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Research article

Note on adaptive prescribed-time stabilization of nonlinear systems with uncertainty

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Abstract: Krishnamurthy et al. investigated the adaptive output feedback control for prescribed-time stability (PTS) of nonlinear uncertain systems on $[0,T)$, where $T>0$. However, there are special constraints on system structure, and the PTS issue is considered on a finite interval $[0,T)$. For systems without required constraints, the existing adaptive output feedback control seems not to be applicable; the PTS issue on $[0,\infty)$ is more practical than that on $[0,T)$. Motivated by the above, an improved observer and controller were proposed to achieve PTS for nonlinearly uncertain systems with lower-triangular linear growth condition on uncertainties. Compared with the existing work, we emphasized the subsequent contributions: 1) Relax the original structure constraint of objective system; 2) achieve PTS on $[T,\infty)$; and 3) keep the proposed controller bounded. The effectiveness of the controller was verified by numerical simulations across varying initial conditions and prescribed times.

Keywords: prescribed-time stabilization; uncertain systems; output feedback control

1. Introduction

The stability control of nonlinear systems has always been a widely concerned problem in the control field. The design goal of asymptotic control is usually to guarantee various asymptotic properties of a closed-loop system, such as ensuring that system states (or output) converge to a desired value (for example, the origin) when t reaches ∞ [1,2]. Asymptotic control of complex nonlinear systems can be achieved through various effective feedback control strategies. For example, Tsinias [3] showed that a linear state feedback control (SFC) could be explicitly constructed to globally stabilize the nonlinear system. Further studies have been reported on the output feedback control (OFC) method [4–9] Specifically speaking, an OFC design was proposed [10], which first designed the observer without considering nonlinearity, utilized a scale gain to control nonlinearity, and extended the results from state feedback to output feedback. Qian and Du [11] proposed a sampled-data OFC scheme to guarantee the global stability of a closed-loop system. For uncertain systems, uncertainties could be effectively suppressed by OFC [12]. Chen et al. [13] achieved global OFC for a class of nonlinear systems, which established a stabilization frame for systems with output uncertainty. Liang et al. [14] introduced two observers with high gains to realize the global adaptive stabilization of smooth OFC for uncertain systems with polynomial growth. However, the above works guarantee only asymptotic convergence (i.e., as $t \to \infty$).

In practical operation, the stabilization of systems needs to be achieved within a finite time frame. Various techniques have been developed for achieving finite-time stabilization (FTS) [15–19]. Although FTS has its own advantages, the settling time depends on system conditions. This leads to the issue of fixed-time stability (FxTS), which can estimate an upper bound on settling time without any initial condition [20–23]. However, in some special cases, settling time is required to be arbitrarily chosen, which makes people pay more attention to prescribed-time stability (PTS). Moreover, the PTS issue has attracted much attention [24–28].

Researchers have reported some literature on PTS for nonlinear systems by OFC. For instance, Holloway and Krstic [29] introduced an OFC to make a linear system converge to zero in a prescribed time. Distinct from the conventional method of scaling the state by the time function, a method of the power of the dynamic scaling parameter was proposed in [30]. Particularly, we emphasize work [31], which extended the SFC method in [30] to OFC based on dual-dynamic high-gain observe-controller technology, and realized the PTS of nonlinear strict feedback systems. It is worth noting that there are special restrictions on the system structure of [31], that is, the upper diagonal terms of the system are needed to meet the advantages of bidirectional cascade, and the uncertain terms are needed to satisfy the special growth conditions. In addition, [31] achieves PTS only on $[0, T)$ without considering $[0, \infty)$, where $T > 0$. Due to these constraints, the control scheme proposed in [31] may fail for uncertain systems without the above structural and growth conditions; moreover, being stable on $[0, T)$ does not guarantee that for $[0, \infty)$. Thus, some questions arise naturally: For a general nonlinear system with uncertainty, is it possible to construct an observer and controller to achieve the PTS under general restrictions on nonlinear terms? If possible, under what conditions can one design such a controller, and how? Furthermore, can the controller be designed to guarantee the PTS on $[0, \infty)$? Answering these questions is the motivation of this paper. Further, we give positive answers for them.

Inspired by the above, we investigate the PTS problem for nonlinear systems with nonlinear uncertainty by OFC. First, the powers of dynamic scaling gain parameters are introduced into the scaling of observer error and observer state estimation. Then, the time scalar transform is introduced to map a prescribed time range to infinite time range, and the time scaling system after the equivalent transformation is used to design the observer-controller structure based on dynamic scaling. Finally, the dynamic scaling gain parameters and controller are designed to make system states converge to 0 in the prescribed time. In addition, the controller is fully bound throughout the process. Compared to existing results, we emphasize the following major contributions.

(i) Unlike [30,31] with uncertain terms satisfying special nonlinear structural conditions, we consider a class of nonlinear systems where the uncertain terms satisfy the lower triangle growth condition. This provides a clearer framework for controller design and stability analysis.

(ii) Different from asymptotic stability [10–14] and FxTS [20–23], the controller proposed here achieves system stability in the prescribed time. Specifically, system output and observer states converge to zero in the prescribed time.

(iii) Unlike [31], for a prescribed time T, we not only design OFC on $[0,T)$, but also on $[0,\infty)$, which enables the system output and observed state to converge to 0 at T and remain 0 thereafter.

Notation: For matrix A, denote its transpose by A^T, and if A is square, $\lambda_i(A)$, $\lambda_{max}(A)$, and $\lambda_{min}(A)$denote its ith eigenvalue, maximal and minimal eigenvalues, respectively. $A>0$ represents A as a symmetric positive definite matrix. $diag\{a_1,a_2,\cdots,a_n\}$ denote a diagonal matrix whose ith diagonal element is a_i. $I_n\in R^{n\times n}$ represents the identity matrix. For vector $X=(x_1,\cdots,x_n)$, its one-norm is denoted by $\|X\|_1=\sum_{i=1}^{n}|x_i|$.

2. Problem description

The system studied here is as follows:

$$\begin{cases}\dot{\zeta}_1=\zeta_2+f_1(\zeta,u,t)\\ \vdots\\ \dot{\zeta}_{n-1}=\zeta_n+f_{n-1}(\zeta,u,t)\,,\\ \dot{\zeta}_n=u+f_n(\zeta,u,t)\\ y=\zeta_1\end{cases}\tag{1}$$

where $\zeta=[\zeta_1,\zeta_2,\cdots,\zeta_n]^T\in R^n$ is state vector, $u\in R$ is control input, $y\in R$ is output, and $F=[f_1,f_2,\cdots,f_n]^T$ is an uncertain continuous vector function caused by external disturbance and internal modeling error or uncertainty, where $f_i(\zeta,u,t):R^n\times R\times R^+\to R, i=1,2,\cdots,n$, are uncertain continuous vector functions.

Next, some definitions, an assumption, and a lemma are listed.

