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Research article

Prescribed-time control for impulsive systems with uncertainties via adaptive control

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Abstract: To present, there has been much research on prescribed-time stability (PTS) of uncertain systems, but the significant impulse factor has not been considered. Therefore, in this paper, the stability control problem of a class of impulsive systems with uncertainties within the prescribed time was studied by the Lyapunov functional approach. The comparison lemma was utilized and iteration was carried out for each impulsive interval to prove the PTS theorem for general impulsive systems with uncertainties. In addition, a time-varying adaptive controller in combination with the backstepping method was constructed for PTS of special impulsive strict-feedback systems with uncertainties, breaking through the dependence of traditional methods on uncertain parameters. Finally, a simulation example was used to verify the effectiveness and feasibility of the proposed method.

Keywords: impulsive systems; prescribed-time stability; adaptive control; backstepping method; Lyapunov functional approach

1. Introduction

Stability, as a core attribute, determines the reliability and controllability of system behavior. Finite-time stability (FTS) had garnered increasing attention from researchers due to its capability of achieving convergence within a bounded time frame [1]. Studies [2–4] extended the theory and application of FTS and provided novel control methods for complex systems. For FTS, convergence time remains dependent on initial conditions, making it impossible to precisely schedule system behavior in practical engineering applications. To solve this, fixed-time stability (FxTS) was introduced to guarantee that a system will attain or maintain a stable state within a settling time frame [5–8]. For instance, [8] investigated the achievement of FxTS for unstable impulsive systems, proposing novel FxTS criteria and delivering more precise estimates of the settling time. Nevertheless, FxTS guarantees a fixed upper bound for the settling time, where this upper bound depends on the system parameters.

For the aforementioned limitations of FTS and FxTS, prescribed-time stability (PTS) has emerged. In contrast to FTS and FxTS, PTS ensures that the system can achieve a stable state within a precisely predetermined time, demonstrating stronger flexibility and more promising engineering application potential. In recent years, numerous scholars have achieved a series of remarkable outcomes for PTS [9–13]. For instance, reference [14] investigated an effective PTS strategy based on an extended state observer. Reference [15] studied the PTS impulsive control for nonlinear systems, where impulses can instantaneously adjust the system states at specific moments. The above studies on PTS do not consider the uncertainty factor. For uncertain systems, some works have been reported on PTS [16–19]. Here, we particularly emphasize [19]. Specifically speaking, rooted in the Lyapunov functional approach, reference [19] investigated the PTS adaptive control for nonlinear systems with unknown parameters, where uncertain terms can more realistically reflect various unknown factors for practical systems and enhance system robustness. However, [19] did not take into account the impact of the impulsive factor. As known, the impulsive phenomenon, as a significant factor, is vital for system stability/instability. Therefore, one may ask: Can PTS assertions be achieved for impulsive differential systems with uncertainties (IDSUs) rooted in the Lyapunov functional approach? Giving a clear answer to this question constitutes the foremost motivation for this research.

Strict-feedback systems (SFSs) are a class of nonlinear systems with a lower triangular structure, which establishes a systematic framework for complex systems, particularly for those involving unknown parameters [20–24]. Reference [25] proposed a smooth control method for uncertain nonlinear SFSs with state constraints, achieving tracking control within a finite time. Reference [26] solved FxTS of SFSs. Reference [27] investigated the PTS of nonlinear SFSs and proposed a new non-scaling design method. Furthermore, there are some works [28,29] on impulsive SFSs. However, as far as we know, research on achieving PTS for impulsive SFSs has not been reported so far. Therefore, filling this gap constitutes the second motivation for conducting this research.

Inspired by the aforementioned insights, this paper aims to investigate PTS for impulsive systems with uncertainties and apply the obtained assertion to impulsive SFSs by using the backstepping method to handle uncertainties. Compared with existing works, the main contributions are as follows:

(1) Rooted in the Lyapunov functional approach, we propose a PTS assertion for nonlinear systems with impulsive effects and uncertainties, establishing explicit theoretical conditions for state convergence, thereby opening up new avenues for PTS analysis of such systems.

(2) For impulsive SFSs with uncertainties (ISFSUs), a time-varying adaptive controller is adopted by the backstepping approach to achieve PTS, effectively enhancing the reliability and stability of the system in complex environments.

Table 1. Notation list.

Notation	Meaning
$\mathcal{Z}_+$	Collection of positive integers.
$\mathbb{R}_+$	Collection of non-negative real numbers.
$\mathbb{R}^{r\times r}$	Collection of $r\times r$ real matrices.
$\mathcal{T}_0$	Collection of $\{z_k,\ k\in\mathcal{Z}_+\}$ (short for $\{z_k\}$) satisfying $0<z_1<\cdots<z_k\to+\infty,\ k\to+\infty$.
$\mathbb{K}$	$\mathbb{K}=\{\alpha\in\mathbb{C}\,(\mathbb{R}_+,\mathbb{R}_+)\mid\alpha(0)=0,\ \alpha(s)>0$ for $s>0$ and α is strictly increasing in $s\}$.
$\mathbb{K}_\infty$	$\mathbb{K}_\infty=\{\alpha\in\mathbb{K}:\alpha(s)\to\infty$ as $s\to\infty\}$.
$\mathcal{M}>0$	Matrix $\mathcal{M}$ is positive definite.
$\|\cdot\|$	Norm of a vector.
$C^1(\mathbb{R};\mathbb{R})$	Collection of real-value functions $\mathrm{B}(r)$ on $\mathbb{R}$ being once continuously differentiable on r.
$\lambda_{\min}(\Gamma^{-1})$	Minimum eigenvalue of matrix Γ^{-1}.
$\lambda_{\max}(\Gamma^{-1})$	Maximum eigenvalue of matrix Γ^{-1}.

2. Preliminaries

Consider a nonlinear IDSU

$$\begin{cases}\dot{r}(z)=\mathcal{H}(z,r(z),\eta(z),\delta),z\neq z_k,z\geq 0\\ r(z)=\mathcal{W}\big(r(z^-)\big),z=z_k\\ r(0)=r_0\end{cases}\tag{1}$$

where $r(z)\in\mathbb{R}^n$, $\dot{r}(z)$ denotes the right-hand derivative of $r(z)$, $\eta(z)\in\mathbb{R}^m$ is the control input vector, $\delta\in\mathbb{R}^r$ is the uncertain parameter vector, $\{z_k\}\in\mathcal{T}_0$ stands for the sequence of impulse instants; $\mathcal{H}(\cdot)$: $\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^r\to\mathbb{R}^n$ is Lipschitz with respect to (w.r.t.) r and continuous w.r.t. z with $\mathcal{H}(z,0,0,\delta)=0$; and $\mathcal{W}(\cdot)$: $\mathbb{R}^n\to\mathbb{R}^n$ is continuous satisfying $\mathcal{W}(r)=0$ if $r=0$. The other suitable conditions are assumed to be satisfied to ensure solution $r(z)$ of IDSU (1) uniquely exists. In addition, the right-continuous property and the left limit property hold for $r(z)$.

