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Research article

Stochastic interest model driven by compound Poisson process and Brownian motion with applications in life contingencies

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Abstract: In this paper, we introduce a class of stochastic interest model driven by a compound Poisson process and a Brownian motion, in which the jumping times of force of interest obeys compound Poisson process and the continuous tiny fluctuations are described by Brownian motion, and the adjustment in each jump of interest force is assumed to be random. Based on the proposed interest model, we discuss the expected discounted function, the validity of the model and actuarial present values of life annuities and life insurances under different parameters and distribution settings. Our numerical results show actuarial values could be sensitive to the parameters and distribution settings, which shows the importance of introducing this kind interest model.

Keywords: compound Poisson process; Brownian motion; force of interest; expected discounted function; life annuity; actuarial present values
JEL classification numbers: G22

1. Introduction

The randomness of interest rates deeply influences the accuracy of actuarial values of life insurances and life annuities, especially for long-term policies. Hence it is important to consider stochastic interest models in actuarial science. So far, there are numerous papers which have investigated stochastic interest models and their applications in actuarial science. Bellhouse and Panjer Bellhouse and Panjer (1980, 1981) computed the moments of insurance and annuity functions based on AR(1) stochastic interest models. Li et al. (2017) also used AR(1) interest models in the survivorship life insurance portfolios. Dhaene (1989) modeled the force of interest as an ARIMA(p,d,q) process in the analysis of the moments of present value functions. Cai and Dickson (2004) used a Markov process to model the interest rates in the research of ruin probability. Beekman and Fuelling (1990, 1991) modeled the accumulated force of interest as an Ornstein-Uhlenbeck process and a Wiener process respectively in the study of continuous-time life annuity. Later, Hoedemakrs et al. (2005) and Dufresne (2007) discussed the distribution of life annuity under Beekman and Fuelling's methods. Zhao et al. (2007) and Zhao and Zhang (2012) expressed the accumulated force of interest by a Wiener Process and a Poisson Process in the research of optimal dividend and pricing perpetual options respectively.

Generally, stochastic interest models based on time series methods require the assumption that the interest rate in one year is fixed, which does not always fit in with market interest rates. The methods by modeling the accumulated force of interest bring convenience to both theoretical analysis and calculation, but the behavior of the force of interest can't be indicated distinctly. So Parker (1994a, 1994b) studied the first three moments of homogeneous portfolios of life insurance and endowment policies by modeling the force of interest directly based on a Wiener process or an Ornstein-Uhlenbeck process. Especially, the modeling method based on Ornstein-Uhlenbeck process is also called as Vasicek model in finance, and has been applied extensively (such as Boulier et al., 2001; Liang et al., 2017, etc.). Parker (1994c) further compared the randomness of annuity of the models describing the accumulated force of interest with that of the ones describing the force of interest directly. Considering stochastic jumps of the force of interest in financial markets, Li et al. (2017) modeled the force of interest directly by a compound Poisson process.

Considering the change of yield rate in insurance company, the following two aspects are involved: stochastic continuous fluctuation from risk-free investments of insurance companies and stochastic jumps from adjustments of market interest rate (epically official rate of interest). So we might combine these two parts together to aim to introduce a class of interest models, in which the force of interest is expressed by a compound Poisson process and a Brownian motion. Our model is a generalization of Parker (1994a, 1994b, 1994c) and Li et al. (2017). It not only considers discrete and continuous changes simultaneously, but also assumes random adjustment range in each jump about the force of interest. These characters can interpret random changes of interest on financial market. In literature, stochastic interest models with jumps have been studied by stochastic differential equation, for example, Brigo and Mercurio (2006), Deng (2015), Li et al. (2016) and Hu and Chen (2016) etc. But the modeling thinking-way and research method are different from the one presented in this paper.

The paper is organized as follows. In Section 2, we introduce the stochastic interest model driven by a compound Poisson process and a Brownian motion. In Section 3, we give an explicit formula of the expected discounted functions of the proposed model under general circumstances, discuss the validity of this model and investigate their properties under different parameter and distribution settings through analytical and numerical analysis. In Section 4, we give actuarial present values of life annuities in discrete and continuous conditions. Lastly, we end this paper with a conclusion.

2. Stochastic interest models driven by compound Poisson process and Brownian motion

Assume that all random variables and stochastic processes under consideration are defined on an appropriate probability space $(\Omega, P, \mathscr{F})$ and are integrable. In this section, we construct a class of stochastic interest models including a discrete part and a continuous part.

For the discrete part, we follow and generalize the idea of Li et al. (2017), and the following three assumptions are satisfied:

(1) The jumping number of the force of interest can be expressed by a Poisson process;
(2) The adjustment direction (rise or fall) of interest rates in every jump of the force of interest is independent of each other; and
(3) The adjustment ranges of interest rates in all jumps are independent and identically distributed.

For the continuous part, Brownian motion is usually used to describe stochastic tiny fluctuations of all kinds of financial assets. Due to the excellent mathematical properties of Brownian motion and the consistency between the change rules of finance assets and stochastic fluctuations of Brownian motion, we choose Brownian motion to describe continuous tiny fluctuations of the force of interest.

