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*Research article*

# Asymptotic behavior of the solution of a stochastic clearing queueing system

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**Abstract:** In this paper, we investigate the asymptotic behavior of the time-dependent solution for the M/G/1 stochastic clearing queueing system operating in a three-phase environment. The mathematical model of this system is characterized by an infinite set of integro-partial differential equations, with boundary conditions that incorporate integral equations. Initially, we employ probability generating functions to demonstrate that 0 is an eigenvalue of the system operator, possessing a geometric multiplicity of one. Subsequently, by invoking Greiner's boundary perturbation method, we establish that all points on the imaginary axis, with the exception of 0, reside within the resolvent set of the system operator. Furthermore, we highlight that 0 also serves as an eigenvalue of the adjoint operator of the system operator, with a geometric multiplicity of unity. As a result, we conclude that the time-dependent solution of the system converges strongly to its steady-state solution.

**Keywords:** stochastic clearing queueing system; asymptotic behavior; boundary perturbation; resolvent set; eigenvalue

## 1. Introduction

The stochastic clearing queueing system has a wide range of applications across various fields such as manufacturing systems, telecommunications, transportation systems, supply chain and inventory management, etc. For example, in manufacturing, stochastic clearing queueing systems can be applied to optimize production lines. For instance, a production line may operate in batches, where all accumulated orders are processed together. The system can be designed to start production only when a certain number of orders are accumulated, or it can be activated randomly when the number of orders is below the threshold to avoid long delays. In telecommunication networks, stochastic clearing queueing systems can be used to manage data packet transmission. Packets arrive randomly and are processed in batches to optimize bandwidth usage and reduce latency. The clearing mechanism helps in managing the queue of packets efficiently.

Many scholars have previously investigated stochastic clearing queuing systems [1–5]. The M/G/1 stochastic clearing queuing system is a specialized model within the realm of stochastic clearing queues. Here, "M" denotes Markovian arrivals, signifying that the arrival process adheres to a Poisson process, characterized by exponentially distributed inter-arrival times. "G" represents a general service time distribution, implying that service durations can follow any distribution, not limited to exponential ones. Last, "1" indicates that there is only one server in the system. In this system, customers arrive following a Poisson process. A single server attends to them, with service times governed by a general distribution. At random intervals, the entire system is cleared, removing all customers in the queue as well as those being served. The clearance times are random and follow a specific distribution.

Zhang et al. [6] were the first to examine the M/G/1 stochastic clearing queueing system in a three-phase environment. They formulated the mathematical model of this system using the method of supplementary variables and explored the steady-state solution and several key stationary performance measures under the following hypothesis:

$$\begin{aligned}
&\lim_{t\to\infty} Q_{j,0}(t) = Q_{j,0},\ j = 1, 3; \quad \lim_{t\to\infty} V_n(t) = V_n,\ n \geq 0,\\
&\lim_{t\to\infty} p_{j,n}(\cdot, t) = p_{j,n}(\cdot),\ j = 1, 3,\ n \geq 1,
\end{aligned}$$

where $Q_{j,0}(t)$ $(j = 1, 3)$ represents the probability that there are no customers in phase $j$ and the server is idle at time $t$; $V_n(t)$ $(n \geq 0)$ denotes the probability that there are $n$ customers in the system and the system is in phase 2; $p_{j,n}(x, t)dx$ $(j = 1, 3;\ n \geq 1)$ is the probability that there are $n$ customers in the system (including the one being served) at time $t$ with the server busy serving a customer whose elapsed service time lies in the interval $[x, x + dx)$ in phase $j$.

Drawing on the theory of partial differential equations (see [7, 8]), we can infer that the aforementioned hypothesis implies the following two hypotheses:

Hypothesis 1): This queueing system has a nonnegative time-dependent solution.

Hypothesis 2): The time-dependent solution of this system converges to its nonzero steady-state solution.

Recently, in our work [9], we established that Hypothesis 1) holds when the service rate of the server is a bounded function, and Hypothesis 2) holds when the service rate is constant. To address whether Hypothesis 2) holds, as demonstrated in [9], we first identify the adjoint operator of the system operator. Subsequently, we analyze the spectrum of this adjoint operator on the imaginary axis. By leveraging the relationship between the spectrum of the operator and its adjoint, we then determine the spectrum of the system operator on the imaginary axis. However, in the aforementioned literature, we did not address whether Hypothesis 2) remains valid when the service rate is a bounded function. In this article, we aim to resolve this outstanding issue.

In this paper, we employ the boundary perturbation method of Greiner [10] and probability generating functions [6] to investigate the asymptotic behavior of the time-dependent solution for the M/G/1 stochastic clearing queueing system in a three-phase environment. To achieve this, we first select an appropriate Banach space to serve as the state space for the system. We then define the maximal operator and boundary operators for this queueing within this state space. Utilizing these operators, we construct the system operator for the given system.

Next, by introducing suitable probability generating functions, we demonstrate that 0 is an eigenvalue of the system operator with a geometric multiplicity of 1. We further define an operator with a boundary condition of 0 for this queueing system using the aforementioned maximal and boundary operators. We then identify the resolvent set of this operator and define the Dirichlet operator. Drawing on a result from Haji and Radl [11], we prove that all points on the imaginary axis, except for 0, belong to the resolvent set of the system operator. Additionally, we directly show that 0 is also an eigenvalue of the adjoint operator of the system, with a geometric multiplicity of 1.

These results collectively indicate that the time-dependent solution of the M/G/1 stochastic clearing queueing system in a three-phase environment strongly converges to its non-zero steady-state solution. In other words, Hypothesis 2) holds true in the sense of strong convergence. In addition, the conclusion of this article includes the main result of [9].

## 2. Mathematical model and its abstract Cauchy problem

According to Zhang et al. [6], the mathematical model of the M/G/1 stochastic clearing queueing system operating in a three-phase environment can be described by the following system of integro-partial differential equations:

$$\frac{dQ_{1,0}(t)}{dt} = -(\lambda_1 + \theta_1)Q_{1,0}(t) + \theta_3 Q_{3,0}(t) + \int_0^\infty p_{1,1}(x,t)\mu_1(x)dx + \theta_3 \sum_{n=1}^{\infty} \int_0^\infty p_{3,n}(x,t)dx, \tag{2.1}$$

$$\frac{dQ_{3,0}(t)}{dt} = -(\lambda_3 + \theta_3)Q_{3,0}(t) + N_0 V_0(t) + \int_0^\infty p_{3,1}(x,t)\mu_3(x)dx, \tag{2.2}$$

$$\frac{dV_0(t)}{dt} = -V_0(t) + \theta_1 Q_{1,0}(t), \tag{2.3}$$

$$\frac{dV_n(t)}{dt} = -V_n(t) + \theta_1 \int_0^\infty p_{1,n}(x,t)dx, \ n \geq 1, \tag{2.4}$$

$$\frac{\partial p_{j,1}(x,t)}{\partial t} + \frac{\partial p_{j,1}(x,t)}{\partial x} = -[\lambda_j + \theta_j + \mu_j(x)]p_{j,1}(x,t), \ j = 1,3, \tag{2.5}$$

$$\frac{\partial p_{j,n}(x,t)}{\partial t} + \frac{\partial p_{j,n}(x,t)}{\partial x} = -[\lambda_j + \theta_j + \mu_j(x)]p_{j,n}(x,t) + \lambda_j p_{j,n-1}(x,t), \ n \geq 2, \tag{2.6}$$

with the following boundary and initial conditions

$$p_{1,1}(0,t) = \lambda_1 Q_{1,0}(t) + \int_0^\infty p_{1,2}(x,t)\mu_1(x)dx, \tag{2.7}$$

$$p_{1,n}(0,t) = \int_0^\infty p_{1,n+1}(x,t)\mu_1(x)dx, \ n \geq 2, \tag{2.8}$$

$$p_{3,1}(0,t) = \lambda_3 Q_{3,0}(t) + \sum_{k=0}^{1} N_k V_{1-k}(t) + \int_0^\infty p_{3,2}(x,t)\mu_3(x)dx, \tag{2.9}$$

$$p_{3,n}(0,t) = \sum_{k=0}^{n} N_k V_{n-k}(t) + \int_0^\infty p_{3,n+1}(x,t)\mu_3(x)dx, \ n \geq 2, \tag{2.10}$$

$$V_n(0) = u_n \geq 0, \ n \geq 0; Q_{j,1}(0) = \overline{u}_j \geq 0, p_{j,n}(x,0) = u_{j,n}(x) \geq 0, \ j = 1,3; \ n \geq 1. \tag{2.11}$$

Here, $(x,t) \in [0,\infty) \times [0,\infty)$, and

$$\bar{u}_1 + \bar{u}_3 + \sum_{n=0}^{\infty} u_n + \sum_{n=1}^{\infty} \int_0^{\infty} u_{1,n}(x)dx + \sum_{n=1}^{\infty} \int_0^{\infty} u_{3,n}(x)dx = 1;$$

$Q_{j,0}(t)$ $(j = 1,3)$ represents the probability that there are no customers in phase $j$ and the server is idle at time $t$; $V_n(t)$ $(n \geq 0)$ denotes the probability that there are $n$ customers in the system and the system is in phase 2; $p_{j,n}(x,t)dx$ $(j = 1,3;\ n \geq 1)$ is the probability that there are $n$ customers in the system (including the one being served) at time $t$ with the server busy serving a customer whose elapsed service time lies in the interval $[x, x+dx)$ in phase $j$; $\lambda_j$ $(j = 1,2,3)$ is the arrival rate of customers when the system is in phase $j$; $\mu_j(x)$ $(j = 1,3)$ is the conditional probability (hazard rate) of completing a service during the interval $(x, x+dx)$ with elapsed time $x$ in phase $j$. It satisfies the conditions:

$$\mu_j(x) \geq 0, \qquad \int_0^{\infty} \mu_j(x)dx = \infty,\ \ j = 1,3,$$

$\theta_j$ $(j = 1,3)$ is the residence rate of the system in phase $j$.

The system operates in three distinct phases. The first and third phases are working phases, while the second phase is a deterministic time phase during which no service is provided. Upon completion of the first phase, the system transitions into the second phase. If a customer arrives during the second phase, they will either enter the system with probability $p$ or leave without joining the system with probability $q = 1 - p$. During this second phase, all customers are unable to receive service for a fixed duration $d$. After the second phase concludes, the system enters the third phase. Once the third phase is completed, the current customer is forced to leave the system without receiving further service, and the system then returns to the first phase to initiate a new service cycle. We assume that $N_r = (r!)^{-1} e^{-\lambda_2 pd} (\lambda_2 pd)^r$ is the probability of $r$ $(r \geq 0)$ arrivals during phase 2.

In this paper, we present our main result under the following assumption:

**Assumption 2.1.** *Let* $\lambda_j > 0, \theta_j > 0, N_0 \in (0,1)$ *and* $\mu_j(x) : [0,\infty) \to [0,\infty)$ *measurable and*

$$0 < \inf_{x\in[0,\infty)} \mu_j(x) \leq \mu_j(x) \leq \sup_{x\in[0,\infty)} \mu_j(x) < \infty,\ \ j = 1,3.$$

We choose the state space as follows:

$$X = \left\{ (V, p_1, p_3) \;\middle|\; \begin{array}{l} V = (Q_{1,0}, Q_{3,0}, V_0, V_1, V_2 \cdots) \in l^1 \\ p_1 = (p_{1,1}, p_{1,2}, \cdots) \in L^1[0,\infty) \times L^1[0,\infty) \times \cdots \\ p_3 = (p_{3,1}, p_{3,2}, \cdots) \in L^1[0,\infty) \times L^1[0,\infty) \times \cdots \\ \|(V, p_1, p_3)\| = |Q_{1,0}| + |Q_{3,0}| + \sum_{n=0}^{\infty} |V_n| \\ + \sum_{n=1}^{\infty} \|p_{1,n}\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|p_{3,n}\|_{L^1[0,\infty)} < \infty \end{array} \right\}.$$

It is not difficult to verify that $X$ is a Banach space.

We define the maximal operator of systems (2.1)–(2.11) as follows:

$$A_m(V, p_1, p_3) = (A_m^{2,1} V + A_m^{2,2} p_1 + A_m^{2,3} p_3, A_m^1 p_1, A_m^3 p_3),$$

and

$$D(A_m) = \left\{ (V, p_1, p_3) \in X \;\middle|\; \begin{array}{l} \frac{dp_{j,n}}{dx} \in L^1[0,\infty),\ j = 1,3,\ n \geq 1,\ p_{j,n} \text{ are absolutely} \\ \text{continuous functions and } \sum_{n=1}^{\infty} \left\| \frac{dp_{j,n}}{dx} \right\|_{L^1[0,\infty)} < \infty \end{array} \right\},$$

where

$$A_m^{2,1}V=\begin{pmatrix}-(\lambda_1+\theta_1) & \theta_3 & 0 & 0 & 0 & \cdots\\ 0 & -(\lambda_3+\theta_3) & N_0 & 0 & 0 & \cdots\\ \theta_1 & 0 & -1 & 0 & 0 & \cdots\\ 0 & 0 & 0 & -1 & 0 & \cdots\\ 0 & 0 & 0 & 0 & -1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}Q_{1,0}\\ Q_{3,0}\\ V_0\\ V_1\\ V_2\\ \vdots\end{pmatrix},$$

$$A_m^{2,2}p_1=\begin{pmatrix}\varphi_2 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ \theta_1\varphi_1 & 0 & 0 & \cdots\\ 0 & \theta_1\varphi_1 & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}p_{1,1}\\ p_{1,2}\\ p_{1,3}\\ p_{1,4}\\ \vdots\end{pmatrix},\quad A_m^{2,3}p_3=\begin{pmatrix}\theta_3\varphi_1 & \theta_3\varphi_1 & \theta_3\varphi_1 & \cdots\\ \varphi_3 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}p_{3,1}\\ p_{3,2}\\ p_{3,3}\\ \vdots\end{pmatrix},$$

$$A_m^{j}p_j=\begin{pmatrix}B_j & 0 & 0 & \cdots\\ \lambda_j & B_j & 0 & \cdots\\ 0 & \lambda_j & B_j & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}p_{j,1}\\ p_{j,2}\\ p_{j,3}\\ \vdots\end{pmatrix},\quad j=1,3,$$

where $\varphi_1 f(x):=\int_0^\infty f(x)dx$, $\varphi_2 f(x):=\int_0^\infty \mu_1(x)f(x)dx$, $\varphi_3 f(x):=\int_0^\infty \mu_3(x)f(x)dx$ and

$$B_jg:=-\frac{dg(x)}{dx}-[\lambda_j+\theta_j+\mu_j(x)]g(x),\quad g\in W^{1,1}[0,\infty).$$

In the following, we choose the boundary space $\partial X$ of $X$ and define the boundary operators $\Psi$ and $\Phi$ of systems (2.1)–(2.11) as follows:

$$\partial X=l^1\times l^1,\quad \Psi: D(A_m)\to\partial X,\quad \Phi: D(A_m)\to\partial X,$$

and

$$\Psi(V,p_1,p_3)=\left(\begin{pmatrix}p_{1,1}(0)\\ p_{1,2}(0)\\ p_{1,3}(0)\\ \vdots\end{pmatrix},\begin{pmatrix}p_{3,1}(0)\\ p_{3,2}(0)\\ p_{3,3}(0)\\ \vdots\end{pmatrix}\right),$$

$$\Phi(V,p_1,p_3)=(\Phi^{1,2}V+\Phi^{1,1}p_1,\Phi^{3,2}V+\Phi^{3,3}p_3),$$

where

$$\Phi^{1,2}V=\begin{pmatrix}\lambda_1 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}Q_{1,0}\\ Q_{3,0}\\ V_0\\ V_1\\ \vdots\end{pmatrix},\quad \Phi^{1,1}=\begin{pmatrix}0 & \varphi_2 & 0 & 0 & \cdots\\ 0 & 0 & \varphi_2 & 0 & \cdots\\ 0 & 0 & 0 & \varphi_2 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}p_{1,1}\\ p_{1,2}\\ p_{1,3}\\ p_{1,4}\\ \vdots\end{pmatrix},$$

$$\Phi^{3,2}V = \begin{pmatrix} 0 & \lambda_3 & N_1 & N_0 & 0 & 0 & \cdots \\ 0 & 0 & N_2 & N_1 & N_0 & 0 & \cdots \\ 0 & 0 & N_3 & N_2 & N_1 & N_0 & \cdots \\ 0 & 0 & N_4 & N_3 & N_2 & N_1 & \cdots \\ 0 & 0 & N_5 & N_4 & N_3 & N_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{pmatrix} \begin{pmatrix} Q_{1,0} \\ Q_{3,0} \\ V_0 \\ V_1 \\ V_2 \\ \vdots \end{pmatrix}, \quad \Phi^{3,3}p_3 = \begin{pmatrix} 0 & \varphi_3 & 0 & 0 & \cdots \\ 0 & 0 & \varphi_3 & 0 & \cdots \\ 0 & 0 & 0 & \varphi_3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{3,1} \\ p_{3,2} \\ p_{3,3} \\ p_{3,4} \\ \vdots \end{pmatrix}.$$

