Quantitative Finance and Economics

http://www.aimspress.com/journal/QFE

QFE, 2(3): 717–732.
DOI:10.3934/QFE.2018.3.717
Received: 27 May 2018
Accepted: 06 August 2018
Published: 27 August 2018

Research article

Ruin probabilities for a double renewal risk model with frequent premium arrivals

Dimitrios G. Konstantinides*

Department of Statistics and Actuarial-Financial Mathematics, University of the Aegean, Karlovassi, GR-83 200 Samos, Greece

* **Correspondence:** Email: konstant@aegean.gr; Tel: +302273082333; Fax: +302273082309.

Abstract: In this paper a double renewal risk model is studied. The claims represent an i.i.d. sequence of random variables and the premiums represent another sequence of random variables with extended negative dependence. The corresponding two arrival processes have different intensities, which correspond to consideration of frequent arrivals of premiums. The ultimate ruin probability is asymptotically estimated when the initial capital tends to infinity.

Keywords: asymptotics; premium process; claim process
JEL codes: G22, G23

1. Introduction

We consider the asymptotics of ruin probabilities in the renewal risk model with constant force of interest. In this model the claim sizes, Z_k, $k = 1, 2, \ldots$, form a sequence of independent, identically distributed (i.i.d.), and non-negative random variables with generic random variable Z and common distribution function B and distribution tail $1 - \overline{B}$, and the inter-occurrence times θ_k, $k = 1, 2, \ldots$, form another sequence of i.i.d. positive random variables with generic random variable θ and common distribution A and distribution tail $1 - \overline{A}$. We assume that the sequences $\{\theta_k,\ k = 1, 2, \ldots\}$ and $\{Z_k,\ k = 1, 2, \ldots\}$ are mutually independent. The arrival times of the successive claims, $\tau_n = \sum_{k=1}^{n} \theta_k$, $n = 1, 2, \ldots$, constitute a renewal counting process

$$N_1(t) = \#\{n \ :\ \tau_n \leq t\}\,, \qquad t \geq 0\,.$$

Therefore, the compound renewal process $S(t) = \sum_{k=1}^{N_1(t)} Z_k$ represents aggregate claims up to time $t \geq 0$, with $S(t) = 0$ if $N_1(t) = 0$.

The premium sizes, Y_k, $k = 1, 2, \ldots$, form a sequence of non-negative random variables with generic random variable Y and common distribution function G, and the inter-arrival times ζ_k, $k = 1, 2, \ldots$, form another sequence of i.i.d. positive random variables with generic random variable ζ and common distribution E. We assume that the sequences $\{\zeta_k,\ k = 1, 2, \dots\}$ and $\{Y_k,\ k = 1, 2, \dots\}$ are mutually independent. The locations of the successive premiums, $\sigma_n = \sum_{k=1}^{n} \zeta_k$, $n = 1, 2, \dots$, constitute a renewal counting process

$$N_2(s) = \#\{n\ :\ \sigma_n \le s\} = N(\Lambda s)\,, \qquad s \ge 0\,.$$

where Λ represents a positive random variable with distribution $Q(q) = \mathbf{P}(\Lambda \le q)$ and $N(t)$ denotes a Poisson counting point process, independent of Λ, with intensity equal to one. Therefore, the compound renewal process $C(s) = \sum_{k=1}^{N_2(s)} Y_k = \sum_{k=1}^{N(\Lambda s)} Y_k$ represents aggregate premiums up to time $s \ge 0$, with $C(s) = 0$ if $N_2(s) = N(\Lambda s) = 0$.

Let $x > 0$ be the initial surplus of the insurance company, let $\delta > 0$ be the constant force of interest (i.e. after time t a capital x becomes $xe^{\delta t}$). Then the total surplus up to time t, denoted by $\widetilde{U}_\delta(t)$, satisfies the equation

$$\widetilde{U}_\delta(t) = xe^{\delta t} + \int_0^t e^{\delta(t-y)} dC(y) - \int_0^t e^{\delta(t-x)} dS(x)\,, \qquad t \ge 0\,.$$

Let consider the discounted surplus through the formula

$$U_\delta(t) := \widetilde{U}_\delta(t) e^{-\delta t} = x + \int_0^t e^{-\delta y} dC(y) - \int_0^t e^{-\delta x} dS(x) = x + \sum_{k=1}^{N(\Lambda t)} Y_k e^{-\delta \sigma_k} - \sum_{n=1}^{N_1(t)} Z_n e^{-\delta \tau_n}\,,$$

for any $t \ge 0$.

Now we introduce a sequence of random variables Y_k^*, $k = 1, 2, \dots$, that are independent from the sequence Y_k, $k = 1, 2, \dots$, and holds $Y_k^* \overset{d}{=} Y_k$, $k = 1, 2, \dots$. This means that for every $k = 1, 2, \dots$ the distributions of Y_k^* and Y_k are identical and equal to G.

Considering as given, that the premium arrivals are much more frequent in comparison with the occurrences of claims, we take as basic time cycle the inter-occurrence times. In practical set up, the premium can be received every week but the claims are expected to occur every year. Let introduce successively the following random variables

$$\begin{aligned}
X_1 &= Z_1 - \sum_{k=1}^{N(\Lambda \tau_1)} Y_k^* \, e^{\delta(\tau_1 - \sigma_k)} = Z_1 - \sum_{k=1}^{N(\Lambda\, \theta_1)} Y_k^* e^{\delta(\theta_1 - \sigma_k)}\,, \\
X_2 &= Z_2 - \sum_{k=N(\Lambda \tau_1)+1}^{N(\Lambda \tau_2)} Y_k^* \, e^{\delta(\tau_2 - \sigma_k)}\,, \\
\dots & \qquad \dots \qquad \dots \\
X_n &= Z_n - \sum_{k=N(\Lambda \tau_{n-1})+1}^{N(\Lambda \tau_n)} Y_k^* \, e^{\delta(\tau_n - \sigma_k)}\,,
\end{aligned} \tag{1.1}$$

through which we obtain

$$U_\delta(t) \overset{d}{=} x - \sum_{n=1}^{N_1(t)} X_n e^{-\delta \tau_n}\,, \tag{1.2}$$

for any $t \ge 0$.

In the actuarial literature, the probability of ultimate ruin is defined to be the probability that the surplus falls below zero. This probability has been extensively investigated.

