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Research article

Evidence of a Bimodal US GDP Growth Rate Distribution: A Wavelet Approach

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Abstract: We present a quantitative characterisation of the fluctuations of the annualized growth rate of the real US GDP per capita at many scales, using a wavelet transform analysis of two data sets, quarterly data from 1947 to 2015 and annual data from 1800 to 2010. The chosen mother wavelet (first derivative of the Gaussian function) applied to the logarithm of the real US GDP per capita provides a robust estimation of the instantaneous growth rate at different scales. Our main finding is that business cycles appear at all scales and the distribution of GDP growth rates can be well approximated by a bimodal function associated to a series of switches between regimes of strong growth rate ρ_{high} and regimes of low growth rate ρ_{low}. The succession of such two regimes compounds to produce a remarkably stable long term average real annualized growth rate of 1.6% from 1800 to 2010 and ≈ 2.0% since 1950, which is the result of a subtle compensation between the high and low growth regimes that alternate continuously. Thus, the overall growth dynamics of the US economy is punctuated, with phases of strong growth that are intrinsically unsustainable, followed by corrections or consolidation until the next boom starts. We interpret these findings within the theory of "social bubbles" and argue as a consequence that estimations of the cost of the 2008 crisis may be misleading. We also interpret the absence of strong recovery since 2008 as a protracted low growth regime ρ_{low} associated with the exceptional nature of the preceding large growth regime.

Keywords: wavelet transform; economic growth; GDP per capita; business cycle; multi-scale analysis *PACS:* 89.65.Gh, 05.45.Tp

1. Introduction

The dynamics of the growth of GDP (gross domestic product) is arguably the most scrutinised metric quantifying the overall economic development of an economy. A weak annual growth rate of GDP, as has been characterising the US and Europe in the years following the financial crisis of 2008, is interpreted as underperformance, which has called for unorthodox monetary policies attempting to fix it (Erber, 2012). In contrast, a strong growth of GDP is usually lauded, because it reflects a rise of living standards and is generally accompanied by decreasing unemployment. But what is meant by "weak" or "strong" growth? Is there a "natural" growth rate? Does past growth rates of GDP imply future growth rates? This last question is particularly relevant in the present context of small growth compared with previous decades in developed countries and the argument by many that we may have shifted to a "new normal" of slower intrinsic growth (Dabla-Norris et al., 2015).

It is well-known that plotting the logarithm of the real US GDP per capita over the last one hundred years looks remarkably linear with a slope estimated between 0.019 and 0.02. In other words, the inflation adjusted GDP per capita exhibits a long term average growth of 1.9 – 2% per year. [*] The occurrence of such a near trend-stationary long run growth covering a period with two world wars, the cold war and its associated proxy wars, the collapse of the Bretton Woods System in 1973, several large bubbles, crashes and recessions and strong changes in interest rate policies, is truly remarkable. It entices one to entertain the possibility of an equilibrium or natural growth rate, which then could be extrapolated in the future. Business cycles would then be viewed as fluctuations around this equilibrium growth rate.

In this paper, we challenge the standard hypothesis that business cycles are merely fluctuations (or transient deviations) around a stationary equilibrium growth rate. By analyzing quarterly data of real US GDP per capita (r-US-GDP-pc) between 1947 until 2015, we find both parametric and non-parametric evidence that the GDP growth rate density is bimodal, with peaks at a high and a low growth rate of $\rho_{\text{high}} \approx 3\%$, and $\rho_{\text{low}} \approx 1\%$, respectively. [†] This leads to the conclusion that the US economy per capita is intrinsically composed of alternation between regimes of strong growth rate ρ_{high}, associated with booms (or bubbles), and regimes of low growth rate ρ_{low} that include plateaus (and recessions). Alternations between those two regimes give rise to business cycles. Only when viewed at larger scales, these two alternating regimes renormalize to an effective long-term growth rate $\rho_{\text{lt}} \approx 2\%$ that is between ρ_{low} and ρ_{high}. [‡]

Our findings have important economic and policy implications. The existence of a well-characterised strong growth regime with average growth rate ρ_{high} often leads to the misleading expectations that it is the normal that reflects a well-functioning economy, while the other mode of low growth ρ_{low} is considered abnormal, often interpreted as due to a surprising shock, bringing considerable dismay and pain, and leading to policy interventions. Our finding of a robust bimodal distribution of GDP growth rates over the whole history of the US suggests that this interpretation is incorrect. Rather than accepting the existence of the long-term growth rate as given and interpreting the deviations from it as perturbations, the bimodal view of the GDP growth suggests a completely different picture. In this representation, the long-term growth rate is the result of a subtle

[*] See the dashed linear fitted lines in figure 3 and figure A1.

[†] Unless stated otherwise, we refer from hereon to the real GDP per capita (r-US-GDP-pc) simply as GDP.

[‡] Throughout this article, we stick to quarterly dataset. In Appendix A we show the analysis for annual GDP data from 1800 until today. The conclusions are similar.

compensation between the high and low growth regimes that alternate continuously. The overall growth dynamics that emerges is that the US economy is growing in a punctuated way, as illustrated in the model developed by Louzoun et al. (2003), following phases of strong growth that are intrinsically unsustainable, followed by corrections or consolidations until the next boom starts. In other words, the approximately long-term growth rate reflects an economy that oscillates between booms and consolidation regimes. Because of the remarkable recurrence of the strong regime and in view of its short-term beneficial effects, economists and policy makers are tempted (and actually incentivised) to form their expectations based on it, possibly catalysing or even creating it in a self-fulfilling prophecy fashion even when the real productivity gains are no more present, as occurred in the three decades before the 2008 crisis (Sornette and Cauwels, 2014).

We suggest that the transient strong growth regimes can be rationalised within the framework of the "social bubble hypothesis" (Sornette, 2008; Gisler and Sornette, 2009, 2010; Gisler et al., 2011), in the sense that they result from collective enthusiasm that are similar to those developing during financial bubbles, which foster collective attitude towards more risk taking. The social bubble hypothesis claims that strong social interactions between enthusiastic supporters weave a network of reinforcing feedbacks that lead to widespread endorsement and extraordinary commitment by those involved, beyond what would be rationalised by a standard cost-benefit analysis. For a time, the economy grows faster than its long-term trend, due to a number of factors that reinforce each other, leading to a phase of creative innovation (e.g. the internet dotcom bubble) or credit based expansion (e.g. the house boom and financialisation of the decade before 2008). These regimes then unavoidably metamorphose into a "hangover", the recovery and strengthening episode until the next upsurge.

Despite the huge attention paid to the analysis of GDP growth rate fluctuations, this work presents, to the best of our knowledge, for the first time a detailed analysis of the bimodal nature of its density distribution. In the next section, we first address the question of bimodality from the perspective of classic, parametric business cycle models. This yields first evidence of a bimodal distribution density. However, the reported results are susceptible to small changes in parameters and noise. Subsequently, we thus turn to a non-parametric analysis. The usage of wavelets as adequate tool for this analysis is motivated in section 3. Section 4 presents the wavelet methodology itself. Results are reported in 5. Section 6 is focused on the evidence supporting the bimodal structure of the distribution of GDP growth rates and section 7 concludes. Appendix A complements the presentation with annual GDP data over the past 200 years.