Now, we define the system operator $\mathcal{A}$ of systems (2.1)–(2.11) by

$$\begin{aligned} &\mathcal{A}(V, p_1, p_3) = A_m(V, p_1, p_3), \\ &D(\mathcal{A}) = \{(V, p_1, p_3) \in D(A_m) | \Psi(V, p_1, p_3) = \Phi(V, p_1, p_3)\}. \end{aligned}$$

Consequently, the systems (2.1)–(2.11) can be written as an abstract Cauchy problem in the Banach space $X$ by

$$\begin{cases} \frac{d(V,p_1,p_3)(t)}{dt} = \mathcal{A}(V, p_1, p_3)(t), \quad \text{for all} \quad t \in (0, \infty), \\ (V, p_1, p_3)(0) = \left( \begin{pmatrix} \overline{u}_1 \\ \overline{u}_3 \\ u_0 \\ u_1 \\ \vdots \end{pmatrix}, \begin{pmatrix} u_{1,1} \\ u_{1,1} \\ u_{1,3} \\ u_{1,4} \\ \vdots \end{pmatrix}, \begin{pmatrix} u_{3,1} \\ u_{3,2} \\ u_{3,3} \\ u_{3,4} \\ \vdots \end{pmatrix} \right). \end{cases} \tag{2.12}$$

Recently, in our work [9], we have obtained the following results.

**Theorem 2.1.** *Let $\lambda_j > 0, \theta_j > 0, N_r > 0$, and $0 < \sup_{x \in [0,\infty)} \mu_j(x) < \infty$, $j = 1, 3$; $r \geq 0$; then system operator $\mathcal{A}$ generates a positive $C_0$-semigroup $e^{\mathcal{A}t}$ of contractions on X. Hence, system (2.12) admits a unique positive time-dependent solution $(V(t), p_1(\cdot, t), p_3(\cdot, t))$ that satisfies*

$$\|(V(t), p_1(\cdot, t), p_3(\cdot, t))\| = 1, \quad \textit{for all} \quad t \in [0, \infty).$$

**Theorem 2.2.** *Let $\mu_j(\cdot) = \mu_j$ be a constant and $\lambda_j, \theta_j, \mu_j > 0, N_r > 0$, $j = 1, 3$; $r \geq 0$, then we have the following results:*

*1) All points in*

$$\left\{ \gamma \in \mathbb{C} \;\middle|\; \begin{aligned} &\sup \left\{ \frac{1}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|} \sup \left\{ \frac{\lambda_1(\theta_1+\mu_1)|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_1|}, \right.\right. \\ &\frac{\lambda_1^2|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_1}, \frac{\lambda_3\theta_1 N_0(\theta_3+\mu_3)}{|\sigma_3|}, \frac{\lambda_3^2\theta_1 N_0}{\Re\sigma_3}, \frac{\theta_1\theta_3(1-N_0)|\gamma+\lambda_3+\theta_3|}{|\sigma_3|-\mu_3}, \\ &\frac{\lambda_3\mu_3\theta_1(1-N_0)|\gamma+\lambda_3+\theta_3|}{\Re\sigma_3(|\sigma_3|-\mu_3)}, \frac{\lambda_1\theta_3(\theta_1+\mu_1)|\gamma+1|}{|\sigma_1|}, \frac{\lambda_1^2\theta_3|\gamma+1|}{\Re\sigma_1}, \frac{\theta_1\theta_3^2(1-N_0)}{|\sigma_3|-\mu_3}, \\ &\frac{\lambda_3(\theta_3+\mu_3)|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{|\sigma_3|}, \frac{\lambda_3^2|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{\Re\sigma_3}, \frac{\lambda_3\mu_3\theta_1\theta_3(1-N_0)}{\Re\sigma_3(|\sigma_3|-\mu_3)}, \\ &\frac{\lambda_3\theta_1 N_0(\theta_1+\mu_1)}{|\sigma_1|}, \frac{\lambda_1\lambda_3\theta_1 N_0}{\Re\sigma_1}, \frac{\lambda_3 N_0(\theta_3+\mu_3)|\gamma+\lambda_1+\theta_1|}{|\sigma_3|}, \frac{\lambda_3^2 N_0|\gamma+\lambda_1+\theta_1|}{\Re\sigma_3}, \\ &\left. \frac{\theta_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_3(|\sigma_3|-\mu_3)} \right\}, \\ &\frac{\theta_3}{|\gamma+1||\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3}{|\gamma+1|\Re\sigma_3(|\sigma_3|-\mu_3)}, \frac{\theta_1+\mu_1}{|\sigma_1|}, \frac{\lambda_1}{\Re\sigma_1}, \frac{\theta_1}{|\sigma_1|-\mu_1}, \frac{\lambda_1\mu_1}{\Re\sigma_1(|\sigma_1|-\mu_1)}, \\ &\left. \frac{\theta_3+\mu_3}{|\sigma_3|}, \frac{\lambda_3}{\Re\sigma_3}, \frac{\theta_3}{|\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3}{\Re\sigma_3(|\sigma_3|-\mu_3)} \right\} < 1, \Re\sigma_j > 0, \ |\sigma_j| > \mu_j \end{aligned} \right\}$$

*are belong to the resolvent set $\rho(\mathcal{A}^*)$. In particular, all points on the imaginary axis except zero belong to $\rho(\mathcal{A})$, where $\sigma_j = \gamma + \lambda_j + \theta_j + \mu_j, 0 < N_0 < 1$, and $\Re\gamma$ is the real part of $\gamma$.*

*2) If $\lambda_j < \theta_j + \mu_j$, then zero is an eigenvalue of $\mathcal{A}$ with geometric multiplicity one.*

*3) Zero is an eigenvalue of $\mathcal{A}^*$ with geometric multiplicity one.*

## 3. Asymptotic behavior of the solution of system (2.12)

According to Theorem 1.96 of [12], if we can demonstrate that the intersection of the point spectrum of the system operator $\mathcal{A}$ and its adjoint operator $\mathcal{A}^*$ with the imaginary axis is a singleton set containing only zero, and that 0 is an eigenvalue of $\mathcal{A}^*$ with geometric multiplicity one, then we can conclude that the time-dependent solution of system (2.12) strongly converges to its steady-state solution.

In this section, we first employ probability generating functions to demonstrate that 0 is an eigenvalue of the system operator $\mathcal{A}$, with a geometric multiplicity is 1. Subsequently, we apply the boundary perturbation method of Greiner [10] to show that all points on the imaginary axis except for 0 fall on the resolvent set of $\mathcal{A}$. Additionally, we directly prove that 0 is also an eigenvalue of the adjoint operator $\mathcal{A}^*$, with a geometric multiplicity of 1. Consequently, the primary findings of this paper are articulated by invoking [12, Theorem 1.96].

**Lemma 3.1.** *Let $\lambda_j > 0, \theta_j > 0$, $N_r \in (0,1)$, $r \geq 0$, and $0 < \mu_j(\cdot) < \infty$, $j = 1, 3$. Then, 0 is an eigenvalue of $\mathcal{A}$ with geometric multiplicity one.*

*Proof.* Consider the equation $\mathcal{A}(V, p_1, p_3) = 0$, which is equivalent to

$$(\lambda_1 + \theta_1)Q_{1,0} = \theta_3 \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x)dx + \theta_3 Q_{3,0} + \int_0^{\infty} p_{1,1}(x)\mu_1(x)dx, \tag{3.1}$$

$$(\lambda_3 + \theta_3)Q_{3,0} = N_0 V_0 + \int_0^{\infty} p_{3,1}(x)\mu_3(x)dx, \tag{3.2}$$

$$V_0 = \theta_1 Q_{1,0}, \tag{3.3}$$

$$V_n = \theta_1 \int_0^{\infty} p_{1,n}(x)dx, \quad n \geq 1, \tag{3.4}$$

$$\frac{dp_{1,1}(x)}{dx} = -[\lambda_1 + \theta_1 + \mu_1(x)]p_{1,1}(x), \tag{3.5}$$

$$\frac{dp_{1,n}(x)}{dx} = -[\lambda_1 + \theta_1 + \mu_1(x)]p_{1,n}(x) + \lambda_1 p_{1,n-1}(x), \quad n \geq 2, \tag{3.6}$$

$$\frac{dp_{3,1}(x)}{dx} = -[\lambda_3 + \theta_3 + \mu_3(x)]p_{3,1}(x), \tag{3.7}$$

$$\frac{dp_{3,n}(x)}{dx} = -[\lambda_3 + \theta_3 + \mu_3(x)]p_{3,n}(x) + \lambda_3 p_{3,n-1}(x), \quad n \geq 2, \tag{3.8}$$

$$p_{1,1}(0) = \lambda_1 Q_{1,0} + \int_0^{\infty} p_{1,2}(x)\mu_1(x)dx, \tag{3.9}$$

$$p_{1,n}(0) = \int_0^{\infty} p_{1,n+1}(x)\mu_1(x)dx, \quad n \geq 2, \tag{3.10}$$

$$p_{3,1}(0) = \lambda_3 Q_{3,0} + N_1 V_0 + N_0 V_1 + \int_0^\infty p_{3,2}(x)\mu_3(x)dx, \tag{3.11}$$

$$p_{3,n}(0) = \sum_{k=0}^{n} N_{n-k} V_k + \int_0^\infty p_{3,n+1}(x)\mu_3(x)dx, \quad n \geq 2. \tag{3.12}$$

By solving the Eqs (3.5)–(3.8), we have

$$p_{j,n}(x) = e^{-(\lambda_j+\theta_j)x - \int_0^x \mu_j(\tau)d\tau} \sum_{k=1}^{n} \frac{(\lambda_j x)^{n-k}}{(n-k)!} p_{j,k}(0), \quad j = 1, 3; \ n \geq 1. \tag{3.13}$$

Due to the difficulty in finding the specific expression of $V_n,\ p_{1,n}(x)$, and $p_{3,n}(x)$ and proving whether 0 is the eigenvalue of the system operator $\mathcal{A}$, we will solve this problem by introducing the probability generating function.

$$p_j(x, z) := \sum_{n=1}^{\infty} p_{j,n}(x) z^n, \quad \overline{V}(z) := \sum_{n=0}^{\infty} V_n z^n,$$

$$p_j(z) := \sum_{n=1}^{\infty} p_{j,n} z^n, \quad p_{j,n} := \int_0^\infty p_{j,n}(x)dx, \quad j = 1, 3.$$

Multiply both sides of Eq (3.4) by $z^n,\ n \geq 1$, then sum $n$ from 0 to positive infinity, and add Eq (3.3) to obtain

$$\begin{aligned} \overline{V}(z) &= V_0 + \sum_{n=1}^{\infty} V_n z^n = V_0 + \theta_1 \sum_{n=1}^{\infty} \int_0^\infty p_{1,n}(x) z^n dx \\ &= \theta_1 Q_{1,0} + \theta_1 p_1(z). \end{aligned} \tag{3.14}$$

Multiply both sides of Eqs (3.6) and (3.8) by $z^n,\ n \geq 2$, then sum $n$ from 0 to positive infinity, and using Eqs (3.5) and (3.7), we have

$$\frac{\partial \sum_{n=1}^{\infty} p_{j,n}(x) z^n}{\partial x} = -\sum_{n=1}^{\infty} [\lambda_j + \theta_j + \mu_j(x)] p_{j,n}(x) z^n + \lambda_j \sum_{n=1}^{\infty} p_{j,n-1}(x) z^n.$$

That is,

$$\begin{aligned} \frac{\partial p_j(x, z)}{\partial x} &= -[\lambda_j + \theta_j + \mu_j(x)] p_j(x, z) + \lambda_j z p_j(x, z) \\ &= -[\theta_j + \lambda_j(1 - z) - \mu_j(x)] p_j(x, z). \end{aligned} \tag{3.15}$$

By solving the above equation, we obtain

$$p_j(x, z) = p_j(0, z) e^{-[\theta_j + \lambda_j(1-z)]x - \int_0^x \mu_j(\tau)d\tau}, \quad j = 1, 3. \tag{3.16}$$

Using the boundary conditions (3.9), (3.10), and Eq (3.16), we have

$$
\begin{aligned}
p_1(0,z) &= \sum_{n=1}^{\infty} p_{1,n}(0)z^n = p_{1,1}(0)z + \sum_{n=2}^{\infty} p_{1,n}(0)z^n \\
&= z\lambda_1 Q_{1,0} + \int_0^{\infty} p_{1,2}(x)\mu_1(x)z dx + \sum_{n=2}^{\infty}\int_0^{\infty} p_{1,n+1}(x)\mu_1(x)z^n dx \\
&= z\lambda_1 Q_{1,0} + \sum_{n=1}^{\infty}\int_0^{\infty} p_{1,n+1}(x)\mu_1(x)z^n dx \\
&= z\lambda_1 Q_{1,0} + \frac{1}{z}\int_0^{\infty}\sum_{n=1}^{\infty} p_{1,n}(x)z^n\mu_1(x)dx - \int_0^{\infty} p_{1,1}(x)\mu_1(x)dx \\
&= z\lambda_1 Q_{1,0} + \frac{1}{z}\int_0^{\infty} p_1(x,z)\mu_1(x)dx - \int_0^{\infty} p_{1,1}(x)\mu_1(x)dx \\
&= z\lambda_1 Q_{1,0} + \frac{1}{z}p_1(0,z)\int_0^{\infty}\mu_1(x)e^{-[\theta_1+\lambda_1(1-z)]x-\int_0^x\mu_1(\tau)d\tau}dx - \int_0^{\infty} p_{1,1}(x)\mu_1(x)dx.
\end{aligned}
$$

That is,

$$
(z - \int_0^{\infty}\mu_1(x)e^{-[\theta_1+\lambda_1(1-z)]x-\int_0^x\mu_1(\tau)d\tau}dx)p_1(0,z) = z^2\lambda_1 Q_{1,0} - z\int_0^{\infty} p_{1,1}(x)\mu_1(x)dx. \tag{3.17}
$$