Definition 1 [32]. (PTS on $[\mathbf{0},\boldsymbol{T})$) The nonlinear system $\dot{\zeta}=S(t,\zeta,F)$ with $S:R^+\times R^n\times R^n\to R^n, t\geq 0$ is said to be prescribed-time stable on $[0,T)$ if for any $\zeta(0)\in R^n$, there exists a constant $T>0$ such that $\lim\limits_{t\to T}\|\zeta(t)\|=0$.

Definition 2 [33]. (PTS on $[\mathbf{0},\infty)$) The nonlinear system $\dot{\zeta}=S(t,\zeta,F)$ with $S:R\times R^n\times R^n\to R^n, t\geq 0$ is said to be prescribed-time stable on $[0,\infty]$ if for any $\zeta(0)\in R^n$, there exists a constant $T>0$ such that $\lim\limits_{t\to T}\|\zeta(t)\|=0$ and $\zeta(t)=0\ \forall t>T$.

Assumption 1. There is a constant b satisfying:

$$|f_i(\zeta,u,t)|\leq b(|\zeta_1|+|\zeta_2|+\cdots+|\zeta_i|), i=1,\cdots,n.\tag{2}$$

Lemma 1 [34]. Let $A_0,A_c\in R^{m\times m}$ satisfying:

$$A_{0(i,1)}=-g_i, i=1,\cdots,m,\quad A_{0(i,i+1)}=1, i=1,\cdots,m-1,$$

$$A_{c(m,i)}=-k_i, i=1,\cdots,m,\quad A_{c(i,i+1)}=1, i=1,\cdots,m-1,$$

with zeros elsewhere. Let C be the $1\times m$ vector $[1,0,\cdots,0]$, $B_0=diag\{b_1,b_2,\cdots,b_m\}$, $B_c=$

$diag\{\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m\}$. Then, matrix $P_0 > 0$, $P_c > 0$, and positive constants v_0, $\tilde{v}_0$, $\underline{v}_0$, $\overline{v}_0$, v_c, $\underline{v}_c$, and $\overline{v}_c$ exist to satisfy:

$$A_0^T P_0 + P_0 A_0 \leq -v_0 I_n - \tilde{v}_0 C^T C, \tag{3}$$

$$\underline{v}_0 I_n \leq \left(B_0 - \frac{I_n}{2}\right)^T P_0 + P_0\left(B_0 - \frac{I_n}{2}\right) \leq -\overline{v}_0 I_n, \tag{4}$$

$$A_c^T P_c + P_c A_c \leq -v_c I_n, \tag{5}$$

$$\underline{v}_c I_n \leq \left(B_c - \frac{I_n}{2}\right)^T P_c + P_c\left(B_c - \frac{I_n}{2}\right) \leq -\overline{v}_c I_n. \tag{6}$$

3. Major results on PTS

3.1. Observer design

Construct the following observer:

$$\begin{cases} \dot{\hat{\zeta}}_1 = \hat{\zeta}_2 - r g_1(\hat{\zeta}_1 - y) - \frac{\dot{r}}{r}(\hat{\zeta}_1 - y) \\ \dot{\hat{\zeta}}_i = \hat{\zeta}_{i+1} - r^i g_i(\hat{\zeta}_1 - y),\ i = 2, \cdots, n-1 \\ \dot{\hat{\zeta}}_n = u - r^n g_n(\hat{\zeta}_1 - y) \end{cases}, \tag{7}$$

in which $r(t) \geq 1,\ \forall t \geq 0$ is a dynamic high-gain scaling parameter, and $g_i > 0$ and $i = 1, \cdots, n$ are parameters chosen later.

The observer errors are defined as:

$$e_i = \hat{\zeta}_i - \zeta_i, 1 \leq i \leq n, \tag{8}$$

and the scaled observer errors are defined as:

$$\epsilon_i = \frac{e_i}{r^{i-1}}, 1 \leq i \leq n. \tag{9}$$

Then,

$$\dot{\epsilon} = r A_0 \epsilon - \frac{\dot{r}}{r} B_0 \epsilon - \Phi, \tag{10}$$

where $\epsilon = [\epsilon_1, \cdots, \epsilon_n]^T$, $\Phi = \left[f_1, \frac{f_2}{r} \cdots, \frac{f_n}{r^{n-1}}\right]^T$, and

$$A_0 = \begin{bmatrix} -g_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -g_{n-1} & 0 & \cdots & 1 \\ -g_n & 0 & \cdots & 0 \end{bmatrix} \in R^{n\times n}, B_0 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & n-1 \end{bmatrix} \in R^{n\times n}.$$

3.2. Construction of output feedback control law

Introduce the scaled observer estimate signals $\eta_2, \cdots, \eta_n$ as:

$$\eta_2 = \frac{\hat{\zeta}_2 + \beta(\xi, y)}{r}, \eta_i = \frac{\hat{\zeta}_i}{r^{i-1}}, i = 3, \cdots, n, \tag{11}$$

with $\beta(\xi, y)$ being of the form:

$$\beta(\xi, y) = \beta_1 \xi y, \tag{12}$$

where $\xi(t) \geq 1, \forall t \geq 0$ is a dynamic adaptive parameter, and β_1 is a parameter designed later. Then,

$$\begin{cases} \dot{\eta}_2 = r\eta_3 - rg_2\epsilon_1 - \frac{\dot{r}}{r}\eta_2 + \frac{\dot{\beta}}{r} \\ \dot{\eta}_i = r\eta_{i+1} - rg_i\epsilon_1 - (i-1)\frac{\dot{r}}{r}\eta_i, \ \ i = 3, \cdots, n-1 . \\ \dot{\eta}_n = \frac{u}{r^{n-1}} - rg_n\epsilon_1 - (n-1)\frac{\dot{r}}{r}\eta_n. \end{cases} \tag{13}$$

Now, design the output feedback control law as:

$$u = -r^n K\eta, \tag{14}$$

with $\eta = [\eta_2, \cdots, \eta_n]^T$, $K = [k_2, \cdots, k_n]$, where $k_i, i = 2, \cdots, n$ are parameters chosen later.

Substituting the control law (14) into Eq (13), yield

$$\dot{\eta} = rA_c\eta - \frac{\dot{r}}{r}B_c\eta - rG\epsilon_1 + D, \tag{15}$$

in which $G = [g_2, \cdots, g_n]^T$, $D = \left[\frac{\dot{\beta}}{r}, 0, \cdots, 0\right]^T$, and

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -k_2 & -k_3 & \cdots & -k_n \end{bmatrix} \in R^{(n-1)\times(n-1)}, B_c = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & n-1 \end{bmatrix} \in R^{(n-1)\times(n-1)}.$$

The observer and controller above aim to establish a framework for PTS by OFC. However, to rigorously guarantee convergence within the prescribed time, we propose an effective method transforming the finite-time interval into an equivalent infinite-time scale as below. Function $\rho(\cdot)[0, T): \rightarrow [0, \infty)$ is introduced with the properties of twice continuous differentiability, monotonically increasing, and [31]

1) $\rho(0) = 0$, $\rho(T) = \infty$.