So the force of interest $\delta_t, t \geq 0$ is expressed by

$$\delta_t = \delta_0 + \sum_{i=0}^{N(t)} Y_i + \sigma B(t), t \in [0, +\infty)\}. \tag{1}$$

where $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ and denotes the jumping number of the force of interest on interval $[0, t]$. $\{Y_i\}_{i=1}^{+\infty}$ are independent and identically distributed random variables and each Y_i expresses the jumping range of the i-th jump of the force of interest. So $\sum_{i=1}^{N(t)} Y_i$, $t > 0$, is a compound Poisson process. The stochastic process $\{B(t), t \geq 0\}$ is a standard Brownian motion which describes the stochastic fluctuations of the force of interest and the constant σ represents the fluctuation intensity.

The jumping range of the force of interest Y_i can be further written as

$$Y_i = I_i \cdot Z_i,$$

where $I_i(i = 1, 2, 3, ...)$ is the direction of the i-th jump of the force of interest with $\{I_i = 1\}$ for a rise and $\{I_i = -1\}$ for a fall. The random variable $I_i(i = 1, 2, 3, ...)$ obeys $P(I_i = 1) = 1 - P(I_i = -1) = p(0 \leq p \leq 1)$, in which p is called the up-jumping probability of the force of interest. The random variable $Z_i(i = 1, 2, 3...)$ is non-negative and can be interpreted as the adjustment range of the i-th jump of the force of interest with identical distribution $F(z) = P(Z_i \leq z)$. Both of $\{I_i(i = 1, 2, 3, ...)\}$ and $\{Z_i(i = 1, 2, 3, ...)\}$ are independent and identically distributed sequences of random variables. Furthermore, the random sequences $\{I_i(i = 1, 2, 3, ...)\}$, $\{Z_i(i = 1, 2, 3, ...)\}$, $\{N(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ are independent of each other.

Remark 1. *If $P(Z_i = 0) = 1$, or $P(Z_i = \alpha) = 1$ and $\sigma = 0$, our interest model will be reduced to one in Parker (1994c) and Li et al. (2017) respectively.*

Here, we give an important property (refer to Ross (1996)) on Poisson process which will be used in the following analysis.

Lemma 1. *For a Poisson process, if the occur times of all the events are considered as unordered random variables under the condition $N(t) = n$, these random variables are distributed independently and uniformly on the time interval* $[0, t]$.

3. Expected discounted function under stochastic interest model

In this section, we will study the accumulated interest force function and the expected discounted function of the stochastic interest model (1).

3.1. Expected discounted function

The accumulated interest force on the time interval $[0, t]$ can be expressed as

$$\begin{aligned} J_0^t &= \int_0^t \delta_s \mathrm{d}s = \int_0^t \left(\delta_0 + \sum_{i=1}^{N(s)} Y_i + \sigma B(s)\right) \mathrm{d}s \\ &= \delta_0 \cdot t + \int_0^t \left(\sum_{i=1}^{N(s)} Y_i\right) \mathrm{d}s + \sigma \int_0^t B(s) \mathrm{d}s = \delta_0 \cdot t + H_1(t) + H_2(t). \end{aligned} \tag{2}$$

(a) In formula (2), by changing the integral direction (refer to Section 2.2 in Li et al. (2017)), we obtain

$$H_1 = \sum_{i=1}^{N(t)} Y_i(t - T_i) = \sum_{i=1}^{N(t)} I_i \cdot Z_i(t - T_i), \tag{3}$$

where T_i $(i = 1, 2, 3, ...)$ denotes the i-th jumping time of the force of interest.
(b) Through stochastic calculus (see Klebaner (2005)), we know

$$H_2 \sim N(0, \sigma^2 t^3/3). \tag{4}$$

So from formulas (2), (3) and (4), we have

$$J_0^t = \delta_0 \cdot t + \sum_{i=1}^{N(t)} I_i \cdot Z_i(t - T_i) + H_2. \tag{5}$$

The discounted function, which is the random present value of one-unit payment at time t, can be expressed as

$$\exp\left(-J_0^t\right) = \exp\left(-\left(\delta_0 \cdot t + H_2(t)\right)\right) \prod_{i=1}^{N(t)} \exp(-I_i \cdot Z_i(t - T_i)). \tag{6}$$

From formula (4) and Section 2 in Parker (1994c), we have,

$$E\left[\exp\left(-\left(\delta_0 \cdot t + H_2(t)\right)\right)\right] = \exp\left(\left(-\delta_0 + t^2\sigma^2/6\right)t\right). \tag{7}$$

Moreover, it can be obtained from the law of total expectation that

$$E\left[\prod_{i=1}^{N(t)} \exp(-I_i \cdot Z_i(t - T_i))\right] = E\left[E\left[\prod_{i=1}^{N(t)} \exp\left(-I_i \cdot Z_i(t - T_i)\right) | N(t)\right]\right]. \tag{8}$$

