Networks and Heterogeneous Media

http://www.aimspress.com/journal/nhm

NHM, 20(2): 566–589.
DOI: 10.3934/nhm.2025025
Received: 11 March 2025
Revised: 09 May 2025
Accepted: 16 May 2025
Published: 28 May 2025

Research article

An integral system of Matukuma type with negative exponents

Changhong Li*

School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland

* **Correspondence:** Email: changhong.li@ucdconnect.ie.

Abstract: We study positive solutions to the integral system

$$\begin{cases} u(x) = \int_{\mathbb{R}^N} \frac{K(y)v^{-p}(y)}{|x-y|^{N-\alpha}}dy & \text{for all } x \in \mathbb{R}^N, \\ v(x) = \int_{\mathbb{R}^N} \frac{L(y)u^{-q}(y)}{|x-y|^{N-\beta}}dy & \text{for all } x \in \mathbb{R}^N, \end{cases}$$

where $p, q > 0$, $\alpha, \beta \in (0, N)$ and $K, L : \mathbb{R}^N \to (0, \infty)$ are continuous functions which satisfy $C_1(1 + |x|)^{-\gamma} \le K(x), L(x) \le C_2(1 + |x|)^{-\gamma}$ in $\mathbb{R}^N$, for some $\gamma > 0$ and constants $C_2 > C_1 > 0$. We discuss the existence, nonexistence, and uniqueness of positive solutions to the above system with respect to α, β, p, q, and γ. We also classify the finite and infinite total mass solutions of the system.

Keywords: general weighted integral system; negative exponent; Matukuma type equation; Riesz potentials; existence and uniqueness of positive solution
Mathematics Subject Classification: 45G15, 45M20, 47H10, 47J05, 35J48

1. Introduction and the main results

In this paper, we are concerned with the study of positive solutions to the following integral system:

$$\begin{cases} u(x) = \int_{\mathbb{R}^N} \frac{K(y)v^{-p}(y)}{|x-y|^{N-\alpha}}dy & \text{for all } x \in \mathbb{R}^N, \\ v(x) = \int_{\mathbb{R}^N} \frac{L(y)u^{-q}(y)}{|x-y|^{N-\beta}}dy & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p, q > 0$, $\alpha, \beta \in (0, N)$, and $K, L : \mathbb{R}^N \to (0, \infty)$ are continuous functions which satisfy:

$$C_1(1 + |x|)^{-\gamma} \le K(x), L(x) \le C_2(1 + |x|)^{-\gamma} \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

for some $\gamma > 0$ and $C_2 > C_1 > 0$.

We investigated positive solutions (u, v) of Eq (1.1), that is, functions $u, v \in C(\mathbb{R}^N)$ such that $u, v > 0$ and satisfying Eq (1.1) pointwise in $\mathbb{R}^N$. Our study is motivated by the Matukuma equation [1, 2] introduced in the 1930s to describe the dynamics of globular clusters of stars. Mathematically, the Matukuma equation reads as

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0 \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

where $p > 1$. Solutions of Eq (1.3) which tend to zero at infinity satisfy the integral equation

$$u(x) = \int_{\mathbb{R}^3} \frac{u(y)^p}{|x-y|(1+|y|^2)} dy \quad \text{for all } x \in \mathbb{R}^3.$$

Also, the quantity

$$\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^p}{1+|x|^2} dx \tag{1.4}$$

is called the total mass of the solution u of Eq (1.3) (see [3, 4]). The extended integral equation

$$u(x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}(1+|y|^\alpha)} dy \quad \text{for all } x \in \mathbb{R}^N, N \geq 3 \tag{1.5}$$

was studied in [5], and the behavior of positive solutions was discussed in [6–8] (see also [9–13]). The case of negative exponents, that is,

$$u(x) = \int_{\mathbb{R}^N} \frac{K(y)u(y)^{-p}}{|x-y|^{N-\alpha}} dy \quad \text{in } \mathbb{R}^N (N \geq 1),$$

was recently investigated in [14] under the conditions $p > 0$, $\alpha \in (0, N)$, and $C_1(1+|x|)^{-\beta} \leq K(x) \leq C_2(1+|x|)^{-\beta}$ in $\mathbb{R}^N$, for some $\beta > 0$ and $C_2 > C_1 > 0$. Integral systems with negative exponents were considered in the case of half-space in [15, 16] and in the case of whole-space in [17–19] (see also [20–24] for further results).

