare no conflict of interest.

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