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Research article

Chebyshev pseudospectral approximation of two dimensional fractional Schrodinger equation on a convex and rectangular domain

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Abstract: In this article, the authors report the Chebyshev pseudospectral method for solving two-dimensional nonlinear Schrodinger equation with fractional order derivative in time and space both. The modified Riemann-Liouville fractional derivatives are used to define the new fractional derivatives matrix at CGL points. Using the Chebyshev fractional derivatives matrices, the given problem is reduced to a diagonally block system of nonlinear algebraic equations, which will be solved using Newton's Raphson method. The proposed methods have shown error analysis without any dependency on time and space step restrictions. Some model examples of the equations, defined on a convex and rectangular domain, have tested with various values of fractional order α and β. Moreover, numerical solutions are demonstrated to justify the theoretical results.

Keywords: nonlinear fractional Schrodinger equation (NFSE); pseudospectral method; modified Riemann-Liouville fractional derivatives; Chebyshev-Gauss-Lobbato points; error analysis
Mathematics Subject Classification: 35C07, 35C11, 35R11

1. Introduction

The fractional partial differential equation was first developed as a pure mathematical theory in the 19th century [36]. Time- and space- nonlinear fractional Schrodinger equation(NFSE) is the fundamental equation of physics for describing nonrelativistic quantum mechanical behavior. This equation was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrodinger. In past years, the time- and space- NFSE has attracted application of various fields such as, electromagnetic waves, quantitative finance, quantum evolution of complex systems and dielectric polarization [6, 9, 28, 29, 35, 37].

Let us consider nonlinear fractional Schrodinger equation

$$i\partial_t^{\alpha} U + \kappa\left(\partial_x^{2\beta} U + \partial_y^{2\beta} U\right) + G(x, y, t)U + \lambda|U|^2 U = F(x, y, t), \quad (x, y) \in \Omega, \quad t \in [0, T], \tag{1.1}$$

with initial condition:

$$U(x, y, 0) = k_1(x, y), \qquad (x, y) \in \Omega,$$

and boundary conditions:

$$\begin{aligned} U(x_L, y, t) &= p_{11}(y, t), \ U(x_R, y, t) = p_{21}(y, t), \qquad t \in [0, 1], \quad (x, y) \in \partial\Omega, \\ U(x, y_L, t) &= p_{31}(x, t), \ U(x, y_R, t) = p_{41}(x, t). \end{aligned}$$

Here, α and β represent the fractional order of derivatives in time and space respectively, with values $0 < \alpha \leq 1$ and $1/2 < \beta \leq 1$. The convex domain Ω is defined as, $\Omega = (x, y) \in [x_L, x_R] \times [y_L, y_R]$, κ and λ are constants, $G(x, y, t)$ is a potential function, $F(x, y, t)$ is complex source term. The function $U(x, y, t)$ is assumed to be a complex wave function of time and space.

In past years, many authors have proposed various numerical methods for the solutions of time- and space- NFSE. These numerical methods are very important for understanding the physical behavior of the equations. Dong [11] has proposed scattering problems for two-dimensional NFSE and further, gives the mathematical expression of the Green's functions. Fan and Qi [14] have introduced the Galerkin finite element method for the solutions of NFSE. Some other authors [15,18,22,23,25,26,42] have used the same methods to studied numerical solutions and error analysis of two-dimensional NFSE. Zhang et al. [43] have proposed Galerkin-Legendre spectral schemes for the numerical solution of space NFSE. Li et al. [17] have discussed the numerical and stability analysis of multi-dimensional time-NFSE using the L1-Galerkin finite element method. Zhao et al. [44] have proposed a fourth-order compact ADI method for the numerical solutions and convergence analysis of two-dimensional space NFSE. In this method, the authors have found fourth-order of accuracy. Li et al. [21] have proposed a fast linearized conservative finite element method for the numerical solution of strongly coupled NFSEs and also prove that the scheme preserved both the mass and energy. Li et al. [24] have used Galerkin finite element method for the unconditional error analysis of NFSE. Cheng et al. [8] has proposed novel compact alternating direction implicit scheme for the numerical analysis of two-dimensional Riesz space fractional nonlinear reaction-diffusion equations. Bhrawy and Zaky [4] have studied a natural generalization of the NFSE, which is changing the variable-order NFSE to fractional quantum phenomena. Chen et al. [7] have introduced symplectic and multi-symplectic methods for the numerical solution of NSE. Further equation (1.1) was solved by several other numerical methods, such as riccati expansion method [1], Legendre-Galerkin spectral method [19], finite element method [13,21], discontinuous Galerkin finite element method [39], momentum representation method [12], compact boundary value method [34], fractional mapping method [40] and finite difference method [20,41].

Pseudospectral methods have been developed for numerical simulation of related differential equations in many fields because of its high accuracy, especially sufficiently smooth problems. Lanczos showed the power of the Fourier series and Chebyshev polynomials in a number of problems where they had not been used before. Later in the 1970s, Orszag introduced spectral methods again, alongside Kreiss, Oliger, and others, for the purpose of solving partial differential equations in fluid mechanics [38]. Many authors have used spectral method to approximate the solution of such equations [2,27,30–33]. In this paper, the authors propose a highly accurate pseudospectral method to approximate the NFSE. For the proposed method, we derive the solution of a nonlinear partial differential equation as a sum of basis functions in both space and time directions. The spectral coefficients of the sum are chosen to satisfy the solution of the nonlinear partial differential equation. The fractional derivatives matrix is considered in the modified Riemann-Liouville formula. It is a new approach to study the nonlinear partial differential equations in time and space and the method is equally important in time as it is in the space direction. The great advantage of time-space pseudospectral methods resides in their high-accuracy for a relatively small number of grid points as compared to standard time-stepping methods.

The structure of the paper is organized as follows. In section 2, we describe some basic definitions and notations. Discretizing and description of the methods are presented in section 3. In section 4, we derive the error analysis for two-dimensional time- and space-NFSE. In section 5, we present numerical solutions and errors by the proposed scheme. In the last section, the conclusion of our work is presented.

2. Preliminary

In this section, we present definition of modified Riemann-Liouville formula of the fractional calculus theory and Sobolev space, which will be useful throughout this paper.

Definition 1: The partial fractional derivatives of order α of a function $\phi_M(z)$, with respect to variable z, in the modified Riemann-Liouville formula, is defined as [10, 16]

$$\phi_M^{'}(z) = \partial_z^\alpha \phi_M(z) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^z (z-\xi)^{-\alpha-1} \left(\phi_M(\xi) - \phi_M(0)\right) d\xi\,, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial z} \int_0^z (z-\xi)^{-\alpha} \left(\phi_M(\xi) - \phi_M(0)\right) d\xi\,, & 0 < \alpha < 1, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial z^n} \int_0^z (z-\xi)^{n-\alpha-1} \left(\phi_M(\xi) - \phi_M(0)\right) d\xi\,, & n-1 \le \alpha < n. \end{cases}$$

Here, z is dummy variable which represents x, y and t as and when required. Γ is the gamma function and $n = \lceil \alpha \rceil + 1$ with $\lceil \alpha \rceil$ denoting the upper integer part of α. Some basic properties of modified Riemann-Liouville fractional derivatives are as follows:

$$\partial_z^\alpha C \;=\; 0, \quad C \text{ is constant,}$$

$$\partial_z^\alpha z^\upsilon = \begin{cases} 0, & for\ \upsilon \in N\ and\ \upsilon < \lceil \alpha \rceil, \\ \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-\alpha)} z^{\upsilon-\alpha}, & for\ \upsilon \in N\ and\ \upsilon \ge \lceil \alpha \rceil. \end{cases}$$

and

$$\partial_z^\alpha \partial_z^\beta z^\upsilon = \partial_z^\beta \partial_z^\alpha z^\upsilon = \partial_z^{\alpha+\beta} z^\upsilon.$$

Construction of Chebyshev fractional differential matrix is given as

$$Q^{(\alpha)} = \begin{bmatrix} \phi_0^{'}(z_0) & \dots & \dots & \phi_0^{'}(z_M) \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \phi_M^{'}(z_0) & \dots & \dots & \phi_M^{'}(z_M) \end{bmatrix}.$$

Definition 2: Sobolev space: Let $p > 0$ be an integer and $1 \leq q \leq \infty$. The set of all function in $L_w^q[a,b]$ such that all distribution derivatives upto order p are also in $L_w^q[a,b]$ is called sobolev space, which are denoted by $W_w^{p,q}[a,b]$ and defined by

$$W_w^{p,q}[a,b] = \left\{U \in L_w^q[a,b] \mid \varDelta^{(\gamma)}U \in L_w^q[a,b], |\gamma| \leq p\right\},$$

where w is a weight function and norm is defined by

$$\|U\|_{W_w^{p,q}[a,b]} = \sum_{|\gamma|\leq p} \|\varDelta^{(\gamma)}U\|_{L_w^q[a,b]}.$$