The aim of the present paper is to discuss various qualitative properties of the positive solutions to Eq (1.1). We start our study of Eq (1.1) with the following nonexistence result which extends and improves Theorem 1.1 in [14].

Theorem 1.1. (Nonexistence)

Let $1 \leq s_1, s_2 \leq \infty$ be such that

$$\max\left\{\alpha + N\left(\frac{p}{s_2} + \frac{1}{s_1}\right), \beta + N\left(\frac{q}{s_1} + \frac{1}{s_2}\right)\right\} > \gamma. \tag{1.6}$$

Then, Eq (1.1) has no positive solutions $(u, v) \in L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$.

We notice that if $\max\{\alpha, \beta\} > \gamma$, then Eq (1.6) is clearly fulfilled and we directly obtain the following corollary.

Corollary 1.2. *Let $1 \leq s_1, s_2 \leq \infty$. If one of the conditions below holds,*

(i) $\max\{\alpha, \beta\} > \gamma$ *and* $1 \leq s_1, s_2 \leq \infty$;

(ii) $\max\{\alpha,\beta\} \geq \gamma$ *and* $1 \leq s_1, s_2 \leq \infty$, $(s_1, s_2) \neq (\infty, \infty)$*;*

then Eq (1.1) *has no positive solutions* $(u, v) \in L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$.

Next, using some integral identities from harmonic analysis, we can construct an exact solution to Eq (1.1).

Theorem 1.3. (Exact solution)
Assume that $\gamma > 2N$ *and let*

$$K(x) = (1 + |x|^2)^{-\frac{\gamma}{2}}, \quad L(x) = (1 + |x|^2)^{-\frac{\gamma}{2}}.$$

Then, there exist $C, D > 0$, *and* $p, q > 0$ *depending on* α, β, γ, *and* N *such that*

$$\begin{cases} u(x) = C(1 + |x|^2)^{-\frac{N-\alpha}{2}} \\ v(x) = D(1 + |x|^2)^{-\frac{N-\beta}{2}} \end{cases} \qquad x \in \mathbb{R}^N, \tag{1.7}$$

is an exact solution to Eq (1.1).

The condition $\gamma > 2N$ is needed to ensure that $pq \neq 1$, which is a restriction in our argument.

Next, we turn to the existence of a solution to Eq (1.1) in a more general framework. Given two positive functions $f, g : \mathbb{R}^N \to (0, \infty)$ we use the symbol $f \simeq g$ to denote that the quotient f/g is bounded between two positive constants on $\mathbb{R}^N$. Our result in this case is stated below.

Theorem 1.4. (Existence)
Assume $\max\{\alpha,\beta\} < \gamma$.

(i) *If* $\max\{\alpha,\beta\} < \gamma < N$, $0 < p < \frac{\gamma-\alpha}{\gamma-\beta}$, *and* $0 < q < \frac{\gamma-\beta}{\gamma-\alpha}$, *then the integral system* (1.1) *has a solution* (u, v) *such that*

$$\begin{cases} u(x) \simeq (1 + |x|)^{-\frac{\gamma-\alpha-p(\gamma-\beta)}{1-pq}}, \\ v(x) \simeq (1 + |x|)^{-\frac{\gamma-\beta-q(\gamma-\alpha)}{1-pq}}. \end{cases}$$

Moreover, $(u, v) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$ *for some* $\mu, \nu \in (0, 1)$.

(ii) *If* $\gamma > N$, $0 < p < \frac{\gamma-N}{N-\beta}$, $0 < q < \frac{\gamma-N}{N-\alpha}$ *and* $pq < 1$, *then the integral system* (1.1) *has a solution* (u, v) *such that*

$$\begin{cases} u(x) \simeq (1 + |x|)^{\alpha-N}, \\ v(x) \simeq (1 + |x|)^{\beta-N}. \end{cases}$$

Moreover, $(u, v) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$ *for some* $\mu, \nu \in (0, 1)$.

The existence of a positive solution to Eq (1.1) is achieved using the Schauder fixed-point theorem. One major difficulty in our approach is the fact that $C(\mathbb{R}^N)$ is not a suitable space to work on, due to its lack of compactness. Instead, we shall use an approximation argument and construct solutions to Eq (1.1) over a sequence of expanding balls in $\mathbb{R}^N$. Further, using some integral estimates which are universal (see Lemma 2.2 below), we are able to obtain the solution to Eq (1.1) through a limit argument. Our argument requires a restriction in the range of exponents p and q as stated in Theorem 1.4 above. Let us point out that the restrictions on p, q in Theorem 1.4(ii) are natural. They appear again in Theorem 1.5 and Theorem 1.6 below.