Definition 1. [19] For $r(0)\in\mathbb{R}^n$, if a time-varying adaptive controller

$$\eta(z)=\eta\left(z,r(z),\widehat{\Delta}\big(z,r(z)\big)\right),\eta\left(z,0,\widehat{\Delta}(z,0)\right)=0\tag{2}$$

$$\dot{\widehat{\Delta}}(z)=\varphi\left(z,r(z),\widehat{\Delta}\big(z,r(z)\big)\right),\varphi\left(z,0,\widehat{\Delta}(z,0)\right)=0\tag{3}$$

exists such that $r(z)$ of IDSU (1) and $\widehat{\Delta}(z)$ are bounded, and $r(z)=0,\forall z\geq\mathbb{Z}_p$, $\mathbb{Z}_p>0$, then IDSU (1) is globally prescribed-time stable ($\mathbb{Z}_p$-GPTS), where $\widehat{\Delta}(z)$ is the estimated value of the uncertain parameter vector Δ which depends on δ. $\widehat{\Delta}(0)=\widehat{\Delta}_0$ and $\widetilde{\Delta}(z)=\Delta(z)-\widehat{\Delta}(z)$ is the estimation error.

Lemma 1. [30] Let $\mathcal{G}(\mathrm{y})\geq 0$ be continuous on $[\mathrm{m},\mathrm{n})$ with a singularity n. If $\lim\limits_{\mathrm{y}\to\mathrm{n}^-}(\mathrm{n}-\mathrm{y})\mathcal{G}(\mathrm{y})=\mathcal{D}$, where $\mathcal{D}>0$ or $\mathcal{D}=+\infty$, then $\int_{\mathrm{m}}^{\mathrm{n}}\mathcal{G}(\mathrm{y})\,d\mathrm{y}=+\infty$.

Lemma 2. [31] If $\dot{\mathbb{T}}(y) = -\mathbb{Q}(y)\mathbb{T}(y) + \mathbb{L}(y)$, then

$$\mathbb{T}(y) = \mathbb{T}(0)e^{-\int_0^y \mathbb{Q}(s)ds} + e^{-\int_0^y \mathbb{Q}(s)ds}\int_0^y \mathbb{L}(s)\, e^{\int_0^s \mathbb{Q}(z)dz} ds.$$

Definition 2. [19] If function $\beta(z)$ is continuous, $\beta(z) > 0$, $\forall z \in [0, \mathbb{Z}_p)$, and $\lim_{z\to\mathbb{Z}_p^-}(\mathbb{Z}_p - z)\beta(z) = \mathcal{Q}$, where $\mathcal{Q} > 0$ or $+\infty$, then $\beta(z)$ is referred to as a prescribed-time adjustment ($\mathbb{Z}_p$-PTA) function.

Remark 1. By Definition 2, one can derive $\lim_{z\to\mathbb{Z}_p^-}\beta(z) = +\infty$. Furthermore, by leveraging Lemma 1, it becomes evident that when $\beta(z)$ is a $\mathbb{Z}_p$- PTA function, and under the assumption of $\lim_{z\to\mathbb{Z}_p^-}(\mathbb{Z}_p - z)\beta(z) = +\infty$, it follows that $\int_0^{\mathbb{Z}_p}\beta(z)dz = +\infty$. In addition, the $\mathbb{Z}_p$-PTA function can be regarded as a special case of the $\mathbb{Z}_p$-finite-time stable function [32]. This is because, on the basis of ensuring FTS, it further provides the capability for precise control and adjustment of the settling time.

Theorem 1. For given $\mathbb{Z}_p > 0$, if there exist functions $\mathrm{B}_1(r(z))$, $\mathrm{B}_2\left(\widetilde{\Delta}(z)\right) \in C^1(\mathbb{R};\mathbb{R})$, and $\varpi_i \in \mathbb{K}_\infty$ $(i = 1, 2, 3, 4)$, such that

$$\text{(C1) } \mathrm{B}(z) = \mathrm{B}_1(r(z)) + \mathrm{B}_2\left(\widetilde{\Delta}(z)\right), \forall r \in \mathbb{R}^n,$$

$$\text{(C2) } \varpi_1(\| r \|) \le \mathrm{B}_1(r) \le \varpi_2(\| r \|), \forall r \in \mathbb{R}^n,$$

$$\text{(C3) } \varpi_3(\| \widetilde{\Delta} \|) \le \mathrm{B}_2(\widetilde{\Delta}) \le \varpi_4(\| \widetilde{\Delta} \|),$$

$$\text{(C4) } \dot{\mathrm{B}}(z) \le -\mathbb{C}\beta(z)\mathrm{B}_1(r(z)),\ z \ne z_k,$$

$$\text{(C5) } \mathrm{B}_1(\mathcal{W}(r)) \le \mathcal{J}\mathrm{B}_1(r),$$

where constants $\mathbb{C} > 0$, $0 < \mathcal{J} < 1$, and $\beta(z)$ is a $\mathbb{Z}_p$-PTA function, then IDSU (1) under controller (2) is $\mathbb{Z}_p$-GPTS.

Proof. The proof is divided into two parts: (A°) We will prove that $r(z)$ and $\widehat{\Delta}(z)$ are bounded. (B°) We will show that IDSU (1) converges to zero within $\mathbb{Z}_p$. For $\mathbb{Z}_p > 0$, an integer Q exists for $z_Q \le \mathbb{Z}_p < z_{Q+1}$.

Part A°: When $z \ne z_k$, from (C4), $\dot{\mathrm{B}}(z) \le 0$; when $z = z_k$, from (C5), one has

$$\begin{aligned}\mathrm{B}(z_k) &\le \mathcal{J}\mathrm{B}_1(r(z_k^-)) + \mathrm{B}_2\left(\widetilde{\Delta}(z_k)\right)\\ &\le \mathrm{B}_1(r(z_k^-)) + \mathrm{B}_2\left(\widetilde{\Delta}(z_k)\right)\\ &= \mathrm{B}(z_k^-).\end{aligned} \tag{4}$$

Therefore, $\mathrm{B}(z)$ is monotonically decreasing for $z \in [0, +\infty)$.

From (C1), one can easily obtain that

$$\mathrm{B}_1(r(z)) \le \mathrm{B}(z) \le \mathrm{B}(0), \mathrm{B}_2\left(\widetilde{\Delta}(z)\right) \le \mathrm{B}(z) \le \mathrm{B}(0). \tag{5}$$

From Eq (5) and (C2), (C3), one has

$$\| r(z) \| \leq \varpi_1^{-1}\left(\mathrm{B}_1\big(r(z)\big)\right) \leq \varpi_1^{-1}\big(\mathrm{B}(0)\big),$$

$$\| \widetilde{\Delta}(z) \| \leq \varpi_3^{-1}\left(\mathrm{B}_2\left(\widetilde{\Delta}(z)\right)\right) \leq \varpi_3^{-1}\big(\mathrm{B}(0)\big),$$

so, both $r(z)$ and $\widetilde{\Delta}(z)$ are bounded. Since $\widetilde{\Delta}(z) = \Delta - \widehat{\Delta}(z)$, it follows that $\widehat{\Delta}(z)$ is also bounded.