For $q = 2$, sobolev norm denoted by $H_w^p[a,b] = W_w^{p,2}[a,b]$ and $L_w^2[a,b]$ is a weighted space defined by

$$L_w^2[a,b] = \left\{U : \quad U \text{is measurable and } \|U\|_w^2 < \infty\right\}.$$

Definition 3: Kronecker products of matrices, let $P = (p_{jk})_{mn}$ and $Q = (q_{jk})_{rs}$ are two matrices, where, m, n, r, s are nonnegative integers. The Kronecker product $P \otimes Q$ is a $mr \times ns$ block matrix denoted by

$$P \otimes Q = \begin{bmatrix} p_{11}Q & p_{12}Q & \dots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \dots & p_{2n}Q \\ \ddots & \ddots & \ddots & \ddots \\ p_{m1}Q & p_{12}Q & \dots & p_{mn}Q \end{bmatrix}_{mr\times ns}$$

3. Pseudospectral method based discretization

We seek a pseudospectral approximation $I_M U$, as a finite linear combination of a chosen set of Langrange basis functions in both space and time-variable with pseudospectral coefficients.

$$\mathcal{I}_M U = \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{M} Ł_i(x)\, Ł_j(y)\, Ł_k(t) U(x_i, y_j, t_k), \tag{3.1}$$

where,

$$Ł_j(z) = \sum_{i=0}^{M} w_j \frac{1}{\hat{h}_i}\phi_i(z_j)\phi_i(z) = \frac{S_M(z)}{(z - z_j)\left[\partial S_M(z)/\partial z|_{z=z_j}\right]}$$

here, $\phi_i(z)$ is a Chebyshev polynomial, z_j is a CGL points,

$$S_M(z) = \prod_{i=0}^{M}(z - z_i),$$

normalization constant

$$\hat{h}_i = \begin{cases} \pi, & i = 0 \text{ and } M, \\ \frac{\pi}{2}, & 0 < i < M. \end{cases}$$

and Chebyshev Gauss-Lobatto weight quadrature

$$w_j = \begin{cases} \frac{\pi}{2M}, & j = 0 \text{ and } M, \\ \frac{\pi}{M}, & 0 < j < M. \end{cases}$$

The spectral approximation given in (3.1) can be expressed as a direct product

$$\mathcal{I}_M U(x,y,t) = (\Psi_{[0:M]}(x) \otimes \Psi_{[0:M]}(y) \otimes \Psi_{[0:M]}(t))^T \mathbf{U}, \quad (3.2)$$

where

$$\begin{aligned} \mathbf{U} &= [U_{000}, \ldots, U_{00M} \mid \ldots \mid U_{0M0}, \ldots, U_{0MM} \mid \ldots \mid U_{MM0}, \ldots, U_{MMM}]^T, \\ \Psi_{[0:M]}(z) &= [Ł_0(z), \ldots\ldots., Ł_M(z)]^T. \end{aligned}$$

Now, we can define fractional spatial derivative of equation (3.2) with respect to x is given by

$$\frac{\partial^\beta \mathcal{I}_M U(x,y,t)}{\partial x^\beta} = (\Psi_{[0:M]}(x) \otimes \Psi_{[0:M]}(y) \otimes \Psi_{[0:M]}(t))^T \left(Q^{(\beta)}_{[0:M,0:M]} \otimes I_{M+1} \otimes I_{M+1}\right) \mathbf{U}. \quad (3.3)$$

Similarly, we can define fractional derivative of Eq. (3.2) with respect to t is given by

$$\frac{\partial^\alpha \mathcal{I}_M U(x,y,t)}{\partial t^\alpha} = (\Psi_{[0:M]}(x) \otimes \Psi_{[0:M]}(y) \otimes \Psi_{[0:M]}(t))^T \left(I_{M+1} \otimes I_{M+1} \otimes Q^{(\alpha)}_{[0:M,0:M]}\right) \mathbf{U}. \quad (3.4)$$

where I_{M+1} denotes identity matrix of size $((M+1) \times (M+1))$.

Next, we define $U(x,y,t)$ and $F(x,y,t)$ into their real and imaginary parts

$$U(x,y,t) = U_1(x,y,t) + iU_2(x,y,t) \text{ and } F(x,y,t) = F_1(x,y,t) + iF_2(x,y,t), \quad (3.5)$$

where $U_1(x,y,t)$, $U_2(x,y,t)$, $F_1(x,y,t)$ and $F_2(x,y,t)$ are real functions. Using Eq. (3.5) in Eq. (1.1), we obtain the following coupled system of equations

$$-\partial_t^\alpha U_2 + \kappa\left(\partial_x^{2\beta} U_1 + \partial_y^{2\beta} U_1\right) + G(x,y,t)U_1 + \lambda\left(U_1^2 + U_2^2\right)U_1 = F_1(x,y,t), \quad (x,y) \in \Omega, \quad t \in [0,T], \quad (3.6)$$

$$\partial_t^\alpha U_1 + \kappa\left(\partial_x^{2\beta} U_2 + \partial_y^{2\beta} U_2\right) + G(x,y,t)U_2 + \lambda\left(U_1^2 + U_2^2\right)U_2 = F_2(x,y,t), \quad (x,y) \in \Omega, \quad t \in [0,T]. \quad (3.7)$$

Further, we consider the following transformations which are used to transform the two-dimensional space $\Omega = (x,y) \in [x_L, x_R] \times [y_L, y_R]$ and time $[0,T]$ in to $[-1,1]$.

$$x \longrightarrow \frac{x_R - x_L}{2}x + \frac{x_R + x_L}{2}, \quad y \longrightarrow \frac{y_R - y_L}{2}y + \frac{y_R + y_L}{2} \quad \text{and} \quad t \longrightarrow \frac{T}{2}t + \frac{T}{2}.$$

We obtain the two dimensional time- and space- NFSE in new space $(x,y) \in [-1,1]^2$ and time interval $t \in [-1,1]$

$$-\partial_t^\alpha U_2 + \kappa\left(\frac{2^{(2\beta-\alpha)}T^\alpha}{(x_R - x_L)^{2\beta}}\partial_x^{2\beta} U_1 + \frac{2^{(2\beta-\alpha)}T^\alpha}{(y_R - y_L)^{2\beta}}\partial_y^{2\beta} U_1\right) + \frac{T^\alpha}{2^\alpha}\left[G(x,y,t)U_1 + \lambda\left(U_1^2 + U_2^2\right)U_1\right] = \frac{T^\alpha}{2^\alpha}F_1(x,y,t),$$

(3.8)

$$\partial_t^\alpha U_1 + \kappa\left(\frac{2^{(2\beta-\alpha)}T^\alpha}{(x_R - x_L)^{2\beta}}\partial_x^{2\beta}U_2 + \frac{2^{(2\beta-\alpha)}T^\alpha}{(y_R - y_L)^{2\beta}}\partial_y^{2\beta}U_2\right) + \frac{T^\alpha}{2^\alpha}\left[G(x,y,t)U_2 + \lambda\left(U_1^2 + U_2^2\right)U_2\right] = \frac{T^\alpha}{2^\alpha}F_2(x,y,t), \tag{3.9}$$

with initial condition:

$$U_1(x,y,-1) = h_1(x,y), \quad U_2(x,y,-1) = h_2(x,y), \qquad x \in [-1,1], \qquad y \in [-1,1] \tag{3.10}$$

and boundary conditions:

$$\begin{aligned}
U_1(-1,y,t) = g_{11}(y,t), \quad U_1(1,y,t) = g_{21}(y,t), \quad t \in [-1,1], \quad y \in [-1,1],\\
U_2(-1,y,t) = g_{12}(y,t), \quad U_2(1,y,t) = g_{22}(y,t), \quad t \in [-1,1], \quad y \in [-1,1],\\
U_1(x,-1,t) = g_{31}(x,t), \quad U_1(x,1,t) = g_{41}(x,t), \quad t \in [-1,1], \quad x \in [-1,1],\\
U_2(x,-1,t) = g_{32}(x,t), \quad U_2(x,1,t) = g_{42}(x,t), \quad t \in [-1,1], \quad x \in [-1,1].
\end{aligned}$$

Further, we consider a mapping for converting the non-homogeneous initial and boundary values to homogeneous initial and boundary values [5].