2. Business Cycles as Markov Autoregressive Processes

First introduced by Hamilton (1989), Markov autoregressive processes have become a popular tool for GDP business cycle analysis. In standardized form, such a process is written as

$$\rho_t = \mu(S_t) + \sum_{i=1}^{n} \phi_i \cdot (\rho_{t-i} - \mu(S_{t-i})) + \epsilon_t \tag{1}$$

with ρ_t the growth rate at time t, S_t describes the regime of the economy at time t, which can take m different states that are reflected in m different values $\mu_1, \ldots, \mu_m$ for $\mu(S_t)$. The parameters $\phi_1, \ldots, \phi_n$ specify the autoregressive characteristics of the process and ϵ_t is Gaussian noise with zero mean and standard deviation σ. The switching from a regime i to a regime j at any time step is determined by a matrix of transition probabilities p_{ij}. In the classic Hamilton (1989) model, $m = 2$ and $n = 4$ and we shall stick here to this choice. The main idea is that there are two states of the economy, a boom state $S_t = 1$ and a recession state $S_t = 0$. From time t to $t + 1$, the economy switches between these two states with probabilities p_{01} and p_{10}, respectively. There is a debate about the nature of the switching probabilities p_{ij}. Hamilton (1989) assumes that the probabilities are independent of the business cycle duration whereas others assume that there is a dependence. A famous model of the second type is by Durland and McCurdy (1994). Duration dependence has been formally tested by Diebold and Rudebusch (1990), who conclude that Hamilton (1989)'s assumption of duration independence is legitimate. In own analysis simple, we therefore stick to the model of Hamilton (1989) to keep the discussion simple.

What is the prediction of these models for the GDP density distribution? In order to answer this question, we first fit the 9 parameters of the model (1) to the 270 datapoints of quarterly r-US-GDP-pc from 1947 to 2015. [§] Using these parameters, we then simulate a synthetic time series over a period of 100,000 years and extract the asymptotic density distribution from the simulated growth rates.

When fitting (1) to the raw GDP data, using the same data set of initially used by Hamilton (1989), we recover the same parameters, which are also close to those obtained for the expanded data from 1947 to 2015. However, the analysis of the GDP density distribution reveals a feature that has been underestimated, namely the fact that the model anchors on the few large negative outliers present in the data. Fitted parameters are reported in figure 1. The maximum likelihood algorithm identifies the few very negative outliers as one regime ($\mu_0 = -1.7$, i.e. -1.7% quarterly, or -6.8% yearly growth rate), and the average 2% yearly growth rate as the second regime ($\mu_1 = 0.5$, i.e. 0.5% quarterly growth). The recession regime is interpreted by the calibration as a rare event (transition probability from average growth regime to recession regime $\approx 3\%$) that immediately jumps back to the normal average growth of roughly 2% yearly growth. This anchoring on the few large negative quarters is clearly revealed by the synthetic GDP density distribution shown in figure 1c.

To obtain results that are more in line with actual business cycle fluctuations, we discard all datapoints below -4% and above 8% yearly growth (below -1% and above 2% growth per quarter). This specific choice for the cut-off range is ad hoc and determined through simple visual inspection. The results are robust to reasonable variations of these two thresholds. Moreover, we overcome this arbitrariness with non-parametric estimates in the subsequent sections. Without the outliers, the maximum likelihood estimates of model (1) converge to more balanced results. The boom and recession regimes are now identified as 0.8% and 3.2% yearly growth, respectively. As is clearly visible in figure 1d, the corresponding stationary growth rate distribution is bimodal. This is in good agreement with non-parametric evidence reported in the subsequent sections. We also observe that the left (recession) peak is larger than the right (boom) peak, indicating an imbalance between the two.

However, the results of this section have to be interpreted with caution, as there may be multiple local optima of the likelihood function. An inspection of the likelihood function (not shown here) also reveals that there are sloppy manifolds in the 9-dimensional parameter space, making the calibration intrinsically difficult with several degenerate solutions (see (Filimonov et al., 2017) and references therein). We thus turn our attention to an alternative, parameter free approach in the next section.

[§] The following parameters have to be fit: $\mu_0 = \mu(S_t = 0), \mu_1 = \mu(S_t = 1), \phi_1, \phi_2, \phi_3, \phi_4, p_{12}, p_{21}$ and σ.