Part B°: According to (C1) and (C4), one can easily calculate that

$$\dot{\mathrm{B}}(z) \leq -\mathbb{C}\beta(z)\left(\mathrm{B}(z) - \mathrm{B}_2\left(\widetilde{\Delta}(z)\right)\right) = -\mathbb{C}\beta(z)\mathrm{B}(z) + \mathbb{C}\beta(z)\mathrm{B}_2\left(\widetilde{\Delta}(z)\right). \tag{6}$$

When "=" holds, Eq (6) is recast as a first-order linear differential equation. When $z \in [0, z_1)$, by Lemma 2, one has

$$\mathrm{B}(z) = \mathrm{B}(0)e^{-\mathbb{C}\int_0^z \beta(s)ds} + e^{-\mathbb{C}\int_0^z \beta(s)ds}\int_0^z \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_0^s \beta(v)dv}ds. \tag{7}$$

By the comparison lemma, we have

$$\mathrm{B}(z) \leq \mathrm{B}(0)e^{-\mathbb{C}\int_0^z \beta(s)ds} + e^{-\mathbb{C}\int_0^z \beta(s)ds}\int_0^z \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_0^s \beta(v)dv}ds. \tag{8}$$

Assume that Eq (8) also holds on $z \in [z_{k-1}, z_k)$. Then when $z \in [z_k, z_{k+1})$, from Eq (4), we have

$$\begin{aligned} \mathrm{B}(z_k) &\leq \mathrm{B}(z_k^-) \\ &\leq \mathrm{B}(0)e^{-\mathbb{C}\int_0^{z_k} \beta(s)ds} + e^{-\mathbb{C}\int_0^{z_k} \beta(s)ds}\int_0^{z_k} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_0^s \beta(v)dv}ds. \end{aligned} \tag{9}$$

From (C4), we can get

$$\begin{aligned} \mathrm{B}(z) &\leq \mathrm{B}(z_k)e^{-\mathbb{C}\int_{z_k}^z \beta(s)ds} + e^{-\mathbb{C}\int_{z_k}^z \beta(s)ds}\int_{z_k}^z \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_{z_k}^s \beta(v)dv}ds \\ &\leq \left[\mathrm{B}(0)e^{-\mathbb{C}\int_0^{z_k} \beta(s)ds} + e^{-\mathbb{C}\int_0^{z_k} \beta(s)ds}\int_0^{z_k} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_0^s \beta(v)dv}ds\right]e^{-\mathbb{C}\int_{z_k}^z \beta(s)ds} \\ &\quad + e^{-\mathbb{C}\int_{z_k}^z \beta(s)ds}\int_{z_k}^z \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_{z_k}^s \beta(v)dv}ds \\ &\leq \mathrm{B}(0)e^{-\mathbb{C}\int_0^z \beta(s)ds} + e^{-\mathbb{C}\int_0^z \beta(s)ds}\int_0^{z_k} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_0^s \beta(v)dv}ds \\ &\quad + e^{-\mathbb{C}\int_{z_k}^z \beta(s)ds}\int_{z_k}^z \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_{z_k}^s \beta(v)dv}ds \\ &\leq \mathrm{B}(0)e^{-\mathbb{C}\int_0^z \beta(s)ds} + e^{-\mathbb{C}\int_0^z \beta(s)ds}\int_0^{z_k} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_0^s \beta(v)dv}ds \end{aligned}$$

$$
\begin{aligned}
&+e^{-\mathbb{C}\int_0^z \beta(s)ds} \cdot e^{\mathbb{C}\int_0^{z_k} \beta(s)ds} \int_{z_k}^{z} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right) e^{\mathbb{C}\int_{z_k}^{s} \beta(v)dv} ds \\
\leq\ & \mathrm{B}(0)e^{-\mathbb{C}\int_0^z \beta(s)ds} + e^{-\mathbb{C}\int_0^z \beta(s)ds} \int_0^{z_k} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right) e^{\mathbb{C}\int_0^{s} \beta(v)dv} ds \\
&+ e^{-\mathbb{C}\int_0^z \beta(s)ds} \int_{z_k}^{z} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right) e^{\mathbb{C}\int_0^{s} \beta(v)dv} ds \\
\leq\ & \mathrm{B}(0)e^{-\mathbb{C}\int_0^z \beta(s)ds} + e^{-\mathbb{C}\int_0^z \beta(s)ds} \left[\int_0^{z_k} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right) e^{\mathbb{C}\int_0^{s} \beta(v)dv} ds\right. \\
&\left.+ \int_{z_k}^{z} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right) e^{\mathbb{C}\int_0^{s} \beta(v)dv} ds\right] \\
\leq\ & \mathrm{B}(0)e^{-\mathbb{C}\int_0^z \beta(s)ds} + e^{-\mathbb{C}\int_0^z \beta(s)ds} \int_0^{z} \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right) e^{\mathbb{C}\int_0^{s} \beta(v)dv} ds.
\end{aligned} \tag{10}
$$

Therefore, Eq (8) holds $\forall z \in [0, +\infty)$.

By Definition 2 and Lemma 1, $\lim_{z\to\mathbb{Z}_p^-} \int_0^z \beta(v)dv = +\infty$. Further, it is easy to obtain that

$$
\begin{aligned}
\lim_{z\to\mathbb{Z}_p^-} \mathrm{B}(z) &= \lim_{z\to\mathbb{Z}_p^-} \left(\mathrm{B}_1\big(r(z)\big) + \mathrm{B}_2\left(\widetilde{\Delta}(z)\right)\right) \\
&\leq \lim_{z\to\mathbb{Z}_p^-} \frac{\int_0^z \mathbb{C}\beta(s)\mathrm{B}_2\left(\widetilde{\Delta}(s)\right)e^{\mathbb{C}\int_0^s \beta(v)dv} ds}{e^{\mathbb{C}\int_0^z \beta(s)ds}} \\
&= \lim_{z\to\mathbb{Z}_p^-} \frac{\mathbb{C}\beta(z)\mathrm{B}_2\left(\widetilde{\Delta}(z)\right)e^{\mathbb{C}\int_0^z \beta(v)dv} ds}{\mathbb{C}\beta(z)e^{\mathbb{C}\int_0^z \beta(s)ds}} \\
&= \lim_{z\to\mathbb{Z}_p^-} \mathrm{B}_2\left(\widetilde{\Delta}(z)\right).
\end{aligned} \tag{11}
$$

Consequently, we obtain $\lim_{z\to\mathbb{Z}_p^-} \mathrm{B}_1\big(r(z)\big) = 0$ and $\lim_{z\to\mathbb{Z}_p^-} r(z) = 0$.

When $\mathbb{Z}_p \neq z_Q$, by the continuity of $r(z)$ at $\mathbb{Z}_p$, $r(\mathbb{Z}_p) = 0$; when $\mathbb{Z}_p = z_Q$, by the continuity of $\mathrm{B}_1(r)$ and $\mathcal{W}(r)$, from (C5), $\mathrm{B}_1\left(r(\mathbb{Z}_p)\right) = \mathrm{B}_1\left(\mathcal{W}\left(r(\mathbb{Z}_p^-)\right)\right) = \mathrm{B}_1\left(\mathcal{W}\left(\lim_{z\to\mathbb{Z}_p^-} r(z)\right)\right) = \mathrm{B}_1\left(\lim_{z\to\mathbb{Z}_p^-} \mathcal{W}\big(r(z)\big)\right) = \lim_{z\to\mathbb{Z}_p^-} \mathrm{B}_1\left(\mathcal{W}\big(r(z)\big)\right) \leq \lim_{z\to\mathbb{Z}_p^-} \mathcal{J}\mathrm{B}_1\big(r(z)\big) = 0$. Thus, $r(\mathbb{Z}_p) = 0$ can be concluded. From Eq (2), it follows that $\eta(\mathbb{Z}_p) = 0$.