By Lemma 1 and the independent assumptions in Section 2, we have

$$
\begin{aligned}
& E\left[\prod_{i=1}^{N(t)} \exp(-I_i \cdot Z_i(t-T_i))|N(t))\right] \\
&= E\left[\prod_{i=1}^{N(t)} \exp(-I_i \cdot Z_i(t-U_i))\right] \\
&= \prod_{i=1}^{N(t)} E\left[\exp(-I_i \cdot Z_i(t-U_i))\right] \\
&= \prod_{i=1}^{N(t)} \left(E\left[\exp\left(-Z_i(t-U_i)\right)\right] p + E\left[\exp\left(Z_i(t-U_i)\right)\right](1-p)\right) \\
&= \prod_{i=1}^{N(t)} \left(\frac{p}{t}\int_0^{+\infty}\int_0^t \exp\left(-z(t-u)\right) \mathrm{d}u\mathrm{d}F(z) + \frac{1-p}{t}\int_0^{+\infty}\int_0^t \exp\left(z(t-u)\right) \mathrm{d}u\mathrm{d}F(z)\right) \\
&= \beta_t^{N(t)},
\end{aligned}
\tag{9}
$$

where the random variables, $U_1, U_2, ..., U_{N(t)}$ are independent and identically distributed on the time interval $[0, t]$ and

$$\beta_t = \frac{1}{t}\int_0^{+\infty} \frac{1}{z}\left(\left(p(1-\exp(-zt)) + (1-p)\left(\exp(zt)-1\right)\right)\right) \mathrm{d}F(z).$$

Based on the foregoing analysis, we obtain the following theorem.

Theorem 1. *Under the stochastic interest model (1), the expected discounted function can be expressed as*

$$E\left[\exp\left(-J_0^t\right)\right] = \exp\left(\left(-\delta_0 + t^2\sigma^2/6 + \lambda(\beta_t - 1)\right)t\right). \tag{10}$$

Proof. See the Appendix. □

Remark 2. *From equation (10) and β_t in formula (9), we can prove that the expected discounted function is an increasing function of σ and a decreasing function of p. We will further demonstrate this property in Section 3.2 through numerical analysis.*

In practice, the jump sizes of interest rate are relatively fixed, for examples, the Federal reserve rate and the Chinas central bank benchmark interest rate etc. In addition, the jump sizes of interest rate are not too big. Hence, we give three special distributions about the jump size as special cases in the following corollary.

Corollary 1. *The expected discounted function $E[\exp(-J_0^t)]$ is influenced by the distribution function $F(z)$ and the up-jumping probability p. There are some special cases of $F(z)$ and p as follows,*

(1) If $P(Z_i = \alpha) = 1$ $(i = 1, 2, 3, ...)$ for a positive constant α, the result of β_t is consistent with that in Li et al. (2017) which is generalized further in this paper, we obtain that

$$\beta_t = \frac{1}{\alpha t}\left(p\left(1-\exp(-\alpha t)\right) + (1-p)\left(\exp(\alpha t)-1\right)\right).$$

(2) If $P(Z_i = \alpha_1) = q = 1 - P(Z_i = \alpha_2)$ $(i = 1, 2, 3, ...)$ for two positive constants α_1 and α_2, that is, $Z_i(i = 1, 2, 3, ...)$ obeys a two-point distribution, we have that

$$\beta_t = \frac{q\,(1-\exp(-\alpha_1 t))}{\alpha_1 t}\,(p + (1-p)\exp(\alpha_1 t)) + \frac{(1-q)(1-\exp(-\alpha_2 t))}{\alpha_2 t}(p + (1-p)\exp(\alpha_2 t)).$$

(3) If $F(z) = z/\theta$ for $z \in [0, \theta]$, that is, the random variable $Z_i(i = 1, 2, 3, ...)$ is uniformly distributed on the time interval $[0, \theta]$, we have that

$$\beta_t = \frac{1}{\theta t}\left(p\int_0^\theta \frac{1}{z}\,(1-\exp(-zt))\,dz + (1-p)\int_0^\theta \frac{1}{z}\,(\exp(zt)-1)\,dz\right).$$

(4) If $p = 0$, the market interest will always jump down at the moments when interest rates jump. Because $\exp(zt) - 1 > zt$, we have that

$$\beta_t = \int_0^\infty \frac{1}{zt}\,(\exp(zt)-1)\,\mathrm{d}F(z) > \int_0^\infty \mathrm{d}F(z) = 1.$$

In this case, the interest rate might be negative if the jumping number of the interest rates on the time interval $[0, t]$ is sufficiently large.

(5) If $p = 1$, the market interest will always jump up at the moments when the interest rates jump. Because $1 - \exp(-zt) < zt$, we have that

$$\beta_t = \int_0^\infty \frac{1}{zt}\,(1-\exp(-zt))\,\mathrm{d}F(z) < \int_0^\infty \mathrm{d}F(z) = 1.$$

In this case, the larger the jumping number of interest rates is, the smaller the expected discounted function is.

Table 1. Values of expected discounted functions when $F(z)$ is an one-point distribution.