2) The derivative of $\rho(t)$ has a positive lower bound on $[0, T)$, i.e., there is a positive constant ρ_0 such that for all $t \in [0, T)$, $\frac{d\rho}{dt} \geq \rho_0$.

3) Setting $\theta = \rho(t)$ and $\psi(\theta) = \frac{d\rho}{dt}$. $\psi(\theta)$ and $\frac{d\psi}{d\theta}$ grows no faster than polynomial level as $\theta \rightarrow \infty$.

Remark 1. Many functions satisfy the above, for example,

$$\rho(t) = \frac{t}{T-t},$$

then,

$$\frac{d\rho}{dt} = \frac{T}{(T-t)^2},\ \ \psi(\theta) = \frac{(1+\theta)^2}{T},\ \ \frac{d\psi}{d\theta} = \frac{2(1+\theta)}{T}.$$

The above conditions imply that ρ is invertible, and that when t goes from 0 to T, θ goes from 0 to ∞. Denoting the inverse function by ρ^{-1}, we have,

$$\zeta(t) = \zeta\big(\rho^{-1}(\theta)\big) = \breve{\zeta}(\theta) = \breve{\zeta}\big(\rho(t)\big). \tag{16}$$

Remark 2. The above temporal scale transformation serves two key purposes: 1) It can map the finite-time problem to an infinite-time horizon; and 2) it further ensures system exact convergence at $t = T$. The transformation preserves the system's stability properties while enabling the controller to account for the prescribed-time requirement.

3.3. PTS on $[0, T)$

In this section, we investigate the PTS of the uncertain nonlinear system (1) on $[0, T)$ by adaptive output feedback. With $T > 0$, the PTS control objective is to design a suitable controller $u(t)$ such that:

(i) The state $\zeta(t)$ and the control $u(t)$ of the system are bounded.

(ii) $\lim_{t\to T^-} \zeta(t) = 0$ and $\lim_{t\to T^-} u(t) = 0$.

Similar to [31], assume the dynamic of r satisfies:

$$\frac{dr}{d\theta} = \lambda\big(\Gamma(y,\xi,\dot{\xi}) + \psi(\theta) - r\big)\big[\Omega\big(r,y,\xi,\dot{\xi}\big) + \tilde{\psi}(\theta)\big], \tag{17}$$

with $r(0) \geq \max\{1, \psi(0)\}$, $\tilde{\psi}(\theta) = \frac{d\psi}{d\theta}$, λ, Γ , and Ω as non-negative functions. Choose function $\lambda: R \to R^+$ as:

$$\lambda(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0 \end{cases}. \tag{18}$$

To clearly state the main result, the following lemma is cited.

Lemma 2 [31]. The inequality $r(\theta) \geq \psi(\theta)$ holds for any θ within the maximum interval of solution of Eq (17).

Proof: We prove by contradiction that $r(\theta) \geq \psi(\theta)$ holds for all $\theta \geq 0$.

When $r > \Gamma(y,\xi,\dot{\xi}) + \psi(\theta)$, it is clear that $r(\theta) > \psi(\theta)$.

When $r \leq \Gamma(y,\xi,\dot{\xi}) + \psi(\theta)$, we have $\frac{dr}{d\theta} = \Omega\big(r,y,\xi,\dot{\xi}\big) + \tilde{\psi}(\theta) > \tilde{\psi}(\theta)$.

Assume that $r(\theta) \geq \psi(\theta)$ does not hold. Then, there exists θ such that $r(\theta) < \psi(\theta)$. Let

$\theta_{min} = inf\{\theta > 0|r(\theta) < \psi(\theta)\}$. Note that $r(0) \geq \psi(0)$, so $\theta_{min} > 0$. At $\theta = \theta_{min}$, we have $r(\theta_{min}) = \psi(\theta_{min})$. Considering that $\Gamma(y, \xi, \dot{\xi}) > 0$, it follows that:

$$r(\theta_{min}) < \Gamma(y, \xi, \dot{\xi}) + \psi(\theta_{min}).$$

Therefore, the dynamics of θ at θ_{min} satisfy:

$$\frac{dr}{d\theta}|_{\theta=\theta_{min}} = \Omega(r, y, \xi, \dot{\xi}) + \tilde{\psi}(\theta) \geq \tilde{\psi}(\theta) = \frac{d\psi}{d\theta}|_{\theta=\theta_{min}}.$$

Since $r(\theta_{min}) = \psi(\theta_{min})$ and $\frac{dr}{d\theta} \geq \tilde{\psi}(\theta)$ at θ_{min}, there exists a θ in the right neighborhood of θ_{min} such that $r(\theta) \geq \psi(\theta)$, which contradicts the assumption. Thus, $r(\theta) \geq \psi(\theta)$ holds.

This proof is essentially similar to [31]. Here, we emphasize the clear logic.

Theorem 1. Given $T > 0$, the control law (14) with Eqs (17) and (18), and

$$\Gamma(y, \xi, \dot{\xi}) = \max\left(1, \frac{8w(y,\xi,\dot{\xi})}{v_c}, \frac{2w(y,\xi,\dot{\xi})}{cv_0}\right), \tag{19}$$

$$\Omega(r, y, \xi, \dot{\xi}) = rw(y, \xi, \dot{\xi}) \max\left(\frac{1}{\underline{v}_c}, \frac{1}{c\underline{v}_0}\right), \tag{20}$$

$$\frac{d\xi}{d\theta} = \tilde{\psi}(\theta), \tag{21}$$

$$\frac{1}{4}\beta_1 = \max\left\{\underline{\beta}, q_1\right\}, \tag{22}$$

is introduced to make system (1) prescribed-time stable on $[0, T)$, where constant $\underline{\beta} > 0$, $\xi(0) \geq \max\{1, \psi(0)\}$.

Proof: For the complexity, the proof is divided into four parts.

1) Construct the synthetic Lyapunov function (LF) of system (1).