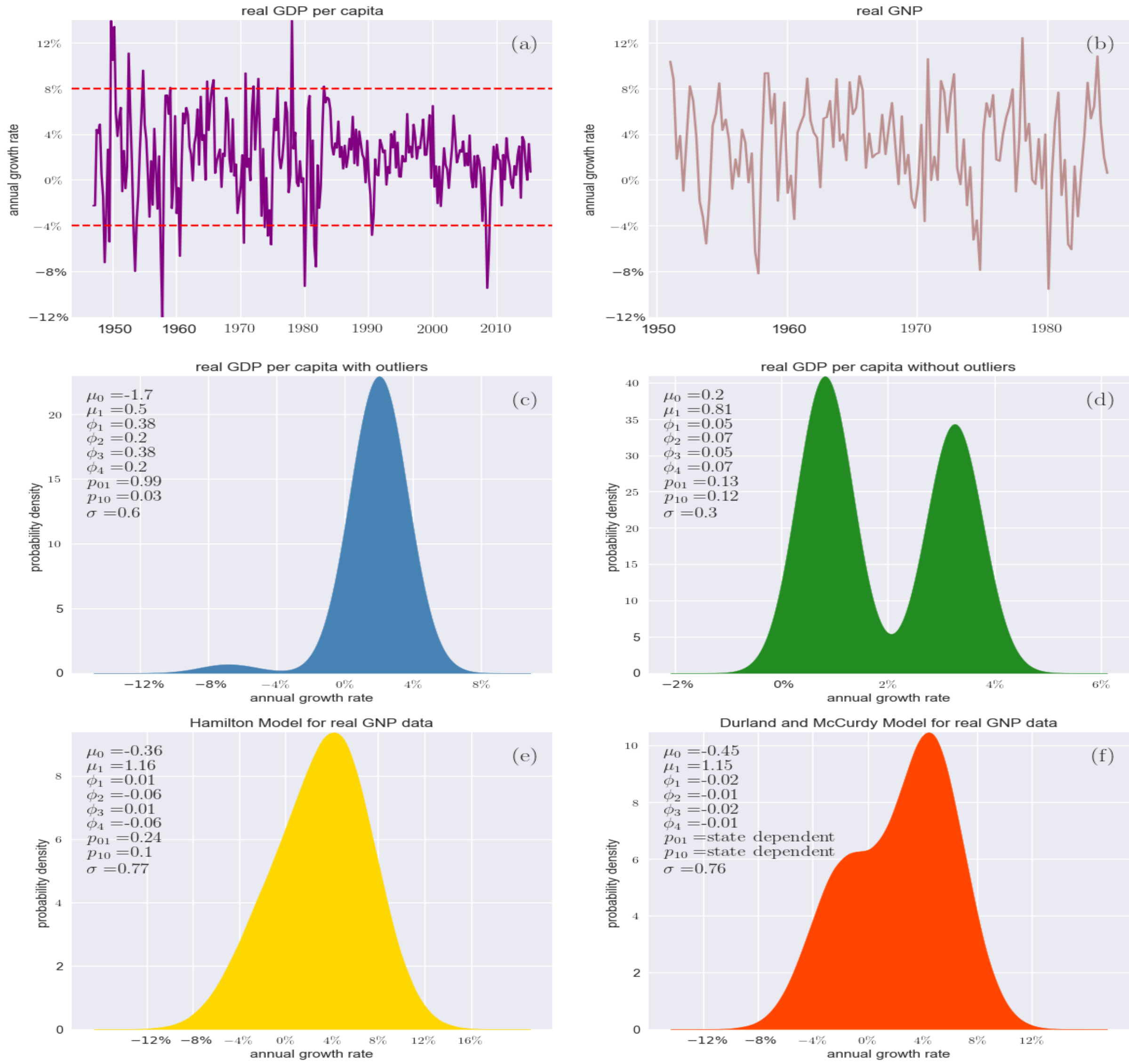

Figure 1. Summary of all results discussed in section 2. For easier interpretation, all quarterly growth rates in this figure have been rescaled to annual values, i.e. multiplied by a factor of four. The top two figures depict quarterly data of real GDP per capita and real GNP growth rates, respectively. The red dashed lines in figure (a) indicate the ad hoc definitions of outliers. For the creation of figure (d), only growth rates within that band have been fitted to the model (1). The resulting asymptotic growth rate distribution is clearly bimodal. This is in contrast to figure (c), where no outliers have been discarded. The bulk of all probability mass forms a unimodal structure, with a small additional hump around -8% for the negative outlier regime. Figures (e) and (f) complement the analysis by showing the predictions of Hamilton (1989) and Durland and McCurdy (1994). While the first distribution is clearly unimodal, the second one exhibits a bimodal structure.

The results of this section are summarized in figure 1. For comparison, we also show the density distribution corresponding to the original fits obtained by Hamilton (1989) and Durland and McCurdy (1994). Interestingly, the two models differ in their prediction: The Hamilton (1989) model converges to a unimodal distribution, whereas the model by Durland and McCurdy (1994) yields a more bimodal structure. It is interesting to see that the two models predict different shapes. One should be cautious in comparing the results of these two models with ours, since they have analyzed nominal GNP data, and not real GDP per capita. We use real GDP per capita since it represents a measure for real innovation and productivity gains. In contrast, the total nominal US GDP contains two additional contributions to its growth: population growth and inflation, both of which do not represent increased income per individual.

3. Scale Free Business Cycles

In order to examine further the modal structure of the USD growth rate distribution, we turn our attention to non-parametric models. The classification of economic growth into phases of booms and busts has been extensively investigated in the business cycle literature. Seminal papers that use non-parametric filters are by Hodrick and Prescott (1997) and Baxter and King (1999).

Parametric or non-parametric, the traditional business cycle literature is concerned with pinning down explicit dates indicating a turning point in an economy. Furthermore, such approaches are often constrained by a priori imposed minimum or maximum business cycle durations, as specified for instance by the NBER business cycle committee (Moore and Zarnowitz,1984). Here, we avoid this approach of preconditioning and let the data decide what are the possible cycles that stand out and justify the identification of a cycle. A first parameter free approach to this problem is to perform a spectral analysis. Figure 2 shows the spectral density $P(f)$ of the $\ln$ (r-US-GDP-pc) for both quarterly data since 1947 and annual data since 1800. We observe a scale-free continuum of frequencies. There are no peak selecting a natural frequency. Instead, all frequencies seem to contribute to the overall GDP dynamics according to a spectral weight following a simple inverse power law function $P(f) \sim 1/f^p$, where $p \approx 1.8$ obtained via a least-squares calibration. In figure 2, we show two lines $P_{RW}(f) \sim 1/f^2$ (spectral density of the random walk) and $P_{1/f}(f) \sim 1/f$ ($1/f$-spectrum) that are often taken as references. This suggests that the fluctuations of the GDP dynamics are close to that of a random walk, albeit with some departure making it less volatilite than a genuine random walk and closer to a stationary (around its long term trend) long-memory process. We stress that this observation, of a smooth continuous spectrum with no special frequency standing out, casts some doubt on the reliability of previous findings on business cycle periods. We are particularly concerned with the procedure consisting in arguing that business cycle periods should be larger than two years, say, to avoid being contaminated by “noise” at short time scales and should be smaller than ten years, say, to avoid the influence of secular trends (Burns and Mitchell, 1946). To us, this looks dangerous because it is obvious that one or two characteristic periods will appear as dominant under these imposed conditions. From a time signal analysis perspective, this corresponds to distorting the spectral density by aliasing by a window constraint.

Given this first evidence that all frequencies are important and essentially undistinguishable, we turn to the wavelet transform as a convenient tool to disentangle frequencies appearing at all scales and determine their times of occurrence.

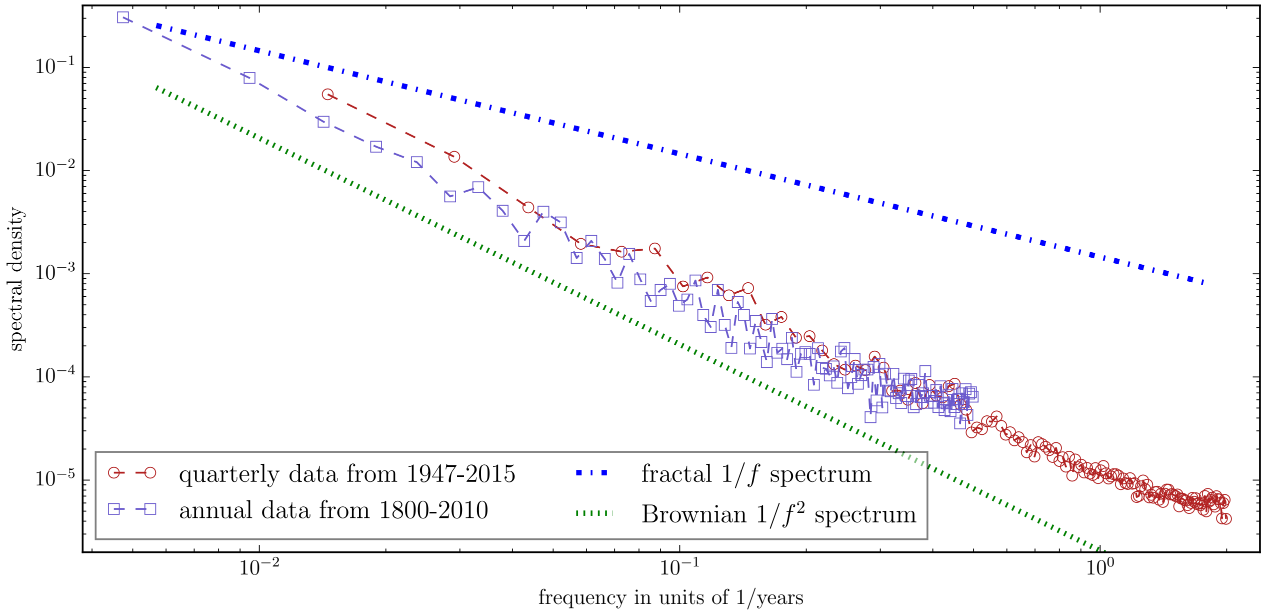

Figure 2. Spectral density of r-US-GDP-pc data. We observe a scale-free continuum of scales with no clear peaks. A least squares fit determines an exponent of ≈ -1.80 for both the quarterly and the annual data set, thus classifying the GDP as a long-memory process.