α	Values of expected discounted functions							
	$p = 0.4$ $\sigma = 0.01$	$p = 0.4$ $\sigma = 0.02$	$p = 0.5$ $\sigma = 0.01$	$p = 0.5$ $\sigma = 0.02$	$p = 0.6$ $\sigma = 0.01$	$p = 0.6$ $\sigma = 0.02$	$p = 0.7$ $\sigma = 0.01$	$p = 0.7$ $\sigma = 0.02$
0.0030	0.7259	0.7631	0.6836	0.7187	0.6438	0.6768	0.6063	0.6374
0.0028	0.7227	0.7598	0.6834	0.7184	0.6462	0.6793	0.6110	0.6423
0.0026	0.7196	0.7565	0.6831	0.7181	0.6485	0.6818	0.6156	0.6472
0.0024	0.7165	0.7532	0.6829	0.7179	0.6509	0.6843	0.6204	0.6522
0.0022	0.7134	0.7500	0.6827	0.7177	0.6533	0.6868	0.6252	0.6572
0.0020	0.7103	0.7468	0.6825	0.7175	0.6557	0.6894	0.6300	0.6623

3.2. Analysis of validity of Stochastic Interest Model

As a general rule, the accumulated interest force function should be increasing with respect to time t which means that the expected discounted function in formula (10) is decreasing. This property is

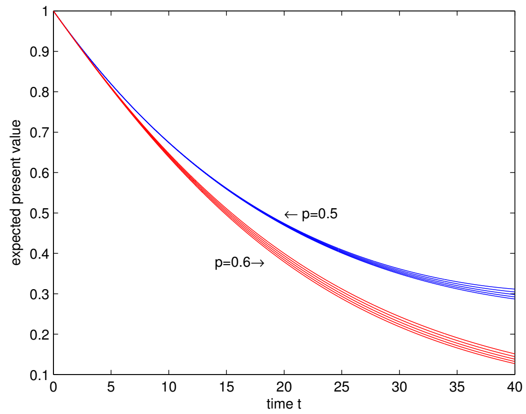

Figure 1. Curves of expected discounted functions for different p and α when $F(z)$ is an one-point distribution.

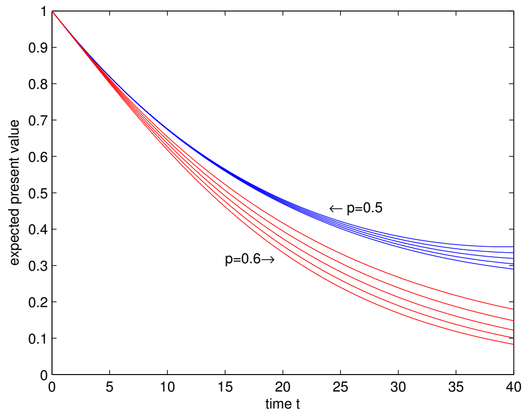

Figure 2. Curves of expected discounted functions for different p and λ when $F(z)$ is an one-point distribution.

called the validity of stochastic interest model which will be analyzed in this section. Now, we will discuss how to restrict the value of the future time t in order to ensure this validity.

Let $f(t) = (\delta_0 + \lambda)t - \sigma^2 t^3/6 + \lambda\beta_t t$, and we should ensure this function is increasing. Obviously, the derivative function of $f(t)$ exists if the function $f(t)$ is finite. In fact, we can find that $\lim_{t\to 0^+}\beta_t = 1$ and $\beta_t \le \int_0^{+\infty} \frac{\exp zt-1}{zt} dF(z)\ \ t \in (0, +\infty)$. Hence, if $F(\cdot)$ is a light-tailed distribution, β_t is finite for any $t \in [0, +\infty)$, and so is the function $f(t)$.

Under the condition of light-tailed distribution, the derivative function can be expressed as

$$f'(t) = (\delta_0 + \lambda) - \sigma^2 t^2/2 - \lambda \int_0^{\infty} \left(pe^{-zt} + (1-p)e^{zt}\right) dF(z).$$

We can find that $f'(0) = \lambda_0 > 0$ and $\lim_{t\to+\infty} f'(t) < 0$, then there is at least one critical value in interval $(0, +\infty)$ which satisfies $f'(t) = 0$ and the first one is denoted as t^*. Hence, this model is valid if time $t \in [0, t^*]$. Now, we will try to find the values of t^* under various cases of the probability distribution of Z_i, $F(\cdot)$ given in Corollary 1 respectively.