By Eqs (3.2), (3.11), (3.12), (3.16) and

$$
\sum_{n=0}^{\infty} N_n z^n = \sum_{n=0}^{\infty} e^{-\lambda_2 pd}\frac{(\lambda_2 pdz)^n}{n!} = e^{-\lambda_2 pd(1-z)},
$$

we calculate that

$$
\begin{aligned}
p_3(0,z) &= \sum_{n=1}^{\infty} p_{3,n}(0)z^n = p_{3,1}(0)z + \sum_{n=2}^{\infty} p_{3,n}(0)z^n \\
&= z\lambda_3 Q_{3,0} + zN_1V_0 + zN_0V_1 + \int_0^{\infty} p_{3,2}(x)z\mu_3(x)dx \\
&+ \sum_{n=2}^{\infty}\left[\sum_{k=0}^{n} N_{n-k}V_k + \int_0^{\infty} p_{3,n+1}(x)\mu_3(x)dx\right]z^n \\
&= z\lambda_3 Q_{3,0} + \sum_{n=1}^{\infty} z^n\sum_{k=0}^{n} N_{n-k}V_k + \sum_{n=1}^{\infty}\int_0^{\infty} p_{3,n+1}(x)z^n\mu_3(x)dx \\
&= z\lambda_3 Q_{3,0} + \overline{V}(z)e^{\lambda_2 pd(1-z)} - V_0N_0 + \frac{1}{z}\int_0^{\infty}\sum_{n=1}^{\infty} p_{3,n}(x)z^n\mu_3(x)d - \int_0^{\infty} p_{3,1}(x)\mu_3(x)dx \\
&= z\lambda_3 Q_{3,0} + \overline{V}(z)e^{\lambda_2 pd(1-z)} - (\lambda_3+\theta_3)Q_{3,0} + \frac{1}{z}\int_0^{\infty} p_3(x,z)\mu_3(x)dx \\
&= -[\theta_3+\lambda_3(1-z)]Q_{3,0} + \overline{V}(z)e^{\lambda_2 pd(1-z)} + \frac{1}{z}p_3(0,z)\int_0^{\infty}\mu_3(x)e^{-[\theta_3+\lambda_3(1-z)]x-\int_0^x\mu_3(\tau)d\tau}dx.
\end{aligned}
$$

That is,

$$\left[z-\int_0^{\infty}\mu_3(x)e^{-[\theta_3+\lambda_3(1-z)]x-\int_0^x\mu_3(\tau)d\tau}dx\right]p_3(0,z)=-z[\theta_3+\lambda_3(1-z)]Q_{3,0}+z\overline{V}(z)e^{\lambda_2pd(1-z)}. \quad (3.18)$$

Notice that when $z\in(-1,1)$, according to Rouche's theorem, we deduce that there exists a unique solution (for detailed proof, please refer to [6, Lemma 1]) to equation

$$z=\int_0^{\infty}\mu_j(x)e^{-[\theta_j+\lambda_j(1-z)]x-\int_0^x\mu_j(\tau)d\tau}dx, \quad j=1,3,$$

we assume that the solution is $\gamma_j$. Then, by the above Eqs (3.17) and (3.18), we obtain $\gamma_1$ and $\gamma_3$ that satisfy the following equations:

$$\gamma_1\lambda_1Q_{1,0}=\int_0^{\infty}p_{1,1}(x)\mu_1(x)dx, \quad (3.19)$$

$$Q_{3,0}=\frac{\overline{V}(\gamma_3)e^{\lambda_2pd(1-\gamma_3)}}{[\theta_3+\lambda_3(1-\gamma_3)]}. \quad (3.20)$$

Using the above Eqs (3.16), (3.19), and (3.20), we have

$$p_1(0,z)=\frac{\lambda_1Q_{1,0}z(z-\gamma_1)}{z-\int_0^{\infty}\mu_1(x)e^{-[\theta_1+\lambda_1(1-z)]x-\int_0^x\mu_1(\tau)d\tau}dx}, \quad (3.21)$$

$$p_3(0,z)=\frac{-z[\theta_3+\lambda_3(1-z)]Q_{3,0}+z\overline{V}(z)e^{\lambda_2pd(1-z)}}{z-\int_0^{\infty}\mu_3(x)e^{-[\theta_3+\lambda_3(1-z)]x-\int_0^x\mu_3(\tau)d\tau}dx}. \quad (3.22)$$

Hence, by Eqs (3.16), (3.21), (3.22), and (3.15), we obtain

$$\begin{aligned}
p_1(1)&=\lim_{z\to1}p_1(z)=\lim_{z\to1}\int_0^{\infty}p_1(x,z)dx\\
&=\lim_{z\to1}\frac{\lambda_1Q_{1,0}z(z-\gamma_1)\int_0^{\infty}e^{-[\theta_1+\lambda_1(1-z)]x-\int_0^x\mu_1(\tau)d\tau}dx}{z-\int_0^{\infty}\mu_1(x)e^{-[\theta_1+\lambda_1(1-z)]x-\int_0^x\mu_1(\tau)d\tau}dx}\\
&=\frac{\lambda_1Q_{1,0}(1-\gamma_1)\int_0^{\infty}e^{-\theta_1x-\int_0^x\mu_1(\tau)d\tau}dx}{1-\int_0^{\infty}\mu_1(x)e^{-\theta_1x-\int_0^x\mu_1(\tau)d\tau}dx},
\end{aligned} \quad (3.23)$$

$$\begin{aligned}
p_3(1)&=\lim_{z\to1}p_3(z)=\lim_{z\to1}\int_0^{\infty}p_3(x,z)dx\\
&=\lim_{z\to1}\frac{[-z(\theta_3+\lambda_3(1-z))Q_{3,0}+z\overline{V}(z)e^{\lambda_2pd(1-z)}]}{z-\int_0^{\infty}\mu_3(x)e^{-[\theta_3+\lambda_3(1-z)]x-\int_0^x\mu_3(\tau)d\tau}dx}\\
&\quad\times\int_0^{\infty}\mu_3(x)e^{-[\theta_3+\lambda_3(1-z)]x-\int_0^x\mu_3(\tau)d\tau}dx\\
&=\frac{[-\theta_3Q_{3,0}+\overline{V}(1)]\int_0^{\infty}\mu_3(x)e^{-\theta_3x-\int_0^x\mu_3(\tau)d\tau}dx}{1-\int_0^{\infty}\mu_3(x)e^{-\theta_3x-\int_0^x\mu_3(\tau)d\tau}dx},
\end{aligned} \quad (3.24)$$

$$\begin{aligned}\overline{V}(1) &= \lim_{z\to 1}\overline{V}(z) = \theta_1 Q_{1,0} + \theta_1 p_1(1)\\ &= \theta_1 Q_{1,0} + \theta_1 \frac{\lambda_1 Q_{1,0}(1-\gamma_1)\int_0^\infty e^{-\theta_1 x - \int_0^x \mu_1(\tau)d\tau}dx}{1-\int_0^\infty \mu_1(x)e^{-\theta_1 x-\int_0^x \mu_1(\tau)d\tau}dx}.\end{aligned} \tag{3.25}$$

Consequently, using Eqs (3.19), (3.20), (3.23)–(3.25) and the normalizing condition $Q_{1,0} + Q_{3,0} + \overline{V}(1) + p_1(1) + p_3(1) = 1$, we can obtain the specific expression for $Q_{1,0}$, which we have omitted here. Notice that $\int_0^\infty \mu_j(x)e^{-\theta_j x-\int_0^x \mu_j(\tau)d\tau}dx < 1,\ \ j = 1, 3.$ Hence, the above discussions mean that 0 is an eigenvalue of $\mathcal{A}$.

In addition, since Eqs (3.1)–(3.4) and (3.13), it is easy to see that the geometric multiplicity of zero is one.

Next, we define the operator $A_0$ with boundary conditions $p_{j,n}(0) = 0, \quad j = 1, 3;\ n \geq 1$ for systems (2.1)–(2.11) as follows:

$$A_0(V, p_1, p_3) = A_m(V, p_1, p_3),$$

$$D(A_0) = \{(V, p_1, p_3) \in D(A_m)|\Psi(V, p_1, p_3) = 0\}.$$

And find the resolvent set of this operator. For this objective, we consider the equation $(\gamma I - A_0)(V, p_1, p_3) = (w, y_1, y_3)$ for all $(V, p_1, p_3) \in X$, where $w = (z_{1,0}, z_{3,0}, w_0, w_1, \cdots)$ and $y_j = (y_{j,1}, y_{j,2}, \cdots)$, $j = 1, 3$. This equation is equivalent to

$$(\gamma + \lambda_1 + \theta_1)Q_{1,0} = z_{1,0} + \theta_3 \sum_{n=1}^{\infty}\int_0^\infty p_{3,n}(x)dx + \theta_3 Q_{3,0} + \int_0^\infty p_{1,1}(x)\mu_1(x)dx, \tag{3.26}$$

$$(\gamma + \lambda_3 + \theta_3)Q_{3,0} = z_{3,0} + N_0 V_0 + \int_0^\infty p_{3,1}(x)\mu_3(x)dx, \tag{3.27}$$

$$(\gamma + 1)V_0 = w_0 + \theta_1 Q_{1,0}, \tag{3.28}$$

$$(\gamma + 1)V_n = w_n + \theta_1 \int_0^\infty p_{1,n}(x)dx, \quad n \geq 1, \tag{3.29}$$

$$\frac{dp_{1,1}(x)}{dx} = -[\gamma + \lambda_1 + \theta_1 + \mu_1(x)]p_{1,1}(x) + y_{1,1}(x), \tag{3.30}$$

$$\frac{dp_{1,n}(x)}{dx} = -[\gamma + \lambda_1 + \theta_1 + \mu_1(x)]p_{1,n}(x) + \lambda_1 p_{1,n-1}(x) + y_{1,n}(x), \quad n \geq 2, \tag{3.31}$$

$$\frac{dp_{3,1}(x)}{dx} = -[\gamma + \lambda_3 + \theta_3 + \mu_3(x)]p_{3,1}(x) + y_{3,1}(x), \tag{3.32}$$

$$\frac{dp_{3,n}(x)}{dx} = -[\gamma + \lambda_3 + \theta_3 + \mu_3(x)]p_{3,n}(x) + \lambda_3 p_{3,n-1}(x) + y_{3,n}(x), \quad n \geq 2, \tag{3.33}$$

$$p_{j,n}(0) = 0, \quad j = 1, 3;\ n \geq 1. \tag{3.34}$$

Solve the above Eqs (3.26)–(3.33), and using boundary conditions $p_{j,n}(0) = 0$, we have

$$\begin{aligned}
Q_{1,0} \;=\;& \frac{(\gamma+1)(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} z_{1,0} \\
&+\frac{(\gamma+1)\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} z_{3,0} \\
&+\frac{\theta_3N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} w_0 \\
&+\frac{(\gamma+1)(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \int_0^\infty p_{1,1}(x)\mu_1(x)dx \\
&+\frac{(\gamma+1)\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \int_0^\infty p_{3,1}(x)\mu_3(x)dx \\
&+\frac{(\gamma+1)(\gamma+\lambda_3+\theta_3)\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \sum_{n=1}^{\infty}\int_0^\infty p_{3,n}(x)dx,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
Q_{3,0} \;=\;& \frac{\theta_1N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} z_{1,0} \\
&+\frac{(\gamma+1)(\gamma+\lambda_1+\theta_1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} z_{3,0} \\
&+\frac{N_0(\gamma+\lambda_1+\theta_1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} w_0 \\
&+\frac{\theta_1N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \int_0^\infty p_{1,1}(x)\mu_1(x)dx \\
&+\frac{(\gamma+1)(\gamma+\lambda_1+\theta_1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \int_0^\infty p_{3,1}(x)\mu_3(x)dx \\
&+\frac{\theta_1\theta_3N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \sum_{n=1}^{\infty}\int_0^\infty p_{3,n}(x)dx,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
V_0 \;=\;& \frac{\theta_1(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} z_{1,0} \\
&+\frac{\theta_1\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} z_{3,0} \\
&+\frac{(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} w_0 \\
&+\frac{\theta_1(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \int_0^\infty p_{1,1}(x)\mu_1(x)dx \\
&+\frac{\theta_1\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \int_0^\infty p_{3,1}(x)\mu_3(x)dx \\
&+\frac{\theta_1\theta_3(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0} \sum_{n=1}^{\infty}\int_0^\infty p_{3,n}(x)dx,
\end{aligned} \tag{3.37}$$

$$V_n = \frac{1}{\gamma+1} w_n + \frac{\theta_1}{\gamma+1}\int_0^\infty p_{1,n}(x)dx, \tag{3.38}$$

$$p_{j,1}(x) = e^{-(\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau}\int_0^x y_{j,1}(\tau)e^{(\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi}d\tau, \quad j=1,3, \tag{3.39}$$

$$\begin{aligned} p_{j,n}(x) &= e^{-(\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau}\int_0^x y_{j,n}(\tau)e^{(\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi}d\tau \\ &\quad +\lambda_j e^{-(\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau}\int_0^x p_{j,n-1}(\tau)e^{(\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi}d\tau, \ n\ge 1. \end{aligned} \tag{3.40}$$

For convenience, we introduce $E_j$ as follows:

$$E_j f(x) := e^{-(\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau}\int_0^x f(\tau)e^{(\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi}d\tau, \quad j=1,3, \tag{3.41}$$

for any $f \in L^1[0,\infty)$. Then, if $(\gamma I - A_0)^{-1}$ exists, then by Eqs (3.35)–(3.40), we have

$$\begin{aligned} (\gamma I - A_0)^{-1}(w,y_1,y_3) = &\left(\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & 0 & 0 & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{\gamma+1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} z_{1,0} \\ z_{3,0} \\ w_0 \\ w_1 \\ w_2 \\ \vdots \end{pmatrix}\right. \\ &+\begin{pmatrix} a_{1,4}\varphi_2 E_1 & 0 & 0 & 0 & \cdots \\ a_{2,4}\varphi_2 E_1 & 0 & 0 & 0 & \cdots \\ a_{3,4}\varphi_2 E_1 & 0 & 0 & 0 & \cdots \\ \frac{\theta_1}{\gamma+1}\varphi_1 E_1 & 0 & 0 & 0 & \cdots \\ \frac{\theta_1}{\gamma+1}\lambda_1\varphi_1 E_1^2 & \frac{\theta_1}{\gamma+1}\varphi_1 E_1 & 0 & 0 & \cdots \\ \frac{\theta_1}{\gamma+1}\lambda_1^2\varphi_1 E_1^3 & \frac{\theta_1}{\gamma+1}\lambda_1\varphi_1 E_1^2 & \frac{\theta_1}{\gamma+1}\varphi_1 E_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}\begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \\ \vdots \end{pmatrix} \\ &+\begin{pmatrix} a_{1,5}\varphi_3 + a_{1,6}\varphi_1 & a_{1,6}\varphi_1 & a_{1,6}\varphi_1 & \cdots \\ a_{2,5}\varphi_3 + a_{2,6}\varphi_1 & a_{2,6}\varphi_1 & a_{2,6}\varphi_1 & \cdots \\ a_{3,5}\varphi_3 + a_{3,6}\varphi_1 & a_{3,6}\varphi_1 & a_{3,6}\varphi_1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} E_3 & 0 & 0 & \cdots \\ \lambda_3 E_3^2 & E_3 & 0 & \cdots \\ \lambda_3^2 E_3^3 & \lambda_3 E_3^2 & E_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} y_{3,1} \\ y_{3,2} \\ y_{3,3} \\ \vdots \end{pmatrix}, \\ &\left.\begin{pmatrix} E_1 & 0 & 0 & \cdots \\ \lambda_1 E_1^2 & E_1 & 0 & \cdots \\ \lambda_1^2 E_1^3 & \lambda_1 E_1^2 & E_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ \vdots \end{pmatrix}, \begin{pmatrix} E_3 & 0 & 0 & \cdots \\ \lambda_3 E_3^2 & E_3 & 0 & \cdots \\ \lambda_3^2 E_3^3 & \lambda_3 E_3^2 & E_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} y_{3,1} \\ y_{3,2} \\ y_{3,3} \\ \vdots \end{pmatrix}\right), \end{aligned} \tag{3.42}$$

where

$$\varphi_1 f(x) = \int_0^\infty f(x)dx, \quad \varphi_2 f(x) = \int_0^\infty \mu_1(x)f(x)dx, \quad \varphi_3 f(x) = \int_0^\infty \mu_3(x)f(x)dx,$$

and

$$a_{1,1}=a_{1,4}=\frac{(\gamma+1)(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.43}$$

$$a_{1,2}=a_{1,5}=\frac{(\gamma+1)\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.44}$$

$$a_{1,3}=\frac{\theta_3N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.45}$$

$$a_{1,6}=\frac{(\gamma+1)(\gamma+\lambda_3+\theta_3)\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.46}$$

$$a_{2,1}=a_{2,4}=\frac{\theta_1N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.47}$$

$$a_{2,2}=a_{2,5}=\frac{(\gamma+1)(\gamma+\lambda_1+\theta_1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.48}$$

$$a_{2,3}=\frac{N_0(\gamma+\lambda_1+\theta_1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.49}$$

$$a_{2,6}=\frac{\theta_1\theta_3N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.50}$$

$$a_{3,1}=\frac{\theta_1(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.51}$$

$$a_{3,2}=a_{3,5}=\frac{\theta_1\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.52}$$

$$a_{3,3}=\frac{(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.53}$$

$$a_{3,4}=\frac{\theta_1(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}, \tag{3.54}$$

$$a_{3,6}=\frac{\theta_1\theta_3(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}. \tag{3.55}$$

**Lemma 3.2.** *Let* $\lambda_j>0,\theta_j>0,N_0\in(0,1)$*, and* $\mu_j(x):[0,\infty)\to[0,\infty)$ *be measurable and*

$$0<\inf_{x\in[0,\infty)}\mu_j(x)\le\mu_j(x)\le\sup_{x\in[0,\infty)}\mu_j(x)<\infty,\ \ j=1,3.$$

*Then,* $S=:\left\{\gamma\in\mathbb{C}\,\middle|\,\Re\gamma+1>0,\Re\gamma+\theta_j+\inf_{x\in[0,\infty)}\mu_j(x)>0\right\}\subset\rho(A_0)$.