$$\begin{aligned}
\Omega_k(x,y,t) = \ & \tfrac{1-t}{2}h_k(x,y) + \tfrac{1-x}{4}g_{1k}(y,t) + \tfrac{1+x}{4}g_{2k}(y,t) + \tfrac{1-y}{4}g_{3k}(x,t) + \tfrac{1+y}{4}g_{4k}(x,t) - \tfrac{(1-t)(1-x)}{8}\\
& g_{1k}(y,-1) - \tfrac{(1-t)(1+x)}{8}g_{2k}(y,-1) - \tfrac{(1-t)(1-y)}{8}g_{3k}(x,-1) - \tfrac{(1-t)(1+y)}{8}g_{4k}(x,-1),
\end{aligned} \tag{3.11}$$

here corner initial and boundary values satisfy,

$$\begin{aligned}
g_{1k}(-1,-1) &= g_{3k}(-1,-1) = h_k(-1,-1), \quad g_{1k}(-1,1) = g_{3k}(-1,1),\\
g_{1k}(1,-1) &= g_{4k}(-1,-1) = h_k(-1,1), \quad g_{1k}(1,1) = g_{4k}(-1,1),\\
g_{2k}(-1,-1) &= g_{3k}(1,-1) = h_k(1,-1), \quad g_{2k}(-1,1) = g_{3k}(1,1),\\
g_{2k}(1,-1) &= g_{4k}(1,-1) = h_k(1,1), \quad g_{2k}(1,1) = g_{4k}(1,1).
\end{aligned}$$

Define new variable $V_k(x,y,t)$.

$$U_k(x,y,t) = V_k(x,y,t) + \Omega_k(x,y,t), \qquad \forall k = 1,2, \tag{3.12}$$

the above equations, can be modified with new variable and obtain the new equations residuals,

$$\begin{aligned}
-\partial_t^\alpha(V_2+\Omega_2) + \kappa\left(\frac{2^{(2\beta-\alpha)}T^\alpha}{(x_R - x_L)^{2\beta}}\partial_x^{2\beta}(V_1+\Omega_1) + \frac{2^{(2\beta-\alpha)}T^\alpha}{(y_R - y_L)^{2\beta}}\partial_y^{2\beta}(V_1+\Omega_1)\right) +\\
\frac{T^\alpha}{2^\alpha}\left[G(x,y,t)(V_1+\Omega_1) + \lambda\left((V_1+\Omega_1)^2 + (V_2+\Omega_2)^2\right)(V_1+\Omega_1)\right] = \frac{T^\alpha}{2^\alpha}F_1(x,y,t),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\partial_t^\alpha(V_1+\Omega_1) + \kappa\left(\frac{2^{(2\beta-\alpha)}T^\alpha}{(x_R - x_L)^{2\beta}}\partial_x^{2\beta}(V_2+\Omega_2) + \frac{2^{(2\beta-\alpha)}T^\alpha}{(y_R - y_L)^{2\beta}}\partial_y^{2\beta}(V_2+\Omega_2)\right) +\\
\frac{T^\alpha}{2^\alpha}\left[G(x,y,t)(V_2+\Omega_2) + \lambda\left((V_1+\Omega_1)^2 + (V_2+\Omega_2)^2\right)(V_2+\Omega_2)\right] = \frac{T^\alpha}{2^\alpha}F_2(x,y,t).
\end{aligned} \tag{3.14}$$

Applying the pseudospectral method in Eqs. (3.13) and (3.14), we get

$$\begin{aligned} H_1(V_1,V_2) &= -B\left(I_{M+1}\otimes I_{M+1}\otimes Q^{(\alpha)}_{[0:M,0:M]}\right)V_2-\partial_t^{\alpha}(\Omega_2)+\\ &\kappa B\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}\left(Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\otimes I_{M+1}\right)+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}\left(I_{M+1}\otimes Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\right)\right]V_1\\ &+\kappa\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}\partial_x^{2\beta}(\Omega_1)+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}\partial_y^{2\beta}(\Omega_1)\right]-\frac{T^{\alpha}}{2^{\alpha}}F_1(x,y,t)+\\ &\frac{T^{\alpha}}{2^{\alpha}}\left[G(x,y,t)(V_1+\Omega_1)+\lambda\left((V_1+\Omega_1)^2+(V_2+\Omega_2)^2\right)(V_1+\Omega_1)\right]=0 \end{aligned} \tag{3.15}$$

$$\begin{aligned} H_2(V_1,V_2) &= B\left(I_{M+1}\otimes I_{M+1}\otimes Q^{(\alpha)}_{[0:M,0:M]}\right)V_1+\partial_t^{\alpha}(\Omega_1)+\\ &\kappa B\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}\left(Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\otimes I_{M+1}\right)+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}\left(I_{M+1}\otimes Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\right)\right]V_2\\ &+\kappa\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}\partial_x^{2\beta}(\Omega_2)+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}\partial_y^{2\beta}(\Omega_2)\right]-\frac{T^{\alpha}}{2^{\alpha}}F_2(x,y,t)+\\ &\frac{T^{\alpha}}{2^{\alpha}}\left[G(x,y,t)(V_2+\Omega_2)+\lambda\left((V_1+\Omega_1)^2+(V_2+\Omega_2)^2\right)(V_2+\Omega_2)\right]=0 \end{aligned} \tag{3.16}$$

where $B=(\Psi_{[0:M]}(x)\otimes\Psi_{[0:M]}(y)\otimes\Psi_{[0:M]}(t))^T$ Using the CGL points in Eqs. (3.15) and (3.16) and obtained the algebraic equations

$$\begin{aligned} &-\left(I_{M+1}\otimes I_{M+1}\otimes Q^{(\alpha)}_{[0:M,0:M]}\right)\mathbf{V}_2-(\Omega_2)_t^{\alpha}+\\ &\kappa\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}\left(Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\otimes I_{M+1}\right)+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}\left(I_{M+1}\otimes Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\right)\right]\mathbf{V}_1\\ &+\kappa\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}(\Omega_1)_x^{2\beta}+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}(\Omega_1)_y^{2\beta}\right]-\frac{T^{\alpha}}{2^{\alpha}}\mathbf{F}_1+\\ &\frac{T^{\alpha}}{2^{\alpha}}\left[\mathbf{G}(\mathbf{V}_1+\Omega_1)+\lambda\left(I_{(\mathbf{V}_1+\Omega_1)}\right)\left((\mathbf{V}_1+\Omega_1)^2+(\mathbf{V}_2+\Omega_2)^2\right)\right]=\mathbf{H}_1, \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\left(I_{M+1}\otimes I_{M+1}\otimes Q^{(\alpha)}_{[0:M,0:M]}\right)\mathbf{V}_1-(\Omega_1)_t^{\alpha}+\\ &\kappa\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}\left(Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\otimes I_{M+1}\right)+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}\left(I_{M+1}\otimes Q^{(2\beta)}_{[0:M,0:M]}\otimes I_{M+1}\right)\right]\mathbf{V}_2\\ &+\kappa\left[\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(x_R-x_L)^{2\beta}}(\Omega_2)_x^{2\beta}+\frac{2^{(2\beta-\alpha)}T^{\alpha}}{(y_R-y_L)^{2\beta}}(\Omega_2)_y^{2\beta}\right]-\frac{T^{\alpha}}{2^{\alpha}}\mathbf{F}_2+\\ &\frac{T^{\alpha}}{2^{\alpha}}\left[\mathbf{G}(\mathbf{V}_2+\Omega_2)+\lambda\left(I_{(\mathbf{V}_2+\Omega_2)}\right)\left((\mathbf{V}_1+\Omega_1)^2+(\mathbf{V}_2+\Omega_2)^2\right)\right]=\mathbf{H}_2, \end{aligned} \tag{3.18}$$

where,$I_{(V_1+\Omega_1)}$ and $I_{(V_2+\Omega_2)}$ are diagonal entry matrix

$$\mathbf{V}_1 = \left[V_{1,001},\ldots,V_{1,00(M)}\mid\ldots\mid V_{1,0(M)1},\ldots,V_{1,0(M)(M)}\mid\ldots\mid V_{1,(M)(M)1},\ldots,V_{1,M(M)(M)}\right]^T,$$