Let us define the ultimate ruin probability as

$$\psi_\delta(x) = \mathbf{P}\left(\inf_{s\geq 0} U_\delta(s) < 0 \;\middle|\; U_\delta(0) = x\right) = \mathbf{P}\left(M_\infty > x\right), \qquad x \geq 0, \tag{1.3}$$

which represents the distribution tail of the supremum

$$M_\infty := \sup_{m\geq 1} \sum_{n=1}^{m} X_n\, e^{-\delta\tau_n} = \sup_{m\geq 1} S_m, \tag{1.4}$$

with $S_m = \sum_{k=1}^{m} X_k\, e^{-\delta\tau_k}$.

Here and henceforth, all limit relationships are for $x \to \infty$ unless stated otherwise and the symbol $\sim$ means that the quotient of both sides tends to 1. The relation $a(x) \sim b(x)$ means $\lim a(x)/b(x) = 1$, while the relation $a(x) \asymp b(x)$ stands for $0 < \liminf a(x)/b(x) \leq \limsup a(x)/b(x) < \infty$. The relation $\limsup a(x)/b(x) \leq 1$ is denoted by $a(x) \lesssim b(x)$.

A real-valued random variable X with distribution $F(x) = \mathbf{P}[X \leq x]$ is heavy tailed, symbolically $\overline{F} \in \mathcal{K}$, if for any $\varepsilon > 0$ the following relation holds

$$\mathbf{E}\left[e^{\varepsilon X}\right] = \int_{-\infty}^{\infty} e^{\varepsilon x}\, F(dx) = \infty.$$

A distribution F is long tailed, symbolically $\overline{F} \in \mathcal{L}$, if for any fixed $y \in \mathbb{R}$ the following relation holds

$$\lim \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1.$$

A distribution F concentrated on $\mathbb{R}_+ = [0, \infty)$ belongs to the subexponential class, symbolically $\overline{F} \in \mathcal{S}$, if for any integer $n \geq 2$ the following relation holds

$$\lim \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n,$$

where $F^{n*}(x)$ denotes the n-fold convolution of F. It is well known that the subexponential distributions are long tailed (see Chistyakov, 1964). More generally, a distribution F, defined on the whole real line $\mathbb{R}$, is called subexpoential if the function $F(x)\, \mathbf{1}_{\{x\in\mathbb{R}_+\}}$ is subexpenential, where $\mathbf{1}_A$ denotes the indicator function of A.

A distribution F belongs to the class of dominatedly-varying tails, symbolically $\overline{F} \in \mathcal{D}$, if for any $y \in (0, 1)$ the following relation holds

$$\limsup \frac{\overline{F}(x\,y)}{\overline{F}(x)} < \infty.$$

The intersection $\mathcal{B} = \mathcal{D} \cap \mathcal{L} = \mathcal{D} \cap \mathcal{S}$ represents a useful subclass of subexponential distributions (see Goldie, 1978).

A distribution F belongs to the class of consistently-varying tails, symbolically $\overline{F} \in \mathcal{C}$, if the following relation holds

$$\lim_{y \uparrow 1} \limsup \frac{\overline{F}(x\,y)}{\overline{F}(x)} = 1\,,$$

or equivalently the following holds

$$\lim_{y \downarrow 1} \limsup \frac{\overline{F}(x\,y)}{\overline{F}(x)} = 1\,.$$

A distribution F belongs to the class of extended regularly varying tails over the indices $(-\beta, -\alpha)$, symbolically $\overline{F} \in ERV(-\beta, -\alpha)$, with $0 \leq \beta \leq \alpha < \infty$ if for any $y \geq 1$ the following relation holds

$$y^{-\alpha} \leq \liminf \frac{\overline{F}(x\,y)}{\overline{F}(x)} \leq \limsup \frac{\overline{F}(x\,y)}{\overline{F}(x)} \leq y^{-\beta}\,.$$

A distribution F belongs to the class of regularly varying tails with index $-\alpha$, symbolically $\overline{F} \in \mathcal{R}_{-\alpha}$, with $\alpha > 0$ if for any $y > 0$ the following relation holds

$$\lim \frac{\overline{F}(x\,y)}{\overline{F}(x)} = y^{-\alpha}\,.$$

It is well known that

$$\mathcal{R}_{-\alpha} \subset ERV(-\beta, -\alpha) \subset \mathcal{C} \subset \mathcal{B} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{K}\,.$$

For a distribution F let us introduce the lower and upper Matuszewska indices (see Chapter 2.1 from (Bingham et al., 1987)) as follows

$$\beta_F = -\lim \frac{\ln M^F(x)}{\ln x}\,, \qquad \alpha_F = -\lim \frac{\ln M_F(x)}{\ln x}\,,$$

where for any $x > 0$ we denote

$$M^F(x) = \limsup_{u \to \infty} \frac{\overline{F}(x\,u)}{\overline{F}(u)}\,, \qquad M_F(x) = \limsup_{u \to \infty} \frac{\overline{F}(x\,u)}{\overline{F}(u)}\,,$$

If $\overline{F} \in ERV(-\beta, -\alpha)$ then $\beta \leq \beta_F \leq \alpha_F \leq \alpha$ and if $\overline{F} \in \mathcal{R}_{-\alpha}$ then $\beta_F = \alpha_F = \alpha$. By the Potter's inequalities (see Proposition 2.2.1 from (Binghan et al., 1987)) if $\overline{F} \in \mathcal{D}$ then for any $\varepsilon > 0$ we obtain $x^{-\alpha_F - \varepsilon} = o\left[\overline{F}(x)\right]$. If $\overline{F} \in \mathcal{D}$ then $\alpha_F < \infty$. To secure the inequality $\beta_F > 0$ we introduce the following class of extended regular variation.

For a subexponential distribution F we say that its tail $\overline{F}$ belongs to the class $\mathcal{A}$ if for every $v > 1$ the following holds

$$\limsup \frac{\overline{F}(v\,x)}{\overline{F}(x)} < 1\,.$$

If $\overline{F} \in \mathcal{A}$ then $\beta_F > 0$.