*Proof.* The proof of this lemma is provided in the appendix.

**Lemma 3.3.** *Let* $\lambda_j>0,\theta_j>0,N_0\in(0,1)$*, and* $\mu_j(x):[0,\infty)\to[0,\infty)$ *be measurable and*

$$0<\inf_{x\in[0,\infty)}\mu_j(x)\le\mu_j(x)\le\sup_{x\in[0,\infty)}\mu_j(x)<\infty,\ \ j=1,3.$$

*If* $\gamma \in S$*, then* $(V, p_1, p_3) \in \ker(\gamma I - A_m)$ *if and only if*

$$\begin{aligned}
Q_{1,0} &= a_{1,4}p_{1,1}(0)\int_0^\infty \mu_1(x)e^{-(\gamma+\lambda_1+\theta_1)x-\int_0^x \mu_1(\tau)d\tau}dx \\
&+a_{1,5}p_{3,1}(0)\int_0^\infty \mu_3(x)e^{-(\gamma+\lambda_3+\theta_3)x-\int_0^x \mu_3(\tau)d\tau}dx \\
&+a_{1,6}\sum_{n=1}^{\infty}\sum_{k=1}^{n} p_{3,n-k+1}(0)\int_0^\infty \frac{(\lambda_3 x)^{k-1}}{(k-1)!}e^{-(\gamma+\lambda_3+\theta_3)x-\int_0^x \mu_3(\tau)d\tau}dx,
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
Q_{3,0} &= a_{2,4}p_{1,1}(0)\int_0^\infty \mu_1(x)e^{-(\gamma+\lambda_1+\theta_1)x-\int_0^x \mu_1(\tau)d\tau}dx \\
&+a_{2,5}p_{3,1}(0)\int_0^\infty \mu_3(x)e^{-(\gamma+\lambda_3+\theta_3)x-\int_0^x \mu_3(\tau)d\tau}dx \\
&+a_{2,6}\sum_{n=1}^{\infty}\sum_{k=1}^{n} p_{3,n-k+1}(0)\int_0^\infty \frac{(\lambda_3 x)^{k-1}}{(k-1)!}e^{-(\gamma+\lambda_3+\theta_3)x-\int_0^x \mu_3(\tau)d\tau}dx,
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
V_0 &= a_{3,4}p_{1,1}(0)\int_0^\infty \mu_1(x)e^{-(\gamma+\lambda_1+\theta_1)x-\int_0^x \mu_1(\tau)d\tau}dx \\
&+a_{3,5}p_{3,1}(0)\int_0^\infty \mu_3(x)e^{-(\gamma+\lambda_3+\theta_3)x-\int_0^x \mu_3(\tau)d\tau}dx \\
&+a_{3,6}\sum_{n=1}^{\infty}\sum_{k=1}^{n} p_{3,n-k+1}(0)\int_0^\infty \frac{(\lambda_3 x)^{k-1}}{(k-1)!}e^{-(\gamma+\lambda_3+\theta_3)x-\int_0^x \mu_3(\tau)d\tau}dx,
\end{aligned} \tag{3.58}$$

$$V_n = \frac{\theta_1}{\gamma+1}\sum_{k=1}^{n} p_{1,n-k+1}(0)\int_0^\infty \frac{(\lambda_1 x)^{k-1}}{(k-1)!}e^{-(\gamma+\lambda_1+\theta_1)x-\int_0^x \mu_1(\tau)d\tau}dx, \tag{3.59}$$

$$p_{j,n}(x) = e^{-(\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau}\sum_{k=1}^{n}\frac{(\lambda_j x)^{k-1}}{(k-1)!}p_{j,n-k+1}(0), \quad j = 1,3;\ n \geq 1, \tag{3.60}$$

$$p_j = (p_{j,1}, p_{j,2}, \cdots) \in l^1, \quad j = 1,3, \tag{3.61}$$

*where the specific expressions for* $a_{k,m}$, $k = 1,2,3;\ m = 4,5,6$ *are given in Eqs (3.43)–(3.55).*

*Proof.* The proof of this lemma is provided in the appendix.

It is clear that the boundary operator $\Psi$ is surjective. In addition, for $\gamma \in S$, we see that the operator

$$\Psi\big|_{\ker(\gamma I - A_m)} : \ker(\gamma I - A_m) \to \partial X,$$

is a reversible operator.

Now, for $\gamma \in S$, we define the Dirichlet operator $D_\gamma$ as follows:

$$D_\gamma := (\Psi\big|_{\ker(\gamma I - A_m)})^{-1} : \partial X \to \ker(\gamma I - A_m).$$

Then, for any $\gamma \in S$, using Lemma 3.3, we obtain the specific expression of the operator $D_\gamma$:

$$D_\gamma(\vec{p}_1(0), \vec{p}_3(0)) = \left( \begin{pmatrix} a_{1,4}\varphi_2 m_{1,1} & 0 & 0 & 0 & \cdots \\ a_{2,4}\varphi_2 m_{1,1} & 0 & 0 & 0 & \cdots \\ a_{3,4}\varphi_2 m_{1,1} & 0 & 0 & 0 & \cdots \\ \frac{\theta_1}{\gamma+1}\varphi_1 m_{1,1} & 0 & 0 & 0 & \cdots \\ \frac{\theta_1}{\gamma+1}\varphi_1 m_{1,2} & \frac{\theta_1}{\gamma+1}\varphi_1 m_{1,1} & 0 & 0 & \cdots \\ \frac{\theta_1}{\gamma+1}\varphi_1 m_{1,3} & \frac{\theta_1}{\gamma+1}\varphi_1 m_{1,2} & \frac{\theta_1}{\gamma+1}\varphi_1 m_{1,1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} p_{1,1}(0) \\ p_{1,2}(0) \\ p_{1,3}(0) \\ p_{1,4}(0) \\ \vdots \end{pmatrix} + \right.$$

$$\begin{pmatrix} a_{1,5}\varphi_3 + a_{1,6}\varphi_1 & a_{1,6}\varphi_1 & a_{1,6}\varphi_1 & \cdots \\ a_{2,5}\varphi_3 + a_{2,6}\varphi_1 & a_{2,6}\varphi_1 & a_{2,6}\varphi_1 & \cdots \\ a_{3,5}\varphi_3 + a_{3,6}\varphi_1 & a_{3,6}\varphi_1 & a_{3,6}\varphi_1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} m_{3,1} & 0 & 0 & \cdots \\ m_{3,2} & m_{3,1} & 0 & \cdots \\ m_{3,3} & m_{3,2} & m_{3,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{3,1}(0) \\ p_{3,2}(0) \\ p_{3,3}(0) \\ \vdots \end{pmatrix}, \tag{3.62}$$

$$\left. \begin{pmatrix} m_{1,1} & 0 & 0 & \cdots \\ m_{1,2} & m_{1,1} & 0 & \cdots \\ m_{1,3} & m_{1,2} & m_{1,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,1}(0) \\ p_{1,2}(0) \\ p_{1,3}(0) \\ \vdots \end{pmatrix}, \begin{pmatrix} m_{3,1} & 0 & 0 & \cdots \\ m_{3,2} & m_{3,1} & 0 & \cdots \\ m_{3,3} & m_{3,2} & m_{3,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{3,1}(0) \\ p_{3,2}(0) \\ p_{3,3}(0) \\ \vdots \end{pmatrix} \right),$$

where

$$m_{j,k} = \frac{(\lambda_j x)^{k-1}}{(k-1)!} e^{-(\gamma+\lambda_j+\theta_j)x - \int_0^x \mu_j(\tau)d\tau}, \quad j = 1, 3;\ k \geq 1.$$

Finally, by the definitions of operators $\Phi$ and $D_\gamma$, we calculate that

$$\begin{aligned} &\Phi D_\gamma(\vec{p}_1(0), \vec{p}_3(0)) \\ &= \left( \begin{pmatrix} \lambda_1\delta_1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \varphi_2[m_{1,2}p_{1,1}(0) + m_{1,1}p_{1,2}(0)] \\ \varphi_2[m_{1,3}p_{1,1}(0) + m_{1,2}p_{1,2}(0) + m_{1,1}p_{1,3}(0)] \\ \varphi_2[m_{1,4}p_{1,1}(0) + m_{1,3}p_{1,2}(0) + m_{1,2}p_{1,3}(0) + m_{1,1}p_{1,4}(0)] \\ \vdots \end{pmatrix}, \right. \\ &\quad \begin{pmatrix} \lambda_3\delta_2 + N_1\delta_3 + N_0\frac{\theta_1}{\gamma+1}\varphi_1 m_{1,1}p_{1,1}(0) \\ N_2\delta_3 + N_1\frac{\theta_1}{\gamma+1}\varphi_1 m_{1,1}p_{1,1}(0) + N_0\frac{\theta_1}{\gamma+1}\sum_{k=1}^{2}\varphi_1 m_{1,k}p_{1,2-k+1}(0) \\ \Upsilon \\ \vdots \end{pmatrix} \\ &\quad \left. + \begin{pmatrix} \varphi_3[m_{3,2}p_{3,1}(0) + m_{3,1}p_{3,2}(0)] \\ \varphi_3[m_{3,3}p_{3,1}(0) + m_{3,2}p_{3,2}(0) + m_{3,1}p_{3,3}(0)] \\ \varphi_3[m_{3,4}p_{3,1}(0) + m_{3,3}p_{3,2}(0) + m_{3,2}p_{3,3}(0) + m_{3,1}p_{3,4}(0)] \\ \vdots \end{pmatrix} \right), \end{aligned} \tag{3.63}$$

where

$$\Upsilon = N_3\delta_3 + N_2\frac{\theta_1}{\gamma+1}\varphi_1 m_{1,1}p_{1,1}(0) + N_1\frac{\theta_1}{\gamma+1}\sum_{k=1}^{2}\varphi_1 m_{1,k}p_{1,2-k+1}(0) + N_0\frac{\theta_1}{\gamma+1}\sum_{k=1}^{3}\varphi_1 m_{1,k}p_{1,3-k+1}(0),$$

$$\delta_1 = a_{1,4}\varphi_2 m_{1,1} p_{1,1}(0) + a_{1,5}\varphi_3 m_{3,1} p_{3,1}(0) + a_{1,6} \sum_{n=1}^{\infty} \varphi_1 m_{3,n} \sum_{k=1}^{\infty} p_{3,k}(0),$$

$$\delta_2 = a_{2,4}\varphi_2 m_{1,1} p_{1,1}(0) + a_{2,5}\varphi_3 m_{3,1} p_{3,1}(0) + a_{2,6} \sum_{n=1}^{\infty} \varphi_1 m_{3,n} \sum_{k=1}^{\infty} p_{3,k}(0),$$

$$\delta_3 = a_{3,4}\varphi_2 m_{1,1} p_{1,1}(0) + a_{3,5}\varphi_3 m_{3,1} p_{3,1}(0) + a_{3,6} \sum_{n=1}^{\infty} \varphi_1 m_{3,n} \sum_{k=1}^{\infty} p_{3,k}(0).$$

Based on a conclusion drawn by Haji and Radl [11], we know that the following result, Lemma 3.4, holds true.

**Lemma 3.4.** *If $\gamma \in \rho(A_0)$ and there exists some $\gamma_0$ such that $1$ is not in the spectrum $\sigma(\Phi D_{\gamma_0})$ of $\Phi D_{\gamma_0}$, then the conclusion*

$$\gamma \in \sigma(\mathcal{A}) \quad \text{if and only if} \quad 1 \in \sigma(\Phi D_\gamma)$$

*holds.*

Hence, using Lemma 3.4 and Nagel [13, p. 297], we have the following result:

**Lemma 3.5.** *Let $\lambda_j > 0, \theta_j > 0, N_r \in (0,1),\ r \geq 0$, and $\mu_j(x) : [0,\infty) \to [0,\infty)$ be measurable and*

$$0 < \inf_{x\in[0,\infty)} \mu_j(x) \leq \mu_j(x) \leq \sup_{x\in[0,\infty)} \mu_j(x) < \infty,\ j = 1, 3.$$

*Then, all points on the imaginary axis, except for 0, fall on the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$.*

*Proof.* If we take $\gamma = ib$, $b \in \mathbb{R} \setminus \{0\}$, $\vec{p}_1(0) = (p_{1,1}(0), p_{1,2}(0), \cdots) \in l^1$ and $\vec{p}_3(0) = (p_{3,1}(0), p_{3,2}(0), \cdots) \in l^1$. The Riemann–Lebesgue Lemma states that for any $f \in L^1[0,\infty)$ (i.e., $f$ is an integrable function on $[0,\infty)$, we have

$$\lim_{b\to\infty} \int_0^\infty f(x)\cos(bx)dx = 0, \qquad \lim_{b\to\infty} \int_0^\infty f(x)\sin(bx)dx = 0.$$

This means that as $b$ approaches infinity, the integrals of $f(x)$ with high-frequency cosine or sine functions tend to zero. Using Euler's formula $e^{-ibx} = \cos(bx) - i\sin(bx)$, the integral $\left|\int_0^\infty f(x)e^{-ibx}dx\right|$ can be split into real and imaginary parts:

$$\left|\int_0^\infty f(x)e^{-ibx}dx\right| = \left|\int_0^\infty f(x)\cos(bx)dx - i\int_0^\infty f(x)\sin(bx)dx\right|. \tag{3.64}$$

Then, according to the Riemann–Lebesgue Lemma, for sufficiently large $|b|$, both of integrals of Eq (3.64) tend to zero.