$$
\begin{aligned}
\mathbf{V}_2 &= \left[V_{2,001},\ldots,V_{2,00(M)} \mid \ldots \mid V_{2,0(M)1},\ldots,V_{2,0(M)(M)} \mid \ldots \mid V_{2,(M)(M)1},\ldots,V_{2,M(M)(M)}\right]^T,\\
(\Omega_1)_t^{\alpha} &= \left[(\Omega_1)_{t,001}^{\alpha},\ldots,(\Omega_1)_{t,00(M)}^{\alpha} \mid \ldots \mid (\Omega_1)_{t,0(M)1}^{\alpha},\ldots,(\Omega_1)_{t,0(M)(M)}^{\alpha} \mid \ldots \mid (\Omega_1)_{t,(M)(M)1}^{\alpha},\ldots,(\Omega_1)_{t,M(M)(M)}^{\alpha}\right]^T,\\
(\Omega_2)_t^{\alpha} &= \left[(\Omega_2)_{t,001}^{\alpha},\ldots,(\Omega_2)_{t,00(M)}^{\alpha} \mid \ldots \mid (\Omega_2)_{t,0(M)1}^{\alpha},\ldots,(\Omega_2)_{t,0(M)(M)}^{\alpha} \mid \ldots \mid (\Omega_2)_{t,(M)(M)1}^{\alpha},\ldots,(\Omega_2)_{t,M(M)(M)}^{\alpha}\right]^T,\\
(\Omega_1)_x^{2\beta} &= \left[(\Omega_1)_{x,001}^{2\beta},\ldots,(\Omega_1)_{x,00(M)}^{2\beta} \mid \ldots \mid (\Omega_1)_{x,0(M)1}^{2\beta},\ldots,(\Omega_1)_{x,0(M)(M)}^{2\beta} \mid \ldots \mid (\Omega_1)_{x,(M)(M)1}^{2\beta},\ldots,(\Omega_1)_{x,M(M)(M)}^{2\beta}\right]^T,\\
(\Omega_1)_y^{2\beta} &= \left[(\Omega_1)_{y,001}^{2\beta},\ldots,(\Omega_1)_{y,00(M)}^{2\beta} \mid \ldots \mid (\Omega_1)_{y,0(M)1}^{2\beta},\ldots,(\Omega_1)_{y,0(M)(M)}^{2\beta} \mid \ldots \mid (\Omega_1)_{y,(M)(M)1}^{2\beta},\ldots,(\Omega_1)_{y,M(M)(M)}^{2\beta}\right]^T,\\
(\Omega_2)_x^{2\beta} &= \left[(\Omega_2)_{x,001}^{2\beta},\ldots,(\Omega_2)_{x,00(M)}^{2\beta} \mid \ldots \mid (\Omega_2)_{x,0(M)1}^{2\beta},\ldots,(\Omega_2)_{x,0(M)(M)}^{2\beta} \mid \ldots \mid (\Omega_2)_{x,(M)(M)1}^{2\beta},\ldots,(\Omega_2)_{x,M(M)(M)}^{2\beta}\right]^T,\\
(\Omega_2)_y^{2\beta} &= \left[(\Omega_2)_{y,001}^{2\beta},\ldots,(\Omega_2)_{y,00(M)}^{2\beta} \mid \ldots \mid (\Omega_2)_{y,0(M)1}^{2\beta},\ldots,(\Omega_2)_{y,0(M)(M)}^{2\beta} \mid \ldots \mid (\Omega_2)_{y,(M)(M)1}^{2\beta},\ldots,(\Omega_2)_{y,M(M)(M)}^{2\beta}\right]^T,\\
\mathbf{F}_1 &= \left[F_{1,001},\ldots,F_{1,00(M)} \mid \ldots \mid F_{1,0(M)1},\ldots,F_{1,0(M)(M)} \mid \ldots \mid F_{1,(M)(M)1},\ldots,F_{1,M(M)(M)}\right]^T,\\
\mathbf{F}_2 &= \left[F_{2,001},\ldots,F_{2,00(M)} \mid \ldots \mid F_{2,0(M)1},\ldots,F_{2,0(M)(M-1)} \mid \ldots \mid F_{2,(M)(M)1},\ldots,F_{2,M(M)(M)}\right]^T,\\
\mathbf{H}_1 &= \left[H_{1,001},\ldots,H_{1,00(M)} \mid \ldots \mid H_{1,0(M)1},\ldots,H_{1,0(M)(M)} \mid \ldots \mid H_{1,(M)(M)1},\ldots,H_{1,M(M)(M)}\right]^T,\\
\mathbf{H}_2 &= \left[H_{2,001},\ldots,H_{2,00(M)} \mid \ldots \mid H_{2,0(M)1},\ldots,H_{2,0(M)(M-1)} \mid \ldots \mid H_{2,(M)(M)1},\ldots,H_{2,M(M)(M)}\right]^T,\\
\mathbf{G} &= \left[G_{001},\ldots,G_{00(M)} \mid \ldots \mid G_{0(M)1},\ldots,G_{0(M)(M-1)} \mid \ldots \mid \mathbf{G}_{(M)(M)1},\ldots,G_{M(M)(M)}\right]^T.
\end{aligned}
$$

Hence,

$$
\mathbf{V} = [\mathbf{V}_1|\mathbf{V}_2], \qquad \mathbf{H}(\mathbf{V}) = [\mathbf{H}_1|\mathbf{H}_2] = 0. \tag{3.19}
$$

The system of nonlinear equations (3.19) can be solved by using Newton Raphson method. The Jacobian matrix of this coupled nonlinear algebraic equations are as

$$
J(V) = \left[\begin{array}{c|c} \frac{\partial H_{1,ijk}}{\partial V_{1,ijk}} & \frac{\partial H_{1,ijk}}{\partial V_{2,ijk}} \\ \hline \frac{\partial H_{2,ijk}}{\partial V_{1,ijk}} & \frac{\partial H_{2,ijk}}{\partial V_{2,ijk}} \end{array}\right], \qquad \forall i,j,k \in [0,...,M].
$$

.

4. Theory of error estimates

In this section, we derive error estimates for two-dimensional nonlinear fractional Schrodinger equation by using time-space Chebyshev pseudospectral method.

Theorem 1: Let $U(x,y,t) \in H_w^p[-1,1]$, where $p > 0$, there exist a constant C, such that the following estimate holds:

$$
\|U - \mathcal{I}_M U\|_{L_w^2[-1,1]} \leq C M^{-p} \|U\|_{H_w^p[-1,1]}.
$$

Proof : We approximate the solutions obtained from truncated Chebyshev expansion are defined by

$$
\mathcal{P}_M U = \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{M} \eta_i(x)\,\eta_j(y)\,\eta_k(t)\,\Gamma_{ijk}, \tag{4.1}
$$

where $\eta_l(z), \forall l \in (i,j,k)$ and $z \in (x,y,t)$ is Chebyshev polynomials, $w(z) = (\sqrt{1-z^2})^{-1}$ is weight function and expansion coefficients are

$$\begin{aligned}\Gamma_{ijk} &= \frac{1}{p_i p_j p_k}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} U\,\eta_i(x)\eta_j(y)\eta_k(t)w(x)w(y)w(t)\,dx\,dy\,dt\,,\\ &= \frac{1}{p_i p_j p_k}\int_{-1}^{1}\int_{-1}^{1}\eta_j(y)w(y)\eta_k(t)w(t)\left[\int_{-1}^{1} U\,\eta_i(x)w(x)\,dx\right]dy\,dt\,. \end{aligned} \tag{4.2}$$

where $p_l\;\; l \in (i,j,k)$ is normalizing constant of orthogonal Chebyshev polynomials.
Next, we define the Chebyshev operator

$$n^2\eta_n(z)w(z) = \mathcal{L}_n\eta_n(z)w(z) = \frac{\partial}{\partial z}\left(-\sqrt{1-z^2}\frac{\partial}{\partial z}(\eta_n(z))\right), \qquad \forall n, \tag{4.3}$$

