Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(2): 231–241.
DOI:10.3934/math.2019.2.231
Received: 29 November 2018
Accepted: 22 February 2019
Published: 07 March 2019

Research article

Direct similarity reductions and new exact solutions of the short pulse equation

Quting Chen and Yadong Shang *

School of Mathematics and Information Science, Guangzhou University, Guangzhou, Guangdong, 510006, P.R. China

* **Correspondence:** Email: gzydshang@126.com; Tel: +02039366863.

Abstract: In this paper, we present some similarity reductions of the short pulse equation(SPE) based on the direct similarity reduction method proposed by Clarkson and Kruskal. These similarity reductions have a more general form than those obtained by using the Lie group method. Especially, we obtain one new similarity reduction which can not be obtained by Lie group method. Furthermore, we derive one new exact analytic solutions by the method of undetermined coefficients.

Keywords: similarity reductions; direct method; short pulse equation; exact solution; the method of undetermined coefficients
Mathematics Subject Classification: 35L70, 35Q58

1. Introduction

The general short pulse equation (SPE) takes the form

$$u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx}, \tag{1}$$

where, $u(x,t)$ represents the magnitude of electric field, while the subscripts t and x denote partial differentiations. The short pulse equation was derived by Schäfer and Wayne [1] as a model equation describing the propagation of ultr-short light pulses in silica optical fibres [1, 2]. The short pulse equation represents an alternative approach in contrast with the slowly varying envelope approximation which leads to the nonlinear Schrödinger equation. The SPE is also known as the cubic Robelo equation that has been derived as an integrable differential equation associated with pseudo-spherical surfaces [3, 4]. In last two decades, many researches have been carry out for the short pulse equation. The Lax pair of the SPE of the Wadati-Konno-Ichikawa type was rediscovered in [5]. The integrability, soliton solutions, bi-Hamiltonian structure, Lie symmetry analysis and conservation laws, and other features of the SPE have been investigated by many authors [6–14]. In recent years, the short pulse equation has been generalized in many different aspects [15–28]. For more new developments, please refer to [29–31] and the references therein.

It is well-known that finding the exact solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical systems. A wealth of methods have been developed to find these exact solutions of a partial differential equation. In the last decade, active research efforts have been made on the derivation of exact solutions for the SPE. In [9], the Lie symmetry analysis and the generalized method are performed for a short pulse equation. In [10], the invariance properties and conservation laws of the nonlinear short pulse equation are studied through Lie symmetry analysis. In [19], the complete group classification is performed on the extended short pulse equation. In 1989, Clarkson and Kruskal [32,33] presented a simple ansatz based method for seeking similarity reductions of nonlinear partial differential equations, which is referred to as the direct method. The direct method has been employed to investigate a number of nonlinear partial differential equations [34–36]. The great advantage of the direct method is that it does not invoke group theory. Several researchers have showed a close relationship between the symmetry group methods and the direct method [37, 38].

The goal of this paper is find some new similarity reductions of the short pulse equation

$$u_{xt} + u + 2uu_x^2 + u^2u_{xx} = 0, \tag{2}$$

by using the direct method due to Clarkson and Kruskal. Since this method does not involve group theory and the constant coefficients in equation can be easily handled, a large number of sophisticate calculations can be avoided. The results of the present paper show that the direct method is more than the Lie group similarity reduction method for symmetry similarity. One new traveling wave solutions are obtained by the method of undetermined coefficients.

The rest of this paper is organized as follows. Section 2 gives reductions of (2) using the direct method as well as the reduced ordinary differential equations. These reductions obtained in the present paper have a more general form than those by using the Lie group method. In addition, we also find a new similarity reduction. In section 3, one new traveling wave solutions will be given by the method of undetermined coefficients.

2. Materials and method

According to the direct method, we seek the similarity reduction of (2) in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)), \tag{3}$$

where $z = z(x,t)$ is the similarity variable.