Next, we show that Eq (1.1) may have a unique solution in some ranges of exponents p and q. This is stated in the following result.

Theorem 1.5. (Uniqueness)
Assume $0 < p < \frac{\gamma-N}{N-\beta}$ *and* $0 < q < \frac{\gamma-N}{N-\alpha}$. *If* $pq < 1$, *then the integral system* (1.1) *has a unique positive solution* (u, v) *in* $C(\mathbb{R}^N) \times C(\mathbb{R}^N)$.

Finally, and in analogy to the case of a single equation (see Eq (1.4)), we say that a solution (u, v) of system (1.1) has *finite total mass* if:

$$\int_{\mathbb{R}^N} K(x)v(x)^{-p}dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^N} L(x)u(x)^{-q}dx < \infty. \tag{1.8}$$

Otherwise, we say that (u, v) has *infinite total mass*.
Our last result in this section is stated below

Theorem 1.6. (Finite and infinite total mass solutions)
Let (u, v) *be a positive solution of Eq* (1.1).

(i) *If* $0 < p < \frac{\gamma-N}{N-\beta}$ *and* $0 < q < \frac{\gamma-N}{N-\alpha}$, *then* (u, v) *has finite total mass. Moreover*

$$\lim_{|x|\to\infty} |x|^{N-\alpha}u(x) = \int_{\mathbb{R}^N} K(y)v(y)^{-p}dy, \tag{1.9}$$

$$\lim_{|x|\to\infty} |x|^{N-\beta}v(x) = \int_{\mathbb{R}^N} L(y)u(y)^{-q}dy. \tag{1.10}$$

(ii) *If either* $p = \frac{\gamma-N}{N-\beta}$, $q > \frac{\gamma-N}{N-\alpha}$ *or* $q = \frac{\gamma-N}{N-\alpha}$, $p > \frac{\gamma-N}{N-\beta}$, *then* (u, v) *has infinite total mass.*

The remainder of this paper is organized as follows. In Section 2, we recall some results related to the Riesz potential and its regularity. Section 3 is devoted to the proof of Theorem 1.1. Further, Theorems 1.3–1.5 are proved in Sections 4, 5, and 6. The method we develop in the article can be used to study integral systems of type (1.1) with different signs of exponents. This will be explained in Section 7. Throughout this paper, $C, c, c_1, c_2, \ldots$ denote positive constants whose values may change from line to line, unless otherwise stated. By B_r, we denote the open ball in $\mathbb{R}^N$ centered at the origin and having radius $r > 0$.

2. Preliminary results

Let $f \in L^1_{loc}(\mathbb{R}^N)$. The Riesz potential of order $\gamma \in (0, N)$ of f is given by

$$I_\gamma f(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\gamma}}dy \quad \text{for all } x \in \mathbb{R}^N.$$

For $\alpha, \beta \in (0, N)$, we define

$$J_{\alpha,\beta} f(x) = \Big(\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}dy, \int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^{N-\beta}}dy\Big) \quad \text{for all } x \in \mathbb{R}^N,$$

and for $n > 0$ we also set

$$J_{\alpha,\beta,n}f(x) = \Big(\int\limits_{B_n} \frac{f(y)}{|x-y|^{N-\alpha}} dy, \int\limits_{B_n} \frac{g(y)}{|x-y|^{N-\beta}} dy \Big) \quad \text{for all } x \in B_n.$$

We recall the following lemma from [25] and [26].

Lemma 2.1. *Assume $s > 1$ is such that $\alpha - 1 < \frac{N}{s} < \alpha$ and $\beta - 1 < \frac{N}{s} < \beta$. Let $\mu = \alpha - \frac{N}{s}$ and $\nu = \beta - \frac{N}{s}$. Then, the Riesz operators*

$$J_{\alpha,\beta} : L^s(\mathbb{R}^N) \to C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$$

and

$$J_{\alpha,\beta,n} : L^s(B_n) \to C^{0,\mu}(\overline{B}_n) \times C^{0,\nu}(\overline{B}_n)$$

are continuous.

From [14], we recall the following integral estimates that extend to those in [27].