2. Dependence modelling

Let consider the sequence of real-valued random variables $\{X_i\,,\ i \in \mathbb{N}\}$. Following Definition 1.1 from the paper (Chen and Yuen, 2009) we say that the $\{X_i\,,\ i \in \mathbb{N}\}$ are pairwise quasi-asymptotically independent, symbolically $\{X\} \in pQAI$, if for any $i \neq j$ holds the limit

$$\lim \mathbf{P}[|X_i| \wedge X_j > x \,\big|\, X_i \vee X_j > 0] = 0\,.$$

Further following the work (Geluk and Tang, 2009) we say that the $\{X_i\,,\ i \in \mathbb{N}\}$ are tail asymptotically independent, symbolically $\{X\} \in TAI$ (or by some authors pairwise strong quasi-asymptotically independent $pSQAI$), if for any $i \neq j$ holds the limit

$$\lim_{x_i \wedge x_j \to \infty} \mathbf{P}[|X_i| > x_i \,\big|\, X_j > x_j] = 0\,.$$

We say that the $\{X_i\,,\ i \in \mathbb{N}\}$ are widely orthant dependent, symbolically $\{X\} \in WOD$ if there exist two finite real sequences $\{g_U(n)\}$ and $\{g_L(n)\}$ for $n \in \mathbb{N}$, such that for any real $x_k\,,\ k = 1, \ldots, n$ both

$$\mathbf{P}\left[\bigcap_{k=1}^{n}\{X_k \leq x_k\}\right] \leq g_L(n) \prod_{k=1}^{n} \mathbf{P}[X_k \leq x_k]\,,$$

$$\mathbf{P}\left[\bigcap_{k=1}^{n}\{X_k > x_k\}\right] \leq g_U(n) \prod_{k=1}^{n} \mathbf{P}[X_k > x_k]\,,$$

hold. This dependent structure was introduced in (Wang et al., 2003).

We say that the $\{X_i\,,\ i \in \mathbb{N}\}$ are extended negatively dependent, symbolically $\{X\} \in END$ if there exists some $M > 0$ such that for any $n \in \mathbb{N}$ and any $x_k\,,\ k = 1, \ldots, n$ both

$$\mathbf{P}\left[\bigcap_{k=1}^{n}\{X_k \leq x_k\}\right] \leq M \prod_{k=1}^{n} \mathbf{P}[X_k \leq x_k]\,,$$

$$\mathbf{P}\left[\bigcap_{k=1}^{n}\{X_k > x_k\}\right] \leq M \prod_{k=1}^{n} \mathbf{P}[X_k > x_k]\,,$$

hold. This notion was introduced in (Liu, 2009).

When in these two relations the value of the constant is $M = 1$ then we say that the $\{X_i\,,\ i \in \mathbb{N}\}$ are negatively quadrant dependent, symbolically $\{X\} \in NQD$ (or by some authors negatively orthant dependent NOD or simply negatively dependent ND).

It is well known the inclusions

$$NQD \subset END \subset WOD \subset TAI \subset pQAI\,.$$

Now, we study the asymptotic behaviour of the distribution tail of the discounted sums in (1.1). By the total probability formula we obtain

$$\mathbf{P}\left(\sum_{k=1}^{N(\Lambda\,\theta_1)} Y_k e^{\delta(\theta_1 - \sigma_k)} > x\right)$$

$$= \int_0^\infty \int_0^\infty \mathbf{P}\left(\sum_{k=1}^{N(qt)} Y_k e^{\delta(t-\sigma_k)} > x \,\middle|\, \theta_1 = t,\ \Lambda = q \right) Q(dq)\, A(dt)$$

$$= \int_0^\infty \int_0^\infty \sum_{n=1}^{\infty} \mathbf{P}\left(\sum_{k=1}^{n} Y_k e^{\delta(t-\sigma_k)} > x \,\middle|\, \theta_1 = t,\ \Lambda = q,\ N(qt) = n \right) \mathbf{P}(N(qt) = n)$$

$$\times Q(dq)\, A(dt)\,.$$

Next, we employ Theorem 2.3.1 from (Ross, 1983) to express the conditional distribution of the random vector $(t - \sigma_1, \ldots, t - \sigma_n)$, given that $N(qt) = n$, as distribution of the random vector $(tU_{(1,n)}, \ldots, tU_{(n,n)})$, where by $U_{(1,n)}, \ldots, U_{(n,n)}$ denote the order statistics of the n uniformly distributed over the interval [0, 1] random variables $U_1, \ldots, U_n$ $(U_{(1,n)} \geq \cdots \geq U_{(n,n)})$. Furthermore, since in the sum $\sum_{k=1}^{n} Y_k\, e^{\delta t U_{(k,n)}}$ the vector $(Y_1, \ldots, Y_n)$ consists of i.i.d. random variables and is independent of $(U_{(1,n)}, \ldots, U_{(n,n)})$, by rearrangement this sum is equal in distribution to the sum $\sum_{k=1}^{n} Y_k\, e^{\delta t U_k}$ with U_k representing uniformly distributed random variables over the interval [0, 1], symbolically $U_k \sim U[0, 1]$,

$$\mathbf{P}\left(\sum_{k=1}^{N(\Lambda\,\theta_1)} Y_k e^{\delta(\theta_1-\sigma_k)} > x \right)$$

$$= \int_0^\infty \int_0^\infty \sum_{n=1}^{\infty} \mathbf{P}\left(\sum_{k=1}^{n} Y_k e^{\delta t U_k} > x \,\middle|\, \theta_1 = t,\ \Lambda = q,\ N(qt) = n \right) \mathbf{P}(N(qt) = n)$$

$$\times Q(dq)\, A(dt) = \int_0^\infty \mathbf{P}\left(\sum_{k=1}^{N(\Lambda t)} Y_k e^{\delta t U_k} > x \,\middle|\, \theta_1 = t \right) A(dt)\,.$$

Let us denote $\mu_t := \mathbf{E}\left[Y\, e^{\delta t U}\right]$ and $\Lambda_t := \Lambda\, t$. Next we use Theorem 4.1 (b) from (Chen et al., 2010) (see further (Schmidli, 1999) and Theorem 3.1 from (Robert and Segers, 2008)) to have the following result:

Lemma 2.1. *If the random variables* $\{Y_k\,,\ k \geq 1\}$ *is a sequence of END random variables with common distribution G, mean value* $\mu > 0$ *and finite exponential moment* $\mathbf{E}[e^{\gamma Y}] < \infty$ *for some* $\gamma > 0$ *and the distribution of* $\Lambda\ Q$ *has regularly varying tail* $\overline{Q} \in C$*, for some* $\alpha > 0$*, then holds the following relation*

$$\mathbf{P}\left(\sum_{k=1}^{N(\Lambda_t)} Y_k e^{\delta t U_k} > x \right) \sim \mathbf{P}\left(N(\Lambda_t)\, \mu_t > x\right)\,,$$

for any $t \in (0,\ \infty)$ *and with* $U_k \sim U[0,\ 1]$ *for any* $k \in \mathbb{N}$.