4. The Wavelet Transform

Originally developed in geophysics to analyze seismic signals (Morlet et al., 1982), the wavelet transform has proven useful for data analysis in a variety of fields such as image processing (Antonini et al., 1992), astrophysics (Slezak et al., 1990), and turbulence (Argoul et al., 1989). In economics, the wavelet transform has many useful applications (Ramsey, 1999; Crowley, 2007), especially in the context of business cycles (Yogo, 2008; Aguiar-Conraria and Joana Soares, 2014; Ardila and Sornette, 2016).

A ψ-wavelet transform W_ψ is simply a projection of a signal $X(\tau)$ onto t-translated and s-dilated versions of ψ (Yiou et al., 2000):

$$W_\psi[X](s,t) = \int_{-\infty}^{\infty} d\tau\, \psi(\tau - t; s)\; X(\tau). \tag{2}$$

We call s the scale and t the time parameter. The analyzing function ψ, called the wavelet, has to be a localized function both in time and frequency domain. Depending on the application, the wavelets must be endowed with several additional properties. See for instance Daubechies (1992) for mathematical details. For our purposes, it is important for the wavelet to be properly normalized. Assuming that $\psi(t; s)$ is approximately zero for values of t outside the interval $[-s, s]$, the wavelet transform has then the following intuitive interpretation: $W_\psi[X](s,t)$ is the weighted average of X over the interval $[t - s, t + s]$. The wavelet transform can thus be seen as a 'mathematical microscope' that resolves local structures of X (determined by the shape of the wavelet $\psi(t; s)$) at 'position' (time) t and at a 'magnification' (scale) s. Denoting by $*$ the convolution operator, expression (2) can also be written compactly as $W_\psi[X](s,t) = [X(\tau) * \psi(\tau; s)](t)$, or, for brevity, just $X * \psi$.

Replacing ψ in (2) by its n-th derivative $\psi^{(n)}$ corresponds to a ψ-analysis of the n-th derivative of the time series $X(t)$ (up to a normalization factor), as a simple integration by parts derivation shows. In this context, $\psi = \psi^{(0)}$ is also called the mother wavelet. Arneodo et al. (1993) show that the overall statistical characterization of complex structures depends only weakly on the choice of the mother wavelet. We will therefore present here only results for the Gaussian mother wavelet $\psi(t; s) = \exp(-t^2/2s^2)/\sqrt{2\pi}s$. We have checked that other real-valued mother wavelets give similar results.

In this article, we use the wavelet transform to quantify the pattern of local slopes (giving the local growth rates) of the analyzed time series (logarithm of the real US GDP per capita). This amounts to replacing ψ in (2) by the first derivative $\psi^{(1)}$ of the Gaussian mother wavelet, up to a normalization. The normalization is chosen such that the wavelet transform of the test signal $X(t) = pt$ with constant slope p gives exactly its slope p for all times t and all scales s. This leads to the following expression for our analyzing mother wavelet used in expression in (2):

$$\psi^{(1)}(t; s) = \frac{t}{\sqrt{2\pi}s^3} \exp\left(-\frac{1}{2}\left(\frac{t}{s}\right)^2\right). \tag{3}$$

Note also that, by construction, the wavelet transform performed with $\psi^{(1)}(t; s)$ of a constant signal is zero, meaning that the wavelet transform is insensitive to the absolute level and only quantifies precisely the local slope at a scale s.

Our approach uses the wavelet transform to achieve two goals in one stroke: (i) estimate the growth rates and (ii) analyse them at multiple scales. This is different from the approach of Aguiar-Conraria and Joana Soares (2014), who first calculated the growth rates by taking the first difference of the logarithm of GDP, and then applied the wavelet transform with a $\psi^{(0)}$ wavelet to the obtained time series of growth rates. Our integrated approach is better suited to minimize aliasing and biases.

In the remainder of this article, all figures are the result of the wavelet transform $X * \psi^{(1)}$ with $\psi^{(1)}$ given by (3). We focus on the quarterly dataset r-US-GDP-pc dataset, and present a similar analysis for the annual data over the much larger extended period between 1800 and 2010 in Appendix A.

5. Wavelet Analysis of the Growth of Real US GDP Per Capita

Plotting the r-US-GDP-pc in a semi-logarithmic plot (figure 3) shows, to a first approximation, a remarkably straight line, suggesting that the GDP grows exponentially as $\exp(\rho_{\rm lt} t)$ with t in units of years and a long-term annual growth rate $\rho_{\rm lt} \approx 2\%$ determined by an ordinary least squares (OLS) fit. This value is often reported in the literature as the average long-term historical growth of real GDP per capita, e.g. (Fernald and Jones, 2014).

Beyond this long term average growth, one can see deviations that occur again and again. Moreover, it is interesting to observe that the long-term growth rate $\rho_{\rm lt}$ represented by the slope of the straight dashed line seems to almost never describe the actual local growth rate of the r-US-GDP-pc. In other words, the average growth rate does not seem to be a good description of the typical growth rates. To quantify these qualitative observations, we perform a wavelet transform analysis of the logarithm of the the r-US-GDP-pc at different times t and different scales s to obtain the local growth rate at time t, averaged over a time interval $[t - s, t + s]$, defined by