Lemma 2.2. *Let $n \geq 1$ be an integer, $\alpha \in (0, N)$, and $\sigma > \alpha$.*

(i) *If $\alpha < \sigma < N$, then there exist $c_2 > c_1 > 0$ independent of n such that for all $x \in B_n$, we have*

$$c_1(1+|x|)^{\alpha-\sigma} \leq \int\limits_{B_n} \frac{dy}{(1+|y|)^{\sigma}|x-y|^{N-\alpha}} \leq c_2(1+|x|)^{\alpha-\sigma}. \tag{2.1}$$

(ii) *If $\sigma > N$, then there exist $c_2 > c_1 > 0$ independent of n such that for all $x \in B_n$, we have*

$$c_1(1+|x|)^{\alpha-N} \leq \int\limits_{B_n} \frac{dy}{(1+|y|)^{\sigma}|x-y|^{N-\alpha}} \leq c_2(1+|x|)^{\alpha-N}. \tag{2.2}$$

(iii) *There exist $c_2 > c_1 > 0$ independent of n such that for all $x \in B_n$, we have*

$$c_1(1+|x|)^{\alpha-N}\log(e+|x|) \leq \int\limits_{B_n} \frac{dy}{(1+|y|)^{N}|x-y|^{N-\alpha}} \leq c_2(1+|x|)^{\alpha-N}\log(e+|x|). \tag{2.3}$$

3. Proof of Theorems

3.1. Proof of Theorem 1.1

Assume that $(u, v) \in L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$ is a positive solution of Eq (1.1) for some $1 \leq s_1, s_2 \leq \infty$.
Case 1: $s_1 = s_2 = \infty$. Then, u and v are bounded, and Eq (1.6) yields $\alpha > \gamma$.

For all $x \in \mathbb{R}^N \setminus B_1$, we deduce

$$u(x) \geq C \int\limits_{\mathbb{R}^N} \frac{K(y)}{|x-y|^{N-\alpha}} dy \geq C \int\limits_{|y|\geq|2x|} \frac{dy}{(1+|y|)^{\gamma}|x-y|^{N-\alpha}}.$$

We notice that

$$|y| \geq |2x| \Rightarrow |x-y| \leq |x| + |y| \leq 2|y|.$$

Together with $\alpha > \gamma$, they yield

$$u(x) \geq C \int_{|y|\geq|2x|} (2|y|)^{\alpha-\gamma-N} dy \geq C \int_{2|x|}^{\infty} t^{\alpha-\gamma-1} dt = \infty. \tag{3.1}$$

This contradicts the assumption that u is bounded, implying that Eq (1.1) has no positive solution in $L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$.

Case 2: $1 \leq s_1, s_2 < \infty$. Without loss of generality, from Eq (1.6) we may assume

$$\alpha + N\left(\frac{p}{s_2} + \frac{1}{s_1}\right) > \gamma.$$

Let $R > 1$. Then,

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^N} \frac{K(y)v^{-p}(y)}{|x-y|^{N-\alpha}} dy \\ &\geq C \int_{|y|<R} \frac{v^{-p}(y)}{(1+|y|)^{\gamma}|x-y|^{N-\alpha}} dy. \end{aligned}$$

If $|x|, |y| < R$, then $|x - y| \leq 2R$ and $1 + |y| \leq 1 + R \leq 2R$. We estimate

$$\begin{aligned} u(x) &\geq C \int_{B_R} \frac{v^{-p}(y)}{2R^{\gamma+N-\alpha}} dy \\ &= CR^{\alpha-\gamma}\left(\frac{1}{R^N} \int_{B_R} v^{-p}(y) dy\right) \\ &\geq CR^{\alpha-\gamma}\left(\frac{1}{R^N} \int_{B_R} (v^{s_2}(y))^{-\frac{p}{s_2}} dy\right). \end{aligned}$$

By Jensen's inequality, we obtain

$$u(x) \geq CR^{\alpha-\gamma}\left(\frac{1}{R^N} \int_{B_R} v^{s_2}(y) dy\right)^{-\frac{p}{s_2}} = CR^{\alpha-\gamma+\frac{Np}{s_2}}\left(\int_{B_R} v^{s_2}(y) dy\right)^{-\frac{p}{s_2}}.$$

This yields

$$\left(\int_{B_R} v^{s_2}(y) dy\right)^{\frac{p}{s_2}} u(x) \geq CR^{\alpha-\gamma+\frac{Np}{s_2}}.$$