Proof. We check the conditions of Theorem 4.1 (b) from (Chen et al., 2010). As far the uniform random variables U_k are bounded and the random premiums Y_k are END we see that the products $Y_k\, e^{\delta t U_k}$ are also END. Indeed, by the fact that $e^{\delta t U_k} \geq 1$ and using Lemma 2.2 from (Chen et al., 2010) we have that for any fixed values of $U_k\,,\ k = 1, \ldots, n$ the products are also END. Applying a total probability argument we obtain the case.

From the fact that the products are non-negative and non-degenerate, we obtain the positive mean value $\mu_t > 0$. Further, as far the Y is light tailed, follows that there is some $\varepsilon \in (0, \gamma)$ such that $\mathbf{E}\left[Y^{\varepsilon}\, e^{\varepsilon \delta t U}\right] < \infty$.

Next, from the fact that $\overline{Q} \in \mathcal{C}$ we obtain that the distribution of $N(\Lambda_t)$ is consistently varying. Indeed, by the notation $N^{\leftarrow}(x) := \inf\{z\ :\ N(z) \geq x\}$ we have $N^{\leftarrow}(0) = 0$, $N^{\leftarrow}(\infty) = \infty$ and for any $y \in (0,\ 1)$ the asymptotic relation

$$\frac{N^{\leftarrow}(y\, x)}{N^{\leftarrow}(x)} \to y\,,$$

from where we get

$$\begin{aligned}\lim_{y\uparrow 1}\limsup \frac{\mathbf{P}[N(\Lambda_t) > y\, x]}{\mathbf{P}[N(\Lambda_t) > x]} &= \lim_{y\uparrow 1}\limsup \frac{\mathbf{P}[\Lambda > N^{\leftarrow}(y\, x)/t]}{\mathbf{P}[\Lambda > N^{\leftarrow}(x)/t]} \\ &= \lim_{y\uparrow 1}\limsup \frac{\mathbf{P}[\Lambda > y\, N^{\leftarrow}(x)/t]}{\mathbf{P}[\Lambda > N^{\leftarrow}(x)/t]} = 1\,,\end{aligned}$$

where the last equality comes from $\overline{Q} \in \mathcal{C}$. Hence, $\mathbf{P}[N(\Lambda_t) > x] \in \mathcal{C}$, but its mean value is finite $\mathbf{E}[N(\Lambda_t)] < \infty$ for any $t \in (0, \infty)$. Finally, from the fact that the distribution of Y is light tailed follows that

$$\mathbf{P}\left(Y e^{\delta t U} > x\right) = o\left(\mathbf{P}\left[N(\Lambda_t) > x\right]\right)\,.$$

Now we just apply Theorem 4.1 (b) from (Chen et al., 2010) to take the required result. □

We observe that $\mu_t < \infty$ and the distribution of Λ_t has a regularly varing tail with index $-\alpha$, exactly as the random variable Λ. We also consider successive epochs $\{\sigma'_k\,,\ k \geq 1\}$ with $\sigma'_0 = 0$ of the Poisson point process $N(t)$, with the corresponding inter-arrival times $\{\zeta'_k\,,\ k \geq 1\}$, where $\zeta'_k = \sigma'_k - \sigma'_{k-1}$.

Lemma 2.2. *In addition to the other conditions of Lemma 2.1, if* $\overline{Q} \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0$*, then holds the relation*

$$\mathbf{P}\left(\sum_{k=1}^{N(\Lambda_t)} Y_k e^{\delta t U_k} > x\right) \sim \mathbf{P}\left(\Lambda_t > \frac{x}{\mu_t}\right)\,,$$

for any $t \in (0, \infty)$ *and with and* $U_k \sim U[0,\ 1]$ *for any* $k \in \mathbb{N}$.

Proof. Following the expression found in Lemma 2.1, for any $\varepsilon > 0$ we can write

$$\begin{aligned}&\mathbf{P}\left(N(\Lambda_t) > \frac{x}{\mu_t}\right) = \int_0^{\infty} \mathbf{P}\left(N(q) > \frac{x}{\mu_t}\,\Big|\, \Lambda_t = q\right) \mathbf{P}(\Lambda_t \in dq) \\ &= \left(\int_0^{x/(\mu_t+\varepsilon)} + \int_{x/(\mu_t+\varepsilon)}^{x/(\mu_t-\varepsilon)} + \int_{x/(\mu_t-\varepsilon)}^{\infty}\right) \mathbf{P}\left(N(q) > \frac{x}{\mu_t}\,\Big|\, \Lambda_t = q\right) \mathbf{P}(\Lambda_t \in dq) \\ &= I_1 + I_2 + I_3\,.\end{aligned}$$

Let us observe that the main term is the last one. Indeed, taking into account the SLLN we obtain the convergence $N(t)/t \stackrel{a.s.}{\to} 1$. So we can write

$$I_3 = \int_{x/(\mu_t-\varepsilon)}^{\infty} \mathbf{P}\left(N(q) > \frac{x}{\mu_t}\,\Big|\, \Lambda_t = q\right) \mathbf{P}(\Lambda_t \in dq)$$

$$\geq \int_{x/(\mu_t-\varepsilon)}^{\infty} \mathbf{P}\left(\frac{N(q)}{q} > \frac{\mu_t-\varepsilon}{\mu_t} \,\Big|\, \Lambda_t = q\right) \mathbf{P}(\Lambda_t \in dq)$$

$$\to \mathbf{P}\left(\Lambda_t > \frac{x}{\mu_t-\varepsilon}\right),$$

as $t \to \infty$. From the other side, the upper bound or the probability gives the same

$$I_3 = \int_{x/(\mu_t-\varepsilon)}^{\infty} \mathbf{P}\left(N(q) > \frac{x}{\mu_t} \,\Big|\, \Lambda_t = q\right) \mathbf{P}(\Lambda_t \in dq) \leq \mathbf{P}\left(\Lambda_t > \frac{x}{\mu_t-\varepsilon}\right),$$

so after leaving the ε to tend to zero we finally obtain

$$I_3 \sim \mathbf{P}\left(\Lambda_t > \frac{x}{\mu_t}\right).$$

Next, we calculate the asymptotics of I_2

$$I_2 = \int_{x/(\mu_t+\varepsilon)}^{x/(\mu_t-\varepsilon)} \mathbf{P}\left(N(q) > \frac{x}{\mu_t} \,\Big|\, \Lambda_t = q\right) \mathbf{P}(\Lambda_t \in dq)$$

$$\leq \mathbf{P}\left(\frac{x}{\mu_t+\varepsilon} \leq \Lambda_t \leq \frac{x}{\mu_t-\varepsilon}\right) = \mathbf{P}\left(\Lambda_t \leq \frac{x}{\mu_t-\varepsilon}\right) - \mathbf{P}\left(\Lambda_t \leq \frac{x}{\mu_t+\varepsilon}\right)$$

$$\sim [(\mu_t+\varepsilon)^{\alpha} - (\mu_t-\varepsilon)^{\alpha}]\mathbf{P}(\Lambda_t > x) = o\left[\mathbf{P}\left(\Lambda_t > \frac{x}{\mu_t}\right)\right],$$

as $\varepsilon \to 0$. So the second term is negligible.