Further, since $r(z) = 0$ is an equilibrium point of IDSU (1), $\mathcal{H}(z, 0, 0, \delta) = 0$, and $\mathcal{W}(0) = 0$, we have $r(z) = 0, \eta(z) = 0, \forall z \geq \mathbb{Z}_p$. Therefore, $\widehat{\Delta}(z) = \widehat{\Delta}(\mathbb{Z}_p)$ and $\widetilde{\Delta}(z) = \widetilde{\Delta}(\mathbb{Z}_p)$ for $\forall z \geq \mathbb{Z}_p$.

Based on Parts A° and B°, IDSU (1) is $\mathbb{Z}_p$-GPTS. This completes the proof.

Remark 2. Compared with [19], our Theorem 1 focuses on the impulsive factor. Based on the Lyapunov functional and combined with the comparison lemma, we conduct a piecewise recursive proof for each impulsive interval. Furthermore, we rigorously address the particular scenario of whether $\mathbb{Z}_p$ coincides with an impulsive point.

Remark 3. Unlike [16,17], which use parameterized Lyapunov equations to handle uncertain parameters, this paper is based on the general Lyapunov functional approach. This method does not rely on parametric design, but uses traditional tools such as inequality scaling and the comparison lemma to handle the influence of unknown parameters, which avoids excessive reliance on parametric design and offers broader applicability and stronger robustness.

Remark 4. [33] employed matrix-valued Lyapunov functions and the comparison principle to analyze the stability of IDSUs. [34] integrated fractional-order calculus with almost periodicity theory to address the existence and robust stability of almost periodic solutions of IDSUs. In contrast, this paper constructs a general Lyapunov functional approach by introducing the $\mathbb{Z}_p$-PTA function, and subsequently proposes a PTS theorem.

Remark 5. In the proof of Theorem 1, z_Q is introduced to address whether $\mathbb{Z}_p$ coincides with an impulsive point or not. Specifically, for fixed value $\mathbb{Z}_p$ and monotonically increasing impulsive sequence $\{z_k\}$, there must exist an integer Q such that $z_Q \leq \mathbb{Z}_p < z_{Q+1}$. Then, we categorize and discuss: When $\mathbb{Z}_p \neq z_Q$, it follows that $r(\mathbb{Z}_p) = 0$; when $\mathbb{Z}_p = z_Q$, $r(\mathbb{Z}_p) = 0$ holds as well. Thus, $r(\mathbb{Z}_p) = 0$ holds, which provides rigorous logical support for establishing PTS in Theorem 1.

3. Main results

An ISFSU is considered:

$$\begin{cases} \dot{r}_i(z) = r_{i+1}(z) + \delta^T \mathcal{H}_i(\bar{r}_i(z)), i = 1, 2, \ldots, n-1, z \neq z_k \\ \dot{r}_n(z) = \eta + \delta^T \mathcal{H}_n(\bar{r}_n(z)), z \neq z_k \\ r(z) = \mathcal{W}(r(z^-)), z = z_k \end{cases} \tag{12}$$

where $r_j \in \mathbb{R}$, $\eta \in \mathbb{R}$, and $\delta \in \mathbb{R}^r$ are the system state, control input, and uncertain parameter vector, respectively; $\bar{r}_i = [r_1, \cdots, r_i]^T \in \mathbb{R}^i$; $r(z) = \bar{r}_n(z)$; and $\mathcal{H}_i: \mathbb{R}^i \to \mathbb{R}^r$ are known smooth functions with $\mathcal{H}_i(0) = 0, i = 1, \cdots, n$.

Theorem 2. For given $\mathbb{Z}_p > 0$ and impulsive ISFSU (12) with condition (C6),

$$\text{(C6)} \ \|\mathcal{W}(r)\|^2 \leq \mathcal{J}\|r\|^2,$$

one can employ a continuous time-varying adaptive controller

$$\begin{aligned} \eta &= -\hat{\delta}^T \mathcal{H}_n + \Xi, \\ \dot{\hat{\delta}} &= \Gamma \xi_n, 0 \leq z < \mathbb{Z}_p \end{aligned} \tag{13}$$

such that ISFSU (12) is $\mathbb{Z}_p$-GPTS, where $\hat{\delta}$ is the estimated value of δ, $\Gamma > 0 \in \mathbb{R}^{r \times r}$, $\xi_n = \xi_{n-1} + \varepsilon_n \left(\mathcal{H}_n - \sum_{j=1}^{n-1} \frac{\partial \Theta_{n-1}}{\partial r_j} \mathcal{H}_j\right)$, $\Xi = \left(\mathcal{H}_n - \sum_{j=1}^{n-1} \frac{\partial \Theta_{n-1}}{\partial r_j} \mathcal{H}_j\right) \Gamma \sum_{j=2}^{n-1} \frac{\partial \Theta_{j-1}}{\partial \hat{\delta}} \varepsilon_j + \frac{\partial \Theta_{n-1}}{\partial \hat{\delta}} \Gamma \xi_n - \frac{\sigma_n}{\mathbb{Z}_p - z} \varepsilon_n - \varepsilon_{n-1} + \frac{\partial \Theta_{n-1}}{\partial z} + \sum_{j=1}^{n-1}(r_{j+1} + \hat{\delta}^T \mathcal{H}_j)$, $\sigma_n > 1$ is to be designed, and for Θ_{n-1}, $\varepsilon_{n-1}, \varepsilon_n$ refer to its proof.

Proof. Choose $\mathrm{B}(z) = \mathrm{B}_1(\varepsilon(z)) + \mathrm{B}_2(\tilde{\delta}(z))$, $\mathrm{B}_1(\varepsilon(z)) = \varepsilon^T(z)\varepsilon(z)$, $\mathrm{B}_2(\tilde{\delta}(z)) = \tilde{\delta}^T(z)\Gamma^{-1}\tilde{\delta}(z)$, $\varepsilon(z) = (\varepsilon_1(z), \varepsilon_2(z), \cdots, \varepsilon_n(z))^T$, $\tilde{\delta} = \delta - \hat{\delta}$. Hence, conditions (C1–C3), in Theorem 1 hold with $\varpi_1(\|\varepsilon\|) = \varpi_2(\|\varepsilon\|) = \|\varepsilon\|^2$, $\varpi_3(\|\tilde{\delta}\|) = \lambda_{\min}(\Gamma^{-1})\|\tilde{\delta}\|^2$, $\varpi_4(\|\tilde{\delta}\|) = \lambda_{\max}(\Gamma^{-1})\|\tilde{\delta}\|^2$.

When $z \neq z_k$ $\left(0 \leq z < \mathbb{Z}_p\right)$, set

$$\begin{cases} \varepsilon_1 = r_1, \\ \varepsilon_i = r_i - \Theta_{i-1}, i = 2, \cdots, n \end{cases} \tag{14}$$

where Θ_{i-1} refers to the virtual controller designed later.

Next, we will design a prescribed-time controller by means of the backstepping method. The design consists of n steps.