Since $v \in L^{s_2}(\mathbb{R}^N)$, the above estimate implies

$$\left(\int_{\mathbb{R}^N} v^{s_2}(y) dy\right)^{\frac{p}{s_2}} u(x) \geq CR^{\alpha-\gamma+\frac{Np}{s_2}} \quad \text{for all } x \in B_R.$$

This yields

$$u(x) \geq CR^{\alpha-\gamma+\frac{Np}{s_2}} \quad \text{in } B_R, \tag{3.2}$$

and then

$$u^{s_1}(x) \geq CR^{(\alpha-\gamma)s_1+\frac{Nps_1}{s_2}},$$

$$\int_{B_R} u^{s_1}(x) \geq CR^{(\alpha-\gamma)s_1+\frac{Nps_1}{s_2}+N}.$$

Since $\alpha - \gamma + N(\frac{p}{s_2} + \frac{1}{s_1}) > 0$ holds, letting $R \to \infty$ in the above estimate, we deduce

$$\int_{\mathbb{R}^N} u^{s_1}(x) = \infty.$$

This contradicts the assumption that $u \in L^{s_1}(\mathbb{R}^N)$, implying that Eq (1.1) has no positive solution in $L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$.

Case 3: $1 \leq s_1 < \infty = s_2$. In this case, Eq (1.6) yields

$$\max\left\{\alpha + \frac{N}{s_1}, \beta + \frac{Nq}{s_1}\right\} > \gamma.$$

If $\alpha + \dfrac{N}{s_1} > \gamma$, as in the estimate Eq (3.1), we obtain

$$u(x) \geq C \int_{|y|\geq|2x|} |y|^{\alpha-\gamma-N} dy = C \int_{2|x|}^{\infty} t^{\alpha-\gamma-1} dt \quad \text{for all } |x| > 1.$$

This further yields $\alpha < \gamma$ and $u(x) \geq C|x|^{\alpha-\gamma}$ for $|x| > 1$. Then, $u(x)^{s_1} \geq C|x|^{(\alpha-\gamma)s_1}$ for all $x \in \mathbb{R}^N \setminus B_1$, which yields $u \notin L^{s_1}(\mathbb{R}^N)$, a contradiction.

If $\beta + \dfrac{Nq}{s_1} > \gamma$, as in the estimate Eq (3.2) which we obtained in Case 2 above, we find

$$v(x) \geq CR^{\beta-\gamma+\frac{Nq}{s_1}} \quad \text{in } B_R,$$

which contradicts the fact that $v \in L^\infty(\mathbb{R}^N)$.

Case 4: $1 \leq s_2 < \infty = s_1$. We use the same argument as in Case 3 above in which we replace s_1 by s_2 and u by v to raise a contradiction.

The proof of Corollary 1.2 follows directly from Theorem 1.1 since either of the conditions (i) and (ii) imply Eq (1.6).

3.2. *Proof of Theorem 1.3*

We will employ the following integral identities, see [28].

$$\int_{\mathbb{R}^N} \frac{dy}{(1+|y|^2)^{\frac{N+\alpha}{2}}|x-y|^{N-\alpha}} = \frac{\pi^{N/2}\Gamma(\frac{\alpha}{2})}{\Gamma\left(\frac{N+\alpha}{2}\right)}(1+|x|^2)^{-\frac{N-\alpha}{2}} \quad \text{for all } x \in \mathbb{R}^N.$$

$$\int_{\mathbb{R}^N} \frac{dy}{(1+|y|^2)^{\frac{N+\beta}{2}}|x-y|^{N-\beta}} = \frac{\pi^{N/2}\Gamma(\frac{\beta}{2})}{\Gamma\left(\frac{N+\beta}{2}\right)}(1+|x|^2)^{-\frac{N-\beta}{2}} \quad \text{for all } x \in \mathbb{R}^N.$$

where Γ stands for the Gamma function. Based on the α, β in Eq (1.1), we set

$$\begin{cases} A(\alpha) = \dfrac{\pi^{N/2}\Gamma(\frac{\alpha}{2})}{\Gamma\Big(\frac{N+\alpha}{2}\Big)}, \\ A(\beta) = \dfrac{\pi^{N/2}\Gamma(\frac{\beta}{2})}{\Gamma\Big(\frac{N+\beta}{2}\Big)}. \end{cases} \tag{3.3}$$