Next, we consider I_1. We remind the well-known relation $\{N(t) > x\} = \left\{\sigma'_{\lfloor x \rfloor} \leq t\right\}$, so we can write

$$I_1 = \int_0^{x/(\mu_t+\varepsilon)} \mathbf{P}\left(N(q) > \frac{x}{\mu_t} \,\Big|\, \Lambda_t = q\right) \mathbf{P}(\Lambda_t \in dq)$$

$$\leq \mathbf{P}\left[N\left(\frac{x}{\mu_t+\varepsilon}\right) > \frac{x}{\mu_t}\right] = \mathbf{P}\left[\sum_{i=1}^{\lfloor x/\mu_t \rfloor} \zeta'_i \leq \frac{x}{\mu_t+\varepsilon}\right].$$

Now for an arbitrarily chosen variable $h > 0$ we apply standard Chernoff inequality

$$I_1 \leq \exp\left\{h\,\frac{x}{\mu_t+\varepsilon}\right\} \mathbf{E}\left[\exp\left\{-h \sum_{i=1}^{\lfloor x/\mu_t \rfloor} \zeta'_i\right\}\right]$$

$$\sim \exp\left\{h\,\frac{x}{\mu_t+\varepsilon}\right\} \left(\mathbf{E}\left[e^{-h\zeta'_1}\right]\right)^{x/\mu_t} \sim \exp\left\{\left(h\,\frac{\mu_t}{\mu_t+\varepsilon} + \ln \mathbf{E}\left[e^{-h\zeta'_1}\right]\right) \frac{x}{\mu_t}\right\}.$$

Now we choose some positive value for h such that the expression

$$v(h) := h\,\frac{\mu_t}{\mu_t+\varepsilon} + \ln \mathbf{E}\left(e^{-h\zeta'_1}\right),$$

becomes negative. This is possible because for $h = 0$ we obtain $v(0) = 0$ and its first derivative becomes negative for small enough h

$$v'(h) := \frac{\mu_t}{\mu_t + \varepsilon} - \frac{\mathbf{E}\left(\zeta_1'\, e^{-h\,\zeta_1'}\right)}{\mathbf{E}\left(e^{-h\,\zeta_1'}\right)},$$

due to the fact that $\mathbf{E}[\zeta_1'] = 1$ by definition of the process $N(t)$. Therefore the term I_1 decays with exponential speed

$$I_1 \sim \exp\left\{v(h)\,\frac{x}{\mu_t}\right\} = o\left[\mathbf{P}\left(\Lambda_t > \frac{x}{\mu_t}\right)\right],$$

which makes the first term also negligible. □

3. Ruin probability in infinite horizon

Next, consider the case with regular varying tails of distributions of the random variables Z and Λ with the same parameter $-\alpha$, symbolically $\overline{B}$, $\overline{Q} \in \mathcal{R}_{-\alpha}$ and we examine the tail of the distribution F

$$\overline{F}(x) = \mathbf{P}[X > x] = \mathbf{P}\left(Z - \sum_{k=1}^{N(\Lambda_\theta)} Y_k^*\, e^{\delta(\theta-\sigma_k)} > x\right), \tag{3.1}$$

From Theorem 3.1 in (Tang and Tsitsiashvili, 2003) we can find easily:

Lemma 3.1. *If* $\overline{F} \in \mathcal{D} \cap \mathcal{A}$, *then*

$$\overline{F}(x) = o(x^{-\beta}), \qquad \forall\, \beta < \beta_F\,,$$

$$x^{\alpha}\overline{F}(x) \to \infty, \qquad \forall\, \alpha > \alpha_F\,,$$

$$0 \le \beta_F \le \alpha_F < \infty,$$

hold.

Now we assume that the joint distribution of (Λ, Z) follows a multivariate regular variation with parameter α and measure ν. This means that there exist some $0 < \alpha < \infty$, some distribution function B with $\overline{B} \in \mathcal{R}_{-\alpha}$, and some Radon measure ν on $[0,\infty]^d \setminus \{\mathbf{0}\}$ satisfying $\nu\left([0,\infty]^d \setminus \{\mathbf{0}\}\right) > 0$ such that the following vague convergence holds:

$$\frac{1}{\overline{B}(x)}\mathbb{P}\left(\frac{(\Lambda, Z)}{x} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \qquad \text{on } [0,\infty]^d \setminus \{\mathbf{0}\}.$$

In this case, we write $(\Lambda, Z) \in \mathrm{MRV}\,(\alpha, B, \nu)$.

We introduce now the event

$$A_{x,t} := \{(Z,\ \Lambda)\ :\ Z - \Lambda_t\,\mu_t > x\} = \left\{(Z,\ \Lambda)\ :\ Z - \Lambda\, t\, \mathbf{E}\left[Y_1\, e^{\delta t U_1}\right] > x\right\},$$

for any $x > 0$.

Lemma 3.2. *The following asymptotic relation is true*

$$\overline{F}(x) \sim \overline{B}(x) E\left[\nu(A_{1,\theta})\right] .$$

Proof. Through Lemma 3.1 we find that $\overline{F}(x)$, given in (3.1), has the following asymptotics

$$\begin{aligned}
\overline{F}(x) &= \int_0^\infty P\left[\sum_{m=1}^{N(\Lambda_\theta)} Y_m^* \, e^{\delta(\theta-\sigma_m)} < z - x\right] B(dz) \\
&= \int_0^\infty \left(1 - \int_0^\infty P\left[\sum_{m=1}^{N(\Lambda_t)} Y_m^* \, e^{\delta\, t\, U_m} \geq z - x \,\middle|\, \theta = t\right] A(dt)\right) B(dz) \\
&\sim \int_0^\infty \left(1 - \int_0^\infty P\left[\Lambda_t \, \mu_t \geq z - x \,\middle|\, \theta = t\right] A(dt)\right) B(dz) \\
&= \int_0^\infty P\left[\Lambda_t \, \mu_t < z - x \,\middle|\, \theta = t\right] A(dt)\, B(dz) \\
&= \int_0^\infty P\left[Z - \Lambda_t \, \mu_t > x \,\middle|\, \theta = t\right] A(dt) .
\end{aligned}$$