For observer errors system (10), choose a LF:

$$V_0 = r\epsilon^T P_0 \epsilon, \tag{23}$$

in which $P_0 > 0$. Using Lemma 1, the time derivative of V_0 along the trajectories of system (10) can be evaluated as:

$$\begin{aligned} \dot{V}_0 &= \dot{r}\epsilon^T P_0 \epsilon + r\left(r\epsilon^T {A_0}^T - \frac{\dot{r}}{r}\epsilon^T {B_0}^T - \Phi^T\right) P_0 \epsilon + r\epsilon^T P_0 \left(rA_0 \epsilon - \frac{\dot{r}}{r} B_0 \epsilon - \Phi\right) \\ &= r^2 \epsilon^T \left({A_0}^T P_0 + P_0 A_0\right)\epsilon - \dot{r}\epsilon^T \left(\left(B_0 - \frac{I_n}{2}\right)^T P_0 + P_0 \left(B_0 - \frac{I_n}{2}\right)\right)\epsilon - 2r\epsilon^T P_0 \Phi \\ &\leq -v_0 r^2 |\epsilon|^2 - \tilde{v}_0 r^2 |\epsilon_1|^2 - \dot{r}\underline{v}_0 |\epsilon|^2 - 2r\epsilon^T P_0 \Phi. \end{aligned} \tag{24}$$

For observer estimate signals system (15), choose another LF:

$$V_c = \frac{1}{2}y^2 + r\eta^T P_c \eta, \tag{26}$$

in which $P_c > 0$. Using Lemma 1, the time-derivative of V_c along the trajectories of system (15) can be evaluated as:

$$\begin{aligned}\dot{V}_c &\le y\dot{y} + \dot{r}\eta^T P_c \eta + r\left(r\eta^T {A_c}^T - \frac{\dot{r}}{r}\eta^T {B_c}^T - rG^T \epsilon_1 + D^T\right) P_c \eta \\ &\quad + r\eta^T P_c \left(rA_c\eta - \frac{\dot{r}}{r}B_c\eta - rG\epsilon_1 + D\right) \\ &= y\dot{y} + r^2\eta^T\left({A_c}^T P_c + P_c A_c\right)\eta - \dot{r}\eta^T\left(\left(B_c - \frac{I_{n-1}}{2}\right)^T P_c + P_c\left(B_c - \frac{I_{n-1}}{2}\right)\right)\eta \\ &\quad -2r^2\eta^T P_c G\epsilon_1 + 2r\eta^T P_c D \\ &\le y\dot{y} - v_c r^2|\eta|^2 - \dot{r}\underline{v}_c|\eta|^2 - 2r^2\eta^T P_c G\epsilon_1 + 2r\eta^T P_c D.\end{aligned} \tag{26}$$

Now, set $V = cV_0 + V_c$, by combining Eqs (24) and (26), we have:

$$\begin{aligned}\dot{V} &\le -cv_0 r^2|\epsilon|^2 - c\tilde{v}_0 r^2|\epsilon_1|^2 - c\dot{r}\underline{v}_0|\epsilon|^2 + y\dot{y} - v_c r^2|\eta|^2 - \dot{r}\underline{v}_c|\eta|^2 \\ &\quad -2cr\epsilon^T P_0\Phi - 2r^2\eta^T P_c G\epsilon_1 + 2r\eta^T P_c D.\end{aligned} \tag{27}$$

Next, we aim to estimate the unknown terms in the right-hand side of Eq (27).

Noting $\zeta_2 = r(\eta_2 - \epsilon_2) - \beta$, we obtain

$$\begin{aligned}y\dot{y} &= yr(\eta_2 - \epsilon_2) - y\beta + yf_1 \\ &\le \frac{1}{v_c}y^2 + \frac{v_c}{4}r^2|\eta|^2 + \frac{1}{cv_0}y^2 + \frac{cv_0}{4}r^2|\epsilon|^2 - y\beta + by^2.\end{aligned} \tag{28}$$

With $\zeta_j = r^{j-1}(\eta_j - \epsilon_j), j = 3, \cdots, n$, Assumption 1, and $r(t) \ge 1$, we get:

$$\begin{aligned}|\Phi_i| &\le \frac{b}{r^{i-1}}|\zeta_1| + \frac{b}{r^{i-1}}|\zeta_2| + \frac{b}{r^{i-1}}\sum_{j=3}^{i}|\zeta_j| \\ &= \frac{b}{r}|\zeta_1| + \frac{b}{r}|\beta| + b\sum_{j=2}^{i}|\eta_j - \epsilon_j| \\ &\le \frac{b}{r}|\zeta_1| + \frac{b}{r}|\xi y\beta_1| + b(\|\eta\|_1 + \|\epsilon\|_1) \\ &\le \frac{b}{r}(1 + |\xi\beta_1|)|y| + \sqrt{n}b(|\eta| + |\epsilon|),\end{aligned} \tag{29}$$

where Φ_i represents the i-th term of Φ. Therefore,

$$|\Phi| \le \sum_{i=1}^{n}|\Phi_i| \le \frac{nb}{r}(1 + |\xi\beta_1|)|y| + n\sqrt{n}b(|\eta| + |\epsilon|), \tag{30}$$

and the term $-2r\epsilon^T P_0\Phi$ can be bounded as:

$$\begin{aligned}|-2r\epsilon^T P_0\Phi| &\le 2\lambda_{max}(P_0)\epsilon^T P_0 nb(1 + |\xi\beta_1|)y \\ &\quad +2\lambda_{max}(P_0)n\sqrt{n}b(|\epsilon||\eta| + |\epsilon|^2) \\ &\le \lambda_{max}^2(P_0)n^2b^2(1 + |\xi\beta_1|)^2|\epsilon|^2 + y^2 \\ &\quad +r\lambda_{max}(P_0)n\sqrt{n}b(|\epsilon|^2 + |\eta|^2 + 2|\epsilon|^2) \\ &\le \lambda_{max}^2(P_0)n^2b^2(1 + |\xi\beta_1|)^2|\epsilon|^2 + y^2 \\ &\quad +3r\lambda_{max}(P_0)n\sqrt{n}b(|\eta|^2 + |\epsilon|^2).\end{aligned} \tag{31}$$

In addition,

$$|-2r^2\eta^T P_c G\epsilon_1| \le \frac{v_c}{4} r^2|\eta|^2 + \frac{4}{v_c} r^2\lambda_{max}^2(P_c)\overline{G}^2|\epsilon_1|^2. \tag{32}$$

Pick $c > 0$, such that

$$c \ge \frac{4\lambda_{max}^2(P_c)\overline{G}^2}{v_c\tilde{v}_0},$$

the inequality (32) reduces to

$$|-2r^2\eta^T P_c G\epsilon_1| \le \frac{v_c}{4} r^2|\eta|^2 + c\tilde{v}_0 r^2|\epsilon_1|^2. \tag{33}$$