Let us take

$$\begin{cases} K(x) = (1+|x|^2)^{-\frac{\gamma}{2}} \\ L(x) = (1+|x|^2)^{-\frac{\gamma}{2}} \end{cases} \qquad \text{for all } x \in \mathbb{R}^N,$$

where $\gamma > 2N$, and next define

$$\begin{cases} p = \dfrac{\gamma - N - \alpha}{N - \beta} > 0, \\ q = \dfrac{\gamma - N - \beta}{N - \alpha} > 0. \end{cases}$$

Since $\gamma > \max\{2N, \alpha + \beta\}$, we have $pq \neq 1$. Next, let $C, D > 0$ be such that $CD^p = A(\alpha)$ and $DC^q = A(\beta)$. We then deduce

$$\begin{cases} C = \Big(\dfrac{A(\beta)^p}{A(\alpha)}\Big)^{1/(pq-1)}, \\ D = \Big(\dfrac{A(\alpha)^q}{A(\beta)}\Big)^{1/(pq-1)}. \end{cases}$$

Next, we claim that

$$\begin{cases} u(x) = C(1+|x|^2)^{-\frac{N-\alpha}{2}}, \quad x \in \mathbb{R}^N, \\ v(x) = D(1+|x|^2)^{-\frac{N-\beta}{2}}, \quad x \in \mathbb{R}^N. \end{cases}$$

is a solution of Eq (1.1). Indeed, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{K(y)v(y)^{-p}}{|x-y|^{N-\alpha}} dy &= D^{-p} \int_{\mathbb{R}^N} \frac{dy}{(1+|y|^2)^{\frac{N+\alpha}{2}} |x-y|^{N-\alpha}} \\ &= D^{-p} A(\alpha)(1+|x|^2)^{-\frac{N-\alpha}{2}} \\ &= C(1+|x|^2)^{-\frac{N-\alpha}{2}} \\ &= u(x) \quad \text{for all } x \in \mathbb{R}^N. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\alpha}} dy &= C^{-q} \int_{\mathbb{R}^N} \frac{dy}{(1+|y|^2)^{\frac{N+\beta}{2}} |x-y|^{N-\beta}} \\ &= C^{-q} A(\beta)(1+|x|^2)^{-\frac{N-\beta}{2}} \\ &= D(1+|x|^2)^{-\frac{N-\beta}{2}} \\ &= v(x) \quad \text{for all } x \in \mathbb{R}^N. \end{aligned}$$

This concludes the proof of our result.

3.3. Proof of Theorem 1.4

The existence of a positive solution to Eq (1.1) will be obtained by combining the Schauder fixed-point theorem with the properties of the Riesz potentials stated in Section 2. We split our discussion into two cases.
(i) Assume $0 < p < \frac{\gamma-\alpha}{\gamma-\beta}$ and $0 < q < \frac{\gamma-\beta}{\gamma-\alpha}$. Let

$$\kappa_1 = \frac{\gamma - \alpha - p(\gamma - \beta)}{1 - pq} > 0 \quad \text{and} \quad \kappa_2 = \frac{\gamma - \beta - q(\gamma - \alpha)}{1 - pq} > 0. \tag{3.4}$$

For $n \geq 1$, we define the closed and convex set $\mathcal{A}_n \subset C(\overline{B}_n) \times C(\overline{B}_n)$ by

$$\mathcal{A}_n = \left\{(u, v) \in C(\overline{B}_n) \times C(\overline{B}_n) : \begin{array}{l} m_1(1 + |x|)^{-\kappa_1} \leq u(x) \leq M_1(1 + |x|)^{-\kappa_1} \\ m_2(1 + |x|)^{-\kappa_2} \leq v(x) \leq M_2(1 + |x|)^{-\kappa_2} \end{array} \text{ in } \overline{B}_n\right\}, \tag{3.5}$$

where $0 < m_1 < M_1$, $0 < m_2 < M_2$ are constants depending on $\alpha, \beta, \gamma, p, q$, and N that will be chosen in Lemma 3.1 below.