Now we employ the multivariate regular variation of the $(Z,\ \Lambda) \in \text{MRV}\,(\alpha, B, \nu)$ to find

$$\begin{aligned}
\overline{F}(x) &= \int_0^\infty P\left[\sum_{m=1}^{\Lambda_\theta} Y_m^* \, e^{\delta(\theta-\sigma_m)} < z - x\right] B(dz) \\
&\sim \int_0^\infty \overline{B}(x)\, \nu\left[A_{1,t}\right] \, A(dt) = \overline{B}(x)\, \mathbf{E}\left(\nu\left[A_{1,\theta}\right]\right) .
\end{aligned}$$

□

Proposition 3.3. *Let the real-valued random variables* $\{X_n\,,\quad n \;=\; 1, 2, \ldots\}$ *be pairwise quasi-asymptotically independent (pQAI) with common distribution* $F(x)$ *with tail* $\overline{F} \in \mathcal{C} \subset \mathcal{D} \cap \mathcal{A}$*, and independent from the random variables* $\{\tau_n\,,\ n = 1,\ 2, \ldots\}$*. Then the asymptotic relation*

$$\psi_\delta(x) \sim \sum_{n=1}^{\infty} \mathbf{P}[e^{-\delta\tau_n} X_n > x] ,$$

holds if either of the following conditions are true:

(i) If $\alpha_F \in (0,\ 1)$ *then for any* $\beta \in (0,\ \beta_F)$ *and for any* $\alpha \in (\alpha_F,\ 1)$ *converges the sum*

$$\sum_{n=1}^{\infty} (\mathbf{E}[e^{-\alpha\,\delta\,\tau_n}] + \mathbf{E}[e^{-\beta\,\delta\,\tau_n}]) < \infty .$$

(ii) If $\alpha_F \geq 1$ *then for any* $\beta \in (0,\ \beta_F)$ *and for any* $\alpha > \alpha_F$ *converges the sum*

$$\sum_{n=1}^{\infty} (\mathbf{E}[e^{-\alpha\,\delta\,\tau_n}] + \mathbf{E}[e^{-\beta\,\delta\,\tau_n}])^{1/\alpha} < \infty .$$

Proof. We follow the argument developed in Theorem 2 from (Yi et al., 2011). However, we omit the condition $F(-x) = o[\overline{F}(x)]$, inspired by Theorem 2.1 from (Ignataviciute et al., 2018).

We begin with the lower asymptotic bound. For any $m \in \mathbb{N}$, under condition (i) we find

$$\begin{aligned}
\psi_\delta(x) &= \mathbf{P}[\sup_{n\geq 1} \sum_{k=1}^{n} e^{-\delta\tau_k} X_k > x] \geq \frac{\mathbf{P}[\sup_{1\leq n\leq m} \sum_{k=1}^{n} e^{-\delta\tau_k} X_k > x]}{\sum_{k=1}^{m} \mathbf{P}[e^{-\delta\tau_k} X_k > x]} \sum_{k=1}^{m} \mathbf{P}[e^{-\delta\tau_k} X_k > x] \\
&\gtrsim \liminf_{x\to\infty} \sum_{k=1}^{m} \mathbf{P}[e^{-\delta\tau_k} X_k > x] \\
&= \liminf_{x\to\infty} \left(\sum_{k=1}^{\infty} \mathbf{P}[e^{-\delta\tau_k} X_k > x] - \sum_{k=m+1}^{\infty} \mathbf{P}[e^{-\delta\tau_k} X_k > x] \right) \\
&= \sum_{k=1}^{\infty} \mathbf{P}[e^{-\delta\tau_k} X_k > x] \left(1 - \limsup_{x\to\infty} \frac{\sum_{k=m+1}^{\infty} \mathbf{P}[e^{-\delta\tau_k} X_k > x]}{\sum_{k=1}^{\infty} \mathbf{P}[e^{-\delta\tau_k} X_k > x]} \right),
\end{aligned}$$

where in the second line we used Theorem 2.1 from (Ignataviciute et al., 2018) in combination with Theorem 1 from (Yi et al., 2011). Further by Theorem 3.3 from (Cline and Samorodnitksy, 1994) we have $\mathbf{P}[e^{-\delta\tau_k} X_k > x] \asymp \mathbf{P}[X_k > x]$ and by Lemma 1 from (Yi et al., 2011) we find $\mathbf{P}[e^{-\delta\tau_k} X_k > x] \leq C\,(\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}])\,\overline{F}(x)$. Hence we apply in the last inequality to obtain

$$\psi_\delta(x) \gtrsim \sum_{k=1}^{\infty} \mathbf{P}[e^{-\delta\tau_k} X_k > x] \left(1 - C \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}]) \right).$$

Next, letting m to tend to infinity, from condition (i), we have the lower asymptotic bound.

For the upper asymptotic bound we see that for any $m \in \mathbb{N}$ and $v \in (0,\ 1)$ is true the inequality

$$\psi_\delta(x) \leq \mathbf{P}\left[\sup_{1\leq n\leq m} \sum_{k=1}^{n} e^{-\delta\tau_k} X_k > (1-v)\,x \right] + \mathbf{P}\left[\sum_{k=m+1}^{\infty} e^{-\delta\tau_k} X_k^+ > v\,x \right] = P_1 + P_2\,.$$

For the first term we find

$$\begin{aligned}
P_1 &\lesssim \limsup \frac{\mathbf{P}\left[\sup_{1\leq n\leq m} \sum_{k=1}^{n} e^{-\delta\tau_k} X_k > (1-v)\,x\right]}{\sum_{k=1}^{m} \mathbf{P}\left[e^{-\delta\tau_k} X_k > (1-v)\,x\right]} \frac{\sum_{k=1}^{m} \mathbf{P}\left[e^{-\delta\tau_k} X_k > (1-v)\,x\right]}{\sum_{k=1}^{m} \mathbf{P}\left[e^{-\delta\tau_k} X_k > x\right]} \\
&\quad \times \sum_{k=1}^{m} \mathbf{P}\left[e^{-\delta\tau_k} X_k > x\right] \leq M^F(1-v) \sum_{k=1}^{\infty} \mathbf{P}\left[e^{-\delta\tau_k} X_k > x\right].
\end{aligned}$$