Substituting (3) into (2) and collecting coefficients of monomials of w and its derivatives yields

$$\begin{aligned}
&(\beta z_x z_t + \alpha^2\beta z_x^2)w'' + (\beta z_{xt} + \beta_x z_t + \beta_t z_x + \alpha^2\beta z_{xx} + 2\alpha^2\beta_x z_x + 4\alpha\beta\alpha_x z_x)w' \\
&+ (\beta + \beta_{xt} + 2\beta\alpha_x^2 + \alpha^2\beta_{xx} + 2\alpha\beta\alpha_{xx} + 4\alpha\alpha_x\beta_x)w \\
&+ (2\alpha\beta_x^2 + \beta^2\alpha_{xx} + 4\beta\alpha_x\beta_x + 2\alpha\beta\beta_x)w^2 + (2\beta\beta_x^2 + \beta^2\beta_{xx})w^3 \\
&+ (\beta^3 z_{xx} + 6\beta^2\beta_x z_x)w'w^2 + 2\beta^3 z_x^2 w'^2 w + \beta^3 z_x^2 w'' w^2 + 2\alpha\beta^2 z_x^2 w^2 \\
&+ (2\alpha\beta^2 z_{xx} + 4\beta^2\alpha_x z_x + 8\alpha\beta\beta_x z_x)w'w + 2\alpha\beta^2 z_x^2 w'' w \\
&+ (\alpha + \alpha_{xt} + 2\alpha\alpha_x^2 + \alpha^2\alpha_{xx}) = 0.
\end{aligned} \tag{4}$$

In order that this equation (4) becomes an ordinary differential equation of for $w(z)$ then the ratios of the coefficients of different derivatives and powers of $w(z)$ have to be functions of z only. This gives an over-determined system of equations for $\alpha(x,t)$, $\beta(x,t)$ and $z(x,t)$, whose solutions yield the desired similarity solutions. Before doing this we will make some remarks about this direct method of seeking similarity reductions.

Remark 1. We will choose the coefficient of $w''w^2$ as the normalizing coefficient and introduce a function $\Gamma(z)$, where Γ is a function of z to be determined.

Remark 2. Whenever we use $\Gamma(z)$ to denote a function to be determined, then it is a function, upon which we can perform any mathematical operation (e.g., differential, integration, logarithm, exponentiation, taking powers, rescaling, etc.) and then also call the resulting function $\Gamma(z)$, without loss of generality.

In the determination of α, β, z, and $\Gamma(z)$, there are following three remarks:

Remark I: If $\alpha(x,t)$ is determined by $\alpha(x,t) = \hat{\alpha}(x,t) + \beta(x,t)\Gamma(z)$, one can take $\Gamma(z) \equiv 0$.

Remark II: If $\beta(x,t)$ has the form $\beta(x,t) = \hat{\beta}(x,t)\Gamma(z)$, one can choose $\Gamma(z) \equiv 1$ by rescaling $w(z) \to \frac{w(z)}{\Gamma(z)}$.

Remark III: If $z(x,t)$ is determined by an implicit form $\Gamma(z) = \hat{z}(x,t)$, one can assume $z = \hat{z}(x,t)$ by the transformation $z \to \Gamma^{-1}(z)$.

The coefficients of w'^2w and $w''w$ yield the common constraint

$$\beta^3 z_x^2 \Gamma(z) = \alpha\beta^2 z_x^2. \tag{5}$$

By using Remark I, we obtain $\Gamma(z) \equiv 0$, and therefore

$$\alpha \equiv 0. \tag{6}$$

Substituting (6) in equation (4), we have:

$$\begin{aligned} &\beta z_x z_t w'' + (\beta z_{xt} + \beta_x z_t + \beta_t z_x)w' + (\beta + \beta_{xt})w + (2\beta\beta_x^2 + \beta^2\beta_{xx})w^3 \\ &+ (\beta^3 z_{xx} + 6\beta^2\beta_x z_x)w'w^2 + 2\beta^3 z_x^2 w'^2 w + \beta^3 z_x^2 w'' w^2 = 0. \end{aligned} \tag{7}$$

To make an ordinary differential equation for $w(z)$, we introduce functions $\Gamma_n(z)$, $n = 1, 2, \ldots, 5$ in the following manner:

$$\begin{aligned} \beta^3 z_x^2 \Gamma_1(z) &= \beta z_x z_t, \\ \beta^3 z_x^2 \Gamma_2(z) &= \beta z_{xt} + \beta_x z_t + \beta_t z_x, \\ \beta^3 z_x^2 \Gamma_3(z) &= \beta + \beta_{xt}, \\ \beta^3 z_x^2 \Gamma_4(z) &= 2\beta\beta_x^2 + \beta^2\beta_{xx}, \\ \beta^3 z_x^2 \Gamma_5(z) &= \beta^3 z_{xx} + 6\beta^2\beta_x z_x. \end{aligned}$$

In order to seek out much more similarity reductions of SPE, we consider the following two cases, respectively.