For all $u, v \in \mathcal{A}_n$, we define

$$J_n(u, v)(x) = \Big(\int_{B_n} \frac{K(y)v(y)^{-p}}{|x - y|^{N-\alpha}} dy, \int_{B_n} \frac{L(y)u(y)^{-q}}{|x - y|^{N-\beta}} dy\Big) \quad \text{for all } x \in \overline{B}_n.$$

By Lemma 2.2 (i) for $\sigma = \gamma - p\kappa_2 > \alpha$ and $\sigma = \gamma - q\kappa_1 > \beta$, respectively, there exists $c_2 > c_1 > 0$ independent of $n \geq 1$ such that

$$c_1(1 + |x|)^{-\kappa_1} \leq \int_{B_n} \frac{K(y)dy}{(1 + |y|)^{-p\kappa_2}|x - y|^{N-\alpha}} \leq c_2(1 + |x|)^{-\kappa_1} \quad \text{for all } x \in B_n, \tag{3.6}$$

and

$$c_1(1 + |x|)^{-\kappa_2} \leq \int_{B_n} \frac{L(y)dy}{(1 + |y|)^{-q\kappa_1}|x - y|^{N-\beta}} \leq c_2(1 + |x|)^{-\kappa_2} \quad \text{for all } x \in B_n. \tag{3.7}$$

Lemma 3.1. *Let*

$$m_1 = \Big(\frac{c_2^p}{c_1}\Big)^{\frac{1}{pq-1}}, \quad M_1 = \Big(\frac{c_1^p}{c_2}\Big)^{\frac{1}{pq-1}}, \quad m_2 = \Big(\frac{c_2^1}{c_1}\Big)^{\frac{1}{pq-1}}, \quad M_2 = \Big(\frac{c_1^q}{c_2}\Big)^{\frac{1}{pq-1}}. \tag{3.8}$$

where c_1, c_2 are defined in Eqs (3.6) *and* (3.7). *If $pq < 1$, then $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$.*

Proof. Since $pq < 1$, it is obvious to see that $m_1 < M_1$ and $m_2 < M_2$. We also have

$$\begin{cases} m_1 = c_1 M_2^{-p}, \\ M_1 = c_2 m_2^{-p}, \\ m_2 = c_1 M_1^{-q}, \\ M_2 = c_2 m_1^{-q}. \end{cases} \tag{3.9}$$

To prove $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$, let $(u, v) \in \mathcal{A}_n$.
Since $v(x) \leq M_2(1 + |x|)^{-\kappa_2}$ in B_n, by Eqs (3.6) and $(3.9)_1$ we have

$$\begin{aligned}\int_{B_n} \frac{K(y)v(y)^{-p}}{|x-y|^{N-\alpha}}dy &\geq M_2^{-p} \int_{B_n} \frac{K(y)}{(1+|x|)^{-p\kappa_2}|x-y|^{N-\alpha}}dy \\ &\geq M_2^{-p}c_1(1+|x|)^{-\kappa_1} \\ &= m_1(1+|x|)^{-\kappa_1} \quad \text{in } B_n.\end{aligned}$$

Also, $v(x) \geq m_2(1 + |x|)^{-\kappa_2}$ in B_n together with Eqs (3.6) and $(3.9)_2$ implies

$$\begin{aligned}\int_{B_n} \frac{K(y)v(y)^{-p}}{|x-y|^{N-\alpha}}dy &\leq m_2^{-p} \int_{B_n} \frac{K(y)}{(1+|x|)^{-p\kappa_2}|x-y|^{N-\alpha}}dy \\ &\leq m_2^{-p}c_2(1+|x|)^{-\kappa_1} \\ &= M_1(1+|x|)^{-\kappa_1} \quad \text{in } B_n.\end{aligned}$$

Similarly, $u(x) \leq M_1(1 + |x|)^{-\kappa_1}$ in B_n combined with Eqs (3.7) and $(3.9)_3$ yields

$$\begin{aligned}\int_{B_n} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\beta}}dy &\geq M_1^{-q} \int_{B_n} \frac{L(y)}{(1+|x|)^{-q\kappa_1}|x-y|^{N-\beta}}dy \\ &\geq M_1^{-q}c_1(1+|x|)^{-\kappa_2} \\ &= m_2(1+|x|)^{-\kappa_2} \quad \text{in } B_n.\end{aligned}$$

Finally, since $u(x) \geq m_1(1 + |x|)^{-\kappa_1}$ in B_n, Eqs (3.7) and $(3.9)_4$ produces

$$\begin{aligned}\int_{B_n} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\beta}}dy &\leq m_1^{-q} \int_{B_n} \frac{L(y)}{(1+|x|)^{-q\kappa_1}|x-y|^{N-\beta}}dy \\ &\leq m_1^{-q}c_2(1+|x|)^{-\kappa_2} \\ &= M_2(1+|x|)^{-\kappa_2} \quad \text{in } B_n.\end{aligned}$$

Hence, $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$.