Since,

$$\begin{aligned} |D| &\le \frac{1}{r}\left|\beta_1\dot{\xi}y + \beta_1\xi\dot{y}\right| \\ &= \frac{1}{r}\left|\beta_1\dot{\xi}y + \xi\beta_1(r(\eta_2-\epsilon_2)-\beta+f_1)\right| \\ &\le \frac{|\beta_1\dot{\xi}y|}{r} + |\xi\beta_1||\eta_2-\epsilon_2| + \frac{1}{r}|\xi\beta_1||b-\beta_1\xi||y|, \end{aligned} \tag{34}$$

therefore,

$$\begin{aligned} |2r\eta^T P_c D| &\le 2\lambda_{max}(P_c)|\eta^T|\cdot\left|\beta_1\dot{\xi}y\right| + 2r\lambda_{max}(P_c)|\eta^T|\cdot|\xi\beta_1|\cdot|\eta_2-\epsilon_2| \\ &\quad +2\lambda_{max}(P_c)|\eta^T|\cdot|\xi\beta_1|\cdot|b-\beta_1\xi|\cdot|y| \\ &\le 2\lambda_{max}(P_c)|\eta^T|\cdot\left|\beta_1\dot{\xi}\right|\cdot|y| + 2r\lambda_{max}(P_c)|\eta^T|\cdot|\xi\beta_1|(|\eta|+|\epsilon|) \\ &\quad +2\lambda_{max}(P_c)|\eta^T|\cdot|\xi\beta_1|\cdot|b-\beta_1\xi|\cdot|y| \\ &\le 2\lambda_{max}^2(P_c)\dot{\xi}^2{\beta_1}^2|\eta|^2 + \frac{1}{2}y^2 + 2r\lambda_{max}(P_c)|\xi\beta_1|\cdot|\eta|^2 \\ &\quad +r\lambda_{max}(P_c)\xi|\beta_1|(|\eta|^2+|\epsilon|^2) + 2\lambda_{max}^2(P_c)\xi^2[\xi\beta_1(b-\beta_1\xi)]^2|\eta|^2 + \frac{1}{2}y^2 \\ &\le 2\lambda_{max}^2(P_c)\dot{\xi}^2{\beta_1}^2|\eta|^2 + 3r\lambda_{max}(P_c)\xi|\beta_1|(|\eta|^2+|\epsilon|^2) \\ &\quad +2\lambda_{max}^2(P_c)\xi^2[\xi\beta_1(b-\beta_1\xi)]^2|\eta|^2 + y^2. \end{aligned} \tag{35}$$

Then, by submitting Eqs (28)–(35) into Eq (27), have

$$\begin{aligned} \dot{V} &\le -\frac{3}{4}cv_0r^2|\epsilon|^2 - \frac{1}{2}v_cr^2|\eta|^2 + q_1y^2 - y\beta \\ &\quad +rw(y,\xi,\dot{\xi})(|\eta|^2+|\epsilon|^2) - \dot{r}c\underline{v}_0|\epsilon|^2 - \dot{r}\underline{v}_c|\eta|^2, \end{aligned} \tag{36}$$

where

$$q_1 = 1 + c + b + \frac{1}{v_c} + \frac{1}{cv_0}, \tag{37}$$

$$\begin{aligned} w(y,\xi,\dot{\xi}) &= c\lambda_{max}^2(P_0)n^2b^2(1+|\xi\beta_1|)^2 + 3c\lambda_{max}(P_0)n\sqrt{n}b \\ &\quad +2\lambda_{max}^2(P_c)\dot{\xi}^2{\beta_1}^2 + 3\lambda_{max}(P_c)\xi|\beta_1| \\ &\quad +2\lambda_{max}^2(P_c)\xi^2[\xi\beta_1(b-\xi\beta_1)]^2. \end{aligned} \tag{38}$$

As discussed above, the temporal scale transformation defined above yields $dt = \frac{d\theta}{\psi(\theta)}$. Hence, from Eq (36),

$$\begin{aligned}\frac{dV}{d\theta} \leq \frac{1}{\psi(\theta)}\{-\tfrac{3}{4}cv_0r^2|\epsilon|^2 - \tfrac{1}{2}v_cr^2|\eta|^2 + q_1y^2 - y\beta \\ +rw(y,\xi,\dot{\xi})(|\eta|^2 + |\epsilon|^2)\} - c\underline{v}_0\frac{dr}{d\theta}|\epsilon|^2 - \underline{v}_c\frac{dr}{d\theta}|\eta|^2.\end{aligned} \tag{39}$$

2) It is be illustrated that V is bounded and furtherly converges to 0 as $\theta \to \infty$.
From Eq (21) $\xi(0) \geq \psi(0)$, we see that $\xi \geq \psi(\theta)$ holds for all $\theta \in [0,\infty)$.
From Eq (22) and $\xi(t) \geq 1$, obtain

$$q_1y^2 - y\beta \leq -\tfrac{3}{4}\xi\beta_1y^2 + (1-\xi)q_1y^2 \leq -\tfrac{3}{4}\xi\beta_1y^2. \tag{40}$$

If $r \geq \Gamma(y,\xi,\dot{\xi}) + \psi(\theta)$, $\frac{dr}{d\theta} = 0$, then we note from Eq (19) that

$$rw(y,\xi,\dot{\xi})|\eta|^2 \leq \frac{v_c}{8}r^2|\eta|^2,$$

$$rw(y,\xi,\dot{\xi})|\epsilon|^2 \leq \frac{1}{2}cv_0r^2|\epsilon|^2,$$

therefore, we have

$$\frac{dV}{d\theta} \leq \frac{1}{\psi(\theta)}\left\{-\tfrac{3}{4}\xi\beta_1y^2 - \tfrac{1}{4}cv_0r^2|\epsilon|^2 - \tfrac{1}{8}v_cr^2|\eta|^2\right\}. \tag{41}$$

If $r < \Gamma(y,\xi,\dot{\xi}) + \psi(\theta)$, $\frac{dr}{d\theta} = \Omega(r,y,\xi,\dot{\xi}) + \tilde{\psi}(\theta)$, using Eqs (20) and (40), we have

$$\begin{aligned}\frac{dV}{d\theta} &\leq \frac{1}{\psi(\theta)}\{-\tfrac{3}{4}\xi\beta_1y^2 - \tfrac{3}{4}cv_0r^2|\epsilon|^2 - \tfrac{1}{4}v_cr^2|\eta|^2 \\ &\quad +rw(y,\xi,\dot{\xi})(|\eta|^2 + |\epsilon|^2)\} - c\underline{v}_0\Omega|\epsilon|^2 - \underline{v}_c\Omega|\eta|^2 \\ &\leq \frac{1}{\psi(\theta)}\{-\tfrac{3}{4}\xi\beta_1y^2 - \tfrac{3}{4}cv_0r^2|\epsilon|^2 - \tfrac{1}{4}v_cr^2|\eta|^2\}.\end{aligned} \tag{42}$$

Therefore,

$$\frac{dV}{d\theta} \leq -\frac{\delta}{\psi(\theta)}\{\xi\beta_1y^2 + cv_0r^2|\epsilon|^2 + v_cr^2|\eta|^2\}, \tag{43}$$

with $\delta = \frac{1}{8}$ when either one of the following conditions hold: $r \geq \Gamma(y,\xi,\dot{\xi}) + \psi(\theta)$ or $r < R(y,\xi,\dot{\xi}) + \psi(\theta)$. With the properties $r \geq \psi(\theta)$ and $\xi \geq \psi(\theta)$, Eq (43) becomes

$$\frac{dV}{d\theta} \leq -\delta\{\beta_1y^2 + cv_0r|\epsilon|^2 + v_cr|\eta|^2\}. \tag{44}$$

Then,

$$\frac{dV}{d\theta} \le -\kappa V, \tag{45}$$

where $\kappa = \min\left\{2\delta\underline{\beta}_1, \frac{\delta v_0}{\lambda_{max}(P_0)}, \frac{\delta v_c}{\lambda_{max}(P_c)}\right\}$.