In addition, for any $0 < f \in L^1[0,\infty)$, we have

$$\left|\int_0^\infty f(x)e^{-ibx}dx\right| \leq \int_0^\infty f(x)dx. \tag{3.65}$$

Then, for any $0 < f \in L^1[0,\infty)$ and $|b| > \mathbb{M}_1$ (where $\mathbb{M}_1$ is some sufficiently large constant), according to Eqs (3.64), (3.65), and the Riemann–Lebesgue Lemma, we can further obtain

$$\begin{aligned}\left|\int_0^\infty f(x)e^{-ibx}dx\right|^2 &= \left(\int_0^\infty f(x)\cos(bx)dx\right)^2 + \left(\int_0^\infty f(x)\sin(bx)dx\right)^2\\ &< \left(\int_0^\infty f(x)dx\right)^2.\end{aligned}\tag{3.66}$$

That is, for any $0 < f \in L^1[0,\infty)$ and $|b| > \mathbb{M}_1$, we have

$$\left|\int_0^\infty f(x)e^{-ibx}dx\right| < \int_0^\infty f(x)dx.\tag{3.67}$$

Furthermore, through tedious calculations, we have found that the following inequality holds true

$$|\lambda_1 a_{1,h}| < 1, \quad |\lambda_3 a_{2,h}| < 1, \quad |(1-N_0)a_{3,h}| < 1, \quad h = 4,5,6,\tag{3.68}$$

where the specific expressions for $a_{k,h}(k = 1,2,3;\ h = 4,5,6)$ are given in Eqs (3.43)–(3.55) and $N_0 = e^{-\lambda_2 pd}(< 1)$ is the probability of zero arrivals during phase 2. In fact,

$$\begin{aligned}&|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3) - \theta_1\theta_3N_0|^2 - \lambda_1^2|(ib+1)(ib+\lambda_3+\theta_3)|^2\\ &= b^6 + b^4[2\lambda_1\theta_1 + \theta_1^2 + (\lambda_3+\theta_3)^2 + 1] + b^2\{2\lambda_1\theta_1 + \theta_1^2 + (\lambda_3+\theta_3)^2\\ &\quad +(2\lambda_1\theta_1 + \theta_1^2)(\lambda_3+\theta_3)^2 + 2(\lambda_1+\theta_1+\lambda_3+\theta_3+1)\theta_1\theta_3N_0\}\\ &\quad +2\lambda_1(\lambda_3+\theta_3)[\theta_1(\lambda_3+\theta_3) - \theta_1\theta_3N_0] + [\theta_1(\lambda_3+\theta_3) - \theta_1\theta_3N_0]^2 > 0\\ &\Rightarrow |\lambda_1 a_{1,4}| = \frac{\lambda_1|(ib+1)(ib+\lambda_3+\theta_3)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3) - \theta_1\theta_3N_0|} < 1,\end{aligned}\tag{3.69}$$

$$\begin{aligned}&|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3) - \theta_1\theta_3N_0|^2\\ &= b^6 + b^4[(\lambda_1+\theta_1)^2 + (\lambda_3+\theta_3)^2 + 1] + b^2\{(\lambda_1+\theta_1)^2 + (\lambda_3+\theta_3)^2\\ &\quad +[(\lambda_1+\theta_1)(\lambda_3+\theta_3)]^2 + 2(\lambda_1+\theta_1+\lambda_3+\theta_3+1)\theta_1\theta_3N_0\}\\ &\quad +[(\lambda_1+\theta_1)(\lambda_3+\theta_3) - \theta_1\theta_3N_0]^2 > \theta_1^2[b^2 + (\lambda_3+\theta_3)^2]\\ &> [\theta_1(1-N_0)]^2[b^2+(\lambda_3+\theta_3)^2] = |\theta_1(1-N_0)(ib+\lambda_3+\theta_3)|^2\\ &\Rightarrow |(1-N_0)a_{3,4}| = \frac{\theta_1(1-N_0)|ib+\lambda_3+\theta_3|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3) - \theta_1\theta_3N_0|} < 1.\end{aligned}\tag{3.70}$$

Using the same method as Eqs (3.69) and (3.70), we can prove that the remaining inequalities in Eq (3.68) also hold true.

Hence, for any $|b| > \mathbb{M}_1$, using Eqs (3.63), (3.67), and (3.68), the formula

$$\sum_{r=0}^\infty N_r = \sum_{r=0}^\infty \frac{e^{-\lambda_2 pd}(\lambda_2 pd)^r}{r!} = 1, \quad \int_0^\infty [\theta_j + \mu_j(x)]e^{-\int_0^x[\theta_j+\mu_j(\tau)]d\tau}dx = 1, \quad j = 1,3,$$

and $\frac{1}{\sqrt{b^2+1}} < 1$, through tedious calculations, we estimate that

$$\|\Phi D_{ib}(\vec{p}_1(0), \vec{p}_3(0))\| \leq \lambda_1|a_{1,4}||\varphi_2 m_{1,1}||p_{1,1}(0)| + \lambda_1|a_{1,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|$$

$$
\begin{aligned}
&+\lambda_1|a_{1,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)|+|\varphi_2 m_{1,2}||p_{1,1}(0)|+|\varphi_2 m_{1,1}||p_{1,2}(0)|+|\varphi_2 m_{1,3}||p_{1,1}(0)|\\
&+|\varphi_2 m_{1,2}||p_{1,2}(0)|+|\varphi_2 m_{1,1}||p_{1,3}(0)|+|\varphi_2 m_{1,4}||p_{1,1}(0)|+|\varphi_2 m_{1,3}||p_{1,2}(0)|+|\varphi_2 m_{1,2}||p_{1,3}(0)|\\
&+|\varphi_2 m_{1,1}||p_{1,4}(0)|+\cdots\\
&+\lambda_3|a_{2,4}||\varphi_2 m_{1,1}||p_{1,1}(0)|+\lambda_3|a_{2,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|+\lambda_3|a_{2,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+N_1|a_{3,4}||\varphi_2 m_{1,1}||p_{1,1}(0)|+N_1|a_{3,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|+N_1|a_{3,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+N_0\frac{\theta_1}{|ib+1|}|\varphi_1 m_{1,1}||p_{1,1}(0)|+|\varphi_3 m_{3,2}||p_{3,1}(0)|+|\varphi_3 m_{3,1}||p_{3,2}(0)|\\
&+N_2|a_{3,4}||\varphi_2 m_{1,1}||p_{1,1}(0)|+N_2|a_{3,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|+N_2|a_{3,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+N_1\frac{\theta_1}{|ib+1|}|\varphi_1 m_{1,1}||p_{1,1}(0)|+N_0\frac{\theta_1}{|ib+1|}\sum_{k=1}^{2}|\varphi_1 m_{1,k}||p_{1,2-k+1}(0)|\\
&+|\varphi_3 m_{3,3}||p_{3,1}(0)|+|\varphi_3 m_{3,2}||p_{3,2}(0)|+|\varphi_3 m_{3,1}||p_{3,3}(0)|\\
&+N_3|a_{3,4}||\varphi_2 m_{1,1}||p_{1,1}(0)|+N_3|a_{3,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|+N_3|a_{3,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+N_2\frac{\theta_1}{|ib+1|}|\varphi_1 m_{1,1}||p_{1,1}(0)|+N_1\frac{\theta_1}{|ib+1|}\sum_{k=1}^{2}|\varphi_1 m_{1,k}||p_{1,2-k+1}(0)|\\
&+N_0\frac{\theta_1}{|ib+1|}\sum_{k=1}^{3}|\varphi_1 m_{1,k}||p_{1,3-k+1}(0)|+|\varphi_3 m_{3,4}||p_{3,1}(0)|+|\varphi_3 m_{3,3}||p_{3,2}(0)|\\
&+|\varphi_3 m_{3,2}||p_{3,3}(0)|+|\varphi_3 m_{3,1}||p_{3,4}(0)|+\cdots\\
\le\ &\lambda_1|a_{1,4}||\varphi_2 m_{1,1}||p_{1,1}(0)|+\lambda_1|a_{1,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|+\lambda_1|a_{1,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+\sum_{n=2}^{\infty}|\varphi_2 m_{1,n}||p_{1,1}(0)|+\sum_{n=1}^{\infty}|\varphi_2 m_{1,n}||p_{1,2}(0)|+\sum_{n=1}^{\infty}|\varphi_2 m_{1,n}||p_{1,3}(0)|\\
&+\sum_{n=1}^{\infty}|\varphi_2 m_{1,n}||p_{1,4}(0)|+\cdots\\
&+\lambda_3|a_{2,4}||\varphi_2 m_{1,1}||p_{1,1}(0)|+\lambda_3|a_{2,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|+\lambda_3|a_{2,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+\sum_{j=1}^{\infty}N_j|a_{3,4}|\varphi_2 m_{1,1}||p_{1,1}(0)|+\sum_{j=1}^{\infty}N_j|a_{3,5}||\varphi_3 m_{3,1}||p_{3,1}(0)|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{\infty} N_j|a_{3,6}|\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|\sum_{k=1}^{\infty}|p_{3,k}(0)| \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,1}||p_{1,1}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,2}||p_{1,1}(0)| \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,3}||p_{1,1}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,4}||p_{1,1}(0)| \\
&+\cdots \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,1}||p_{1,2}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,2}||p_{1,2}(0)| \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,3}||p_{1,2}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,4}||p_{1,2}(0)| \\
&+\cdots \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,1}||p_{1,3}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,2}||p_{1,3}(0)| \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,3}||p_{1,3}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,4}||p_{1,3}(0)| \\
&+\cdots \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,1}||p_{1,4}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,2}||p_{1,4}(0)| \\
&+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,3}||p_{1,4}(0)|+\sum_{j=0}^{\infty} N_j\frac{\theta_1}{\sqrt{b^2+1}}|\varphi_1 m_{1,4}||p_{1,4}(0)| \\
&+\cdots \\
&+\sum_{n=2}^{\infty}|\varphi_3 m_{3,n}||p_{3,1}(0)|+\sum_{n=1}^{\infty}|\varphi_3 m_{3,n}||p_{3,2}(0)|+\sum_{n=1}^{\infty}|\varphi_3 m_{3,n}||p_{3,3}(0)| \\
&+\sum_{n=1}^{\infty}|\varphi_3 m_{3,n}||p_{3,4}(0)|+\cdots \\
<&\left(\theta_3\sum_{n=1}^{\infty}|\varphi_1 m_{3,n}|+\sum_{n=1}^{\infty}|\varphi_3 m_{3,n}|\right)\sum_{k=1}^{\infty}|p_{3,k}(0)| \\
&+\left(\sum_{n=1}^{\infty}|\varphi_2 m_{1,n}|+\theta_1\sum_{n=1}^{\infty}|\varphi_1 m_{1,n}|\right)\sum_{k=1}^{\infty}|p_{1,k}(0)| \\
=&\left(\theta_3\sum_{n=1}^{\infty}\left|\int_0^{\infty}\frac{(\lambda_3 x)^{n-1}}{(n-1)!}e^{-(ib+\lambda_3+\theta_3)x-\int_0^x\mu_3(\tau)d\tau}dx\right|\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\left|\int_0^{\infty}\mu_3(x)\frac{(\lambda_3 x)^{n-1}}{(n-1)!}e^{-(ib+\lambda_3+\theta_3)x-\int_0^x\mu_3(\tau)d\tau}dx\right|\Bigg)\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+\Bigg(\sum_{n=1}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda_1 x)^{n-1}}{(n-1)!}e^{-(ib+\lambda_1+\theta_1)x-\int_0^x\mu_1(\tau)d\tau}dx\right|\\
&+\theta_1\sum_{n=1}^{\infty}\left|\int_0^{\infty}\frac{(\lambda_1 x)^{n-1}}{(n-1)!}e^{-(ib+\lambda_1+\theta_1)x-\int_0^x\mu_3(\tau)d\tau}dx\right|\Bigg)\sum_{k=1}^{\infty}|p_{1,k}(0)|\\
\le&\Bigg(\theta_3\sum_{n=1}^{\infty}\int_0^{\infty}\frac{(\lambda_3 x)^{n-1}}{(n-1)!}e^{-(\lambda_3+\theta_3)x-\int_0^x\mu_3(\tau)d\tau}dx\\
&+\sum_{n=1}^{\infty}\int_0^{\infty}\mu_3(x)\frac{(\lambda_3 x)^{n-1}}{(n-1)!}e^{-(\lambda_3+\theta_3)x-\int_0^x\mu_3(\tau)d\tau}dx\Bigg)\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+\Bigg(\sum_{n=1}^{\infty}\int_0^{\infty}\mu_1(x)\frac{(\lambda_1 x)^{n-1}}{(n-1)!}e^{-(\lambda_1+\theta_1)x-\int_0^x\mu_1(\tau)d\tau}dx\\
&+\theta_1\sum_{n=1}^{\infty}\int_0^{\infty}\frac{(\lambda_1 x)^{n-1}}{(n-1)!}e^{-(\lambda_1+\theta_1)x-\int_0^x\mu_3(\tau)d\tau}dx\Bigg)\sum_{k=1}^{\infty}|p_{1,k}(0)|\\
=&\Bigg(\theta_3\int_0^{\infty}\sum_{n=1}^{\infty}\frac{(\lambda_3 x)^{n-1}}{(n-1)!}e^{-(\lambda_3+\theta_3)x-\int_0^x\mu_3(\tau)d\tau}dx\\
&+\int_0^{\infty}\mu_3(x)\sum_{n=1}^{\infty}\frac{(\lambda_3 x)^{n-1}}{(n-1)!}e^{-(\lambda_3+\theta_3)x-\int_0^x\mu_3(\tau)d\tau}dx\Bigg)\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+\Bigg(\int_0^{\infty}\mu_1(x)\sum_{n=1}^{\infty}\frac{(\lambda_1 x)^{n-1}}{(n-1)!}e^{-(\lambda_1+\theta_1)x-\int_0^x\mu_1(\tau)d\tau}dx\\
&+\theta_1\int_0^{\infty}\sum_{n=1}^{\infty}\frac{(\lambda_1 x)^{n-1}}{(n-1)!}e^{-(\lambda_1+\theta_1)x-\int_0^x\mu_3(\tau)d\tau}dx\Bigg)\sum_{k=1}^{\infty}|p_{1,k}(0)|\\
=&\int_0^{\infty}[\theta_3+\mu_3(x)]e^{-\int_0^x[\theta_3+\mu_3(\tau)]d\tau}dx\sum_{k=1}^{\infty}|p_{3,k}(0)|\\
&+\int_0^{\infty}[\theta_1+\mu_1(x)]e^{-\int_0^x[\theta_1+\mu_1(\tau)]d\tau}dx\sum_{k=1}^{\infty}|p_{1,k}(0)|\\
=&\|\vec{p}_1(0),\vec{p}_3(0)\|.
\end{aligned}
\tag{3.71}
$$

That is, $\|\Phi D_{ib}\|<1$. Inequality (3.71) means that when $|b|>\mathbb{M}_1$, the spectral bound $r(\Phi D_{ib})\le\|\Phi D_{ib}\|<1$. That is, we have $1\notin\sigma(\Phi D_{ib})$ if $|b|>\mathbb{M}_1$. Therefore, by Lemma 3.4, we deduce that

$$\{ib\mid |b|>\mathbb{M}_1\}\subset\rho(\mathcal{A}),\quad \{ib\mid |b|<\mathbb{M}_1\}\subset\sigma(\mathcal{A})\cap i\mathbb{R}. \tag{3.72}$$

On the other hand, since the semigroup $e^{\mathcal{A}t}$ is positive uniformly bounded by Theorem 2.1, by [13, Corollary 2.3, p. 297] we see that $\sigma(\mathcal{A})\cap i\mathbb{R}$ is imaginary additively cyclic, which shows that if

$ib \in \sigma(\mathcal{A}) \cap i\mathbb{R}$, then $ibn \in \sigma(\mathcal{A}) \cap i\mathbb{R}$ for all integer $n$. Hence, from which, together with Eq (3.72) and Lemma 3.1, we conclude that $ib \in \sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$.

It is easy to see that the dual space $X^*$ of $X$ is as follows (see [9]):

$$X^* = \left\{ (V^*, p_1^*, p_3^*) \left| \begin{array}{l} V^* = (Q_{1,0}^*, Q_{3,0}^*, V_0^*, V_1^*, \cdots) \in l^\infty, \\ p_1^* = (p_{1,1}^*, p_{1,2}^*, p_{1,3}^*, \cdots) \in L^\infty[0,\infty) \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ p_3^* = (p_{3,1}^*, p_{3,2}^*, p_{3,3}^*, \cdots) \in L^\infty[0,\infty) \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ \|(V^*, p_1^*, p_3^*)\| = \sup\Big\{ |Q_{1,0}^*|, |Q_{3,0}^*|, \sup_{j\geq 0} |V_j^*|, \\ \sup_{j\geq 1} \|p_{1,j}^*\|_{L^\infty[0,\infty)}, \sup_{j\geq 1} \|p_{3,j}^*\|_{L^\infty[0,\infty)} \Big\} < \infty \end{array} \right. \right\}.$$

Clearly, $X^*$ also is a Banach space.