Using Eq. (4.3) in Eq. (4.2), we obtain

$$\Gamma_{ijk} = \frac{1}{p_i p_j p_k}\int_{-1}^{1}\int_{-1}^{1}\eta_j(y)w(y)\eta_k(t)w(t)\int_{-1}^{1}\frac{1}{i^2}U\left[\left\{\frac{\partial}{\partial x}\left(-\sqrt{1-x^2}\frac{\partial}{\partial x}(\eta_i(x))\right)\right\}dx\right]dy\,dt\,,$$

$$\Gamma_{ijk} = \frac{1}{p_i p_j p_k}\int_{-1}^{1}\int_{-1}^{1}\eta_j(y)w(y)\eta_k(t)w(t)\left[\frac{1}{i^2}S_1\right]dy\,dt, \tag{4.4}$$

where

$$\chi_1 = \int_{-1}^{1} U\left[\frac{\partial}{\partial x}\left(-\sqrt{1-x^2}\frac{\partial}{\partial x}(\eta_i(x))\right)\right]dx\,,$$

using integration by parts:

$$\chi_1 = -U\sqrt{1-x^2}\frac{\partial}{\partial x}(\eta_i(x))\,\Big|_{x=-1}^{x=1} + \int_{-1}^{1}\frac{\partial}{\partial x}U\sqrt{1-x^2}\frac{\partial}{\partial x}(\eta_i(x))\,dx$$

$$\chi_1 = \int_{-1}^{1}\frac{\partial}{\partial x}U\sqrt{1-x^2}\frac{\partial}{\partial x}(\eta_i(x))\,dx$$

again applying integration by parts, we obtain

$$\begin{aligned}\chi_1 &= \frac{\partial}{\partial x}U\sqrt{1-x^2}\eta_i(x)\,\Big|_{x=-1}^{x=1} + \int_{-1}^{1}\frac{\partial}{\partial x}\left(-\sqrt{1-x^2}\frac{\partial}{\partial x}U\right)\eta_i(x)\,dx\,,\\ \chi_1 &= \int_{-1}^{1}\frac{\partial}{\partial x}\left(-\sqrt{1-x^2}\frac{\partial}{\partial x}U\right)\eta_i(x)\,dx\end{aligned}$$

here, $\mathcal{L}_1 = \frac{\partial}{\partial x}\left(-\sqrt{1-x^2}\frac{\partial}{\partial x}\right)$ is Chebyshev operator,

$$\chi_1 = \int_{-1}^{1}[\mathcal{L}_1 U]\,\eta_i(x)w(x)\,dx\,, \tag{4.5}$$

putting the value of χ_1 in eq(4.4), we get

$$\begin{aligned}\Gamma_{ijk} &= \frac{1}{p_i p_j p_k}\int_{-1}^{1}\int_{-1}^{1}\eta_j(y)w(y)\eta_k(t)w(t)\left[\frac{1}{i^2}\int_{-1}^{1}[\mathcal{L}_1 U]\,\eta_i(x)w(x)\,dx\right]dy\,dt,\\ &= \frac{1}{p_i p_j p_k i^2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[\mathcal{L}_1 U\,]\,\eta_i(x)\eta_j(y)\eta_k(t)w(x)w(y)w(t)\,dx\,dy\,dt\,.\end{aligned}$$

Similarly integrate for space y and time t and obtain

$$\begin{aligned}\Gamma_{ijk} &= \frac{1}{p_i p_j p_k (ijk)^2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 U\,]\,\eta_i(x)\eta_j(y)\eta_k(t)w(x)w(y)w(t)\,dx\,dy\,dt,\\ &= \frac{1}{p_i p_j p_k (ijk)^2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[\mathcal{L}U\,]\,\eta_i(x)\eta_j(y)\eta_k(t)w(x)w(y)w(t)\,dx\,dy\,dt.\end{aligned}$$

In this case the order of $\mathcal{L}U$ is 6 as $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are second order operator.
Repeatation of procedure for higher order concludes the results *i.e.*,

$$\Gamma_{ijk} = \frac{1}{p_i p_j p_k (ijk)^{2m}}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}[\mathcal{L}^m U\,]\eta_i(x)\eta_j(y)\eta_k(t)w(x)w(y)w(t)\,dx\,dy\,dt. \tag{4.6}$$

Then the Cauchy-Schwartz inequality implies

$$|\Gamma_{ijk}|^2 \le C\frac{1}{(ijk)^{4m}}\|\mathcal{L}^m U\,\|^2_{L^2_w[-1,1]}, \tag{4.7}$$

where C is generic constant. Therefore, from Sobolev norm

$$\|\mathcal{L}U\,\|^2_{L^2_w[-1,1]} \le \|U\,\|^2_{H^6_w[-1,1]}.$$

By induction,

$$\|\mathcal{L}^m U\,\|^2_{L^2_w[-1,1]} \le \|U\,\|^2_{H^{6m}_w[-1,1]}. \tag{4.8}$$

From Eq. 4.7 and Eq. 4.8, we get

$$|\Gamma_{ijk}|^2 \le C\frac{1}{(ijk)^{4m}}\|U\,\|^2_{H^{6m}_w[-1,1]}. \tag{4.9}$$

Further, let us consider the discrete approximation

$$\mathcal{I}_M U = \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{M}Ł_i(x)\,Ł_j(y)\,Ł_k(t)V(x_i, y_j, t_k) = \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{M}\eta_i(x)\,\eta_j(y)\,\eta_k(t)\,\Pi_{ijk}, \tag{4.10}$$

where Π_{ijk} is discrete expansion coefficient and defined by

$$\Pi_{ijk} = \frac{1}{p_i p_j p_k}\sum_{l=0}^{M}\sum_{m=0}^{M}\sum_{n=0}^{M}\eta_i(x_l)\,\eta_j(y_m)\eta_k(t_n)w_l w_m\,w_n U(x_l, y_m, t_n). \tag{4.11}$$

Under the assumption for sufficient smoothness functions, the aliasing error is

$$\Pi_{ijk} = \Gamma_{ijk} + \frac{1}{p_i p_j p_k} \sum_{p>M}^{\infty} \sum_{q>M}^{\infty} \sum_{r>M}^{\infty} \left[\eta_i(x), \eta_p(x)\right]_w \left[\eta_j(y), \eta_q(x)\right]_w \left[\eta_k(t), \eta_r(t)\right]_w \Gamma_{pqr}, \tag{4.12}$$

where $[.,.]_w$ represents the discrete weighted inner product.
We know that,

$$U = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \eta_i(x)\eta_j(y)\eta_k(t)\Gamma_{ijk}. \tag{4.13}$$

Using Eq. (4.10), Eq. (4.12) and Eq. (4.13) and we get

$$\|U - \mathcal{I}_M U\|^2_{L^2_w[-1,1]} = \|U - \mathcal{P}_M U\|^2_{L^2_w[-1,1]} + \|B_M U\|^2_{L^2_w[-1,1]}, \tag{4.14}$$

where the aliasing error

$$\begin{aligned} B_M U \quad = \quad & \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{k=0}^{M} \frac{1}{p_i p_j p_k} \sum_{p>M}^{\infty} \sum_{q>M}^{\infty} \sum_{r>M}^{\infty} \left[\eta_i(x), \eta_p(x)\right]_w \left[\eta_j(y), \eta_q(y)\right]_w \left[\eta_k(t), \eta_r(t)\right]_w \\ & \eta_i(x)\eta_j(y)\eta_k(t)\Gamma_{ijk}, \end{aligned}$$

to simplify the expression, first interchange the summations

$$\begin{aligned} B_M U \quad = \quad & \sum_{p>M}^{\infty} \sum_{q>M}^{\infty} \sum_{r>M}^{\infty} \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{k=0}^{M} \frac{1}{p_i p_j p_k} \left[\eta_i(x), \eta_p(x)\right]_w \left[\eta_j(y), \eta_q(y)\right]_w \left[\eta_k(t), \eta_r(t)\right]_w \\ & \eta_i(x)\eta_j(y)\eta_k(t)\Gamma_{ijk}. \end{aligned} \tag{4.15}$$

Since $\eta_i(x), \eta_j(y)$ and $\eta_k(t)$ are orthogonal, therefore, we observe that the value of Eq. (4.15) is zero due to the range of summations.

$$\|U - \mathcal{I}_M U\|^2_{L^2_w[-1,1]} = \|U - \mathcal{P}_M U\|^2_{L^2_w[-1,1]}. \tag{4.16}$$

Subtract the Eq. (4.13) and Eq. (4.1), we get

$$(U - \mathcal{P}_M U) = \sum_{i>M}^{\infty} \sum_{j>M}^{\infty} \sum_{k>M}^{\infty} \eta_i(x)\, \eta_j(y)\, \eta_k(t)\, \Gamma_{ijk},$$

taking $L^2_w[-1,1]$ norm both side and we get

$$\begin{aligned} \|U - \mathcal{P}_M U\|^2_{L^2_w[-1,1]} \quad &= \quad \sum_{i>M}^{\infty} \sum_{j>M}^{\infty} \sum_{k>M}^{\infty} \|\eta_i(x)\eta_j(y)\eta_k(t)\Gamma_{ijk}\|^2_{L^2_w[-1,1]}, \\ &= \quad \sum_{i>M}^{\infty} \sum_{j>M}^{\infty} \sum_{k>M}^{\infty} \|\eta_i(x)\eta_j(y)\eta_k(t)\|^2_{L^2_w[-1,1]} |\Gamma_{ijk}|^2, \\ &= \quad \sum_{i>M}^{\infty} \sum_{j>M}^{\infty} \sum_{k>M}^{\infty} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \eta_i^2(x)\eta_j^2(y)\eta_k^2(t) w(x)w(y)w(t)\, dx\, dy\, dt\, |\Gamma_{ijk}|^2 \end{aligned}$$

$$= \sum_{i>M}^{\infty}\sum_{j>M}^{\infty}\sum_{k>M}^{\infty}\int_{-1}^{1}\eta_i^2(x)w(x)\,dx\ \int_{-1}^{1}\eta_j^2(y)w(y)\,dy\ \int_{-1}^{1}\eta_k^2(t)w(t)\,dt\,|\Gamma_{ijk}|^2$$
$$= \sum_{i>M}^{\infty}\sum_{j>M}^{\infty}\sum_{k>M}^{\infty}\left[\eta_i(x),\eta_i(x)\right]_w\left[\eta_j(y),\eta_j(y)\right]_w\left[\eta_k(t),\eta_k(t)\right]_w|\Gamma_{ijk}|^2,$$