$$\rho(s, t) = \ln(\text{r-US-GDP-pc}) * \psi^{(1)} . \tag{4}$$

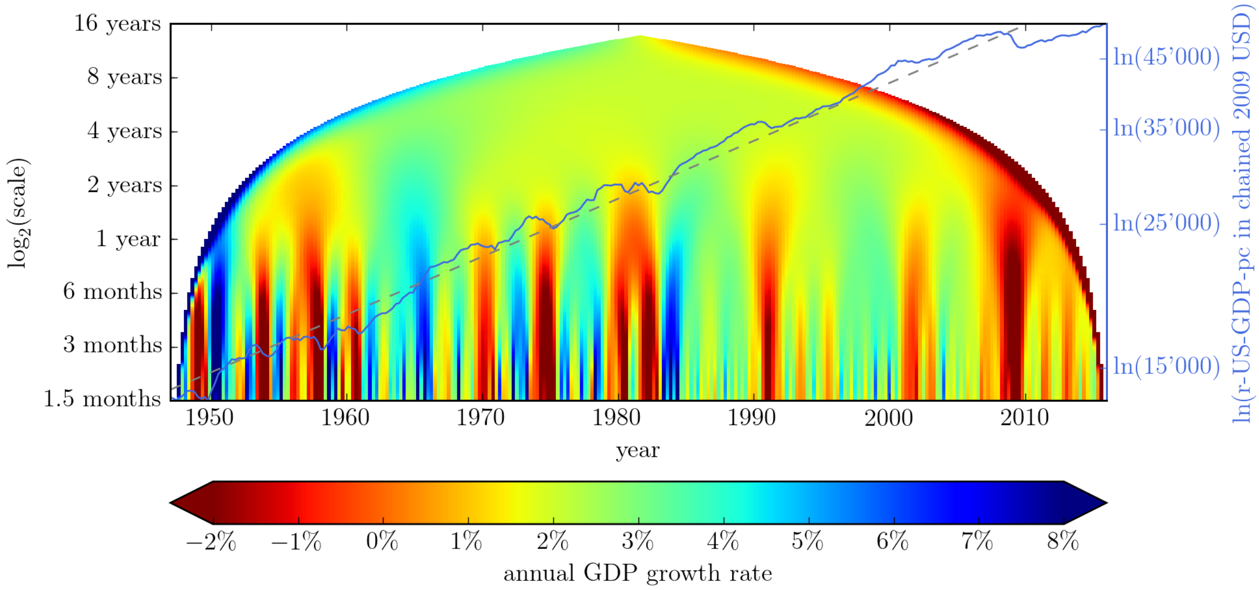

Figure 3. Wavelet transform $\ln(\text{r-US-GDP-pc}) * \psi^{(1)}$ **of the logarithm of the quarterly real US GDP per capita data measured in chained 2009 US dollar over the period from 1947 to 2015 and represented by the continuous dark line (right vertical axis). An ordinary least squares fit determines a long-term annualized growth rate** ρ_{lt} **of approximately** 2%**, shown as the dashed line. The left vertical axis plots the scale** s **of the wavelet analysis, corresponding to an interval of analysis approximately equal to** $2s$**. The color scale encodes the value of the local annualized growth rates at different times and scales. The nonlinear conical shape of the envelop is due to edge-effects in the wavelet transform. The quarterly growth rates have been rescaled to annual values.**

The results are encoded with the color scale for the annualized growth rates in figure 3 over the period from 1947 to 2015 shown on the horizontal axis. The left vertical axis plots the scale s of the wavelet analysis, corresponding to an interval of analysis approximately equal to $2s$. For scales at and lower than $s \approx 4$ years (i.e. averaged over approximately 8 years), one can first observe a hierarchy of branches associated with alternating warm (low or negative growth rates) and cold (positive and strong growth rates) colors. As one goes to smaller and smaller time scales, more fine structures of alternating colors (growth rates) can be seen. At the larger scales, $s \geqslant 4$ years, the color settles to the green value, recovering the known, and also directly determined by OLS, long term growth $\rho_{\text{lt}} \approx 2\%$.

Because the continuous wavelet transform (2) contains a lot of redundant information (a function $X(t)$ of one variable t is transformed into a function $W_\psi[X](s, t)$ of two variables s and t), it is standard to compress the wavelet map shown in figure 3 into a so-called "skeleton". The skeleton of $W_\psi[X](s, t)$ is the set of all local maxima and minima of $W_\psi[X](s, t)$ considered as a function of t, for fixed scale *s*. The skeleton forms a set of connected curves in the time-scale space, called the extrema lines. Geometrically, each such skeleton line corresponds to either a crest or valley bottom of the three-dimensional representation of the wavelet function $W_{\psi}[X](s,t)$. A crest can be viewed as the typical value of the growth rate of a locally surging r-US-GDP-pc. The bottom of a valley is similarly the typical value of the growth rate of a locally slowing down or contracting r-US-GDP-pc. The skeleton structure thus serves as a proxy of (half-) business cycles.

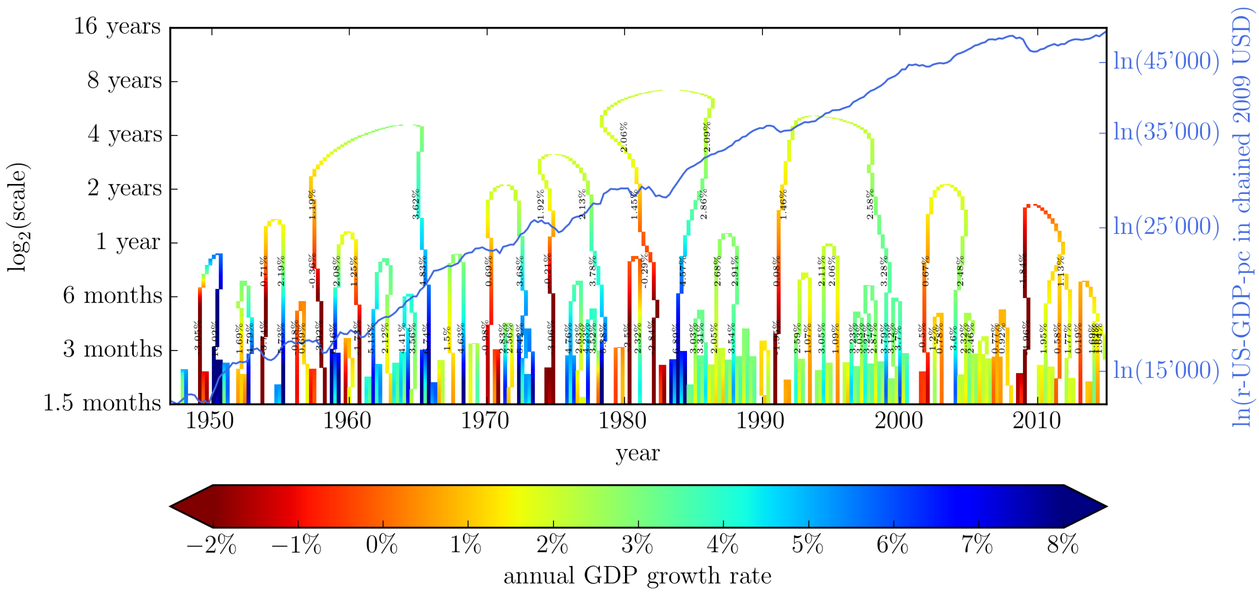

Figure 4. Skeleton structure of the wavelet transform $\ln(\textbf{r-GDP-pc}) * \psi^{(1)}$ **for quarterly real US GDP per capita data measured in chained 2009 US dollar corresponding to figure 3.**

As is clearly visible in figure 4, business cycles are emerging at all scales. There is thus no such thing as "the" business cycle, but rather a continous hierarchy of overlaid business cycle fluctuations. One can observe clearly the hierarchy of alternating growth regimes, which combine into an overall growth of $\approx 2\%$ at large scales. Written along each skeleton line in the figure, we give the values of the local annualized growth rates at four scale levels, 3 months, 6 months, 18 months and 3 years. The structure of the skeleton lines, their colors and the values of the local annualized growth rates confirm the existence of ubiquitous shifting regimes of slow and strong growths. In economics, it has been popular to detrend a time series for its long term growth rate. Our results make this detrending appear somewhat arbitrary, as there is no discrete distinction between short-, medium- and long-term growth.

6. Evidence for a Robust Bimodal Structure of Distributions of US GDP Growth Rates

The nature of the shifting regimes of slow and strong growths can be quantified further by constructing the probability density distributions (pdf) of annualized GDP growth rates at different fixed scales, both from the entire wavelet transform (figure 3) and from the skeleton structure (figure 4). The obtained pdf's for four different scales (6, 9, 15 and 30 months) are depicted in figure 5. They have been obtained using a Gaussian kernel estimations with width equal to 0.002. We have checked the robustness of these pdf's by changing the width of the kernels within a factor of two.