Case 1. $\beta_x = 0$:

Taking $\beta_x = 0$ in the above formulas yields

$$\beta^3 z_x^2 \Gamma_1(z) = \beta z_x z_t, \tag{8}$$

$$\begin{aligned}\beta^3 z_x^2 \Gamma_2(z) &= \beta z_{xt} + \beta_t z_x, &(9)\\ \beta^3 z_x^2 \Gamma_3(z) &= \beta, &(10)\\ \Gamma_4(z) &= 0, &(11)\\ \beta^3 z_x^2 \Gamma_5(z) &= \beta^3 z_{xx}. &(12)\end{aligned}$$

Taking into account Remark II, from (10), we get

$$\beta = z_x^{-1}. \tag{13}$$

Rewriting (12) and take notice of $\beta_x = 0$, we get

$$\Gamma_5(z) = 0. \tag{14}$$

Due to (14) and (12), we have

$$z = \theta(t)x + \sigma(t). \tag{15}$$

Substituting (13) and (15) in (8), (9), we get

$$\begin{aligned}\Gamma_1(z) &= \theta(t)[\theta'(t)x + \sigma'(t)], &(16)\\ \Gamma_2(z) &= 0. &(17)\end{aligned}$$

(16) could be rewritten as

$$\Gamma_1(z) = \theta'(t)z + \theta(t)\sigma'(t) - \sigma(t)\theta'(t). \tag{18}$$

In this case, (7) becomes

$$[\theta'(t)z + \theta(t)\sigma'(t) - \sigma(t)\theta'(t)]w''(z) + w(z) + 2w'^2(z)w(z) + w''(z)w^2(z) = 0. \tag{19}$$

In order to the equation (19) is an ordinary differential equation of $w(z)$, $\theta(t)$ and $\sigma(t)$ need to satisfy the system

$$\begin{aligned}\theta'(t) &= a_1, &(20)\\ \theta\sigma'(t) - \sigma\theta'(t) &= c_1, &(21)\end{aligned}$$

where a_1, c_1 are arbitrary constants.

Solving the one order ordinary differential equations (20) and (21) yields

$$\begin{aligned}\theta &= a_1 t + a_2, &(22)\\ \sigma &= b_1 t + b_2, &(23)\end{aligned}$$

where a_1, a_2, b_1 and b_2 are all arbitrary constants.

In summary, in this case we obtain the similarity transformation and similarity variables of the SPE (2) given by

$$\begin{aligned}u(x,t) &= \frac{1}{a_1 t + a_2} w(z), &(24)\\ z(x,t) &= a_1 xt + a_2 x + b_1 t + b_2, &(25)\end{aligned}$$

where $w(z)$ satisfies

$$(a_1 z + a_2 b_1 - a_1 b_2)w'' + w + 2w'^2 w + w'' w^2 = 0. \tag{26}$$

Especially, setting $a_1 = b_2 = 0$, $a_2 = 1$ in (24) and (25), we obtain the traveling wave reduction

$$u = w(z), \qquad z = x + b_1 t, \tag{27}$$

and the SPE can be reduced to the following ODE:

$$b_1 w'' + w + 2w'^2 w + w'' w^2 = 0. \tag{28}$$

This similarity reduction is the same as the case (i) of Liu and Li [9] obtained by Lie symmetry analysis.

Taking $a_1 = 1$, $a_2 = b_1 = b_2 = 0$ in (24) and (25), we have the following similarity variables

$$u = \frac{1}{t} w(z), \qquad z = xt, \tag{29}$$

and the SPE is reduced to the following ODE:

$$z w'' + w + 2w'^2 w + w'' w^2 = 0. \tag{30}$$

This similarity reduction is the same as the case (ii) of Liu and Li [9] obtained by Lie symmetry analysis.

Compared with the literature [9], we can see that the similarity reduction obtained by the direct method here have more general form. It is shown that the similarity reduction of the SPE obtained by the Lie group method can be regarded as a special case of those obtained by using the direct method developed by Clarkson and Kruskal. Our result indicates that the direct method is more general than the Lie method for symmetry reductions.