**Lemma 3.6.** *The adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is as follows:*

$$\mathcal{A}^*(V^*, p_1^*, p_3^*) = \left( \begin{pmatrix} -(\lambda_1+\theta_1) & 0 & \theta_1 & 0 & 0 & \cdots \\ \theta_3 & -(\lambda_3+\theta_3) & 0 & 0 & 0 & \cdots \\ 0 & N_0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_{1,0}^* \\ Q_{3,0}^* \\ V_0^* \\ V_1^* \\ V_2^* \\ \vdots \end{pmatrix} \right.$$

$$+ \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,1}^*(0) \\ p_{1,2}^*(0) \\ p_{1,3}^*(0) \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \lambda_3 & 0 & 0 & 0 & \cdots \\ N_1 & N_2 & N_3 & N_4 & \cdots \\ N_0 & N_1 & N_2 & N_3 & \cdots \\ 0 & N_0 & N_1 & N_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{3,1}^*(0) \\ p_{3,2}^*(0) \\ p_{3,3}^*(0) \\ p_{3,4}^*(0) \\ \vdots \end{pmatrix},$$

$$\begin{pmatrix} \mu_1(x) & 0 & 0 & \theta_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \theta_1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \theta_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_{1,0}^* \\ Q_{3,0}^* \\ V_0^* \\ V_1^* \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \mu_1(x) & 0 & 0 & 0 & \cdots \\ 0 & \mu_1(x) & 0 & 0 & \cdots \\ 0 & 0 & \mu_1(x) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,1}^*(0) \\ p_{1,2}^*(0) \\ p_{1,3}^*(0) \\ p_{1,4}^*(0) \\ \vdots \end{pmatrix}$$

$$+ \begin{pmatrix} \zeta_1 & \lambda_1 & 0 & \cdots \\ 0 & \zeta_1 & \lambda_1 & \cdots \\ 0 & 0 & \zeta_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,1}^*(x) \\ p_{1,2}^*(x) \\ p_{1,3}^*(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} \theta_3 & \mu_3(x) & 0 & 0 & \cdots \\ \theta_3 & 0 & 0 & 0 & \cdots \\ \theta_3 & 0 & 0 & 0 & \cdots \\ \theta_3 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_{1,0}^* \\ Q_{3,0}^* \\ V_0^* \\ V_1^* \\ \vdots \end{pmatrix}$$

$$\left. + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \mu_3(x) & 0 & 0 & 0 & \cdots \\ 0 & \mu_3(x) & 0 & 0 & \cdots \\ 0 & 0 & \mu_3(x) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{3,1}^*(0) \\ p_{3,2}^*(0) \\ p_{3,3}^*(0) \\ p_{3,4}^*(0) \\ \vdots \end{pmatrix} + \begin{pmatrix} \zeta_3 & \lambda_3 & 0 & \cdots \\ 0 & \zeta_3 & \lambda_3 & \cdots \\ 0 & 0 & \zeta_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{3,1}^*(x) \\ p_{3,2}^*(x) \\ p_{3,3}^*(x) \\ \vdots \end{pmatrix} \right),$$

*where* $\zeta_j = \frac{d}{dx} - [\lambda_j + \theta_j + \mu_j(x)],\ \ j = 1, 3$ *and*

$$D(\mathcal{A}^*) = \left\{ (q_I^*, q_w^*) \in X^* \;\middle|\; \begin{array}{c} p_{j,n}^*(\cdot)\ (n \geq 1) \text{ are absolutely continuous} \\ \text{functions and satisfy } p_{j,n}^*(\infty) = \alpha,\ \alpha \text{ is} \\ \text{a nonzero constant which is irrelevant to } n \end{array} \right\}.$$

*Proof.* For any $(V, p_1, p_3) \in D(\mathcal{A})$ and $(V^*, p_1^*, p_3^*) \in D(\mathcal{A}^*)$, we calculate that

$$\begin{aligned}
&\langle \mathcal{A}(V, p_1, p_3), (V^*, p_1^*, p_3^*) \rangle \\
&= \left[ -(\lambda_1 + \theta_1) Q_{1,0} + \theta_3 \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x) dx + \theta_3 Q_{3,0} + \int_0^{\infty} \mu_1(x) p_{1,1}(x) dx \right] Q_{1,0}^* \\
&\quad + \left[ -(\lambda_3 + \theta_3) Q_{3,0} + N_0 V_0 + \int_0^{\infty} \mu_3(x) p_{3,1}(x) dx \right] Q_{3,0}^* \\
&\quad + (-V_0 + \theta_1 Q_{1,0}) V_0^* + \sum_{n=1}^{\infty} \left[ -V_n + \theta_1 \int_0^{\infty} p_{1,n}(x) dx \right] V_n^* \\
&\quad + \int_0^{\infty} \left[ -\frac{dp_{1,1}(x)}{dx} - (\lambda_1 + \theta_1 + \mu_1(x)) p_{1,1}(x) \right] p_{1,1}^*(x) dx \\
&\quad + \sum_{n=2}^{\infty} \int_0^{\infty} \left[ -\frac{dp_{1,n}(x)}{dx} - (\lambda_1 + \theta_1 + \mu_1(x)) p_{1,n}(x) + \lambda_1 p_{1,n-1}(x) \right] p_{1,n}^*(x) dx \\
&\quad + \int_0^{\infty} \left[ -\frac{dp_{3,1}(x)}{dx} - (\lambda_3 + \theta_3 + \mu_3(x)) p_{3,1}(x) \right] p_{3,1}^*(x) dx \\
&\quad + \sum_{n=2}^{\infty} \int_0^{\infty} \left[ -\frac{dp_{3,n}(x)}{dx} - (\lambda_3 + \theta_3 + \mu_3(x)) p_{3,n}(x) + \lambda_1 p_{3,n-1}(x) \right] p_{3,n}^*(x) dx \\
&= -(\lambda_1 + \theta_1) Q_{1,0} Q_{1,0}^* + \theta_3 \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n} Q_{1,0}^*(x) dx + \theta_3 Q_{3,0} Q_{1,0}^* \\
&\quad + \int_0^{\infty} \mu_1(x) p_{1,1}(x)(x) Q_{1,0}^* dx \\
&\quad - (\lambda_3 + \theta_3) Q_{3,0} Q_{3,0}^* + N_0 V_0 Q_{3,0}^* + \int_0^{\infty} \mu_3(x) p_{3,1}(x) Q_{3,0}^* dx \\
&\quad + \theta_1 Q_{1,0} V_0^* - \sum_{n=0}^{\infty} V_n V_n^* + \theta_1 \sum_{n=1}^{\infty} \int_0^{\infty} p_{1,n}(x) V_n^* dx \\
&\quad + \sum_{n=1}^{\infty} p_{1,n}(0) p_{1,n}^*(0) + \sum_{n=1}^{\infty} \int_0^{\infty} p_{1,n}(x) \left[ \frac{dp_{1,n}^*(x)}{dx} - (\lambda_1 + \theta_1 + \mu_1(x)) p_{1,n}^*(x) \right] dx \\
&\quad + \lambda_1 \sum_{n=1}^{\infty} \int_0^{\infty} p_{1,n}(x) p_{1,n+1}^*(x) dx \\
&\quad + \sum_{n=1}^{\infty} p_{3,n}(0) p_{3,n}^*(0) + \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x) \left[ \frac{dp_{3,n}^*(x)}{dx} - (\lambda_3 + \theta_3 + \mu_3(x)) p_{3,n}^*(x) \right] dx
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_3 \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x) p_{3,n+1}^*(x)dx \\
= &-(\lambda_1+\theta_1)Q_{1,0}Q_{1,0}^* + \theta_3 \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n} Q_{1,0}^*(x)dx + \theta_3 Q_{3,0}Q_{1,0}^* \\
&+ \int_0^{\infty} \mu_1(x) p_{1,1}(x) Q_{1,0}^* dx \\
&-(\lambda_3+\theta_3)Q_{3,0}Q_{3,0}^* + N_0 V_0 Q_{3,0}^* + \int_0^{\infty} \mu_3(x) p_{3,1}(x) Q_{3,0}^* dx \\
&+\theta_1 Q_{1,0} V_0^* - \sum_{n=0}^{\infty} V_n V_n^* + \theta_1 \sum_{n=1}^{\infty} \int_0^{\infty} p_{1,n}(x) V_n^* dx \\
&+\lambda_1 Q_{1,0} p_{1,1}^*(0) + \sum_{n=1}^{\infty} \int_0^{\infty} \mu_1(x) p_{1,n+1}(x) p_{1,n}^*(0)dx \\
&+ \sum_{n=1}^{\infty} \int_0^{\infty} p_{1,n}(x) \left[ \frac{dp_{1,n}^*(x)}{dx} - (\lambda_1+\theta_1+\mu_1(x)) p_{1,n}^*(x) \right] dx \\
&+\lambda_1 \sum_{n=1}^{\infty} \int_0^{\infty} p_{1,n}(x) p_{1,n+1}^*(x)dx \\
&+\lambda_3 Q_{3,0} p_{3,1}^*(0) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} N_k V_{n-k} p_{3,n}^*(0) + \sum_{n=1}^{\infty} \int_0^{\infty} \mu_3(x) p_{3,n+1}(x) p_{3,n}^*(0)dx \\
&+ \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x) \left[ \frac{dp_{3,n}^*(x)}{dx} - (\lambda_3+\theta_3+\mu_3(x)) p_{3,n}^*(x) \right] dx \\
&+\lambda_3 \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x) p_{3,n+1}^*(x)dx \\
= &\langle (V, p_1, p_3), A^*(V^*, p_1^*, p_3^*) \rangle. 
\end{aligned} \tag{3.73}
$$

Eq (3.73) means that Lemma 3.6 holds true.

**Lemma 3.7.** *Let $\lambda_j > 0, \theta_j > 0$, $N_r \in (0,1)$, $r \geq 0$, and $0 < \mu_j(\cdot) < \infty$, $j = 1,3$. Then, 0 is an eigenvalue of $\mathcal{A}^*$ with geometric multiplicity 1.*

*Proof.* We consider the equation $\mathcal{A}^*(V^*, p_1^*, p_3^*) = 0$. That is,

$$-(\lambda_1+\theta_1)Q_{1,0}^* + \theta_1 V_0^* = 0, \tag{3.74}$$

$$\theta_3 Q_{1,0}^* - (\lambda_3+\theta_3)Q_{3,0}^* + \lambda_3 p_{3,1}^*(0) = 0, \tag{3.75}$$

$$N_0 Q_{3,0}^* - V_0^* + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) = 0, \tag{3.76}$$

$$-V_n^* + \sum_{k=0}^{\infty} N_k p_{3,k+n}^*(0) = 0, \quad n \geq 1, \tag{3.77}$$

$$\frac{dp_{1,1}^*(x)}{dx} - [\lambda_1 + \theta_1 + \mu_1(x)]p_{1,1}^*(x) + \lambda_1 p_{1,2}^*(x) + \theta_1 V_1^* + \mu_1(x)Q_{1,0}^* = 0, \tag{3.78}$$

$$\frac{dp_{1,n}^*(x)}{dx} - [\lambda_1 + \theta_1 + \mu_1(x)]p_{1,n}^*(x) + \lambda_1 p_{1,n+1}^*(x) + \theta_1 V_n^* + \mu_1(x)p_{1,n-1}^*(0) = 0, \quad n \geq 2, \tag{3.79}$$

$$\frac{dp_{3,1}^*(x)}{dx} - [\lambda_3 + \theta_3 + \mu_3(x)]p_{3,1}^*(x) + \lambda_3 p_{3,2}^*(x) + \theta_3 Q_{1,0}^* + \mu_3(x)Q_{3,0}^* = 0, \tag{3.80}$$

$$\frac{dp_{3,n}^*(x)}{dx} - [\lambda_3 + \theta_3 + \mu_3(x)]p_{3,n}^*(x) + \lambda_3 p_{3,n+1}^*(x) + \theta_3 Q_{1,0}^* + \mu_3(x)p_{3,n-1}^*(0) = 0, \quad n \geq 2, \tag{3.81}$$

$$p_{j,n}^*(\infty) = \alpha, \quad j = 1, 3; \; n \geq 1. \tag{3.82}$$

It is easy to see that

$$(V^*, p_1^*, p_3^*) = \left( \begin{pmatrix} \alpha \\ \alpha \\ \vdots \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ \vdots \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ \vdots \end{pmatrix} \right) \in D(\mathcal{A}^*),$$

is a nonzero solution of Eqs (3.74)–(3.82). In addition, Eqs (3.74)–(3.81) are equivalent to

$$Q_{1,0}^* = \frac{\theta_1(\lambda_3 + \theta_3)}{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3) - N_0\theta_1\theta_3} \left( \frac{\lambda_3 N_0}{\lambda_3 + \theta_3} p_{3,1}^*(0) + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \right), \tag{3.83}$$

$$Q_{3,0}^* = \frac{\theta_1\theta_3}{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3) - N_0\theta_1\theta_3} \left( \frac{\lambda_3 N_0}{\lambda_3 + \theta_3} p_{3,1}^*(0) + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \right), \tag{3.84}$$

$$V_0^* = \frac{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3)}{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3) - N_0\theta_1\theta_3} \left( \frac{\lambda_3 N_0}{\lambda_3 + \theta_3} p_{3,1}^*(0) + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \right), \tag{3.85}$$

$$V_n^* = \sum_{k=0}^{\infty} N_k p_{3,k+n}^*(0), \quad n \geq 1, \tag{3.86}$$

$$\begin{aligned} p_{1,2}^*(x) \quad &= -\frac{1}{\lambda_1} \left[ \frac{dp_{1,1}^*(x)}{dx} - (\lambda_1 + \theta_1 + \mu_1(x))p_{1,1}^*(x) + \theta_1 \sum_{k=0}^{\infty} N_k p_{3,k+1}^*(0) \right. \\ &\left. + \frac{\theta_1(\lambda_3 + \theta_3)\mu_1(x)}{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3) - N_0\theta_1\theta_3} \left( \frac{\lambda_3 N_0}{\lambda_3 + \theta_3} p_{3,1}^*(0) + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \right) \right], \end{aligned} \tag{3.87}$$

$$\begin{aligned} p_{1,n+1}^*(x) = -\frac{1}{\lambda_1} \left[ \frac{dp_{1,n}^*(x)}{dx} - (\lambda_1 + \theta_1 + \mu_1(x))p_{1,n}^*(x) + \theta_1 \sum_{k=0}^{\infty} N_k p_{3,k+n}^*(0) \right] \\ -\frac{\mu_1(x)}{\lambda_1} p_{3,n-1}^*(0), \quad n \geq 2, \end{aligned} \tag{3.88}$$

$$\begin{aligned} p_{3,2}^*(x) = -\frac{1}{\lambda_3} &\left[ \frac{dp_{3,1}^*(x)}{dx} - (\lambda_3 + \theta_3 + \mu_3(x))p_{3,1}^*(x) \right. \\ &+ \frac{\theta_1\theta_3(\lambda_3 + \theta_3)}{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3) - N_0\theta_1\theta_3} \left( \frac{\lambda_3 N_0}{\lambda_3 + \theta_3} p_{3,1}^*(0) + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \right) \\ &\left. + \frac{\theta_1\theta_3\mu_3(x)}{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3) - N_0\theta_1\theta_3} \left( \frac{\lambda_3 N_0}{\lambda_3 + \theta_3} p_{3,1}^*(0) + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \right) \right], \end{aligned} \tag{3.89}$$

$$p^*_{3,n+1}(x) = -\frac{1}{\lambda_3}\left[\frac{dp^*_{3,n}(x)}{dx} - (\lambda_3 + \theta_3 + \mu_3(x))p^*_{3,n}(x) + \mu_3(x)p^*_{3,n-1}(0) \right.$$
$$\left. + \frac{\theta_1\theta_3(\lambda_3 + \theta_3)}{(\lambda_1 + \theta_1)(\lambda_3 + \theta_3) - N_0\theta_1\theta_3}\left(\frac{\lambda_3 N_0}{\lambda_3 + \theta_3}p^*_{3,1}(0) + \sum_{k=1}^{\infty} N_k p^*_{3,k}(0)\right)\right], \quad n \geq 2. \tag{3.90}$$

Consequently, Eqs (3.83)–(3.90) mean that the geometric multiplicity of 0 is one.