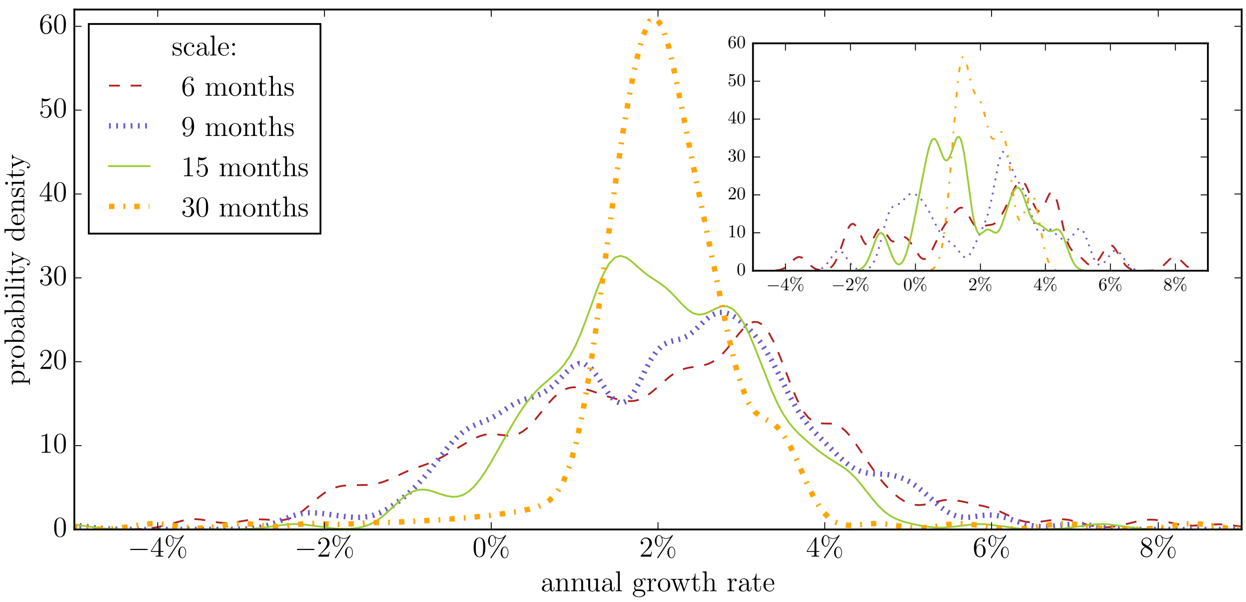

Figure 5. Gaussian kernel estimations (with width equal to 0.002) of the probability density distributions (pdf) of the local annualized growth rates of r-US-GDP-pc at four different scales indicated in the inset in the top-left. The main panel represents the distributions extracted from the wavelet transform shown in figure 3, while the top-right inset shows the pdf's obtained from the skeleton values shown in figure 4.

The pdf's extracted from the wavelet transform shown in figure 3 and from the skeleton values shown in figure 4 exhibit the same structures. First, the pdf's at the largest scale of 30 months peak at the annualized growth rate of $\approx 2\%$, recovering the OLS value reported above (shown as the dashed line in figure 3). Second, as we go down to smaller scales, already at the scale of 15 months, and more pronounced at the scale of 9 and 6 months, a clear bimodal structure emerges (decorated by higher frequency structures, associated with the width of the estimating kernel). Denoting the two main peaks of the bimodal density extracted from the full wavelet transform (the skeleton gives similar results) at scale s by $\rho_{\text{low}}(s)$ and $\rho_{\text{high}}(s)$ respectively, we obtain

$$\rho_{\text{low}}(6\text{ months}) \approx 1\% \lesssim \rho_{\text{low}}(9\text{ months}) \approx 1.1\% \lesssim \rho_{\text{low}}(15\text{ months}) \approx 1.5\% \lesssim \rho_{\text{lt}} \approx 2\% \qquad (5)$$

and

$$\rho_{\text{high}}(6\text{ months}) \approx 3.1\% \gtrsim \rho_{\text{high}}(9\text{ months}) \approx 2.8\% \approx \rho_{\text{high}}(15\text{ months}) \approx 2.8\% \gtrsim \rho_{\text{lt}} \approx 2\%. \qquad (6)$$

The pleasant stability for the estimates $\rho_{\text{low}}(6\text{ months}) \approx \rho_{\text{low}}(9\text{ months})$ and $\rho_{\text{high}}(6\text{ months}) \approx \rho_{\text{high}}(9\text{ months}) \approx \rho_{\text{high}}(15\text{ months})$ suggests that real US GDP per capita can be modelled as an alternation of slow growth around a typical value of 1% and strong growth around a typical value of 3%, which bracket the long-term average growth rate $\rho_{\text{lt}} \approx 2\%$. This bimodality and the appearance of business cycles at all scales constitute the main results of our article.

Table 1. *p*-values for Silverman test, applied to quarterly growth rate distributions at different scales.

	6 months scale	*9 months scale*	*15 months scale*	*30 months scale*
H_1	0.05	0.16	0.22	0.94
H_2	0.31	0.62	0.47	0.09

Finally, we examine the significance of our results from a statistical point of view. How reliable is the observed bimodality, considering that there are only 270 datapoints? How likely is it to observe a bimodal structure, when sampling 270 datapoints from a unimodal density? What is the probability of observing a tri-or multi-modal structure, when sampling from a bimodal density? These questions are formally addressed in the statistical test developed by Silverman (1981). This allows us to test the null hypothesis, H_k, that the density underlying the data has k modes, against the alternative that the density has more than k modes. Here, we test the quarterly growth rates obtained from the wavelet analysis for both unimodality ($k = 1$) and bimodality ($k = 2$). The p-value associated with H_k gives the probability of observing such a sample distribution, assuming that underlying density is truly k-modal. Large p-values for test H_k indicate thus strong statistical evidence for k-modality. The p-values, summarized in table 1, are in good agreement with our expectations. At smaller scales, the bimodality hypothesis H_2 is clearly preferred over a unimodal distribution. At larger scales, this relationship tips over in favor for the unimodal hypothesis H_1.

In conclusion, we have presented evidence that the real US GDP per capita growth rate distribution exhibits a bimodal structure. This is, to the best of our knowledge, the first time that the bimodal character of this distribution has been analyzed explicitly. We stress again the subtle, but very important distinction between the interpretation of business cycles when the GDP distribution is unimodal versus bimodal. A unimodal distribution peaked at 2% represents the case where a 2% growth is the norm, in the sense that the bulk of probability mass is concentrated there. In contrast, a bimodal distribution represents the case where a 2% growth rate is uncommon, in the sense of small probability mass in that regime.

Appendix A presents the wavelet transform, skeleton structure and growth rate distributions for annual r-US-GDP-pc data starting in 1800 till 2010. The important conclusion is that the previous observations presented above for quarterly data from 1950 to 2015 are broadly confirmed when using annual data over this much longer period.

7. Discussion

We have presented a quantitative characterisation of the fluctuations of the annualized growth rate of the real US GDP per capita growth at many scales, using a wavelet transform analysis of two data sets, quarterly data from 1947 to 2015 and annual data from 1800 to 2010. We stress that our use of the chosen mother wavelet (first derivative of the Gaussian function) applied to the logarithm of the real US GDP per capita provides a robust estimation of the instantaneous growth rate at different scales. Our main finding is that business cycles appear at all scales and the distribution of GDP growth rates can be well approximated by a bimodal function associated to a series of switches between regimes of strong growth rate ρ_{high} and regimes of low growth rate ρ_{low}. The succession of alternations of these two regimes compounds to produce a remarkably stable long term average real annualized growth rate of 1.6% from 1800 to 2010 and ≈ 2.0% since 1950.