Finally putting the value of discrete inner product and we get

$$||U-\mathcal{P}_M U||^2_{L^2_w[-1,1]} = \sum_{i>M}^{\infty}\sum_{j>M}^{\infty}\sum_{k>M}^{\infty} p_i p_j p_k |\Gamma_{ijk}|^2, \tag{4.17}$$

Combining the Eq. (4.9) and Eq. (4.17)and we get

$$||U-\mathcal{P}_M U||^2_{L^2_w[-1,1]} \le C||U||^2_{H^{6m}_w[-1,1]}\sum_{i>M}^{\infty}\sum_{j>M}^{\infty}\sum_{k>M}^{\infty} p_i p_j p_k (ijk)^{-4m},$$

$$\Rightarrow ||U-\mathcal{P}_M U||^2_{L^2_w[-1,1]} \le CM^{-12m}||U\,||^2_{H^{6m}_w[-1,1]},$$
$$\Rightarrow ||U-\mathcal{P}_M U||_{L^2_w[-1,1]} \le CM^{-6m}||U\,||_{H^{6m}_w[-1,1]}. \tag{4.18}$$

Take $p = 6m$ in Eq. (4.18)

$$\Rightarrow ||U-\mathcal{P}_M U||_{L^2_w[-1,1]} \le CM^{-p}||U\,||_{H^{p}_w[-1,1]}. \tag{4.19}$$

Finally, putting in Eq. (4.16), we obtain

$$||U-\mathcal{I}_M U||_{L^2_w[-1,1]} \le CM^{-p}||U\,||_{H^{p}_w[-1,1]}. \tag{4.20}$$

The theorem is complete.
Further, we discuss the bounds of the given Schrodinger operator. We consider the error function $E(x,y,t) = U(x,y,t) - \mathcal{I}_M U(x,y,t)$, where, $\mathcal{I}_M U(x,y,t)$ is pseudospectral approximation and $U(x,y,t)$ is exact solution. Moreover, the pseudospectral approximation satisfies the given problem, *i.e.*,

$$R_1(x,y,t) = -\partial_t^{\alpha}\mathcal{I}_M U_2 + \kappa\left(\partial_x^{2\beta}\mathcal{I}_M U_1 + \partial_y^{2\beta}\mathcal{I}_M U_1\right) + G(x,y,t)I_M U_1 + \lambda\left(\mathcal{I}_M U_1^2 + \mathcal{I}_M U_2^2\right)\mathcal{I}_M U_1 - F_1(x,y,t),$$

$$R_2(x,y,t) = \partial_t^{\alpha}\mathcal{I}_M U_1 + \kappa\left(\partial_x^{2\beta}\mathcal{I}_M U_2 + \partial_y^{2\beta}\mathcal{I}_M U_2\right) + G(x,y,t)\mathcal{I}_M U_2 + \lambda\left(\mathcal{I}_M U_1^2 + \mathcal{I}_M U_2^2\right)\mathcal{I}_M U_2 - F_2(x,y,t),$$

Adding both equations, we get

$$R(x,y,t) = i\partial_t^{\alpha}\mathcal{I}_M U + \kappa\left(\partial_x^{2\beta}\mathcal{I}_M U + \partial_y^{2\beta}\mathcal{I}_M U\right) + G(x,y,t)\mathcal{I}_M U + \lambda|\mathcal{I}_M U|^2\mathcal{I}_M U - F(x,y,t). \tag{4.21}$$

where $R(x,y,t) = R_1(x,y,t) + iR_2(x,y,t)$ is the residual function.
Further, we can define

$$R(x,y,t) = i\partial_t^{\alpha}(U-\mathcal{I}_M U) + \kappa\left(\partial_x^{2\beta}(U-\mathcal{I}_M U) + \partial_y^{2\beta}(U-\mathcal{I}_M U)\right)$$
$$+G(x,y,t)(U-\mathcal{I}_M U) + \lambda|(U-\mathcal{I}_M U)|^2(U-\mathcal{I}_M U),$$

$$R(x,y,t) = i\partial_t^{\alpha} E + \kappa\left(\partial_x^{2\beta} E + \partial_y^{2\beta} E\right) + G(x,y,t)E + \lambda|E|^2 E, \tag{4.22}$$

where $E = E_1 + iE_2$.
From theorem 1, we get

$$||U - I_M U||_{L_w^2[-1,1]} = ||E||_{L_w^2[-1,1]} \leq CM^{-p}||U||_{H_w^p[-1,1]}. \tag{4.23}$$

Further, Eq. (4.23) can be written in derivative form.

$$||\frac{\partial^k E}{\partial z^k}||_{L_w^2[-1,1]} \leq CM^{-p}||\frac{\partial^k U}{\partial z^k}||_{H_w^p[-1,1]}, \quad z \in [x,y,t], \quad k \geq 0. \tag{4.24}$$

Using sobolev norms

$$\begin{aligned} ||\frac{\partial^k U}{\partial z^k}||_{H_w^p[-1,1]} &= \sum_{|\gamma|\leq p} ||\frac{\partial^{(\gamma)}}{\partial z^{(\gamma)}}\frac{\partial^k U}{\partial z^k}||_{L_w^2[-1,1]}, = \sum_{|\gamma|\leq p} ||\frac{\partial^{(\gamma)+k}}{\partial z^{(\gamma)+k}} U||_{L_w^2[-1,1]}, \\ &= ||U(x,y,t)||_{H_w^{p+k}[-1,1]}. \end{aligned} \tag{4.25}$$

Using Eq. (4.25), we obtain

$$||\frac{\partial^k E}{\partial z^k}||_{L_w^2[-1,1]} \leq CM^{-p}||U||_{H_w^{p+k}[-1,1]}. \tag{4.26}$$

taking $L_w^2[-1,1]$ norm both side in Eq. (4.22) and we get

$$\begin{aligned} ||R(x,y,t)||_{L_w^2[-1,1]} &= ||i\partial_t^{\alpha} E + \kappa\left(\partial_x^{2\beta} E + \partial_y^{2\beta} E\right) + G(x,y,t)E + \lambda|E|^2 E||_{L_w^2[-1,1]}, \\ &\leq ||\partial_t^{\alpha} E||_{L_w^2[-1,1]} + |\kappa|\left(||\partial_x^{2\beta} E||_{L_w^2[-1,1]} + ||\partial_y^{2\beta} E||_{L_w^2[-1,1]}\right) \\ &\quad + ||G(x,y,t)||||E||_{L_w^2[-1,1]} + |\lambda|||E||^3_{L_w^2[-1,1]}. \end{aligned} \tag{4.27}$$

Using Eq. (4.26) in Eq. (4.27), we get

$$\begin{aligned} ||R(x,y,t)||_{L_w^2[-1,1]} &\leq CM^{-p}||U||_{H_w^{p+\alpha}[-1,1]} + |\kappa|\left(2CM^{-p}||U||_{H_w^{p+2\beta}[-1,1]}\right) \\ &\quad + ||G(x,y,t)||CM^{-p}||U||_{H_w^p[-1,1]} + |\lambda|\left(CM^{-p}||U||_{H_w^p[-1,1]}\right)^3, \\ &\leq CM^{-p}||U||_{H_w^{p+\alpha}[-1,1]} + |\kappa|\left(2CM^{-p}||U||_{H_w^{p+2\beta}[-1,1]}\right) \\ &\quad + ||G(x,y,t)||CM^{-p}||U||_{H_w^p[-1,1]} + |\lambda|CM^{-p}\left(||U||_{H_w^p[-1,1]}\right)^3, \\ &\leq CM^{-p}||U||_{H_w^{p+\alpha}[-1,1]} + |\kappa|\left(2CM^{-p}||U||_{H_w^{p+2\beta}[-1,1]}\right) \\ &\quad + ||G(x,y,t)||CM^{-p}||U||_{H_w^p[-1,1]} + |\lambda|CM^{-p}\left(||U||_{H_w^p[-1,1]}\right)^3. \end{aligned} \tag{4.28}$$

Thus,

$$\begin{aligned} &||R(x,y,t)||_{L_w^2[-1,1]} \leq \\ &CM^{-p}\left[||U||_{H_w^{p+\alpha}[-1,1]} + |\kappa|\left(2||U||_{H_w^{p+2\beta}[-1,1]}\right) + ||G(x,y,t)||||U||_{H_w^p[-1,1]} + |\lambda|\left(||U||_{H_w^p[-1,1]}\right)^3\right]. \end{aligned} \tag{4.29}$$

5. Numerical results

In this section, we present numerical results for two-dimensional time- and space-NFSE using the Chebyshev pseudospectral method. We give three examples in this section. To demonstrate the errors in pseudospectral approximation, we consider the errors in the L_2 norms, defined by

$$L_2 = \||U|^k - |U|\|_2$$

where $|U|^k$ and $|U|$ are the modulus of pseudospectral approximation and modulus of analytical solution, respectively.