(1) In case that $P(Z_i = \alpha) = 1$ $(i = 1, 2, 3, ...)$ for a positive constant α, we have

$$f'(t) = (\delta_0 + \lambda) - \sigma^2 t^2/2 - \lambda\left(pe^{-\alpha t} + (1-p)e^{\alpha t}\right).$$

(2) In case that $P(Z_i = \alpha_1) = q = 1 - P(Z_i = \alpha_2)$ $(i = 1, 2, 3, ...)$ for two positive constants α_1 and α_2, we have

$$f'(t) = (\delta_0 + \lambda) - \sigma^2 t^2/2 - \lambda\left[q\left(pe^{-\alpha_1 t} + (1-p)e^{\alpha_1 t}\right) + (1-q)\left(pe^{-\alpha_2 t} + (1-p)e^{\alpha_2 t}\right)\right].$$

(3) In case that $F(z) = z/\theta$ for $z \in [0, \theta]$, we have that

$$f'(t) = (\delta_0 + \lambda) - \sigma^2 t^2/2 - \frac{\lambda}{\theta t}\left[p\left(1 - e^{-\theta t}\right) + (1-p)\left(e^{\theta t} - 1\right)\right].$$

For each case above, since there are exponential functions part and power function part in equation $f'(t) = 0$, it is very difficult to solve this equation directly. However, we can obtain the value of t^* by numerical approach and then can use the interest model in formula (1) when $t \in (0, t^*)$. For instance, if $\delta_0 = 0.04, \lambda = 2, \sigma = 0.01, \alpha = 0.0025$ and $p = 0.6$ under case (1), we can obtain $t^* = 37.01$; and if $\delta_0 = 0.04, \lambda = 2, \sigma = 0.01, \theta = 0.004$ and $p = 0.6$ under case (3), we can obtain $t^* = 35.08$.

3.3. Numerical analysis

In this subsection, we analyze the changes of the expected discounted function under different assumptions in Corollary 1. We consider three types of distribution function $F(z)$, including one-point distributions, two-point distributions and uniform distributions. The parameter δ_0 is assumed to be 0.04. Firstly, we suppose $\lambda = 2$ and $t = 10$ to calculate the values of the expected discounted functions, shown in Tables 1–3. In addition, the range of the parameters of $F(z)$ is chosen based on the condition of jump amplitudes of major market interest rates (such as the Federal reserve rate and the China's central bank benchmark interest rate).

Table 2. Values of expected discounted functions when $F(z)$ is a two-point distribution.

α_1	α_2	q	Values of expected discounted functions					
			$p = 0.4$ $\sigma = 0.01$	$p = 0.5$ $\sigma = 0.01$	$p = 0.5$ $\sigma = 0.02$	$p = 0.6$ $\sigma = 0.01$	$p = 0.6$ $\sigma = 0.02$	$p = 0.7$ $\sigma = 0.02$
0.001	0.003	0.40	0.7136	0.6829	0.7179	0.6535	0.6870	0.6157
0.001	0.003	0.50	0.7106	0.6827	0.7177	0.6560	0.6896	0.6625
0.001	0.003	0.60	0.7076	0.6825	0.7175	0.6584	0.6922	0.6677
0.001	0.004	0.40	0.7233	0.6839	0.7189	0.6466	0.6798	0.6427
0.001	0.004	0.50	0.7186	0.6835	0.7186	0.6502	0.6835	0.6502
0.001	0.004	0.60	0.7139	0.6832	0.7182	0.6538	0.6873	0.6577
0.002	0.003	0.40	0.7196	0.6832	0.7182	0.6486	0.6818	0.6473
0.002	0.003	0.50	0.7181	0.6831	0.7181	0.6497	0.6831	0.6497
0.002	0.003	0.60	0.7165	0.6830	0.7180	0.6509	0.6843	0.6522
0.002	0.004	0.40	0.7294	0.6841	0.7192	0.6417	0.6746	0.6328
0.002	0.004	0.50	0.7262	0.6839	0.7189	0.6440	0.6771	0.6376
0.002	0.004	0.60	0.7230	0.6836	0.7186	0.6464	0.6795	0.6425

It can be observed from Table 1, Table 2 and Table 3 that the values of the expected discounted functions become larger with increasing σ or decreasing p under three distribution assumptions. This result verifies Remark 2. Furthermore, we can obtain that the values of the expected discounted functions become larger with increasing α under the assumption $P(Z_i = \alpha) = 1$, or with increasing q, decreasing both α_1 and α_2 under the two-point distribution assumption for Z_i, or with decreasing θ under the uniform distribution assumption for Z_i if $p \leq 0.5$. The values change in the opposite direction if $p \geq 0.6$.

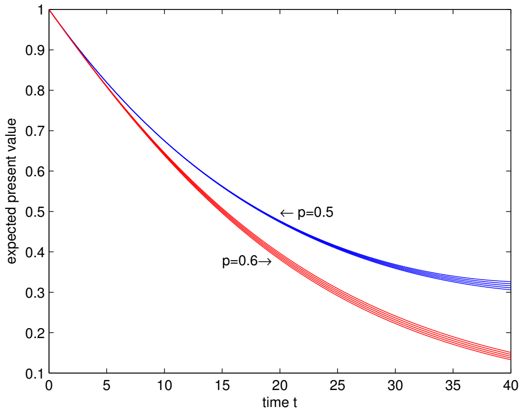

Figure 3. Curves of expected discounted functions for different p and q when $F(z)$ is a two-point distribution.