Step 1: From Eq (14), yield

$$\dot{\varepsilon}_1 = \dot{r}_1 = r_2 + \delta^T \mathcal{H}_1 = \Theta_1 + \varepsilon_2 + \delta^T \mathcal{H}_1. \tag{15}$$

Design Θ_1 as

$$\Theta_1 = -\frac{\sigma_1}{\mathbb{Z}_p - z} \varepsilon_1 - \hat{\delta}^T \mathcal{H}_1, \tag{16}$$

where $\sigma_1 > n$. Then, from Eqs (15) and (16),

$$\begin{aligned} \dot{\varepsilon}_1 &= -\frac{\sigma_1}{\mathbb{Z}_p - z} \varepsilon_1 - \hat{\delta}^T \mathcal{H}_1 + \varepsilon_2 + \delta^T \mathcal{H}_1 \\ &= -\frac{\sigma_1}{\mathbb{Z}_p - z} \varepsilon_1 + \Lambda_1, \end{aligned} \tag{17}$$

where $\Lambda_1 = \tilde{\delta}^T \mathcal{H}_1 + \varepsilon_2$.

Define $\mathcal{R}_1 = \|\varepsilon_1\|^2 + \mathrm{B}_2$, and get

$$\begin{aligned} \dot{\mathcal{R}}_1 &= 2\varepsilon_1 \dot{\varepsilon}_1 + 2\tilde{\delta}^T \Gamma^{-1} \dot{\hat{\delta}} \\ &= 2\varepsilon_1 \left(-\frac{\sigma_1}{\mathbb{Z}_p - z} \varepsilon_1 + \tilde{\delta}^T \mathcal{H}_1 + \varepsilon_2 \right) - 2\tilde{\delta}^T \Gamma^{-1} \dot{\hat{\delta}} \\ &= -2\frac{\sigma_1}{\mathbb{Z}_p - z} \|\varepsilon_1\|^2 + 2\varepsilon_1 \varepsilon_2 + 2\varepsilon_1 \tilde{\delta}^T \mathcal{H}_1 - 2\tilde{\delta}^T \Gamma^{-1} \dot{\hat{\delta}} \\ &= -2\frac{\sigma_1}{\mathbb{Z}_p - z} \|\varepsilon_1\|^2 + 2\varepsilon_1 \varepsilon_2 + 2\tilde{\delta}^T \left(\xi_1 - \Gamma^{-1} \dot{\hat{\delta}} \right), \end{aligned} \tag{18}$$

where $\xi_1 = \varepsilon_1 \mathcal{H}_1$.

Step 2: From Eq (14), we have

$$\begin{aligned} \dot{\varepsilon}_2 &= r_3 + \delta^T \mathcal{H}_2 - \dot{\Theta}_1 \\ &= \Theta_2 + \varepsilon_3 + \delta^T \mathcal{H}_2 - \frac{\partial \Theta_1}{\partial \hat{\delta}} \dot{\hat{\delta}} - \frac{\partial \Theta_1}{\partial r_1} (r_2 + \delta^T \mathcal{H}_1) - \frac{\partial \Theta_1}{\partial z}. \end{aligned} \tag{19}$$

Then Θ_2 is designed as

$$\Theta_2 = -\frac{\sigma_2}{\mathbb{Z}_p - z} \varepsilon_2 - \varepsilon_1 - \hat{\delta}^T \mathcal{H}_2 + \frac{\partial \Theta_1}{\partial \hat{\delta}} \Gamma \xi_2 + \frac{\partial \Theta_1}{\partial r_1} \left(r_2 + \hat{\delta}^T \mathcal{H}_1 \right) + \frac{\partial \Theta_1}{\partial z}, \tag{20}$$

where $\sigma_2 > n - 1$ and $\xi_2 = \xi_1 + \varepsilon_2 \left(\mathcal{H}_2 - \frac{\partial \Theta_1}{\partial r_1} \mathcal{H}_1 \right)$. From Eqs (19) and (20),

$$\dot{\varepsilon}_2 = -\frac{\sigma_2}{\mathbb{Z}_p - z}\varepsilon_2 - \varepsilon_1 - \hat{\delta}^T\mathcal{H}_2 + \frac{\partial\Theta_1}{\partial\hat{\delta}}\Gamma\xi_2 + \frac{\partial\Theta_1}{\partial r_1}\left(r_2 + \hat{\delta}^T\mathcal{H}_1\right) + \frac{\partial\Theta_1}{\partial z} + \varepsilon_3 + \delta^T\mathcal{H}_2 - \frac{\partial\Theta_1}{\partial\hat{\delta}}\dot{\hat{\delta}}$$
$$-\frac{\partial\Theta_1}{\partial r_1}\left(r_2 + \delta^T\mathcal{H}_1\right) - \frac{\partial\Theta_1}{\partial z} \tag{21}$$
$$= -\frac{\sigma_2}{\mathbb{Z}_p - z}\varepsilon_2 + \Lambda_2,$$

where $\Lambda_2 = \varepsilon_3 - \varepsilon_1 + \tilde{\delta}^T\mathcal{H}_2 + \frac{\partial\Theta_1}{\partial\hat{\delta}}\left(\Gamma\xi_2 - \dot{\hat{\delta}}\right) - \frac{\partial\Theta_1}{\partial r_1}\tilde{\delta}^T\mathcal{H}_1$.

Define $\mathcal{R}_2 = \mathcal{R}_1 + \|\varepsilon_2\|^2$, and get

$$\dot{\mathcal{R}}_2 = \dot{\mathcal{R}}_1 + 2\varepsilon_2\dot{\varepsilon}_2$$
$$= -2\frac{\sigma_1}{\mathbb{Z}_p - z}\|\varepsilon_1\|^2 + 2\varepsilon_1\varepsilon_2 + 2\tilde{\delta}^T\left(\xi_1 - \Gamma^{-1}\dot{\hat{\delta}}\right) + 2\varepsilon_2\left[-\frac{\sigma_2}{\mathbb{Z}_p - z}\varepsilon_2 + \varepsilon_3 - \varepsilon_1 + \tilde{\delta}^T\mathcal{H}_2\right.$$
$$\left. + \frac{\partial\Theta_1}{\partial\hat{\delta}}\left(\Gamma\xi_2 - \dot{\hat{\delta}}\right) - \frac{\partial\Theta_1}{\partial r_1}\tilde{\delta}^T\mathcal{H}_1\right]$$
$$= -2\frac{\sigma_1}{\mathbb{Z}_p - z}\|\varepsilon_1\|^2 + 2\varepsilon_1\varepsilon_2 + 2\tilde{\delta}^T\left(\xi_1 - \Gamma^{-1}\dot{\hat{\delta}}\right) - 2\frac{\sigma_2}{\mathbb{Z}_p - z}\|\varepsilon_2\|^2 + 2\varepsilon_2\varepsilon_3 - 2\varepsilon_1\varepsilon_2 \tag{22}$$
$$+ 2\varepsilon_2\tilde{\delta}^T\mathcal{H}_2 + 2\varepsilon_2\frac{\partial\Theta_1}{\partial\hat{\delta}}\left(\Gamma\xi_2 - \dot{\hat{\delta}}\right) - 2\varepsilon_2\frac{\partial\Theta_1}{\partial r_1}\tilde{\delta}^T\mathcal{H}_1$$
$$= -\frac{2}{\mathbb{Z}_p - z}\sum_{j=1}^{2}\sigma_j\left\|\varepsilon_j\right\|^2 + 2\tilde{\delta}^T\left(\xi_2 - \Gamma^{-1}\dot{\hat{\delta}}\right) + 2\varepsilon_2\varepsilon_3 + 2\varepsilon_2\frac{\partial\Theta_1}{\partial\hat{\delta}}\left(\Gamma\xi_2 - \dot{\hat{\delta}}\right).$$