We thus infer that the robust constant growth rate since 1950 cannot be taken as evidence for a purely exogenous "natural" rate of innovations and productivity growth. It is rather a remarkable output of the succession of booms and corrections that punctuate the economic history of the US since more than 200 years. Our results suggest that alternating growth regimes are intrinsic to the dynamics of the US economy and appear at all scales. These alternating regimes can be identified as generalized business cycles, occurring at the scale of the whole economy.

Such business cycles may be briefly rationalised as follows. During the high growth regime, a number of positive feedback loops are in operation, such as deregulation, enhanced credit creation, the belief in a "new economy" and so on. This creates a transient boom, perhaps accelerating itself and leading to financial and social bubbles (Geraskin, 2013; Sornette, 2014; Yukalov, 2015; Sornette, 2008; Gisler, 2009; Gisler, 2011). This over heating of the economy then turns out not to be sustainable and leads to a correction and consolidation phase, the low growth regime. Then, the next strong growth regime starts and so on.

Our findings suggest that strong growth cannot be dissociated from periods of recessions, corrections or plateaus, which serve as a consolidation phase before the next boom starts. However, because of the remarkable recurrence of the strong regime and in view of its short-term beneficial effects, economists and policy makers tend to form expectations of strong continuous growth. Such way of thinking may lead to conclusions that, we argue, may have little merit. Consider the estimation of the US Federal Reserve Bank of Dallas (Atkinson et al., 2013) that the cost of the 2008 crisis, assuming output eventually returning to its pre-crisis trend path, is an output loss of $6 trillion to $14 trillion US dollars. These enormous numbers are based on the integration of the difference between the extrapolation of hypothetical GDP trajectories expected from a typical return to pre-crisis growth compared with the realised GDP per capita. In the light of our findings, we argue that it is incorrect to extrapolate to the pre-crisis growth rate, which is by construction abnormally high, and much higher than the long term growth rate. In addition, one should take into account the fact that the base rate after a crisis should be low or even negative, for the consolidation to work. Moreover, the duration of the boom years may have direct impact on that of the recovery period. In this vein,Sornette (2014) have argued that this 2008 crisis is special, as it is the culmination of a 30 year trend of accelerating financialization, deregulation and debt growth. Our present results impel the reader to ponder what is the "natural" growth rate and avoid naive extrapolations.

Using a simple generic agent-based model of growth, Louzoun et al. (2003) have identified the existence of a trade off between either low and steady growth or large growth associated with high volatility. Translating this insight to the US economy and combining with the reported empirical evidence, the observed growth features shown in the present paper seem to reveal a remarkable stable relationship between growth and its fluctuations over many decades, if not centuries. Perhaps, this is anchored in the political institutions as well as in the psychology of policy makers and business leaders over the long term that transcend the short-term vagaries of political power sharing and geopolitics. It is however important to include in these considerations the fact that the US is unique compared with other developed countries, having benefitted enormously from the two world wars in particular (compared with the destruction of the French, Japanese and UK empires and the demise of the economic dominance of European powers).

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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Appendix A. Analysis of annual US GDP data

In this appendix, we present wavelet transform, skeleton structure and growth rate distributions for annual r-US-GDP-pc data starting in 1800 till 2010. The important conclusion is that the previous observations presented in the main text for quarterly data from 1950 to 2015 are broadly confirmed when using annual data over this much longer period.

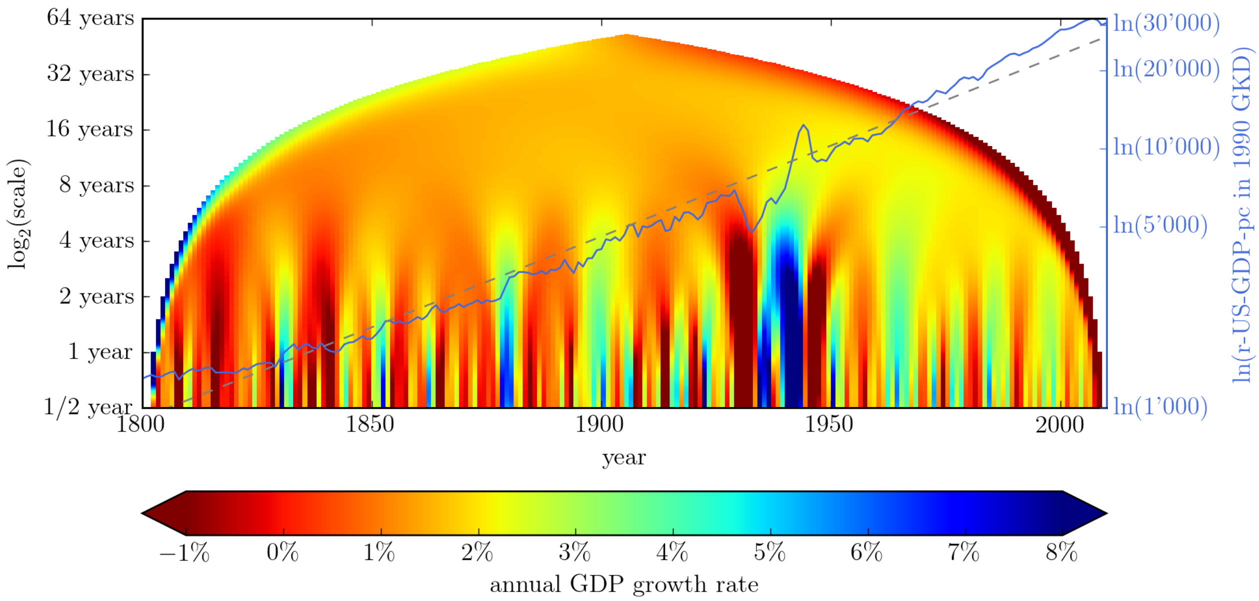

Figure A1. Wavelet transform ln(**r-US-GDP-pc**) $* \psi^{(1)}$ **of the logarithm of the annual real US GDP per capita data measured in 1990 Geary-Khamis dollar and represented by the continuous dark line (right vertical axis). An ordinary least squares fit determines a long-term annualized growth rate** ρ_{lt} **of approximately** 1.6%**.**

Figure A1 shows the wavelet transform ln(r-US-GDP-pc) $* \psi^{(1)}$ of the logarithm of the annual real US GDP per capita data measured in 1990 Geary-Khamis dollar from 1800 to 2010 and represented by the continuous dark line (right vertical axis). An ordinary least squares fit determines a long-term annualized growth rate ρ_{lt} of approximately 1.6%. This value is smaller than the average growth rate of 2% determined for the period from 1950 to 2015. This smaller value is a compromise, given the rather clear long term upward curvature presented by the continuous curve shown in figure A1, expressing a tendency for the growth rate to grow itself (Johanssen and Sornette, 2001). Indeed, one can see that ln(r-US-GDP-pc) departs more and more from the dashed straight line with a larger slope after 1950, in line with the observations shown in figure 3. Figure A2 depicts the skeleton structure extracted from figure A1 and figure A3 plots the distribution of growth rates of the r-US-GDP-pc extracted from annual real US GDP per capita data since 1800. As for the quarterly GDP data, the long term growth rate of $\approx$ 1.5% is recovered as the peak of the distribution at the 8 year scale. The bimodal structure is less clean, due to the fact that annual sampling of GDP growth rates is bound to average over the time scales during which the transitions between the different regimes occur. Nevertheless, one can observe two main peaks, except for the largest time scale of 8