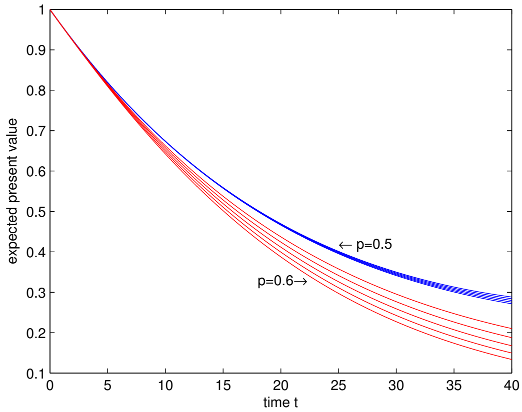

Figure 4. Curves of expected discounted functions for different p and λ when $F(z)$ is a two-point distribution.

Table 3. Values of expected discounted functions when $F(z)$ is an uniform distribution.

θ	Values of expected discounted functions							
	$p = 0.4$ $\sigma = 0.01$	$p = 0.4$ $\sigma = 0.02$	$p = 0.5$ $\sigma = 0.01$	$p = 0.5$ $\sigma = 0.02$	$p = 0.6$ $\sigma = 0.01$	$p = 0.6$ $\sigma = 0.02$	$p = 0.7$ $\sigma = 0.01$	$p = 0.7$ $\sigma = 0.02$
0.0040	0.7107	0.7471	0.6828	0.7178	0.6560	0.6897	0.6303	0.6626
0.0035	0.7068	0.7431	0.6825	0.7175	0.6590	0.6928	0.6364	0.6690
0.0030	0.7030	0.7391	0.6823	0.7172	0.6621	0.6960	0.6425	0.6755
0.0025	0.6993	0.7352	0.6821	0.7170	0.6652	0.6993	0.6488	0.6821
0.0020	0.6957	0.7313	0.6819	0.7168	0.6684	0.7027	0.6552	0.6887

Remark 3. *In the above tables, we only display partial results due to limited space. We also find that there is an equilibrium up-jumping probability p^* under any one of these three assumptions when the time $t = 10$, which satisfies $0.5 < p^* < 0.6$. The change regulation for $p < p^*$ and $p > p^*$ is similar to that when $p = 0.5$ and $p = 0.6$ respectively. In fact, p^* is the solution of equation $\beta_t = 1$ which is related to $F(z)$ and t. So it can be expressed as $p^*(F(z), t)$.*

In Figures 1–6, we present the expected discounted functions as functions of time under different settings. In Figure 1, the upper five curves show the expected discounted functions for $\alpha = 0.0028$, 0.0026, 0.0024, 0.0022 and 0.0020 from top to bottom under the one-point distribution assumption when $p = 0.5$, while the lower five curves show those functions from bottom to top when $p = 0.6$; in Figure 3, the upper five curves show the expected discounted functions for $q = 0.40$, 0.45, 0.50, 0.55 and 0.60 from top to bottom under the two-point distribution assumption when $p = 0.5$, while the lower five curves show those functions from bottom to top when $p = 0.6$; in Figure 5, the upper five curves show the expected discounted functions for $\theta = 0.0040, 0.0035$, $0.0030, 0.0025$ and 0.0020 from top to bottom under the uniform distribution assumption when $p = 0.5$, while the lower five curves show those functions from bottom to top when $p = 0.6$; and in Figure 2, Figure 4 and Figure 6, every upper

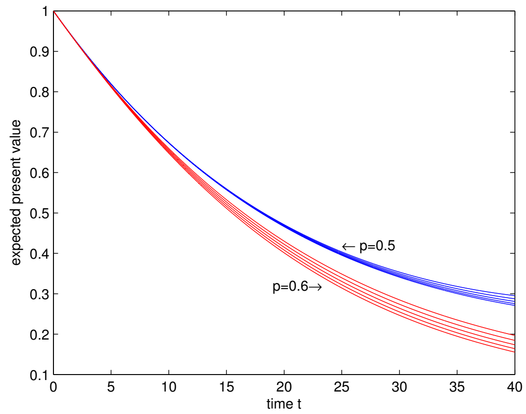

Figure 5. Curves of expected discounted functions for different p and θ when $F(z)$ is an uniform distribution.

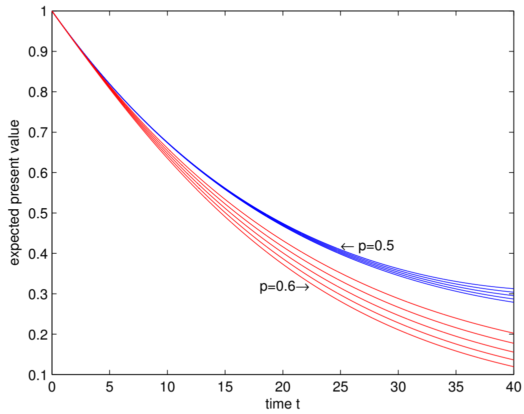

Figure 6. Curves of expected discounted functions for different p and λ when $F(z)$ is an uniform distribution.

five curves show the expected discounted functions from bottom to top for $\lambda = 0.5$, 1, 1.5, 2 and 2.5 with other parameters keep constant respectively when $p = 0.5$ under three distribution assumptions respectively and the lower five curves show the corresponding expected discounted functions from bottom to top respectively when $p = 0.6$.