5.1. Example 1

Let us consider the time- and space- NFSE on an ellipse convex region $\Omega = \left\{(x,y)|\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$

$$i\frac{\partial^\alpha U}{\partial t^\alpha} + \kappa\left\{\frac{\partial^{2\beta}U}{\partial x^{2\beta}} + \frac{\partial^{2\beta}U}{\partial y^{2\beta}}\right\} + G(x,y,t)U + \lambda|U|^2U = F(x,y,t), \qquad (x,y) \in \Omega, \quad t \in [0,T],$$

with $\lambda = \kappa = 1$, $G(x,y,t) = t\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$.
The source term is

$$\begin{aligned} F(x,y,t) &= \left[i\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right]\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2 - \frac{t + i(t^2+1)}{2\cos(\beta\pi)}\left[h(x,a) + h(y,b)\right] \\ &\quad + \left[t + i(t^2+1)\right]t\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^3 + \left[t^2 + (t^2+1)^2\right]\left[t + i(t^2+1)\right]\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^6. \end{aligned}$$

With $s_1(a) = \frac{\Gamma 5}{a^4\Gamma(5-2\beta)}$, $s_2(a) = \frac{\Gamma 4}{a^4\Gamma(4-2\beta)}$ and $s_3(a) = \frac{\Gamma 3}{a^4\Gamma(3-2\beta)}$, the function $h(x,a), h(y,b)$ can be given as

$$\begin{aligned} h(x,a) &= s_1(a)\left[(x-x_L)^{4-2\beta} + (x_R-x)^{4-2\beta}\right] + 4s_2(a)\left[x_L(x-x_L)^{3-2\beta} - x_R(x_R-x)^{3-2\beta}\right] + \\ &\quad 4s_3(a)x_L^2(x-x_L)^{2-2\beta} + 4s_3(a)x_R^2(x_R-x)^{2-2\beta}. \end{aligned}$$

Initial condition

$$U(x,y,0) = i\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2,$$

boundary conditions

$$U(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \in [0,T],$$

and the exact solution is

$$U(x,y,t) = \left[t + i(t^2+1)\right]\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2,$$

where, $x_L = -\frac{a}{b}\sqrt{b^2-y^2}$, $x_R = \frac{a}{b}\sqrt{b^2-y^2}$, $y_L = -\frac{b}{a}\sqrt{a^2-x^2}$, $y_R = \frac{b}{a}\sqrt{a^2-x^2}$.

In this example, numerical solutions of the proposed method have obtained over fractional-order derivative $0 < \alpha \leq 1$ and $1/2 < \beta \leq 1$. The tabulated results of the proposed method with different fractional-order derivatives α and β are presented in Table 1. The numerical results of the proposed method achieved better accuracy as the number of grid points in both space and time axis is increased. Figure 1 has shown that surface plot of the proposed method at time $T = 1.0$ with $a = 1/2, b = 1/4, \alpha = 0.8$ and $\beta = 0.85$. Contour plot also has shown the physical behavior of the equation. Moreover, Figure 2 has also shown that surface plot of the proposed method at time $T = 1.0$ with $a = 1, b = 1, \alpha = 0.8$ and $\beta = 0.85$. The numerical results are more accurate and consistent with our theoretical results.

Table 1. Numerical solutions of proposed method with different α, β and grids points M for example 1.

M	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
	L_2	L_2	L_2	L_2	L_2
			$\beta = 0.70$		
16	4.5324e-03	2.6634e-03	5.7714e-03	4.3257e-03	7.5687e-05
32	1.8317e-03	1.1843e-03	2.3455e-03	1.7289e-03	3.0442e-05
64	8.3699e-04	4.9184e-04	1.1883e-03	7.9882e-04	1.3977e-05
128	5.7443e-05	3.3756e-05	7.3146e-05	5.4823e-05	9.5925e-07
256	1.1066e-05	6.5028e-06	1.4091e-05	1.0561e-05	1.8479e-07
			$\beta = 0.85$		
16	9.5437e-03	8.6147e-03	6.3341e-03	6.1617e-03	5.4673e-05
32	3.8385e-03	3.4649e-03	2.5476e-03	2.4783e-03	2.1990e-05
64	1.7624e-04	1.5909e-04	1.1697e-04	1.1379e-04	1.0096e-06
128	1.2096e-05	1.0918e-05	8.0278e-04	7.8093e-04	6.9292e-06
256	2.3301e-05	2.1033e-05	1.5465e-05	1.5044e-05	1.3349e-07
			$\beta = 1.00$		
16	4.9563e-04	3.8725e-04	4.2852e-04	5.8512e-04	8.2624e-05
32	1.9934e-04	1.5575e-04	1.7235e-04	2.3534e-04	3.3232e-06
64	9.1527e-04	7.1512e-04	7.9134e-04	1.0805e-04	1.5258e-06
128	6.2815e-05	4.9080e-05	5.4310e-05	7.4157e-05	1.0472e-07
256	1.2101e-06	9.4548e-06	1.0462e-06	1.4286e-06	2.0173e-08

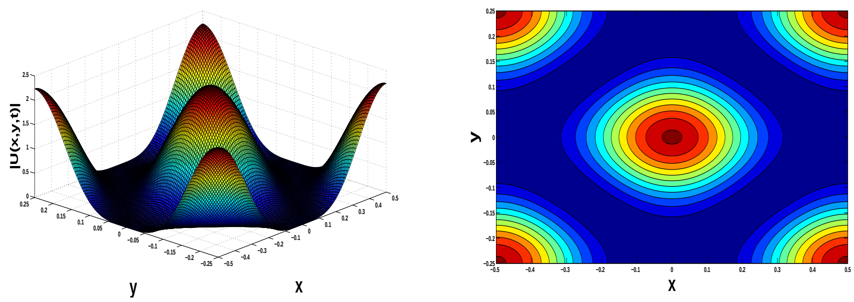

Figure 1. Numerical solutions of example 1 at time $T = 1.0$ with $a = 1/2, b = 1/4, \alpha = 0.8$ and $\beta = 0.85$.

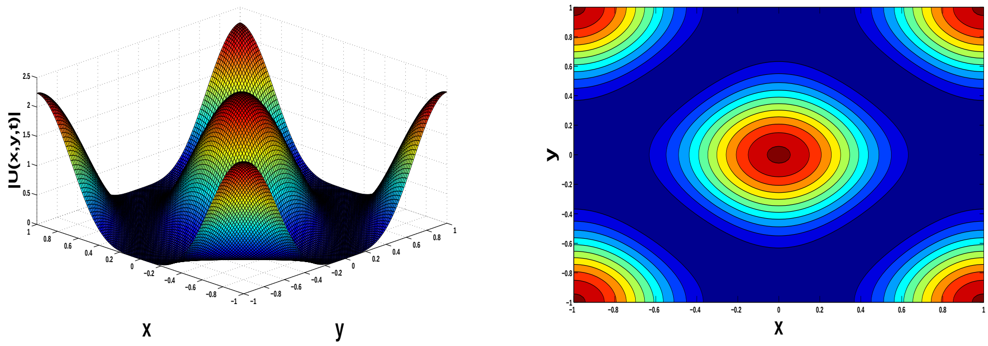

Figure 2. Numerical solutions of example 1 at time $T = 1.0$ with $a = 1, b = 1, \alpha = 0.8$ and $\beta = 0.85$.