Step $i(i = 3, \cdots, n-1)$: From Eq (14), we have

$$\dot{\varepsilon}_i = \varepsilon_{i+1} + \Theta_i + \delta^T\mathcal{H}_i - \frac{\partial\Theta_{i-1}}{\partial z} - \frac{\partial\Theta_{i-1}}{\partial\hat{\delta}}\dot{\hat{\delta}} - \sum_{j=1}^{i-1}\frac{\partial\Theta_{i-1}}{\partial r_j}\left(r_{j+1} + \delta^T\mathcal{H}_j\right). \tag{23}$$

For Eq (23), design Θ_i as

$$\Theta_i = -\frac{\sigma_i}{\mathbb{Z}_p - z}\varepsilon_i - \varepsilon_{i-1} - \hat{\delta}^T\mathcal{H}_i + \frac{\partial\Theta_{i-1}}{\partial z} + \sum_{j=1}^{i-1}\frac{\partial\Theta_{i-1}}{\partial r_j}\left(r_{j+1} + \hat{\delta}^T\mathcal{H}_j\right) + \frac{\partial\Theta_{i-1}}{\partial\hat{\delta}}\Gamma\xi_i$$
$$+\left(\mathcal{H}_i - \sum_{j=1}^{i-1}\frac{\partial\Theta_{i-1}}{\partial r_j}\mathcal{H}_j\right)\Gamma\sum_{j=2}^{i-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j, \tag{24}$$

where $\sigma_i > n - i + 1$ and $\xi_i = \xi_{i-1} + \varepsilon_i\left(\mathcal{H}_i - \sum_{j=1}^{i-1}\frac{\partial\Theta_{i-1}}{\partial r_j}\mathcal{H}_j\right)$. Then

$$\dot{\varepsilon}_i = -\frac{\sigma_i}{\mathbb{Z}_p - z}\varepsilon_i + \Lambda_i, \tag{25}$$

where $\Lambda_i = \tilde{\delta}^T\mathcal{H}_i - \sum_{j=1}^{i-1}\frac{\partial\Theta_{i-1}}{\partial r_j}\tilde{\delta}^T\mathcal{H}_j + \left(\mathcal{H}_i - \sum_{j=1}^{i-1}\frac{\partial\Theta_{i-1}}{\partial r_j}\mathcal{H}_j\right)\Gamma\sum_{j=2}^{i-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j + \frac{\partial\Theta_{i-1}}{\partial\hat{\delta}}\left(\Gamma\xi_i - \dot{\hat{\delta}}\right) + \varepsilon_{i+1} - \varepsilon_{i-1}$.

Define $\mathcal{R}_i = \mathcal{R}_{i-1} + \|\varepsilon_i\|^2$, and get

$$\dot{\mathcal{R}}_i = -\frac{2}{\mathbb{Z}_p - z}\sum_{j=1}^{i}\sigma_j\left\|\varepsilon_j\right\|^2 + 2\varepsilon_i\varepsilon_{i+1} + 2\tilde{\delta}^T\left(\xi_i - \Gamma^{-1}\dot{\hat{\delta}}\right) + 2\sum_{j=2}^{i}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j\left(\Gamma\xi_i - \dot{\hat{\delta}}\right). \tag{26}$$

Step n: From Eq (14), we have

$$\dot{\varepsilon}_n = \eta + \delta^T\mathcal{H}_n - \dot{\Theta}_{n-1} = \eta + \delta^T\mathcal{H}_n - \frac{\partial\Theta_{n-1}}{\partial z} - \frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\dot{\hat{\delta}} - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\left(r_{j+1} + \delta^T\mathcal{H}_j\right). \tag{27}$$

From adaptive controller (13),

$$\begin{aligned}\dot{\varepsilon}_n &= -\hat{\delta}^T\mathcal{H}_n - \frac{\sigma_n}{\mathbb{Z}_p - z}\varepsilon_n - \varepsilon_{n-1} + \frac{\partial\Theta_{n-1}}{\partial z} + \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\left(r_{j+1} + \hat{\delta}^T\mathcal{H}_j\right) + \delta^T\mathcal{H}_n \\ &\quad + \frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\Gamma\xi_n + \left(\mathcal{H}_n - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\mathcal{H}_j\right)\Gamma\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j - \frac{\partial\Theta_{n-1}}{\partial z} - \frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\dot{\hat{\delta}} \\ &\quad - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\left(r_{j+1} + \delta^T\mathcal{H}_j\right) \\ &= -\frac{\sigma_n}{\mathbb{Z}_p - z}\varepsilon_n + \Lambda_n,\end{aligned} \tag{28}$$

where $\Lambda_n = \frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\left(\Gamma\xi_n - \dot{\hat{\delta}}\right) + \left(\mathcal{H}_n - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\mathcal{H}_j\right)\Gamma\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\tilde{\delta}^T\mathcal{H}_j + \tilde{\delta}^T\mathcal{H}_n - \varepsilon_{n-1}$.

Define $\mathcal{R}_n = \mathcal{R}_{n-1} + \|\varepsilon_n\|^2 = \mathrm{B}(\varepsilon(z))$, and get