From Figures 1–6, we further observe that the differences between the values of expected discounted functions under different settings become significantly large as the parameter p or the time t increases. This implies that the expected discounted functions is sensitive to $F(z)$ and λ for large p and t. Hence, more emphasis is required on the selection of parameters in the up-cycle of interest rate.

4. Application in life contingencies

In this section, we will apply the proposed stochastic interest model (1) to discrete life annuity and continuous life annuity, life insurance payable at the end of the year of death and life insurance payable at the moment of death. The actuarial present values (APVs) for these life annuities and life insuriance will be shown under different stochastic interest assumptions.

Following Bowers et al. (1997), the symbol (x) denotes a life-age-x. The future lifetime and the curate-future-lifetime of (x) are indicated by $T(x)$ and $K(x)$ respectively.

4.1. APVs of life annuities

In insurance science, there are two types of discrete life annuities which are discrete life annuity-due and discrete life annuity-immediate. Without loss of generality, we only consider the former. In the nomenclature, an n-year temporary life annuity-due of one per year is the one that pays one unit amount at the beginning of each year while the annuitant (x) survives during the next n years and the actuarial present value of this life annuity is denoted by $\ddot{a}_{x:\overline{n}|}$.

Referring to Chapter 5 in Bowers et al. (1997), the actuarial present value of this life annuity $\ddot{a}_{x:\overline{n}|}$

can be expressed as

$$\ddot{a}_{x:\overline{n}|} = E\left[\sum_{k=0}^{n-1} \exp\left(-\int_0^k \delta_t dt\right) P(K(x) \geq k)\right] = \sum_{k=0}^{n-1} E\left[\exp\left(-\int_0^k \delta_t dt\right)\right] {}_k p_x, \tag{11}$$

where ${}_k p_x$ is the probability that the annuitant (x) will attain age $x + k$. Combining (11) with (10), we obtain

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \exp\left(\left(-\delta_0 + k^2\sigma^2/6 + \lambda(\beta_k - 1)\right) k\right) {}_k p_x. \tag{12}$$

Next, we consider the n-year temporary continuous life annuity for the annuitant (x), and the corresponding actuarial present value is denoted by $\overline{a}_{x:\overline{n}|}$. From Chapter 5 in Bowers et al. (1997), we have that

$$\overline{a}_{x:\overline{n}|} = \int_0^n \overline{a}_{\overline{t}|} \mathrm{d}F_{T(x)}(t) + \overline{a}_{\overline{n}|} \cdot {}_n p_x$$

and

$$\overline{a}_{\overline{t}|} = E\left[\int_0^t \exp(-J_0^u) du\right] = \int_0^t E\left[\exp(-J_0^u)\right] du.$$

So, we obtain the expression of $\overline{a}_{x:\overline{n}|}$ though Fubini's theorem as follows,

$$\overline{a}_{x:\overline{n}|} = \int_0^{+\infty} \exp\left(\left(-\delta_0 + t^2\sigma^2/6 + \lambda(\beta_t - 1)\right) t\right) {}_t p_x dt. \tag{13}$$

Table 4. APVs of discrete life annuity when $F(z)$ is an one-point distribution.

α	actuarial present value			
	$p = 0.5$ $\sigma = 0.01$	$p = 0.5$ $\sigma = 0.02$	$p = 0.6$ $\sigma = 0.01$	$p = 0.6$ $\sigma = 0.02$
0.0030	14.4406	15.6128	13.6018	14.6124
0.0028	14.4325	15.6028	13.6474	14.6663
0.0026	14.4249	15.5935	13.6937	14.7212
0.0024	14.4179	15.5848	13.7409	14.7772
0.0022	14.4114	15.5769	13.7889	14.8341
0.0020	14.4056	15.5697	13.8378	14.8921

4.2. APVs of continuous life insurances

In this section, we analyse the APVs of life insurances by taking two whole life insurances for example which are the whole life insurance paying one unit at the end of the year of death and the one paying one unit at the moment of death for some insured (x). Following standard symbols in Bowers et al. (1997), A_x denotes the actuarial present value of the whole life insurance payable at the end of the year of death and $\overline{A}_x$ denotes the one of payable at the moment of death.

Firstly, we discuss the expression of A_x.

$$
\begin{aligned}
A_x &= E\left[\exp\left(-\int_0^{K(x)+1}\delta_t dt\right)\right] \\
&= \sum_{k=0}^{+\infty} E\left(-\int_0^{k+1}\delta_t dt\right)\cdot P(K(x)=k) \\
&= \sum_{k=0}^{+\infty} \exp\left(\left(-\delta_0+(k+1)^2\sigma^2/6+\lambda(\beta_{k+1}-1)\right)(k+1)\right)\cdot {}_k p_x \cdot p_{x+k}.
\end{aligned}
$$

Secondly, we show the expression of $\overline{A}_x$ under our stochastic interest model.

$$
\begin{aligned}
\overline{A}_x &= E\left[\exp\left(-\int_0^{T(x)}\delta_t dt\right)\right] \\
&= \int_0^{+\infty} E\left(-\int_0^{u}\delta_t dt\right) dF_{T(x)}(u) \\
&= \int_u^{+\infty} \exp\left(\left(-\delta_0+u^2\sigma^2/6+\lambda(\beta_u-1)\right)u\right)\cdot f_{T(x)}(u)du.
\end{aligned}
$$

Table 5. APVs of discrete life annuity when $F(z)$ is a two-point distribution.