The solutions of the closed-loop system exist for $\theta \in [0, \infty)$. From Eq (45), it follows that V is uniformly bounded and V tends to 0 exponentially as $\theta \to \infty$.

3) It will be proved that u tends to 0 exponentially as $\theta \to \infty$.

From the definition of V, it follows that ζ_1, $\sqrt{r}|\epsilon|$ and $\sqrt{r}|\eta|$ tend to 0 exponentially as $\theta \to \infty$.

By Eq (21) and the conditions imposed on $\psi(\theta)$, $\xi(\rho^{-1}(\theta))$, and $\dot{\xi}(\rho^{-1}(\theta))$ are polynomially upper-bounded in θ. Noting that ξ and $\dot{\xi}$ appear polynomially in the definition of w, it follows that w and $\Gamma(y, \xi, \dot{\xi})$ grow no faster than polynomial level in θ. From Eq (34), we have either $r \le \Gamma(y, \xi, \dot{\xi}) + \psi(\theta)$ or $\dot{r} = 0$. Note that $\psi(\theta)$ and $\tilde{\psi}(\theta)$ are polynomially upper-bounded in θ. Hence, $\Gamma(y, \xi, \dot{\xi}) + \psi(\theta)$ and $r(\rho^{-1}(\theta))$ grow no faster than polynomial level as a function of θ.

Since $\sqrt{r}|\epsilon|$ and $\sqrt{r}|\eta|$ go to 0 exponentially as $\theta \to \infty$, r grows no faster than polynomial level in θ. Hence, $|\epsilon|$ and $|\eta|$ tend to 0 exponentially as $\theta \to \infty$. From Eq (14), u tends to 0 exponentially as $\theta \to \infty$.

4) The PTS of the closed-loop system will be proved.

Since $\zeta_2 = r(\eta_2 - \epsilon_2) - \beta$ and $\zeta_i = r^{i-1}(\eta_i - \epsilon_i)$, $i = 3, \cdots, n$, from the above proof, we know that all $\zeta_1, \cdots, \zeta_n$ tend to 0 exponentially as $\theta \to \infty$. Hence, ζ tend to 0 exponentially as $\theta \to \infty$. Since $\theta \to \infty$ corresponds to $t \to T$, the above properties hold as $t \to T$. Therefore, ζ and u tend to 0 as $t \to T$, i.e., PTS is attained, which completes the proof.

Remark 3. Note that q_1 and $w(y, \xi, \dot{\xi})$ in Eqs (37) and (38) involve only known quantities and functions.

Remark 4. Theorem 1 gives a positive answer to the aim of the prescribed-time control objective as stated above. It is worth noting that the controller scheme here is similar to [31], but the constraints here are distinct from those in [31]. In [31], the upper diagonal terms are needed to meet the advantages of the bidirectional cascade, and the uncertain terms are needed to satisfy the special growth conditions. Our system here weakens the dependence on the upper diagonal terms and the bidirectional cascading dominance and adopts the lower triangular linear growth constraint and direct input control, which not only retains the effectiveness of the original control method but also extends the applicability of uncertain nonlinear systems.

Remark 5. Note that ζ_1 converges to 0 exponentially as $t \to T$. Also, from Eq (14) and the fact that ξ grows no faster than polynomial level, it follows that β converges to 0 as $t \to T$. Hence, from Eq (13), it is seen that the observer state signals $\hat{\zeta}_2, \cdots, \hat{\zeta}_n$ also converge to 0 as $t \to T$. Hence, ζ, u, and $\hat{\zeta} = [\hat{\zeta}_2, \cdots, \hat{\zeta}_n]^T$ all converge to 0 as t approaches the prescribed time T. Furthermore, based on Eqs (22), (28), (36) and (38), it can be concluded that $\epsilon_i (i = 1, \ldots, n)$, $\eta_j (j = 2, \ldots, n)$ are bounded over $[0, T)$ and converge to 0 as $t \to T$.

Remark 6. In order to implement the PTS control for system (1) on $[0, T)$ under Assumption 1, we present the concise control algorithm as below.

Step 1: System testing. For system (1), test whether the nonlinear term satisfies Assumption 1 or not. If yes, determine parameter b.

Step 2: Design of high-gain observer (7). Herein, the dynamic gain parameter $r(t)$ is determined by system (17). Further, give the observer error system (10).

Step 3: Design of controller (14). Herein, $\eta(t), \Gamma, \Omega, \xi, \beta_1$ are determined by systems (13), (18)–(22), respectively.

Step 4: Choice of g_i, k_i and other parameters. Choose parameters g_i, k_i, v_0, $\tilde{v}_0$, $\underline{v}_0$, $\overline{v}_0$, v_c, $\underline{v}_c$, and $\overline{v}_c$ to satisfy Lemma 1.

3.4. PTS on $[0,\infty)$

Theorem 1 illustrates that the closed-loop system (1) can reach stability within a prescribed time T, which does not take into account the case of after T. In this section, we show that this conclusion not only achieves PTS within $[0,T)$, but also maintains astable state within $[T,\infty)$. Hence, the PTS control objective here is to design a suitable controller $\tilde{u}(t)$ such that,

(i) $\lim_{t\to T^-}\zeta(t)=0$, and $\zeta(t)\equiv 0$ when $t\in[T,\infty)$.

(ii) $\lim_{t\to T^-}\tilde{u}(t)=0$, and $\tilde{u}(t)\equiv 0$ when $t\in[T,\infty)$.

Theorem 2. Given $T>0$, the following control scheme:

$$\tilde{u}(t)=\begin{cases}-r^nK\eta, & t\in[0,T)\\ 0, & t\in[T,\infty)\end{cases}, \tag{46}$$

is applied to make system (1) achieve PTS on $[0,\infty)$, where the dynamic of r is designed in Eq (34), K and η are defined in Eq (16).

Proof: Due to the domination nature of our design, the proof of this theorem on $[0,T)$ is the same as Theorem 1. When $t\in[T,\infty)$, from Eq (46), we have $\tilde{u}(t)\equiv 0$. For the fact $\lim_{t\to T^-}\tilde{u}(t)=0$, from Theorem 1, it follows that $\tilde{u}(t)$ is continuous at $t=T$ and becomes 0 as $t\to T^-$ and thereafter, e.g., $\tilde{u}(t)$ is bounded and continuous everywhere for $t\in[0,\infty)$.

Since $\zeta_i(t)\to 0$ and $\tilde{u}(t)\to 0$ as $t\to T^-$, according to Assumption 2, we can get $f_i(\zeta(t),u(t),t)\to 0$ and therefore $\dot{\zeta}_i(t)\to 0$ $(i=1,\ldots,n)$ as $t\to T^-$, then it is concluded that each state $\zeta_i(t)$ converges to 0 within prescribed time T.