$$\begin{aligned}\dot{\mathcal{R}}_n &= \dot{\mathcal{R}}_{n-1} + 2\varepsilon_n\dot{\varepsilon}_n \\ &= -\frac{2}{\mathbb{Z}_p - z}\sum_{j=1}^{n-1}\sigma_j\left\|\varepsilon_j\right\|^2 + 2\tilde{\delta}^T\left(\xi_{n-1} - \Gamma^{-1}\dot{\hat{\delta}}\right) + 2\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j\left(\Gamma\xi_{n-1} - \dot{\hat{\delta}}\right) \\ &\quad + 2\varepsilon_{n-1}\varepsilon_n + 2\varepsilon_n\left[-\frac{\sigma_n}{\mathbb{Z}_p - z}\varepsilon_n + \tilde{\delta}^T\mathcal{H}_n - \varepsilon_{n-1} + \frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\left(\Gamma\xi_n - \dot{\hat{\delta}}\right)\right. \\ &\quad \left. - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\tilde{\delta}^T\mathcal{H}_j + \left(\mathcal{H}_n - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\mathcal{H}_j\right)\Gamma\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j\right] \\ &= -\frac{2}{\mathbb{Z}_p - z}\sum_{j=1}^{n-1}\sigma_j\left\|\varepsilon_j\right\|^2 + 2\tilde{\delta}^T\left(\xi_{n-1} - \Gamma^{-1}\dot{\hat{\delta}}\right) + 2\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j\left(\Gamma\xi_{n-1} - \dot{\hat{\delta}}\right) \\ &\quad + 2\varepsilon_{n-1}\varepsilon_n - 2\frac{\sigma_n}{\mathbb{Z}_p - z}\|\varepsilon_n\|^2 + 2\tilde{\delta}^T\varepsilon_n\mathcal{H}_n - 2\varepsilon_{n-1}\varepsilon_n + 2\frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\varepsilon_n\left(\Gamma\xi_n - \dot{\hat{\delta}}\right) \\ &\quad - 2\varepsilon_n\sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\tilde{\delta}^T\mathcal{H}_j + 2\varepsilon_n\left(\mathcal{H}_n - \sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\mathcal{H}_j\right)\Gamma\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j \\ &= -\frac{2}{\mathbb{Z}_p - z}\sum_{j=1}^{n}\sigma_j\left\|\varepsilon_j\right\|^2 + 2\tilde{\delta}^T\left(\xi_{n-1} - \Gamma^{-1}\dot{\hat{\delta}} + \varepsilon_n\mathcal{H}_n - \varepsilon_n\sum_{j=1}^{n-1}\frac{\partial\Theta_{n-1}}{\partial r_j}\mathcal{H}_j\right) \\ &\quad + 2\frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\varepsilon_n\left(\Gamma\xi_n - \dot{\hat{\delta}}\right) + 2\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j\left(\Gamma\xi_{n-1} - \dot{\hat{\delta}} + \Gamma\xi_n - \Gamma\xi_{n-1}\right) \\ &= -\frac{2}{\mathbb{Z}_p - z}\sum_{j=1}^{n}\sigma_j\left\|\varepsilon_j\right\|^2 + 2\tilde{\delta}^T\left(\xi_n - \Gamma^{-1}\dot{\hat{\delta}}\right) + 2\frac{\partial\Theta_{n-1}}{\partial\hat{\delta}}\varepsilon_n\left(\Gamma\xi_n - \dot{\hat{\delta}}\right) \\ &\quad + 2\sum_{j=2}^{n-1}\frac{\partial\Theta_{j-1}}{\partial\hat{\delta}}\varepsilon_j\left(\Gamma\xi_n - \dot{\hat{\delta}}\right) \\ &= -\frac{2}{\mathbb{Z}_p - z}\sum_{j=1}^{n}\sigma_j\left\|\varepsilon_j\right\|^2 \\ &\le -2\beta(z)\mathrm{B}_1\end{aligned} \tag{29}$$

where $\beta(z)=\frac{\sigma}{\mathbb{Z}_p-z}$ is a $\mathbb{Z}_p$-PTA function and $\sigma=\min\limits_{i\epsilon\{1,\cdots,n\}}\{\sigma_i\}>1$. Hence, condition (C4) in Theorem 1 holds for impulsive ISFSU (12).

When $z=z_k$, since there is no control on impulsive ISFSU (12) at z_k, let $\Theta_i(z_k)=0$. Further,

$$\begin{cases}\varepsilon_1(z_k)=r_1(z_k)=\mathcal{W}\big(r_1(z_k^-)\big),\\ \varepsilon_2(z_k)=r_2(z_k)=\mathcal{W}\big(r_2(z_k^-)\big),\\ \qquad\cdots\\ \varepsilon_n(z_k)=r_n(z_k)=\mathcal{W}\big(r_n(z_k^-)\big),\end{cases}\tag{30}$$

and from (C6), we have

$$\begin{aligned}\mathrm{B}_1\big(\varepsilon(z_k)\big)&=r^T(z_k)r(z_k)\\&=\big\|\mathcal{W}\big(r(z_k^-)\big)\big\|^2\\&\leq\mathcal{J}\|r(z_k^-)\|^2\\&\leq\mathcal{J}\mathrm{B}_1\big(\varepsilon(z_k^-)\big).\end{aligned}\tag{31}$$

Thus, condition (C5) in Theorem 1 holds.

Based on Theorem 1, we conclude that impulsive ISFSU (12) is $\mathbb{Z}_p$-GPTS.

Remark 6. In the proof of Theorem 2, when $z\neq z_k$, $\Theta_i(z)$ are designed recursively based on the backstepping technique, progressively constructing a Lyapunov functional and deriving the specific form of controller $\eta(z)$; when $z=z_k$, $\Theta_i(z)$ is used to verify that condition (C5) of Theorem 1 can still be satisfied here.

Remark 7. Similarly to Remark 2, compared with [19], this paper constrains the state jump at impulse instants z_k through condition (C6). When $z\neq z_k$, controllers are designed via the backstepping method, ultimately guaranteeing system stability.

Remark 8. Compared with [28], this paper further addresses the uncertain parameters in ISFSU (12) and conducts research on PTS. Distinct from [29], this paper employs the backstepping technique to design the controller, and achieves PTS by adjusting functions.

Remark 9. Unlike [23,24,35] on adaptive control, this paper focuses on the PTS issue, requiring system states to strictly converge to zero within $\mathbb{Z}_p$ that can be arbitrarily specified and are independent of the initial conditions. Meanwhile, system uncertainties are addressed through a designed adaptive law rather than relying on fuzzy basis functions to estimate unknown terms, thereby reducing computational redundancy and estimation errors while lowering controller complexity. Furthermore, although [23,24,35] and this paper use the backstepping method to design controllers, this paper further takes into account the impact of impulsive effects, making the design more suitable for practical applications.

Remark 10. For uncertain systems, [36] implements a controller design using a data-driven hybrid iteration algorithm; whereas this paper uses the backstepping method to design an adaptive prescribed-time control strategy. Despite these differences, the innovative data-driven controller design method is particularly important to us.

4. Simulation results

Consider an ISFSU

$$\begin{cases} \dot{r}_1(z) = r_2(z) + \delta r_1(z), z \neq z_k, \\ \dot{r}_2(z) = r_3(z), z \neq z_k, \\ \dot{r}_3(z) = \eta + \delta r_3^2(z), z \neq z_k, \\ r_i(z) = \mathcal{A} r_i(z^-), z = z_k, i = 1, 2, 3, \end{cases} \tag{32}$$

with initial conditions IC1 $= (1.0, -2.0, 1.1)$, IC2 $= (-0.8, 2.1, -1.0)$, IC3 $= (0.6, -1.0, -1.2)$, $\mathcal{A} = 0.8$, $\delta = 0.9$, and impulsive points $z_k = 0.2, 0.4, 0.6, \cdots$. When the prescribed time is chosen as $\mathbb{Z}_p = 3$, as illustrated in Figure 1, when there is no controller η, the state trajectory of ISFSU (32) fails to reach stability within $\mathbb{Z}_p$. Hence, it is necessary to employ the appropriate controller η to achieve $\mathbb{Z}_p$-GPTS.