For the second term we can obtain

$$P_2 \lesssim C\, M^{F_1}(v) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}]) \sum_{k=1}^{\infty} \mathbf{P}\left[e^{-\delta\tau_k} X_k > x\right].$$

Indeed, from the elementary inequality $|a+b|^r \leq |a|^r + |b|^r$ for any $r \in (0,\ 1)$ and any $a,\, b \in \mathbb{R}$, we can see due to Lemma 1 and Lemma 2 from (Yi et al., 2011)

$$P_2 \leq \sum_{k=m+1}^{\infty} \mathbf{P}\left[e^{-\delta\tau_k} X_k^+ > v\,x\right] + \mathbf{P}\left[\sum_{k=m+1}^{\infty} e^{-\delta\tau_k} X_k^+ \mathbf{1}_{\{e^{-\delta\tau_k} X_k^+ \leq v\,x\}} > v\,x\right]$$

$$\leq C_1\,\overline{F}(vx) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}])$$

$$+\frac{1}{(v\,x)^{\alpha}} \left(\mathbf{E}\left[\sum_{k=m+1}^{\infty} e^{-\delta\tau_k} X_k^+ \mathbf{1}_{\{e^{-\delta\tau_k} X_k^+ \leq v\,x\}}\right]\right)^{\alpha}$$

$$\leq C_1\,\overline{F}(vx) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}])$$

$$+\frac{1}{(v\,x)^{\alpha}} \sum_{k=m+1}^{\infty} \mathbf{E}\left[\left(e^{-\delta\tau_k} X_k^+ \mathbf{1}_{\{e^{-\delta\tau_k} X_k^{\leq} v\,x\}}\right)^{\alpha}\right]$$

$$\leq C_1\,\overline{F}(vx) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}]) + C_2 \sum_{k=m+1}^{\infty} \mathbf{P}\left[e^{-\delta\tau_k} X_k > v\,x\right]$$

$$\leq (C_1 + C_2)\,\overline{F}(vx) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}])\,.$$

Hence, using again the weak equivalence $\mathbf{P}[e^{-\delta\tau_k} X_k > x] \asymp \mathbf{P}[X_k > x]$ we get

$$P_2 \lesssim (C_1 + C_2)\,M^F(v) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}]) \sum_{k=1}^{\infty} \mathbf{P}\left[e^{-\delta\tau_k} X_k > x\right].$$

After substitution we have

$$\psi_\delta(x) \lesssim \left(M^F(1-v) + (C_1 + C_2)\,M^F(v) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}])\right)$$

$$\times \sum_{k=1}^{\infty} \mathbf{P}\left[e^{-\delta\tau_k} X_k > x\right].$$

Now letting $m \to \infty$ and $v \downarrow 0$ we get the lower asymptotic bound.

Under the condition (*ii*) we apply Minkowski inequality in evaluation of P_2

$$P_2 \leq C_1\,\overline{F}(vx) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}])$$

$$+\frac{1}{(v\,x)^{\alpha}} \left(\mathbf{E}\left[\sum_{k=m+1}^{\infty} e^{-\delta\tau_k} X_k^+ \mathbf{1}_{\{e^{-\delta\tau_k} X_k^+ \leq v\,x\}}\right]\right)^{\alpha}$$

$$\leq C_1\,\overline{F}(vx) \sum_{k=m+1}^{\infty} (\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}] \vee \mathbf{E}[e^{-\beta_F\,\delta\tau_k}])$$

$$+\frac{1}{(v\,x)^{\alpha}}\sum_{k=m+1}^{\infty}\mathbf{E}\left[\left(e^{-\delta\tau_k}\,X_k^{+}\,\mathbf{1}_{\{e^{-\delta\tau_k}\,X_k^{\le}v\,x\}}\right)^{\alpha}\right]$$

$$\le C_1\,\overline{F}(vx)\sum_{k=m+1}^{\infty}(\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}]\vee\mathbf{E}[e^{-\beta_F\,\delta\tau_k}])+C_2\sum_{k=m+1}^{\infty}\mathbf{P}\left[e^{-\delta\tau_k}\,X_k>v\,x\right]$$

$$\le(C_1+C_2)\,\overline{F}(vx)\sum_{k=m+1}^{\infty}(\mathbf{E}[e^{-\alpha_F\,\delta\tau_k}]\vee\mathbf{E}[e^{-\beta_F\,\delta\tau_k}])\,.$$

□

Remark 3.4. We observe that given that $\{\tau_n\,,\ n\ge 1\}$ is a renewal sequence, conditions (*i*) and (*ii*) in Proposition 3.3 are fulfilled. Indeed, in this case we can write

$$\mathbf{E}\left[e^{-\alpha\,\delta\,\tau_n}\right]=\left(\mathbf{E}\left[e^{-\alpha\,\delta\,\theta_1}\right]\right)^n\,,\qquad \mathbf{E}\left[e^{-\beta\,\delta\,\tau_n}\right]=\left(\mathbf{E}\left[e^{-\beta\,\delta\,\theta_1}\right]\right)^n\,,$$

and taking into account that $\mathbf{E}\left[e^{-\alpha\,\delta\,\theta_1}\right]<1$ and $\mathbf{E}\left[e^{-\beta\,\delta\,\theta_1}\right]<1$ we get that the geometric series converge automatically.

Hence, using the property of class $\mathcal{L}$, we obtain the following simplification of conditions in Proposition 3.3.

Corollary 3.5. *If the sequence $\{\tau_n\}$ represents a renewal point process and there exists a constant $C<\infty$ such that the inequality*

$$\sum_{k=1}^{N(\Lambda\,\theta_n)} e^{-\delta\,\sigma_k}\,Y_k^{*}\le C\,, \tag{3.2}$$

holds almost surely, and the positive random variables $\{Z_n\,,\ \ n\ =\ 1,2,\ldots\}$ be pairwise quasi-asymptotically independent (pQAI) with common distribution $\overline{B}\in\mathcal{C}$ then the following asymptotic relation is true

$$\psi_\delta(x)\sim\sum_{n=1}^{\infty}\mathbf{P}[e^{-\delta\tau_n}\,Z_n>x]\,. \tag{3.3}$$

Proof. Since we have the sequence $\{\tau_n\}$ is renewal we can write

$$e^{-\delta\,\tau_n}\,X_n=e^{-\delta\,\tau_n}\,Z_n-\sum_{k=1}^{N(\Lambda\,\theta_n)} e^{-\delta\,\sigma_k}\,Y_k^{*}\,,$$

Therefore, from the condition $\overline{B}\in\mathcal{C}$ and the inequality (3.2) we find that $\overline{F}\in\mathcal{C}$.