On $[0,T)$, according to Eq (45) in Theorem 1 and $\tilde{u}(t)=0$ on $[T,\infty)$, it can be obtained that $\dot{V}<0$ on $[0,\infty)$, so V is monotonically decreasing. Considering that $V\geq 0, V\to 0(t\to T^-)$, so $V=0$ when $t\geq T$. According to Eqs (8), (9), (11), (13) and (18), V is actually composed of ζ_1^2, $\sum_{i=2}^n\hat{\zeta}_i^2$, $\sum_{i=1}^n(\hat{\zeta}_i-\zeta)^2$ and its non-negative coefficients, so there is $\zeta_i=0$, $\hat{\zeta}_i=0$, $i=1,\cdots,n$ when $t\geq T$. That is, the new PTS control objective is achieved, which completes the proof.

Remark 7. Notably, the control law $\tilde{u}(t)$ converges to zero as $t\to T^-$ and remains zero for $t\geq T$; therefore, the system can continuously operate beyond T. It is worth mentioning that in Eq (46), $\tilde{u}(t)=0$ is set for $t\geq T$, which seems to involve a control switching at $t=T$; however, this design does not cause discontinuity to the control signal at $t=T$ for $\lim_{t\to T^-}\tilde{u}(t)=0$. In fact, $\tilde{u}(t)$ here is bounded and continuous for $t\in[0,\infty)$, including $t=T$, reduces to 0 at $t=T$ and remains 0 thereafter.

Remark 8. Compared with [31], and other related literature, our work has highlighted the following differences:

1) Changes of system structural constraints: In the literature (such as [30,31,34]), the control design usually requires the system to satisfy specific structural conditions, such as bidirectional cascade advantages or special growth conditions. These constraints limit the applicable scope of the

existing control method. One of the differences of our work lies in changing these structural constraints and requiring only the nonlinear uncertainty term to satisfy the lower triangular linear growth condition (i.e., Assumption 1).

2) PTS over the full time horizon: Compared with [31] and other literature considering only the stability of $[0, T)$, our work not only realizes the PTS on $[0, T)$, but also considers the control after T. It is necessary to consider the stability after T. Being stable on $[0, T)$ does not mean that it can be stable after T. There have been some studies on stabilization after T. For example, in [35–37], controllers are set on both $[0, T)$ and $[T, \infty)$ to realize the PTS of the controlled system on $[0, \infty)$.

Remark 9. We assume that the nonlinear term satisfies Assumption 1. For more general nonlinear systems, especially when the nonlinear term does not satisfy Assumption 1, the method may not be directly applied. In the future, researchers can focus on a wider range of applicable conditions. In addition, the design of the adaptive controller in this paper requires complex variables and parameters, which can be optimized further.

Remark 10. Similarly to Remark 6, we can give the control algorithm for the PTS of system (1) on $[0, \infty)$. Specifically, we can follow the steps of Remark 6 except for the controller (14) in Step 3 is replaced by the controller (46).

4. Numerical example

Consider the following three-order system having the following form

$$\begin{cases} \dot{\zeta}_1 = \zeta_2 + \zeta_1 \sin(\zeta_2) \\ \dot{\zeta}_2 = \zeta_3 + \zeta_1 \sin(\zeta_2{}^2) + \zeta_2 \\ \dot{\zeta}_3 = u + 3(\zeta_1\zeta_2\zeta_3)^{\frac{1}{3}} \\ y = \zeta_1 \end{cases}. \tag{47}$$

Here, $|f_1| = |\zeta_1 \sin(\zeta_2)| \le |\zeta_1|$, $|f_2| = |\zeta_1 \sin(\zeta_2{}^2) + \zeta_2| \le |\zeta_1| + |\zeta_2|$, $|f_3| = 3|\zeta_1\zeta_2\zeta_3|^{\frac{1}{3}} \le |\zeta_1| + |\zeta_2| + |\zeta_3|$. Assumption 1 is satisfied with $b = 1$. The output y is assumed to be measured.

For $u = 3$, one has:

$$A_0 = \begin{bmatrix} -g_1 & 1 & 0 \\ -g_2 & 0 & 1 \\ -g_3 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}, A_c = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_2 \end{bmatrix}, B_c = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}.$$

The reduced-order observer is designed as:

$$\begin{cases} \dot{\hat{\zeta}}_1 = \hat{\zeta}_2 - g_1 r(\hat{\zeta}_1 - y) - \frac{\dot{r}}{r}(\hat{\zeta}_1 - y) \\ \dot{\hat{\zeta}}_2 = \hat{\zeta}_3 - g_2 r^2(\hat{\zeta}_1 - y) \\ \dot{\hat{\zeta}}_3 = u - g_3 r^3(\hat{\zeta}_1 - y) \end{cases}. \tag{48}$$

Then, the control input is designed as $u = -r^3(k_2\eta_2 + k_3\eta_3)$. Using the constructive procedure in [32], with g_1=1.12, g_2=2.09, g_3=3.06, $\tilde{l}_0 = 0.5$, v_0=2.47 $\underline{l}_0$=0.5, $P_0 > 0$ can satisfy Lemma 1.

Also, with $k_2 = 1$, $k_3 = 2$, l_c=2, $\underline{l}_c$=1.5, a $P_c > 0$ can be found to satisfy Lemma 1. The prescribed time is specified as $T = 1$. $\xi(0) = 2$, $r(0) = 5$, $\hat{\zeta}(0) = (0.5, -1.5, 2.5)$.

In order to comprehensively verify the performance of the proposed adaptive output feedback controller, three comparative experiments are designed here, focusing on examining the system response characteristics under different initial conditions and settling times.

Case 1: Initial condition $\zeta(0) = (-1,0,5)$, settling time $T = 1$. The response curves of the system states $[\zeta_1, \zeta_2, \zeta_3]$, observer states $[\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3]$ and control input $u(t)$ are shown in Figures 1–3.

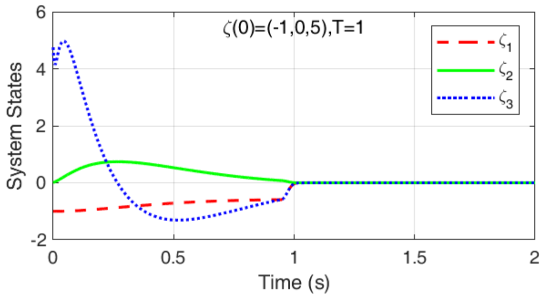

Figure 1. States of system (47).

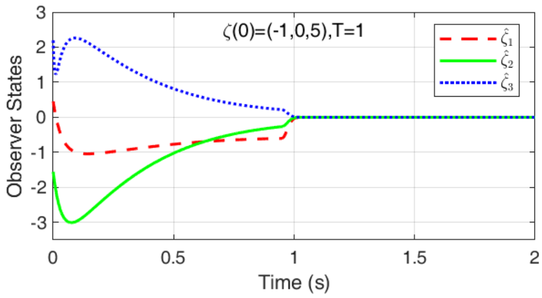

Figure 2. Observer States of system (47).