Based on the design procedure outlined in Section 3, we can formulate the prescribed-time adaptive controller as follows:

$$\eta = -\frac{\sigma_3}{\mathbb{Z}_p - z}\varepsilon_3 + \Xi, \dot{\hat{\delta}} = \xi, z \in [0, \mathbb{Z}_p), \tag{33}$$

where $\Xi = \frac{\partial \Theta_2}{\partial r_1}(r_2 + \hat{\delta}^T r_1) + \frac{\partial \Theta_2}{\partial r_2} r_3 + \frac{\partial \Theta_2}{\partial z} + \frac{\partial \Theta_2}{\partial \hat{\delta}}\xi - \varepsilon_2 \frac{\partial \Theta_1}{\partial \hat{\delta}}\left(r_3 - \frac{\partial \Theta_2}{\partial r_1} r_1\right) - \varepsilon_2 - \hat{\delta} r_3$, $\xi = \varepsilon_1 r_1 - \varepsilon_2 \frac{\partial \Theta_1}{\partial r_1} r_1 + \varepsilon_3\left(r_3^2 - \frac{\partial \Theta_2}{\partial r_1} r_1\right)$. The design parameters are selected as $\sigma_1 = \sigma_2 = \sigma_3 = 5$. The simulation results are illustrated in Figures 2–4. It can be observed from Figure 2 that, under different initial conditions, the controlled system (32) converges to zero within $\mathbb{Z}_p$. As shown in Figure 3 and 4, the controller (33) is continuous and also converges to zero within $\mathbb{Z}_p$, while the parameter estimation $\hat{\delta}$ remains bounded.

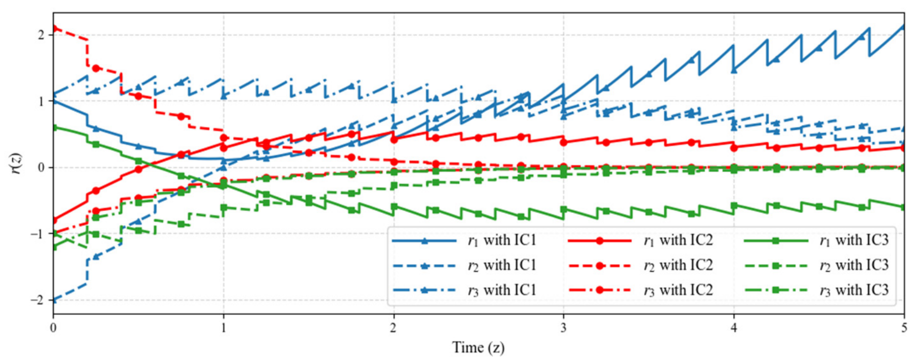

Figure 1. Response of r of ISFSU (32) without a controller under different initial conditions.

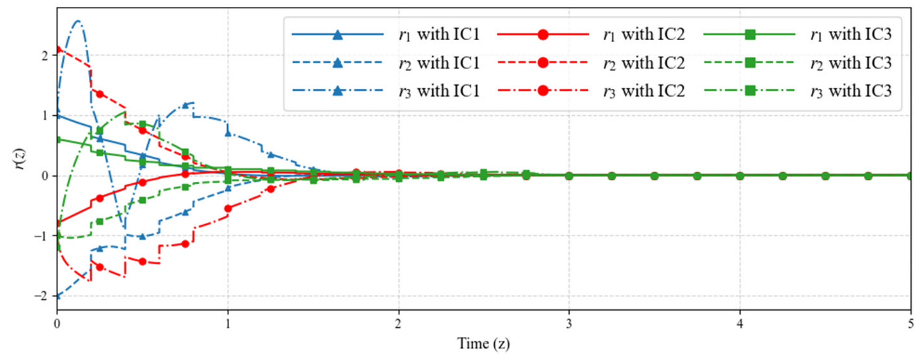

Figure 2. Responses of r of ISFSU (32) under the control input (33) with different initial conditions.

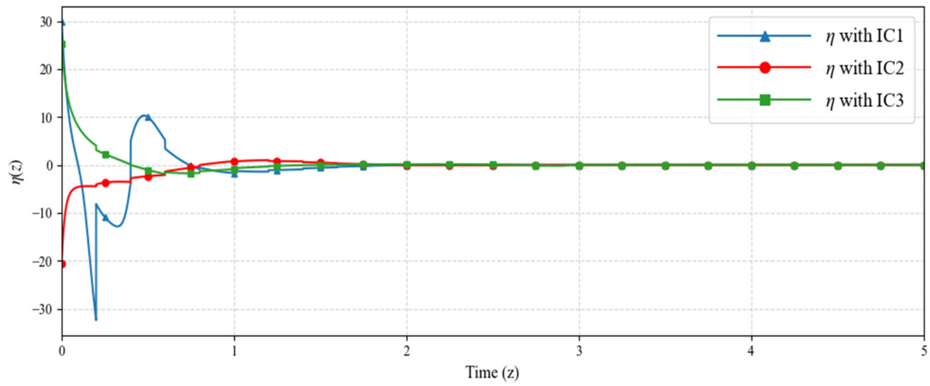

Figure 3. Responses of η with different initial conditions.

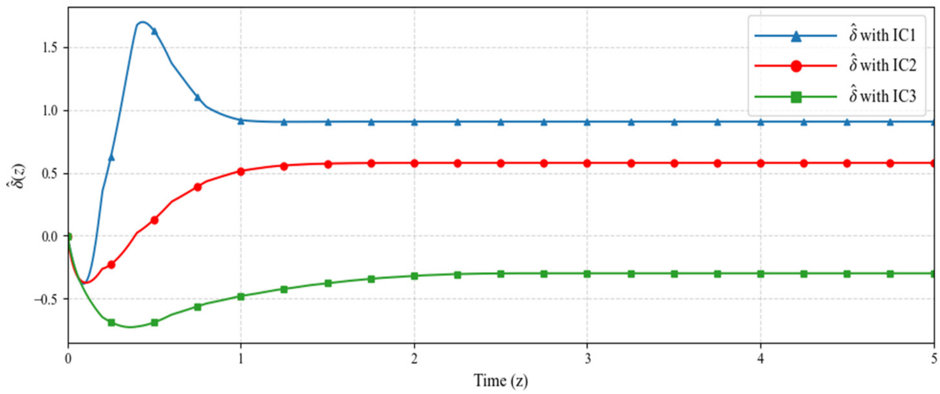

Figure 4. Responses of $\hat{\delta}$ under the control input (33) with different initial conditions.

5. Conclusions

This research proposes a prescribed-time control method for nonlinear IDSUs via adaptive control. First, a PTS theorem is put forward for IDSUs, and its proof is completed by constructing a Lyapunov functional. Then, for an ISFSU, a feedback controller is designed using the backstepping method, and a time-varying gain control law is constructed, avoiding the influence of parameterization, and enabling the system to be $\mathbb{Z}_p$-GPTS.

In future work, we plan to generalize condition (C4) to some broader cases, such as $\dot{\mathrm{B}}(z) \leq -\mathbb{C}\beta(z)\mathrm{B}\big(r(z)\big)$ or $\dot{\mathrm{B}}(z) \leq -a\mathrm{B}_1^{\mathrm{p}}\big(r(z)\big) + b\mathrm{B}_1^{\mathrm{q}}\big(r(z)\big)$ with proper parameters a, b, p, q, $z \neq z_k$. These developments relax the strict constraints on $\mathrm{B}(z)$, allowing it to exhibit more complex functional forms, with the aim of extending applicability to a broader class of systems. Furthermore, we will design an appropriate controller that will adapt to the above developments to ensure system stability and control performance.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

Chenrong Niu: Investigation, Methodology, Writing, Validation; Chunyan Zhang: Investigation, Validation; Liping Du: Writing, Software; Lichao Feng: Conceptualization, Investigation, Supervision, Funding acquisition.

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