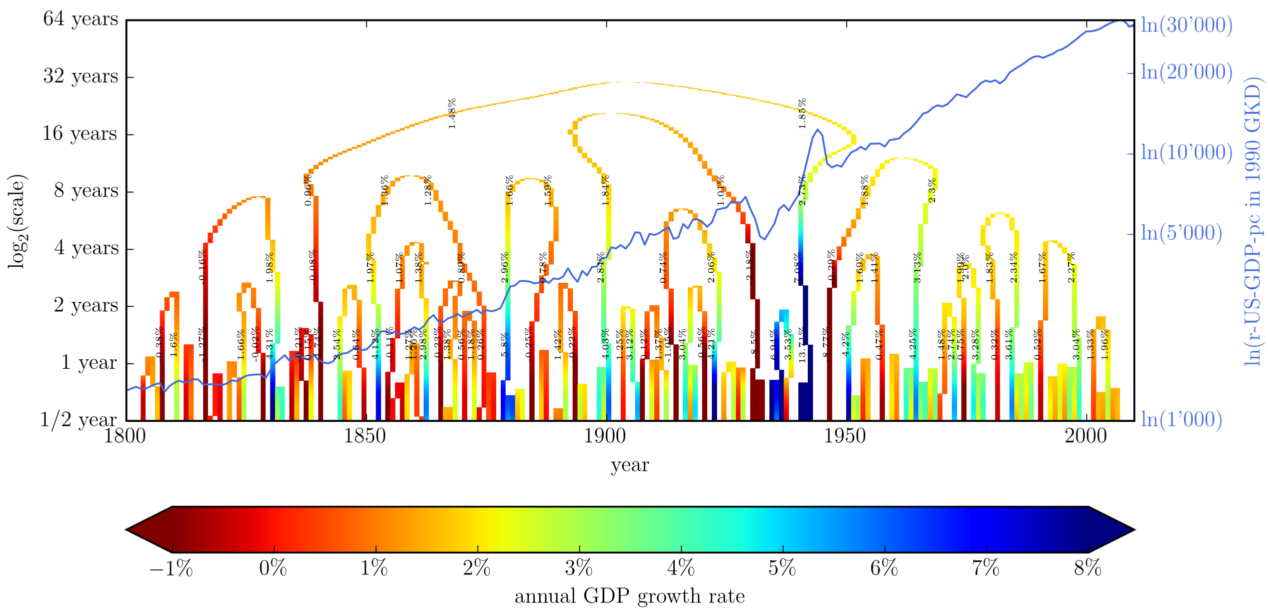

Figure A2. Skeleton structure of $\ln(\text{r-GDP-pc}) * \psi^{(1)}$ for annual real US GDP per capita data measured in 1990 Geary-Khamis dollar corresponding to figure A1.

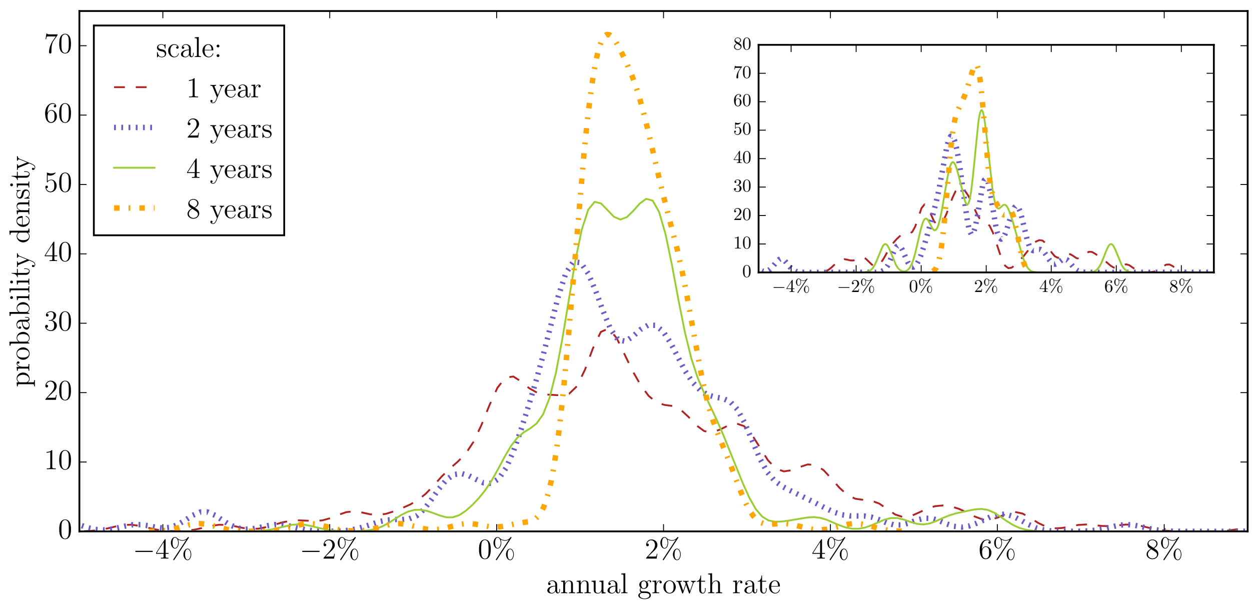

Figure A3. Gaussian kernel estimations (with width equal to 0.002) of the probability density distributions (pdf) of the local annualized growth rates of r-US-GDP-pc sampled annually from 1800 to 2010 at four different scales indicated in the inset in the top-left. The main panel represents the distributions extracted from the wavelet transform shown in figure A1, while the top-right inset shows the pdf's obtained from the skeleton values shown in figure A2.

Table A1. ***p*****-values for Silverman test, applied to annual growth rate distributions at different scales.**

	6 months scale	*9 months scale*	*15 months scale*	*30 months scale*
H_1	0.08	0.05	0.00	0.36
H_2	0.19	0.13	0.38	0.12

years. Moreover, both at the 1 and 2 year scales, the estimated probability density functions (pdf) exhibit a positive skewness, with an asymmetric tail to the right side of large positive growth rates. This means that a considerable part of the probability mass is concentrated at high growth regimes above ρ_{lt}. Quantitatively, the values of the growth rates corresponding to the two main peaks of the pdf's are: $\rho_{\text{low}}(1, 2, 4 \text{ years}) = 0.2\%, 1\%, 1.2\%$ and $\rho_{\text{high}}(1, 2, 4\text{years}) = 1.4\%, 1.9\%, 1.8\%$. The rather low value $\rho_{\text{high}}(1\text{year}) = 1.4\%$ should be considered together with the evidence of the strong positive skewness noted above: at the annual sampling rate, the granularity of the data is too coarse to recover the clean picture of the quarterly data due to overlapping intervals. However, we find that the 50%-quantile growth rate is 1.3%, while the expectation value of the growth rates conditional on growth rates above 1.3% is equal to 3.3%, much larger than ρ_{lt}. The results are thus in broad agreement with those presented in figure 5 for quarterly data. The *p*-values obtained from the Silverman (1981) test, reported in table A1, furthermore support the bimodality hypothesis.