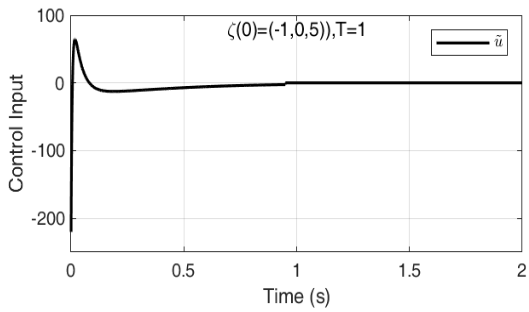

Figure 3. Control input of system (47).

Case 2: Initial condition $\zeta(0) = (1, -2,3)$, settling time $T = 1$. The response curves of the system states $[\zeta_1, \zeta_2, \zeta_3]$, observer states $[\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3]$ and control input $u(t)$ are shown in Figures 4–6.

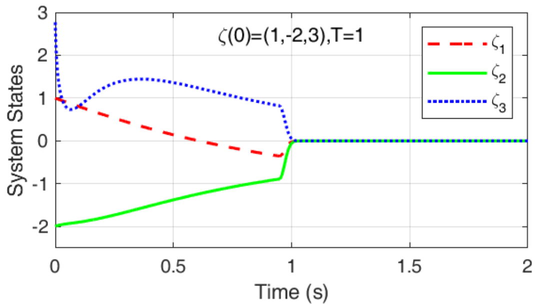

Figure 4. States of system (47).

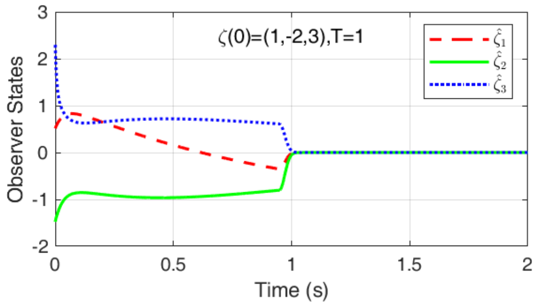

Figure 5. Observer States of system (47).

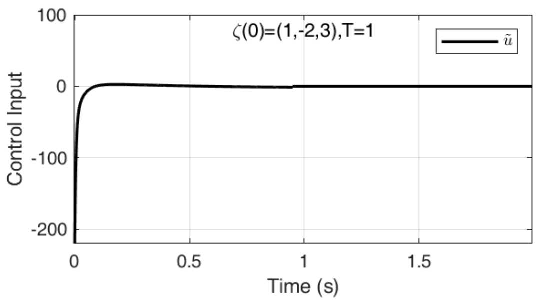

Figure 6. Control input of system (47).

Case 3: Initial condition $\zeta(0) = (1, -2,3)$, settling time $T = 1.5$. The response curves of the system states $[\zeta_1, \zeta_2, \zeta_3]$, observer states $[\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3]$ and control input $u(t)$ are shown in Figures 7–9.

In addition, the original states of system (47) without a controller are shown in Figure 10, which illustrates that system (47) without a controller is not PTS.

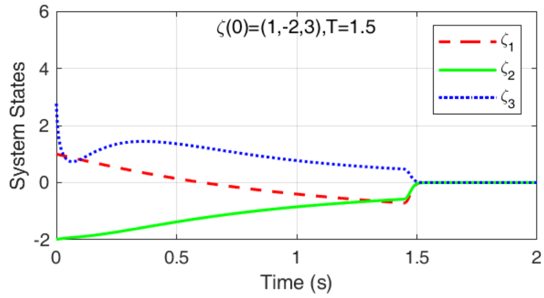

Figure 7. States of system (47).

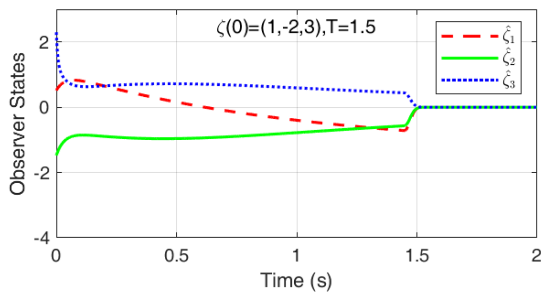

Figure 8. Observer States of system (47).

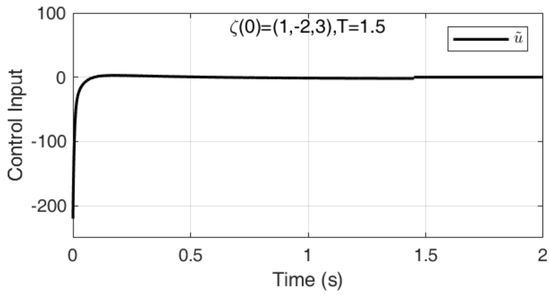

Figure 9. Control input of system (47).

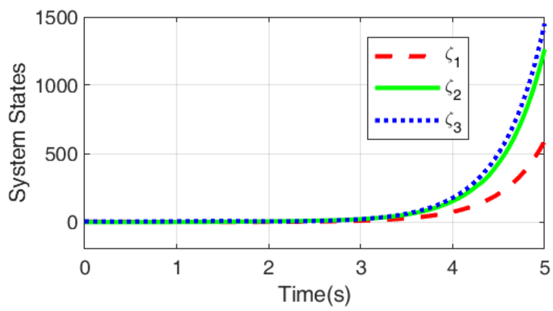

Figure 10. States of system (47) without control.

Compared with Figures 1–6, with the same prescribed time $T = 1$, under different initial conditions, the states of controlled system (47) and its observer states can achieve stability within T

and remain stable afterward, while the controller remains bounded. Figures 4–9 show that, with the same initial condition $\zeta(0) = (1, -2,3)$, system states and observer states achieve stability within different prescribed time and remain stable thereafter, with the controller being also bounded in all cases.

Remark 11. Compared with FTS and FxTS, our proposed control scheme answers not only "whether convergence occurs" but also "when it occurs". Furthermore, in contrast to the PTS works that focus solely on stabilization over $[0, T)$, our work guarantees stability both during $[0, T)$ and beyond T, thus delivering a more comprehensive control solution. In addition, a comparative study on the simulation of $[0, T)$ and $[0, \infty)$ has been achieved in our previous work [36], where the controller design method in $[0, T)$ is the same.

5. Conclusions

We propose an adaptive output feedback control strategy to achieve prescribed-time stabilization of uncertain nonlinear systems. Through dynamic high-gain scaling and time-scale transformation, the designed controller not only ensures that the system state and control input converge to zero at any prescribed time, but also remain stable after the prescribed time. For other nonlinear systems, especially those with dissatisfaction due to the lower triangular growth conditions, the method in this paper may not be directly applied. In the future, researchers can relax the restriction on uncertainty terms and consider a more generally applicable scheme.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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