Case 2. $\beta_x \neq 0$:

In this case, we consider the original constraints:

$$\begin{aligned}
\beta^3 z_x^2 \Gamma_1(z) &= \beta z_x z_t, && (31)\\
\beta^3 z_x^2 \Gamma_2(z) &= \beta z_{xt} + \beta_x z_t + \beta_t z_x, && (32)\\
\beta^3 z_x^2 \Gamma_3(z) &= \beta + \beta_{xt}, && (33)\\
\beta^3 z_x^2 \Gamma_4(z) &= 2\beta\beta_x^2 + \beta^2 \beta_{xx}, && (34)\\
\beta^3 z_x^2 \Gamma_5(z) &= \beta^3 z_{xx} + 6\beta^2 \beta_x z_x. && (35)
\end{aligned}$$

Applying Remark II to (31), we obtain

$$\beta = (\frac{z_t}{z_x})^{\frac{1}{2}}, \tag{36}$$

and

$$\Gamma_1(z) = 1. \tag{37}$$

Substituting (36) in (35), we arrive

$$z_x \Gamma_5(z) = 3\frac{z_{xt}}{z_t} - 2\frac{z_{xx}}{z_x}. \tag{38}$$

Because $\Gamma_5(z)$ is a function of z, we can divide it into the sum of two functions of z as

$$\Gamma_5(z) = \Gamma_6(z) + \Gamma_7(z), \tag{39}$$

where $\Gamma_6(z)$ and $\Gamma_7(z)$ are functions to be determined. In this way, we can assume that

$$z_x\Gamma_6(z) = 3\frac{z_{xt}}{z_t}, \tag{40}$$

$$z_x\Gamma_7(z) = -2\frac{z_{xx}}{z_x}, \tag{41}$$

which still satisfies (38).

Integrating (40) with respect to x once, then integrating with respect to t once and using the Remark 2 gives rise to

$$\Gamma_6(z) = \sigma(x) + \gamma_1(t), \tag{42}$$

where $\sigma(x)$ and $\gamma_1(t)$ are functions of integration.

Integrating (41) with respect to x twice and using the Remark 2 gives rise to

$$\Gamma_7(z) = \theta(t)x + \gamma_2(t), \tag{43}$$

where $\sigma(x)$ and $\gamma_2(t)$ are functions of integration.

From (39), we have

$$\Gamma_5(z) = \theta(t)x + \sigma(x) + \gamma(t), \tag{44}$$

where $\gamma(t) = \gamma_1(t) + \gamma_2(t)$ is a function of integration.

By Remark III, we have

$$z = \theta(t)x + \sigma(x) + \gamma(t). \tag{45}$$

Substituting (45) in (31)-(35), we obtain

$$\begin{aligned}
\Gamma_1(z) &= 1,\\
\Gamma_2(z) &= \frac{\theta'}{(\theta'x+\gamma')(\theta+\sigma')} + \frac{\theta''x+\gamma''}{2(\theta'x+\gamma')^2} - \frac{\sigma''}{2(\theta+\sigma')^2},\\
\Gamma_3(z) &= \frac{1}{(\theta'x+\gamma')(\theta+\sigma')} - \frac{1}{2(\theta'x+\gamma')^2}\cdot\left[-\frac{\theta''}{\theta+\sigma'} + \frac{\theta'^2+\theta''\sigma''x+\sigma''\gamma''}{(\theta+\sigma')^2} - \frac{2\theta'\sigma''(\theta'x+\gamma')}{(\theta+\sigma')^3}\right]\\
&\quad -\frac{1}{4(\theta'x+\gamma')^3(\theta+\sigma')}\left[\theta' - \frac{\sigma''(\theta'x+\gamma')}{\theta+\sigma'}\right]\left[\frac{\theta''x+\gamma''}{\theta+\sigma'} - \frac{\theta'(\theta'x+\gamma')}{(\theta+\sigma')^2}\right],\\
\Gamma_4(z) &= \frac{5\sigma''^2}{4(\theta+\sigma')^4} + \frac{\theta'^2}{4(\theta'x+\gamma')^2(\theta+\sigma')^2}\\
&\quad -\frac{1}{2(\theta'x+\gamma')(\theta+\sigma')^2}(\theta'\sigma'''x + 3\theta'\sigma'' + \sigma'''\gamma'),\\
\Gamma_5(z) &= \frac{3\theta'}{(\theta'x+\gamma')(\theta+\sigma')} - \frac{2\sigma''}{(\theta+\sigma')^2}.
\end{aligned}$$