Finally, using Lemmas 3.1, 3.5, and 3.7 and Theorem 1.96 of [12], we obtain the following desired result.

**Theorem 3.1.** *Let* $\lambda_j > 0, \theta_j > 0, N_r \in (0, 1),\ r \geq 0,$ *and* $\mu_j(x) : [0, \infty) \to [0, \infty)$ *be measurable and*

$$0 < \inf_{x\in[0,\infty)} \mu_j(x) \leq \mu_j(x) \leq \sup_{x\in[0,\infty)} \mu_j(x) < \infty,\ \ j = 1, 3.$$

*Then, the time-dependent solution of system (2.12) strongly converges to its steady-state solution, that is,*

$$\lim_{t\to\infty} \|(V(t), p_1(\cdot, t), p_3(\cdot, t)) - \langle (V^*, p_1^*(\cdot), p_3^*(\cdot)), (V, p_1, p_3)(0))\rangle (V, p_1(\cdot), p_3(\cdot))\| = 0, \tag{3.91}$$

*where* $(V, p_1(\cdot), p_3(\cdot))$ *and* $(V^*, p_1^*(\cdot), p_3^*(\cdot))$ *are the eigenvectors in Lemmas 3.1 and 3.7, respectively, and* $(V, p_1, p_3)(0))$ *is the initial value of system (2.12).*

*Proof.* Theorem 2.1 establishes that the operator $\mathscr{A}$ generates a uniformly bounded $C_0-$semigroup on the Banach space $X$. Furthermore, leveraging Lemmas 3.5, 3.1, and 3.7, we can readily deduce that

$$\sigma_p(\mathscr{A}) \cap i\mathbb{R} = \sigma_p(\mathscr{A}^*) \cap i\mathbb{R} = \{0\},$$

and that the set $\{\gamma \in \mathbb{C} \mid \gamma = ib, b \neq 0, b \in \mathbb{R}\}$ is a subset of the resolvent set $\rho(\mathscr{A})$. Additionally, zero is an eigenvalue of $\mathscr{A}^*$ with geometric multiplicity one.

Consequently, invoking Theorem 1.96 from [12], we conclude that the time-dependent solution of the system (2.12) converges strongly to its steady-state solution. Specifically, the limit Eq (3.91) holds true.

## 4. Conclusions

In this paper, we investigate the asymptotic behavior of the M/G/1 stochastic clearing queueing model in a three-phase environment, specifically when the service rate of the server is a bounded function. By employing probability generating functions and the boundary perturbation method of Greiner, we demonstrate that all points on the imaginary axis, except for 0, fall within the resolvent set of the system operator. Additionally, we highlight that 0 is an eigenvalue of the adjoint operator of the system operator, with a geometric multiplicity of 1. This finding implies that the time-dependent solution of the system strongly converges to its steady-state solution.

This theoretical result provides a solid foundation for understanding the long-term behavior of the system. However, the implications of strong convergence for practical system performance metrics, such as queue-length distributions, transient behavior, and the rate of convergence, are equally important for real-world applications. Strong convergence implies that over time, the system's transient behavior will closely approximate its steady-state behavior. This means that for sufficiently large times, the queue-length distribution and other performance metrics will be well-represented by their steady-state counterparts. In practical settings, this suggests that after an initial transient period, the system will exhibit stable performance characteristics that can be effectively estimated using steady-state analysis.

However, for the more general case of the service rate function, where $0 \leq \mu_j(\cdot) \leq \infty$, we have not yet determined whether the above results still hold. Furthermore, it remains unclear whether the time-dependent solution exponentially converges to its steady-state solution. To address the exponential stability of this system, we need to investigate the spectrum of the system operator on the left half of the complex plane, as discussed in [14, 15]. These topics are among our future research endeavors.

The approach outlined in this paper is specifically tailored for queuing systems that are characterized through partial differential equations [16]. It is not applicable to the queuing systems discussed in [17,18]. There have been extensive studies on the asymptotic behavior of semigroups (see [7, 8, 12, 19]). which are also of significant interest for our future research.

## Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflicts of interest

The authors declare there is no conflict of interest.

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## A. Appendix: The proof of Lemmas 3.2 and 3.3

**The proof of Lemma 3.2.** For all $f \in L^1[0,\infty)$, we compute

$$\begin{aligned}
&\int_0^\infty |E_j f(x)|dx = \int_0^\infty \left| e^{-(\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau} \int_0^x f(\tau) e^{(\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi} d\tau \right| dx \\
&\leq \int_0^\infty e^{-(\Re\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau} \int_0^x |f(\tau)| e^{(\Re\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi} d\tau dx \\
&= \int_0^\infty \frac{-1}{\Re\gamma+\lambda_j+\theta_j+\mu_j(x)} \int_0^x |f(\tau)| e^{(\Re\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi} d\tau d e^{-(\Re\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau} \\
&= \frac{-1}{\Re\gamma+\lambda_j+\theta_j+\mu_j(x)} e^{-(\Re\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau} \int_0^x |f(\tau)| e^{(\Re\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi} d\tau \Big|_{x=0}^{x=\infty} \\
&\quad + \int_0^\infty \frac{1}{\Re\gamma+\lambda_j+\theta_j+\mu_j(x)} e^{-(\Re\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau} e^{(\Re\gamma+\lambda_j+\theta_j)x+\int_0^x \mu_j(\tau)d\tau} |f(x)| dx \\
&= \lim_{x\to\infty} \tfrac{-1}{\Re\gamma+\lambda_j+\theta_j+\mu_j(x)} e^{-(\Re\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau} \int_0^x |f(\tau)| e^{(\Re\gamma+\lambda_j+\theta_j)\tau+\int_0^\tau \mu_j(\xi)d\xi} d\tau \\
&\quad + \int_0^\infty \frac{1}{\Re\gamma+\lambda_j+\theta_j+\mu_j(x)} |f(x)| dx \\
&= \int_0^\infty \frac{1}{\Re\gamma+\lambda_j+\theta_j+\mu_j(x)} |f(x)| dx \\
&\leq \frac{1}{\Re\gamma+\lambda_j+\theta_j+\inf_{x\in[0,\infty)}\mu_j(x)} \|f\|_{L^1[0,\infty)}, \quad j=1,3.
\end{aligned}$$

This means that

$$\|E_j\| \leq \frac{1}{\Re\gamma+\lambda_j+\theta_j+\inf_{x\in[0,\infty)}\mu_j(x)}, \quad j=1,3. \tag{A.1}$$

Then, for any $(w, y_1, y_3) \in X$, using the inequalities $\|\varphi_1\| \leq 1$, $\|\varphi_2\| \leq \sup_{x\in[0,\infty)}\mu_1(x)$, $\|\varphi_3\| \leq \sup_{x\in[0,\infty)}\mu_3(x)$ and Eq (3.56), we estimate

$$\begin{aligned}
&\|(\gamma I - A_0)^{-1}(w, y_1, y_3)\| \\
&= \|(a_{1,1}+a_{2,1}+a_{3,1})z_{1,0} + (a_{1,2}+a_{2,2}+a_{3,2})z_{3,0} + (a_{1,3}+a_{2,3}+a_{3,3})w_0 \\
&\quad + \frac{1}{\gamma+1}\sum_{n=1}^\infty w_n + (a_{1,4}+a_{2,4}+a_{3,4})\varphi_2 E_1 y_{1,1} \\
&\quad + \frac{\theta_1}{\gamma+1}[\varphi_1 E_1 y_{1,1} + (\lambda_1\varphi_1 E_1^2 y_{1,1} + \varphi_1 E_1 y_{1,2}) \\
&\quad + (\lambda_1^2\varphi_1 E_1^3 y_{1,1} + \lambda_1\varphi_1 E_1^2 y_{1,2} + \varphi_1 E_1 y_{1,3}) \\
&\quad + (\lambda_1^3\varphi_1 E_1^4 y_{1,1} + \lambda_1^2\varphi_1 E_1^3 y_{1,2} + \lambda_1\varphi_1 E_1^2 y_{1,3} + \varphi_1 E_1 y_{1,4}) + \cdots] \\
&\quad + (a_{1,5}+a_{2,5}+a_{3,5})\varphi_3 E_3 y_{3,1} + (a_{1,6}+a_{2,6}+a_{3,6})[\varphi_1 E_3 y_{3,1} + (\lambda_3\varphi_1 E_3^2 y_{3,1} + \varphi_1 E_3 y_{3,2}) \\
&\quad + (\lambda_3^2\varphi_1 E_3^3 y_{3,1} + \lambda_3\varphi_1 E_3^2 y_{3,2} + \varphi_1 E_3 y_{3,3}) \\
&\quad + (\lambda_3^3\varphi_1 E_3^4 y_{3,1} + \lambda_3^2\varphi_1 E_3^3 y_{3,2} + \lambda_3\varphi_1 E_3^2 y_{3,3} + \varphi_1 E_3 y_{3,4}) + \cdots]
\end{aligned}$$

$$
\begin{aligned}
&+(E_1y_{1,1}+\lambda_1E_1^2y_{1,1}+E_1y_{1,2}+\lambda_1^2E_1^3y_{1,1}+\lambda_1E_1^2y_{1,2}+E_1y_{1,3}\\
&+\lambda_1^3E_1^4y_{1,1}+\lambda_1^2E_1^3y_{1,2}+\lambda_1E_1^2y_{1,3}+\cdots)\\
&+(E_3y_{3,1}+\lambda_3E_3^2y_{3,1}+E_3y_{3,2}+\lambda_3^2E_3^3y_{3,1}+\lambda_3E_3^2y_{3,2}+E_3y_{3,3}\\
&+\lambda_3^3E_3^4y_{3,1}+\lambda_3^2E_3^3y_{3,2}+\lambda_3E_3^2y_{3,3}+\cdots)\|\\
\le\ &|a_{1,1}+a_{2,1}+a_{3,1}||z_{1,0}|+|a_{1,2}+a_{2,2}+a_{3,2}||z_{3,0}|+|a_{1,3}+a_{2,3}+a_{3,3}||w_0|\\
&+\frac{1}{|\gamma+1|}\sum_{n=1}^{\infty}|w_n|+|a_{1,4}+a_{2,4}+a_{3,4}|\|\varphi_2\|\|E_1\|\|y_{1,1}\|_{L^1[0,\infty)}\\
&+\frac{\theta_1}{|\gamma+1|}\Big[\|\varphi_1\|\|E_1\|\|y_{1,1}\|_{L^1[0,\infty)}+\lambda_1\|\varphi_1\|\|E_1\|^2\|y_{1,1}\|_{L^1[0,\infty)}\\
&+\|\varphi_1\|\|E_1\|\|y_{1,2}\|_{L^1[0,\infty)}+\lambda_1^2\|\varphi_1\|\|E_1\|^3\|y_{1,1}\|_{L^1[0,\infty)}\\
&+\lambda_1\|\varphi_1\|\|E_1\|^2\|y_{1,2}\|_{L^1[0,\infty)}+\|\varphi_1\|\|E_1\|\|y_{1,3}\|_{L^1[0,\infty)}\\
&+\lambda_1^3\|\varphi_1\|\|E_1\|^4\|y_{1,1}\|_{L^1[0,\infty)}+\lambda_1^2\|\varphi_1\|\|E_1\|^3\|y_{1,2}\|_{L^1[0,\infty)}\\
&+\lambda_1\|\varphi_1\|\|E_1\|^2\|y_{1,3}\|_{L^1[0,\infty)}+\|\varphi_1\|\|E_1\|\|y_{1,4}\|_{L^1[0,\infty)}+\cdots\Big]\\
&+|a_{1,5}+a_{2,5}+a_{3,5}|\|\varphi_3\|\|E_3\|\|y_{3,1}\|_{L^1[0,\infty)}\\
&+|a_{1,6}+a_{2,6}+a_{3,6}|\Big[\|\varphi_1\|\|E_3\|\|y_{3,1}\|_{L^1[0,\infty)}+\lambda_3\|\varphi_1\|\|E_3\|^2\|y_{3,1}\|_{L^1[0,\infty)}\\
&+\|\varphi_1\|\|E_3\|\|y_{3,2}\|_{L^1[0,\infty)}+\lambda_3^2\|\varphi_1\|\|E_3\|^3\|y_{3,1}\|_{L^1[0,\infty)}\\
&+\lambda_3\|\varphi_1\|\|E_3\|^2\|y_{3,2}\|_{L^1[0,\infty)}+\|\varphi_1\|\|E_3\|\|y_{3,3}\|_{L^1[0,\infty)}\\
&+\lambda_3^3\|\varphi_1\|\|E_3\|^4\|y_{3,1}\|_{L^1[0,\infty)}+\lambda_3^2\|\varphi_1\|\|E_3\|^3\|y_{3,2}\|_{L^1[0,\infty)}\\
&+\lambda_3\|\varphi_1\|\|E_3\|^2\|y_{3,3}\|_{L^1[0,\infty)}+\|\varphi_1\|\|E_3\|\|y_{3,4}\|_{L^1[0,\infty)}+\cdots\Big]\\
&+\|E_1\|\|y_{1,1}\|_{L^1[0,\infty)}+\lambda_1\|E_1\|^2\|y_{1,1}\|_{L^1[0,\infty)}+\|E_1\|\|y_{1,2}\|_{L^1[0,\infty)}\\
&+\lambda_1^2\|E_1\|^3\|y_{1,1}\|_{L^1[0,\infty)}+\lambda_1\|E_1\|^2\|y_{1,2}\|_{L^1[0,\infty)}+\|E_1\|\|y_{1,3}\|_{L^1[0,\infty)}\\
&+\lambda_1^3\|E_1\|^4\|y_{1,1}\|_{L^1[0,\infty)}+\lambda_1^2\|E_1\|^3\|y_{1,2}\|_{L^1[0,\infty)}+\lambda_1\|E_1\|^2\|y_{1,3}\|_{L^1[0,\infty)}\\
&+\cdots\\
&+\|E_3\|\|y_{3,1}\|_{L^1[0,\infty)}+\lambda_3\|E_3\|^2\|y_{3,1}\|_{L^1[0,\infty)}+\|E_3\|\|y_{3,2}\|_{L^1[0,\infty)}\\
&+\lambda_3^2\|E_3\|^3\|y_{3,1}\|_{L^1[0,\infty)}+\lambda_3\|E_3\|^2\|y_{3,2}\|_{L^1[0,\infty)}+\|E_3\|\|y_{3,3}\|_{L^1[0,\infty)}\\
&+\lambda_3^3\|E_3\|^4\|y_{3,1}\|_{L^1[0,\infty)}+\lambda_3^2\|E_3\|^3\|y_{3,2}\|_{L^1[0,\infty)}+\lambda_3\|E_3\|^2\|y_{3,3}\|_{L^1[0,\infty)}\\
&+\cdots\\
=\ &|a_{1,1}+a_{2,1}+a_{3,1}||z_{1,0}|+|a_{1,2}+a_{2,2}+a_{3,2}||z_{3,0}|\\
&+|a_{1,3}+a_{2,3}+a_{3,3}||w_0|+\frac{1}{|\gamma+1|}\sum_{n=1}^{\infty}|w_n|+|a_{1,4}+a_{2,4}+a_{3,4}|\|\varphi_2\|\|E_1\|\|y_{1,1}\|_{L^1[0,\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta_1}{|\gamma+1|}\|\varphi_1\|\sum_{k=1}^{\infty}\lambda_1^{k-1}\|E_1\|^k\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}+|a_{1,5}+a_{2,5}+a_{3,5}|\|\varphi_3\|\|E_3\|\|y_{3,1}\|_{L^1[0,\infty)}\\
&+|a_{1,6}+a_{2,6}+a_{3,6}|\|\varphi_1\|\sum_{k=1}^{\infty}\lambda_3^{k-1}\|E_3\|^k\sum_{n=1}^{\infty}\|y_{3,n}\|_{L^1[0,\infty)}\\
&+\sum_{k=1}^{\infty}\lambda_1^{k-1}\|E_1\|^k\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}+\sum_{k=1}^{\infty}\lambda_3^{k-1}\|E_3\|^k\sum_{n=1}^{\infty}\|y_{3,n}\|_{L^1[0,\infty)}\\
\leq\ & |a_{1,1}+a_{2,1}+a_{3,1}||z_{1,0}|+|a_{1,2}+a_{2,2}+a_{3,2}||z_{3,0}|\\
&+|a_{1,3}+a_{2,3}+a_{3,3}||w_0|+\frac{1}{\Re\gamma+1}\sum_{n=1}^{\infty}|w_n|\\
&+|a_{1,4}+a_{2,4}+a_{3,4}|\frac{\sup_{x\in[0,\infty)}\mu_1(x)}{\Re\gamma+\lambda_1+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}\|y_{1,1}\|_{L^1[0,\infty)}\\
&+\frac{\theta_1}{\Re\gamma+1}\frac{1}{\Re\gamma+\lambda_1+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}\\
&\times\sum_{k=1}^{\infty}\left(\frac{\lambda_1}{\Re\gamma+\lambda_1+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}\right)^{k-1}\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}\\
&+|a_{1,5}+a_{2,5}+a_{3,5}|\frac{\sup_{x\in[0,\infty)}\mu_3(x)}{\Re\gamma+\lambda_3+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}\|y_{3,1}\|_{L^1[0,\infty)}\\
&+|a_{1,6}+a_{2,6}+a_{3,6}|\frac{1}{\Re\gamma+\lambda_3+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}\\
&\times\sum_{k=1}^{\infty}\left(\frac{\lambda_3}{\Re\gamma+\lambda_3+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}\right)^{k-1}\sum_{n=1}^{\infty}\|y_{3,n}\|_{L^1[0,\infty)}\\
&+\frac{1}{\Re\gamma+\lambda_1+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}\\
&\times\sum_{k=1}^{\infty}\left(\frac{\lambda_1}{\Re\gamma+\lambda_1+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}\right)^{k-1}\sum_{n=1}^{\infty}\|y_{1,n}\|_{L^1[0,\infty)}\\
&+\frac{1}{\Re\gamma+\lambda_3+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}\\
&\times\sum_{k=1}^{\infty}\left(\frac{\lambda_3}{\Re\gamma+\lambda_3+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}\right)^{k-1}\sum_{n=1}^{\infty}\|y_{3,n}\|_{L^1[0,\infty)}\\
\leq\ & \sup\Bigg\{|a_{1,1}+a_{2,1}+a_{3,1}|+|a_{1,2}+a_{2,2}+a_{3,2}|+|a_{1,3}+a_{2,3}+a_{3,3}|+\frac{1}{\Re\gamma+1},\\
&|a_{1,4}+a_{2,4}+a_{3,4}|\frac{\sup_{x\in[0,\infty)}\mu_1(x)}{\Re\gamma+\lambda_1+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}
\end{aligned}
$$