α_1	α_2	q	actuarial present values			
			$p=0.5$ $\sigma=0.01$	$p=0.5$ $\sigma=0.02$	$p=0.6$ $\sigma=0.01$	$p=0.6$ $\sigma=0.01$
0.001	0.003	0.40	14.4182	15.5852	13.7949	14.8415
0.001	0.003	0.50	14.4126	15.5783	13.8441	14.8999
0.001	0.003	0.60	14.4070	15.5714	13.8937	14.9587
0.001	0.004	0.40	14.4477	15.6215	13.6606	14.6825
0.001	0.004	0.50	14.4371	15.6085	13.7311	14.7661
0.001	0.004	0.60	14.4266	15.5955	13.8025	14.8507
0.002	0.003	0.40	14.4265	15.5955	13.6952	14.7231
0.002	0.003	0.50	14.4230	15.5912	13.7188	14.7510
0.002	0.003	0.60	14.4196	15.5869	13.7424	14.7790
0.002	0.004	0.40	14.4562	15.6319	13.5628	14.5666
0.002	0.004	0.50	14.4477	15.6215	13.6079	14.6199
0.002	0.004	0.60	14.4392	15.6111	13.6532	14.6735

4.3. Numerical analysis

In this section, under different parameter and distribution settings, we calculate the APVs of the 20-year temporary discrete life annuity-due under the one-point distribution and the two-point distribution

Table 6. APVs of continuous life annuity when $F(z)$ is an uniform distribution.

θ	actuarial present values			
	$p = 0.5$ $\sigma = 0.01$	$p = 0.5$ $\sigma = 0.02$	$p = 0.6$ $\sigma = 0.01$	$p = 0.6$ $\sigma = 0.02$
0.0040	14.1057	15.3728	13.5084	14.6499
0.0035	14.0963	15.3610	13.5716	14.7259
0.0030	14.0881	15.3508	13.6365	14.8040
0.0025	14.0812	15.3422	13.7032	14.8844
0.0020	14.0757	15.3351	13.7718	14.9671

assumption and the APVs of the 20-year temporary continuous life annuity for the annuitant (30) under the uniform distribution of death assumption (refer to Section 3.7 in Bowers et al. (1997)). Note that the numerical analysis on life insurances will not be given because of the similarity. The mortality rate is from China Life Insurance Mortality Table (2010-2013) (CL5, Pension life table for male). The results are shown in Tables 4–6. Comparing Tables 4–6 with Tables 1–3, we find that the APVs of life annuity change about parameters following the same law of the values of expected discounted functions in Section 3.3. This is an inevitable conclusion from the relation between the APVs of life annuities and the values of expected discounted functions.

5. Conclusions

In this paper, we introduce a new stochastic interest model in which the force of interest is expressed by a compound Poisson process and a Brownian motion. The advantage of this model is that the random jumping behavior and the continuous tiny random fluctuations are described simultaneously and the adjustment ranges of the force of interest in the random jump part of the proposed model are governed by a random variable sequence, which generalizes the modeling methods in Parker (1994a, 1994b, 1994c) and Li et al. (2017). In addition, we derive the expected discounted functions of the proposed model in general circumstances and further discuss the cases under the one point distribution, the two-point distribution and the uniform distribution assumption on random jumping amplitudes Z_i respectively. We also use the proposed model to study two types of common life annuities. Our numerical analysis shows that both the values of expected discounted functions and the APVs of life annuities are influenced distinctly by the change of the interest model parameters under different distribution assumptions of Z_i. Especially when the up-jumping probability p is sufficiently large, the influence of parameters in discrete part of interest model is totally opposite to that when the up-jumping probability p is sufficiently small.

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Conflict of Interest

All authors declare no conflict of interest.

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Appendix

Proof of Theorem 1

Based on the law of total expectation, substituting (9) into (8) yields that

$$\begin{aligned} E\left[\prod_{i=1}^{N(t)} \exp(-I_i \cdot Z_i(t-T_i))\right] &= E\left[\beta_t^{N(t)}\right] \\ &= \sum_{n=0}^{+\infty} \beta_t^n \frac{\exp(-\lambda t)(\lambda t)^n}{n!} \\ &= \exp(-\lambda t) \sum_{n=0}^{+\infty} \frac{(\beta_t \lambda t)^n}{n!} \\ &= \exp\left(\lambda t(\beta_t - 1)\right). \end{aligned}$$

From (6), (7) and the above formula, we have

$$E\left[\exp(-J_0^t)\right] = \exp\left((-\delta_0 + t^2\sigma^2/6)t\right) \cdot \exp\left(\lambda t(\beta_t - 1)\right) = \exp\left(\left(-\delta_0 + t^2\sigma^2/6 + \lambda(\beta_t - 1)\right)t\right).$$

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