From $\Gamma_n(z)$, $n = 1, 2, \ldots, 5$, we can see both $z_x = \theta + \sigma'$ and $z_t = \theta'x + \gamma'$ are indispensable in denominators, so we assume that

$$z = z_x z_t = \theta\theta'x + \theta'\sigma'x + \sigma'\gamma' + \theta\gamma'. \tag{46}$$

Comparing (45) with (46), setting

$$\theta'(t)\left[\theta(t) + \sigma'(x)\right] = \theta(t), \tag{47}$$

$$\sigma'(x)\gamma'(t) = \sigma(x), \tag{48}$$
$$\theta(t)\gamma'(t) = \gamma(t). \tag{49}$$

Taking notice of (49), we find that $\theta(t)$ cannot be equal to zero (otherwise, there will be $\gamma(t) = 0$, so there will be $z = \sigma(x)$ from (45), and therefore there will be $\beta = 0$ from (36)). From (47) we find that

$$\theta'(t) \neq 0, \tag{50}$$
$$\sigma'(x) = c_0, \tag{51}$$

where c_0 is a constant. Solving (51), yields

$$\sigma(x) = c_0 x + c_3, \tag{52}$$

where c_3 is a constant. Substituting (52) into (48), yields

$$c_0\gamma'(t) = c_0 x + c_3. \tag{53}$$

Due to (53), there will be $c_0 = c_3 = 0$, so

$$\sigma(x) = 0. \tag{54}$$

Considering (47) and (50), then the one order ordinary differential equations (45) and (46) yields

$$\theta(t) = t + c_1, \tag{55}$$
$$\gamma(t) = c_2 t + c_1 c_2, \tag{56}$$

where c_1 and c_2 are two arbitrary constants.

In this case, we obtain the similarity reduction of the SPE given by

$$u(x,t) = \sqrt{\frac{x+c_2}{t+c_1}}w(z), \tag{57}$$
$$z(x,t) = xt + c_1 x + c_2 t + c_1 c_2, \tag{58}$$

where c_1 and c_2 are two arbitrary constants, and $w(z)$ satisfies

$$w'' + \frac{1}{z}w' + (\frac{1}{z} - \frac{1}{4z^2})w + \frac{1}{4z^2}w^3 + \frac{3}{z}w'w^2 + 2w'^2 w + w''w^2 = 0. \tag{59}$$

As far as our known, this is a new similarity reduction that have not been reported in any other paper by Lie symmetry analysis. It is shown that there exists exact solutions of the short pulse equation which can be obtained using the direct method developed by Clarkson and Kruskal, but which are not obtained by using the Lie symmetry similarity method. Our results indicate again that the direct method is more general than the Lie method for symmetry reductions.

3. Exact traveling wave solutions of SPE

In this section, we look for the traveling wave solutions of the SPE equation.

In section 2, we have got the similarity reduction $u(x,t) = \varphi(\xi), \xi(x,t) = x + ct$, where $c > 0$ is the propagating wave velocity. And the traveling solution satisfies the following ordinary differential equation (ODE)

$$c\varphi'' + \varphi + 2\varphi'^2\varphi + \varphi''\varphi^2 = 0, \tag{60}$$

where $\varphi' = \frac{d\varphi}{d\xi}$. Let $X = \varphi, Y = \varphi'$, Eq. (60) can be converted to an equivalent planar system

$$X' = Y, Y' = -\frac{(1+2Y^2)X}{c+X^2}. \tag{61}$$

Making the transformation

$$d\xi = (b_1 + X^2)d\zeta, \tag{62}$$

system (61) becomes

$$\frac{dX}{d\zeta} = (c+X^2)Y, \;\; \frac{dY}{d\zeta} = -X(1+2Y^2). \tag{63}$$

In the literature [6], the author considered the solitary wave solutions of the SPE. In the literature [7,8], Parkes obtained loop-soliton solutions and some periodic traveling wave solutions of the SPE. Liu and Li [9] solved the traveling wave equation by the bifurcation theory method of dynamical systems and obtained some traveling wave solutions. Xie, Hong and Gao in [11] investigated the periodic solution of the SPE by using the dynamical system theory. Ma and Li [14] got some new by Jacobi elliptic function solutions by a direct symbolic computation method combined with variable transformations.