$$+\frac{\theta_1}{\Re\gamma+1}\frac{1}{\Re\gamma+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}+\frac{1}{\Re\gamma+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)},$$

$$|a_{1,5}+a_{2,5}+a_{3,5}|\frac{\sup_{x\in[0,\infty)}\mu_3(x)}{\Re\gamma+\lambda_3+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}$$

$$+|a_{1,6}+a_{2,6}+a_{3,6}|\frac{1}{\Re\gamma+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}+\frac{1}{\Re\gamma+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}\Bigg\}\|(w,y_1,y_3)\|$$

$$=\sup\Bigg\{\frac{|(\gamma+1)(\gamma+\lambda_3+\theta_3)+\theta_1N_0+\theta_1(\gamma+\lambda_3+\theta_3)|}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}$$

$$+\frac{|(\gamma+1)\theta_3+(\gamma+1)(\gamma+\lambda_1+\theta_1)+\theta_1\theta_3|}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}$$

$$+\frac{|\theta_3N_0+(\gamma+\lambda_1+\theta_1)N_0+(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}+\frac{1}{\Re\gamma+1},$$

$$\frac{|(\gamma+1)(\gamma+\lambda_3+\theta_3)+\theta_1N_0+\theta_1(\gamma+\lambda_3+\theta_3)|}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}\frac{\sup_{x\in[0,\infty)}\mu_1(x)}{\Re\gamma+\lambda_1+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}$$

$$+\frac{\theta_1}{\Re\gamma+1}\frac{1}{\Re\gamma+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)}+\frac{1}{\Re\gamma+\theta_1+\inf_{x\in[0,\infty)}\mu_1(x)},$$

$$\frac{|(\gamma+1)\theta_3+(\gamma+1)(\gamma+\lambda_1+\theta_1)+\theta_1\theta_3|}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}\frac{\sup_{x\in[0,\infty)}\mu_3(x)}{\Re\gamma+\lambda_3+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}$$

$$+\frac{|(\gamma+1)(\gamma+\lambda_3+\theta_3)\theta_3+\theta_1\theta_3N_0+\theta_1\theta_3(\gamma+\lambda_3+\theta_3)|}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}\frac{1}{\Re\gamma+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}$$

$$+\frac{1}{\Re\gamma+\theta_3+\inf_{x\in[0,\infty)}\mu_3(x)}\Bigg\}\|(w,y_1,y_3)\|. \tag{A.2}$$

Inequality (A.2) means that the result of Lemma 3.2 holds true.

**The proof of Lemma 3.3.** We assume that $(V,p_1,p_3)\in\ker(\gamma I-A_m)$, then $(\gamma I-A_m)(V,p_1,p_3)=0$. That is,

$$(\gamma+\lambda_1+\theta_1)Q_{1,0}=\theta_3\sum_{n=1}^{\infty}\int_0^\infty p_{3,n}(x)dx+\theta_3Q_{3,0}+\int_0^\infty p_{1,1}(x)\mu_1(x)dx, \tag{A.3}$$

$$(\gamma+\lambda_3+\theta_3)Q_{3,0}=N_0V_0+\int_0^\infty p_{3,1}(x)\mu_3(x)dx, \tag{A.4}$$

$$(\gamma+1)V_0=\theta_1Q_{1,0}, \tag{A.5}$$

$$(\gamma+1)V_n=\theta_1\int_0^\infty p_{1,n}(x)dx,\quad n\geq 1, \tag{A.6}$$

$$\frac{dp_{1,1}(x)}{dx}=-[\gamma+\lambda_1+\theta_1+\mu_1(x)]p_{1,1}(x), \tag{A.7}$$

$$\frac{dp_{1,n}(x)}{dx}=-[\gamma+\lambda_1+\theta_1+\mu_1(x)]p_{1,n}(x)+\lambda_1p_{1,n-1}(x),\quad n\geq 2, \tag{A.8}$$

$$\frac{dp_{3,1}(x)}{dx}=-[\gamma+\lambda_3+\theta_3+\mu_3(x)]p_{3,1}(x), \tag{A.9}$$

$$\frac{dp_{3,n}(x)}{dx} = -[\gamma + \lambda_3 + \theta_3 + \mu_3(x)]p_{3,n}(x) + \lambda_3 p_{3,n-1}(x), \quad n \geq 2. \tag{A.10}$$

By solving Eqs (A.7)–(A.10), we have

$$p_{j,n}(x) = e^{-(\gamma+\lambda_j+\theta_j)x - \int_0^x \mu_i(\tau)d\tau} \sum_{k=1}^{n} \frac{(\lambda_j x)^{k-1}}{(k-1)!} p_{j,n-k+1}(0), \quad j = 1, 3; \; n \geq 1. \tag{A.11}$$

By Eqs (A.3)–(A.6), we have

$$Q_{1,0} = \frac{\theta_3}{\gamma + \lambda_1 + \theta_1} \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x)dx + \frac{\theta_3}{\gamma + \lambda_1 + \theta_1} Q_{3,0} + \frac{1}{\gamma + \lambda_1 + \theta_1} \int_0^{\infty} p_{1,1}(x)\mu_1(x)dx, \tag{A.12}$$

$$Q_{3,0} = \frac{N_0}{\gamma + \lambda_3 + \theta_3} V_0 + \frac{1}{\gamma + \lambda_3 + \theta_3} \int_0^{\infty} p_{3,1}(x)\mu_3(x)dx, \tag{A.13}$$

$$V_0 = \frac{\theta_1}{\gamma + 1} Q_{1,0}, \tag{A.14}$$

$$V_n = \frac{1}{\gamma + 1} \int_0^{\infty} p_{1,n}(x)dx, \quad n \geq 1. \tag{A.15}$$

Then, from Eqs (A.11)–(A.15), by a simple calculation, we can obtain that Eqs (3.56)–(3.59) in Lemma 3.3 hold true.

Moreover, since $(V, p_1, p_3) \in \ker(\gamma I - A_m)$, using the Sobolev imbedding theorem [20, Theorem 4.12], we estimate

$$\begin{aligned} \sum_{n=1}^{\infty} |p_{j,n}(0)| &\leq \sum_{n=1}^{\infty} \|p_{j,n}\|_{L^\infty[0,\infty)} \\ &\leq \sum_{n=1}^{\infty} \left( \|p_{j,n}\|_{L^1[0,\infty)} + \left\| \frac{dp_{j,n}}{dx} \right\|_{L^1[0,\infty)} \right) < \infty, \quad j = 1, 3. \end{aligned} \tag{A.16}$$

Hence, we conclude that Eqs (3.56)–(3.61) in Lemma 3.3 hold true.

On the other hand, if the Eqs (3.56)–(3.61) holds true, then for any positive constant $\mathbb{M}_0$ and $k \geq 1$, using the formula

$$\int_0^{\infty} e^{-\mathbb{M}_0 x} x^k dx = \frac{k!}{\mathbb{M}_0^{k+1}},$$

we estimate

$$\begin{aligned} \|p_{j,n}\|_{L^1[0,\infty)} &= \int_0^{\infty} \left| e^{-(\gamma+\lambda_j+\theta_j)x - \int_0^x \mu_j(\tau)d\tau} \sum_{k=1}^{n} \frac{(\lambda_j x)^{k-1}}{(k-1)!} p_{j,n-k+1}(0) \right| dx \\ &\leq \sum_{k=1}^{n} \frac{\lambda_j^{k-1}}{(k-1)!} |p_{j,n-k+1}(0)| \int_0^{\infty} x^{k-1} e^{-[\Re\gamma + \lambda_j + \theta_j + \inf_{x\in[0,\infty)} \mu_j(x)]x} dx \\ &\leq \sum_{k=1}^{n} \frac{\lambda_j^{k-1}}{(k-1)!} |p_{j,n-k+1}(0)| \frac{(k-1)!}{(\Re\gamma + \lambda_j + \theta_j + \inf_{x\in[0,\infty)} \mu_j(x))^k} \\ &= \sum_{k=1}^{n} \frac{\lambda_j^{k-1}}{(\Re\gamma + \lambda_j + \theta_j + \inf_{x\in[0,\infty)} \mu_j(x))^k} |p_{j,n-k+1}(0)|. \end{aligned} \tag{A.17}$$

To sum $n$ from 1 to positive infinity on both sides of the above inequality and using the condition $\Re\gamma + \theta_j + \inf_{x\in[0,\infty)} \mu_j(x) > 0$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \|p_{j,n}\|_{L^1[0,\infty)} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\lambda_j^{k-1}}{(\Re\gamma + \lambda_j + \theta_j + \inf_{x\in[0,\infty)} \mu_j(x))^k} |p_{j,n-k+1}(0)| \\
&= \sum_{k=1}^{\infty} \frac{\lambda_j^{k-1}}{(\Re\gamma + \lambda_j + \theta_j + \inf_{x\in[0,\infty)} \mu_j(x))^k} \sum_{n=1}^{\infty} |p_{j,n}(0)| \\
&= \frac{1}{\Re\gamma + \theta_j + \inf_{x\in[0,\infty)} \mu_j(x)} \sum_{n=1}^{\infty} |p_{j,n}(0)| < \infty, \quad j = 1, 3.
\end{aligned} \tag{A.18}$$

By taking the derivative of $x$ on both sides of Eq (A.18), we can obtain

$$\begin{aligned}
\frac{dp_{j,1}(x)}{dx} &= -[\gamma + \lambda_j + \theta_j + \mu_j(x)]p_{j,1}(0)e^{-(\gamma+\lambda_j+\theta_j)x-\int_0^x \mu_j(\tau)d\tau} \\
&= -[\gamma + \lambda_j + \theta_j + \mu_j(x)]p_{j,1}(x), \quad j = 1, 3,
\end{aligned} \tag{A.19}$$

$$\frac{dp_{j,n}(x)}{dx} = -[\gamma + \lambda_j + \theta_j + \mu_j(x)]p_{j,n}(x) + \lambda_j p_{j,n-1}(x), \quad j = 1, 3;\ n \geq 1. \tag{A.20}$$

Combining the Eqs (A.18)–(A.20), we derive

$$\sum_{n=1}^{\infty} \left\| \frac{dp_{j,n}}{dx} \right\|_{L^1[0,\infty)} \leq \left[ |\gamma| + 2\lambda_j + \theta_j + \sup_{x\in[0,\infty)} \mu_j(x) \right] \sum_{n=1}^{\infty} \|p_{j,n}\|_{L^1[0,\infty)}. \tag{A.21}$$

Therefore, Eqs (A.18)–(A.21) imply that $\gamma \in \rho(A_0)$ and

$$(V, p_1, p_3) \in \ker(\gamma I - A_m).$$

## B. Appendix: List of notations

**Table 1.** List of notations.

| Symbol | First occurrence & Description |
|---|---|
| $Q_{j,0}(t)$ | Page 2: The probability that there are no customers in phase $j$ ($j = 1, 3$) and the server is idle at time $t$. |
| $V_n(t)$ | Page 2: The probability that there are $n$ ($n \geq 0$) customers in the system and the system is in phase 2. |
| $p_{j,n}(x, t)dx$ | Page 2: The probability that there are $n$ ($n \geq 1$) customers in the system at time $t$ with the server busy serving a customer whose elapsed service time lies in the interval $[x, x + dx)$ in phase $j$ ($j = 1, 3$). |
| $\lambda_j$ | Page 4: The arrival rate of customers when the system is in phase $j$ ($j = 1, 2, 3$). |
| $\mu_j(x)$ | Page 4: The conditional probability of completing a service during the interval $(x, x + dx)$ with elapsed time $x$ in phase $j$ ($j = 1, 3$). |
| $\theta_j$ | Page 4: The residence rate of the system in phase $j$ ($j = 1, 3$). |
| $N_r$ | Page 5: The probability of $r$ ($r \geq 0$) arrivals during phase 2. |
| $X$ | Page 5: The state space of systems (2.1)–(2.11). |
| $X^*$ | Page 24: The dual space of $X$. |
| $\partial X$ | Page 6: The boundary space of $X$. |
| $A_m$ | Page 5: The maximal operator of systems (2.1)–(2.11). |
| $D(A_m)$ | Page 5: The domain of operator $A_m$. |
| $\Phi, \Psi$ | Page 6: The boundary operators of systems (2.1)–(2.2). |
| $\mathcal{A}$ | Page 7: The system operator of systems (2.1)–(2.11). |
| $\mathcal{A}^*$ | Page 8: The adjoint operator of $\mathcal{A}$. |
| $A_0$ | Page 12: The system operator with zero boundary conditions for systems (2.1)–(2.11). |
| $\rho(\mathcal{A})$ | Page 8: The resolvent set of $\mathcal{A}$. |
| $\sigma(\mathcal{A})$ | Page 18: The spectrum set of $\mathcal{A}$. |
| $\sigma_p(\mathcal{A})$ | Page 29: The point spectrum set of $\mathcal{A}$. |
| $\Re\gamma$ | Page 8: The real part of $\gamma$. |
| $D_\gamma$ | Page 16: The Dirichlet operator. |

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