Now from the double inequality $Z_n-C\,\prod_{k=1}^{n}e^{\delta\,\theta_n}\le X_n\le Z_n$ we find that the sequence $\{X_n\,,\ n\ge 1\}$ is also pQAI with common distribution $\overline{F}\in\mathcal{C}$. So we can apply the Proposition 3.3 to obtain (3.3). □

Now, we are ready to provide the final asymptotic expression for the ruin probability $\psi(x)$.

Theorem 3.6. *Let the random variables* $\{Y_k\,,\ k \geq 1\}$ *be a sequence of END random variables with common distribution G, mean value* $\mu > 0$ *and finite exponential moment* $\mathbf{E}[e^{\gamma Y_1}] < \infty$ *for some* $\gamma > 0$ *and the distribution of* Λ *be of regularly varying tail* $\overline{Q} \in \mathcal{R}_{-\alpha}$ *with* $\alpha > \alpha_F$*. If* $\{X_k\,,\ k \geq 1\}$ *is independent of* $\{Y_k\,,\ k \geq 1\}$ *and satisfies the conditions of Proposition 3.3, then holds the relation*

$$\psi_\delta(x) \sim \frac{\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}{1-\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}\,\mathbf{E}\left[\nu\left(A_{1,\theta}\right)\right]\,\overline{B}(x)\,. \tag{3.4}$$

Proof. From the formulas (1.2) and (1.3) we conclude that

$$x - \sum_{n=1}^{\infty} X_n\, e^{-\delta\,\tau_n} \leq \widetilde{U}_\delta(t) \leq x - \sum_{n=1}^{\infty} X_n\, e^{-\delta\,\tau_n}\, \mathbf{1}_{\{\tau_n \leq t\}}\,, \tag{3.5}$$

and further taking into account the regular variation of the distribution $\overline{F} \in \mathcal{R}_{-\alpha}$, applying Theorem 2.1 from (Resnick and Willekens, 1991) (or Lemma 1 from (Tang, 2005)) we obtain

$$\psi_\delta(x) \leq P\left[\sum_{n=1}^{\infty} X_n\, e^{-\delta\tau_n} > x\right] = \overline{F}(x)\,\sum_{n=1}^{\infty} \mathbf{E}\left[X_n\, e^{-\alpha\delta\tau_n}\right] = \overline{F}(x)\,\frac{\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}{1-\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}\,. \tag{3.6}$$

Next, for the lower bound we find

$$\begin{aligned}\psi_\delta(x) &\geq P\left[\sup_{t\geq 0}\sum_{n=1}^{\infty} X_n\, e^{-\delta\tau_n}\,\mathbf{1}_{\{\tau_n\leq t\}} > x\right] = P\left[\sum_{n=1}^{\infty} X_n\, e^{-\delta\tau_n} > x\right]\\ &\sim \overline{F}(x)\,\sum_{n=1}^{\infty}\mathbf{E}\left[e^{-\alpha\delta\tau_n}\right] = \overline{F}(x)\,\frac{\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}{1-\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}\,.\end{aligned}$$

So with combination of the previous bounds we have

$$\psi_\delta(x) \sim \overline{F}(x)\,\frac{\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}{1-\mathbf{E}\left[e^{-\alpha\delta\theta}\right]}\,. \tag{3.7}$$

Finally after substitution from Lemma 3.2 we conclude the result. □

Remark 3.7. *For* $\nu\left[A_{1,\theta}\right] = 1$ *we find the asymptotic formula from Theorem 1 in (Tang, 2005).*

Acknowledgments

I would like to thank Prof. S. Kou for the suggestion of this topic and to two anonymous referees for their several constructive comments.

Conflict of interest

The author declares no conflict of interest.

References

Bingham NH, Goldie CM, Teugels JL (1987) *Regular Variation*, New York: Cambridge University Press.

Chen Y, Chen A, Ng KW (2010) The strong law of large numbers for extended negatively dependent random variables. *J Appl Probab* 46: 908–922.

Chen Y, Yuen KC (2009) Sums of pairwise quasi-asymptotically independent random variables with consistent variation. *Stoch Models* 25: 76–89.

Chistyakov VP (1964) A theorem on sums of independent positive random variables and its applications to branching random processes. *Theory Probab Appl* 9: 640–648.

Cline DBH, Samorodnitksy G (1994) Subexponentiality of the product of independent random variables. *Stochast Proc Appl* 49: 75–98.

Ignataviciute E, Ragulina O, Siaulys J (2018) Expectation of the truncated randomly weighted sums with dominatingly varying summands. *Lithuanian Math J* (submitted).

Gao Q, Wang Y (2010) Randomly weighted sums with dominated varying-tailed increments and application to risk theory. *J Korean Stat Society* 39: 305–314.

Geluk J, Tang Q (2009) Asymptotic tail probabilities of sums of dependent subexponential random variables. *J Teoret Probab* 22: 871–882.

Goldie CM (1978) Subexponential distributions and dominated-variation tails. *J Appl Probab* 15: 440–442.

Konstantinides DG (2017) *Risk Theory: A Heavy Tail Approach.*, New Jersey: World Scientific.

Liu L (2009) Precise large deviations for dependent random variables with heavy tails. *Stat Probab Lett* 79: 1290–1298.

Nelsen RB (2006) *An Introduction to Copulas*, Berlin: Springer.

Resnick SI, Willekens E (1991) Moving averages with random coefficients and random coefficient autoregressive models. *Stoch Models* 7: 511–525.

Robert CY, Segers JH (2008) Tails of random sums of a heavy-tailed number of light-tailed terms. *Insur Math Econ* 43: 85–92.

Ross SM (1983) *Stochastic Processes*, New York: Wiley.

Schmidli H (1999) Compound sums and subexponentiality. *Bernoulli* 5: 999–1012.

Tang QH (2005) Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation. *Scand Actuar J* 1: 1–5.

Tang QH, Tsitsiashvili G (2003a) Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stochastic Process Appl* 108: 299–325.

Tang QH, Tsitsiashvili G (2003b) Randomly weighted sums of subexponential random variables with application to ruin theory. *Extremes* 6: 171–188.

Wang KY, Wang YB, Gao QW (2003) Uniform asymptotics for the finte-time ruin probability of a dependent risk model with a constant interest rate. *Methodol Comput Appl Probab* 15: 109–124.

Yi L, Chen Y, Su C (2011) Approximation of the tail probability of randomly weighted sums of dependent random variables with dominated variation. *J Math Anal Appl* 376: 365–372.

AIMS Press