Next, we will get the traveling wave solution of the SPE (2) by the method of undetermined coefficients.

Now let us search for the solution of the SPE (2) in form of

$$u(x,t) = Ax + Bt + C, \tag{64}$$

where A, B and C are three constants to be determined.

Substituting (64) into (2) yields

$$Ax + Bt + C + 2A^3x + 2A^2Bt + 2A^2C = 0. \tag{65}$$

Due to x and t are independent variables, so A, B and C satisfy a set of nonlinear algebraic equations

$$A + 2A^3 = 0, \tag{66}$$
$$B + 2A^2B = 0, \tag{67}$$
$$C + 2A^2C = 0. \tag{68}$$

Solving this system of nonlinear algebraic equations, one obtain $A = \pm\frac{1}{\sqrt{2}}i$, while B and C are arbitrary constants.

Then there is a solution of (2)

$$u(x,t) = \pm\frac{1}{\sqrt{2}}(xi + Bt) + C, \tag{69}$$

where B and C are arbitrary constants. Although this solution is simple, it has not been reported in any other literature. It is expected that this special solution can be applied for the numerical simulations.

4. Conclusion

In the present paper, we establish some similarity reductions of the short pulse equation(SPE) based on the direct similarity reduction method proposed by Clarkson and Kruskal. Some of these similarity reductions have a more general form than those obtained by using the Lie group method. Especially, we obtain some new similarity reductions which can not be obtained by Lie group symmetry analysis method. Finally, we derive one new exact analytic solutions which may be studied for the numerical simulations.

Acknowledgments

This work is supported the Program for the Innovation Research Grant for the postgraduates of Guangzhou University (NO:2017GDJC-D08). The authors wish to express their gratitude to the anonymous referees for the careful reading, the helpful suggestions and comments of the paper.

Conflict of interest

The authors declare no conflict of interest.

References

1. T. Schäfer, C. E. Wayne, *Propagation of ultra-short optical pulses in cubic nonlinear media*, Physica D, **196** (2004), 90–105.
2. Y. Chung, C. K. R. T. Jones, T. Schäfer, et al. *Ultra-short pulses in linear and nonlinear media*, Nonlinearity, **18** (2005), 1351–1374.
3. M. L. Robelo, *On equations which describe pseudospherical surfaces*, Stud. Appl. Math., **81** (1989), 221–248.
4. R. Beals, M. Rabelo, K. Tenenblat, *Bäcklund transformations and inverse scattering solutions for some pseudospherical surface equations*, Stud. Appl. Math., **81** (1989), 125–151.
5. A. Sakovich, S. Sakovich, *The short pulse equation is integrable*, J. Phys. Soc. Jpn., **74** (2005), 239–241.
6. A. Sakovich, S. Sakovich, *Solitary wave solutions of the short pulse equation*, J. Phys. A, **39** (2006), L361–L367.
7. E. J. Parkes, *A note on loop-soliton solutions of the short pulse equation*, Phys. Lett. A, **374** (2010), 4321–4323.
8. E. J. Parkes, *Some periodic and solitary traveling-wave solutions of the short-pulse equation*, Chaos Soliton. Fract., **38** (2008), 154–159.
9. H. Z. Liu, J. B. Li, *Lie symmetry analysis and exact solutions for the short pulse equation*, Nonlinear Anal. Theor., **71** (2009), 2126–2133.
10. K. Fakhar, G. W. Wang, A. H. Kara, *Symmetry reductions and conservation laws of the short pulse equation*, Optik, **127** (2016), 10201–10207.