5.2. Example 2

In this example, let us consider the time- and space- NFSE on a rectangular region $\Omega = [0,1]\times[0,1]$

$$i\frac{\partial^{\alpha}U}{\partial t^{\alpha}} + \kappa\left\{\frac{\partial^{2\beta}U}{\partial x^{2\beta}} + \frac{\partial^{2\beta}U}{\partial y^{2\beta}}\right\} + G(x,y,t)U + \lambda|U|^2U = F(x,y,t), \qquad (x,y)\in\Omega, \quad t\in[0,T],$$

with $\lambda = \kappa = 1$, $G(x,y,t) = txy(1-x)(1-y)$.
The source term is

$$\begin{aligned} F(x,y,t) \;=\; & \left[i\frac{15t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{30t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{30t^{2-\alpha}}{\Gamma(3-\alpha)}\right]x^2(1-x)^2y^2(1-y)^2 + 3375[t^2+(1+t)^4] \\ & [t+i(1+t)^2]x^6(1-x)^6y^6(1-y)^6 + 15[t+i(1+t)^2]tx^3(1-x)^3y^3(1-y)^3 - \\ & \frac{15[t+i(1+t)^2]y^2(1-y)^2}{\cos(\beta\pi)}[h_1(x,\beta)+h_2(x,\beta)] - \frac{15[t+i(1+t)^2]x^2(1-x)^2}{\cos(\beta\pi)} \\ & [h_1(y,\beta)+h_2(y,\beta)]. \end{aligned}$$

where,

$$\begin{aligned} h_1(x,\beta) &= \frac{x^{2-2\beta}}{\Gamma(3-2\beta)}\left[1 - \frac{6x}{3-2\beta} + \frac{12x^2}{(3-2\beta)(3-4\beta)}\right], \\ h_2(x,\beta) &= \frac{(1-x)^{2-2\beta}}{\Gamma(3-2\beta)}\left[1 - \frac{6(1-x)}{3-2\beta} + \frac{12(1-x)^2}{(3-2\beta)(3-4\beta)}\right]. \end{aligned}$$

Initial condition

$$U(x,y,0) = 15ix^2(1-x)^2y^2(1-y)^2,$$

boundary conditions

$$U(x,y,t) = 0, \quad (x,y)\in\partial\Omega, \quad t\in[0,T],$$

and the exact solution is

$$U(x,y,t) = 15[t+i(1+t)^2]x^2(1-x)^2y^2(1-y)^2.$$

For computation purpose we have chosen space interval $\Omega = [0, 1]^2$ with time $T = 1$. Numerical solutions of the proposed method have been computed with different fractional-order derivative $0 < \alpha \leq 1$ and $1/2 < \beta \leq 1$. Figure 3 has shown the numerical solution of the proposed method and the contour plot has shown the physical behaviour of the proposed method. In Table 2, the tabulated results of the proposed method with different fractional-order derivatives α and β are presented. The numerical results of the proposed method achieved better accuracy as the number of grid points in both space and time directions are increased. Moreover, it demonstrated that the numerical method is more efficient.

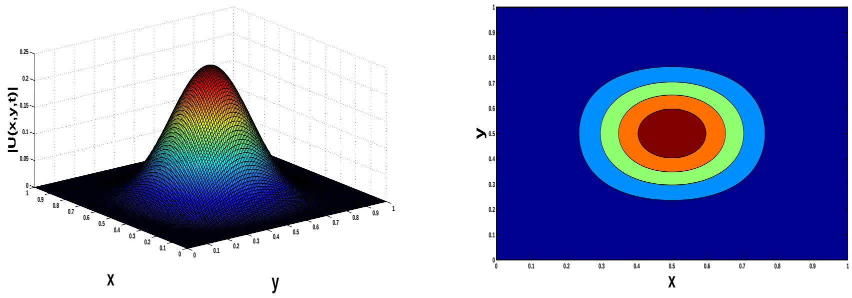

Figure 3. Numerical solutions of example 2 at time $T = 1.0$ with $\alpha = 0.8$ and $\beta = 0.85$.

Table 2. Numerical solutions of proposed method with different α, β and grids points M for example 2.

M	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
	L_2	L_2	L_2	L_2	L_2
			$\beta = 0.70$		
16	1.9236e-04	3.2736e-04	4.4412e-04	5.6202e-04	3.5931e-05
32	7.7368e-04	1.3167e-04	1.7863e-04	2.2605e-04	1.4452e-05
64	3.5523e-04	6.0453e-04	8.2014e-04	1.0379e-04	6.6353e-05
128	2.4379e-05	4.1489e-05	5.6287e-05	7.1230e-05	4.5538e-06
256	4.6965e-05	7.9926e-05	1.0843e-05	1.3722e-05	8.7727e-06
			$\beta = 0.85$		
16	2.9351e-05	5.4325e-05	3.4143e-05	4.2313e-05	6.5295e-05
32	1.1805e-05	2.1850e-05	3.4331e-06	4.2546e-06	6.5655e-06
64	5.4202e-06	1.0032e-05	7.8814e-07	9.7673e-07	1.5072e-06
128	3.7199e-07	6.8851e-07	1.0818e-07	1.3407e-07	2.0688e-07
256	7.1661e-07	1.3264e-07	2.0840e-07	2.5827e-07	3.9855e-08
			$\beta = 1.00$		
16	7.1124e-05	3.5341e-05	2.3774e-05	3.4571e-05	6.6325e-05
32	5.1516e-06	3.5536e-06	2.3905e-06	3.4762e-06	6.6690e-06
64	1.6418e-06	8.1579e-07	5.4878e-07	7.9802e-07	1.5310e-06
128	2.2535e-07	1.1198e-07	7.5327e-08	1.0954e-07	2.1015e-08
256	4.3413e-08	2.1572e-08	1.4511e-08	2.1102e-08	4.0484e-09

5.3. Example 3

In this example, let us consider the time- NFSE with non-homogeneous boundary value

$$i\frac{\partial^\alpha}{\partial t^\alpha}U + 0.5\frac{\partial^2}{\partial x^2}U + 0.5\frac{\partial^2}{\partial y^2}U - \mid U \mid^2 U = h(x,y,t), \quad (x,y,t) \in [0,1]\times[0,1]\times[0,1], \tag{5.1}$$

where $h(x,y,t) = \frac{t^{3-\alpha}(6-i(t^6+1)\Gamma(4-\alpha)t^\alpha)(e^{-i(x+y)})}{\Gamma(4-\alpha)}$,
with initial condition:

$$U(x,y,0) = 0, \qquad (x,y) \in [0,1]\times[0,1],$$

and boundary conditions:

$$\begin{aligned} U(0,y,t) &= t^3 e^{i(y)}, \ U(1,y,t) = t^3 e^{i(1+y)}, \qquad t \in [0,1], \quad (x,y) \in [0,1]\times[0,1], \\ U(x,0,t) &= t^3 e^{i(x)}, \ U(x,1,t) = t^3 e^{i(x+1)}. \end{aligned}$$

The exact solution of this problem is

$$U(x,y,t) = t^3 e^{i(x+y)}. \tag{5.2}$$

In this example, error norms for two dimensional nonlinear time fractional Schrodinger equation has been calculated with different values of grids point and different fractional order derivative $0 < \alpha \leq 1$. In Table 3, it can be seen that the accuracy of the numerical results are increased along with the number of grid points and also achieved good order of accuracy. Further, proposed methods have avheived better accuracy as compared to [3] at $\alpha = 0.50$ with different grid points. Moreover, proposed method has obtained 14^{th} order of accuracy.

Table 3. Numerical solutions of example 3 for different value of α and grids points M .

M	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1.00$	$[3]\alpha = 0.50$
	L_2	L_2	L_2	L_2	MAE
04	1.8374e-05	2.4281e-05	4.2328e-05	7.6581e-06	1.55e-03
06	2.1641e-07	1.8207e-07	3.3021e-07	3.2801e-08	3.58e-06
08	1.2258e-09	3.2740e-09	1.5839e-09	1.4142e-10	4.38e-09
10	8.9053e-12	3.6506e-12	1.0107e-13	9.7058e-14	1.08e-10

6. Conclusion

In this work, we have discussed a highly accurate fully discrete time-space Chebyshev pseudospectral method for the two-dimensional time- and space-NFSE, defined on a convex and rectangular domain. The new fractional derivative matrix has been established using a modified Riemann-Liouville derivative formula at CGL points for a different order of fractional derivatives. We presented the error analysis without any dependency on time and space step restrictions of the schemes. The proposed method supports the theoretical results. To demonstrate the performance, the method has been employed on three different model problems on a convex and rectangular domain and obtained a good order of accuracy. Reported numerical results are highly accurate which shows the efficiency of the proposed method.

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Conflict of interest

The authors declare no conflict of interest.

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