11. S. L. Xie, X. C. Hong, B. Gao, *The periodic traveling-wave solutions of the short-pulse equation*, Appl. Math. Comput., **218** (2011), 2542–2548.
12. Y. Matsuno, *Multiloop soliton and multibreather solutions of the short pulse model equation*, J. Phy. Soc. Jpn., **76** (2007), 084003.
13. Y. Matsuno, *Periodic solutions of the short pulse model equation*, J. Math. Phys., **49** (2008), 073508.
14. Y. L. Ma, B. Q. Li, *Some new Jacobi elliptic function solutions for the short-pulse equation via a direct symbolic computation method*, J. Appl. Math. Comput., **40** (2012), 683–690.
15. B. F. Feng, *Complex short pulse and coupled complex short pulse equations*, Physica D, **297** (2015), 62–75.
16. B. F. Feng, L. M. Ling, Z. N. Zhu, *Defocusing complex short-pulse equation and its multi-dark-soliton solution*, Phys. Rev. E, **93** (2016), 052227.
17. L. M. Ling, B. F. Feng, Z. N. Zhu, *Multi-soliton, multi-breather and higher order rogue wave solutions to the complex short pulse equation*, Physica D, **327** (2016), 13-29.
18. S. Sakovich, *Transformation and integrability of a generalized short pulse equation*, Comm. Nonlinear Sci. Numer. Simulat., **39** (2016), 21–28.
19. S. F. Shen, B. F. Feng, Y. Ohta, *From the real and complex coupled dispersionless equations to the real and complex short pulse equations*, Stud. Appl. Math., **136** (2016), 64–88.
20. S. F. Shen, B. F. Feng, Y. Ohta, *A modified complex short pulse equation of defocusing type*, J. Nonlinear Math. Phy., **24** (2017), 195–209.
21. Q. L. Zha, *The interaction solitons for the complex short pulse equation*, Comm. Nonlinear Sci. Numer. Simulat., **47** (2017), 379–393.
22. R. K. Gupta, V. Kumar, R. Jiwari, *Exact and numerical solutions of coupled short pulse equation with time-dependent coefficients*, Nonlinear Dyn., **79** (2015), 455–464.
23. B. F. Feng, K. I. Maruno, Y. Ohta, *Integrable discretizations of the short pulse equation*, J. Phys. A: Math. Theor., **43** (2010), 085203.
24. J. C. Brunelli, *The short pulse hierarchy*, J. Math. Phys., **46** (2005), 123507.
25. H. Z. Liu, J. B. Li, L. Liu, *Complete group classification and exact solutions to the extended short pulse equation*, Int. J. Nonlin. Mechs., **47** (2012), 694–698.
26. W. G. Rui, *Different kinds of exact solutions with two-loop character of the two-component short pulse equations of the first kind*, Comm. Nonlinear Sci. Numer. Simulat., **18** (2013), 2667–2678.
27. B. F. Feng, *An integrable coupled short pulse equation*, J. Phys. A: Math. Theor., **45** (2012), 1262–1275.
28. V. Kumar, R. K. Gupta, R. Jiwari, *Comparative study of travelling wave and numerical solutions for the coupled short pulse (CSP) equation*, Chin. Phys. B, **22** (2013), 050201.
29. B. Q. Li, Y. L. Ma, *Periodic solutions and solitons to two complex short pulse (CSP) equations in optical fiber*, Optik, **144** (2017), 149–155.
30. Z. Popowicz, *Lax representations for matrix short pulse equations*, J. Math. Phys., **58** (2017), 103506.

31. A. N. W. Hone, V. Novikov, J. P. Wang, *Generalizations of the short pulse equation*, Lett. Math. Phys., **108** (2018), 927–947.
32. P. A. Clarkson, M. D. Kruskal, *New similarity reductions of the Boussinesq equation*, J. Math. Phys., **30** (1989), 2201–2213.
33. P. A. Clarkson, *New similarity solutions for the modified Boussinesq equation*, J. Phys. A: Math. Gen., **22** (1989), 2355–2367.
34. S. Y. Lou, *A note on the new similarity reductions of the Boussinesq equation*, Phys. Lett. A, **151** (1990), 133–135.
35. C. Z. Qu, F. M. Mahomed, *An extension of the direct method via an application*, Quaest. Math., **24** (2001), 111–122.
36. D. J. Arrigo, P. Broadbridge, J.M. Hill, *Nonclassical symmetry solutions and the methods of Bluman-Cole and Clarkson-Kruskal*, J. Math. Phys., **34** (1993), 4692–4703.
37. B. M. Vaganan, R. Asokan, *Direct similarity analysis of generalized Burgers equations and perturbation solutions of Euler-Painlevé transcendents*, Stud. Appl. Math., **111** (2003), 435–451.
38. M. C. Nucci, P. A. Clarkson, *The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh-Nagumo equation*, Phys. Lett. A, **164** (1992), 49–56.

AIMS Press