From the definition of κ_1 and κ_2 in Eq (3.4), we can easily check that

$$\alpha < \gamma - p\kappa_2 \quad \text{and} \quad \beta < \gamma - q\kappa_1.$$

Thus, we can select $s > 1$ such that

$$\alpha - 1 < \frac{N}{s} < \alpha < \gamma - p\kappa_2 \quad \text{and} \quad \beta - 1 < \frac{N}{s} < \beta < \gamma - q\kappa_1.$$

Let $\mu = \alpha - \frac{N}{s} \in (0, 1)$ and $\nu = \beta - \frac{N}{s} \in (0, 1)$. By Lemma 2.1, we obtain that

$$J_n = J_{\alpha,\beta,n} : \mathcal{A}_n \subset L^s(B_n) \times L^s(B_n) \to C^{0,\mu}(B_n) \times C^{0,\nu}(B_n) \quad \text{is continuous}$$

and $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$.

Hence, we can use the Schauder fixed-point theorem for J_n, which implies the existence of $(u_n, v_n) \in \mathcal{A}_n$ such that

$$(u_n, v_n) = \Big(\int_{B_n} \frac{K(y)v_n(y)^{-p}}{|x-y|^{N-\alpha}} dy, \int_{B_n} \frac{L(y)u_n(y)^{-q}}{|x-y|^{N-\beta}} dy \Big) \quad \text{for all } x \in \overline{B}_n.$$

Next, we argue as follows.

- Since $\{(u_n, v_n)\}$ is bounded in $C^{0,\mu}(\overline{B}_1) \times C^{0,\nu}(\overline{B}_1)$. Since the embedding $C^{0,\mu}(\overline{B}_1) \times C^{0,\nu}(\overline{B}_1) \hookrightarrow C(\overline{B}_1) \times C(\overline{B}_1)$ is compact, there exists a subsequence $\{(u_n^1, v_n^1)\}_{n\geq 1}$ of $\{(u_n, v_n)\}_{n\geq 1}$ which converges in $C(\overline{B}_1) \times C(\overline{B}_1)$.

- Since $\{(u_n^1, v_n^1)\}_{n\geq 2}$ is bounded in $C^{0,\mu}(\overline{B}_2) \times C^{0,\nu}(\overline{B}_2)$ and the embedding $C^{0,\mu}(\overline{B}_2) \times C^{0,\nu}(\overline{B}_2) \hookrightarrow C(\overline{B}_2) \times C(\overline{B}_2)$ is compact, there exists a subsequence $\{(u_n^2, v_n^2)\}_{n\geq 2}$ of $\{(u_n^1, v_n^1)\}_{n\geq 2}$ which converges in $C(\overline{B}_2) \times C(\overline{B}_2)$.

- Since $\{(u_n^2, v_n^2)\}_{n\geq 3}$ is bounded in $C^{0,\mu}(\overline{B}_3) \times C^{0,\nu}(\overline{B}_3)$ and the embedding $C^{0,\mu}(\overline{B}_3) \times C^{0,\nu}(\overline{B}_3) \hookrightarrow C(\overline{B}_3) \times C(\overline{B}_3)$ is compact, we can find a subsequence $\{(u_n^3, v_n^3)\}_{n\geq 3}$ of $\{(u_n^2, v_n^2)\}_{n\geq 3}$ which converges in $C(\overline{B}_3) \times C(\overline{B}_3)$.

- Inductively, we deduce that, for all $k \geq 1$, there exists a subsequence $\{(u_n^k, v_n^k)\}_{n\geq k}$ of $\{(u_n^{k-1}, v_n^{k-1})\}_{n\geq k}$ which converges in $C(\overline{B}_k) \times C(\overline{B}_k)$.

Let $\{(U_n, V_n)\} = \{(u_n^n, v_n^n)\}_{n\geq 1}$, which converges to a certain $(U, V) \in C(\mathbb{R}^N) \times C(\mathbb{R}^N)$ and fulfills

$$\begin{cases} U_n(x) = \displaystyle\int_{\mathbb{R}^N} \frac{K(y)V_n^{-p}(y)}{|x-y|^{N-\alpha}} dy & \text{for all } x \in \overline{B}_n, \\ V_n(x) = \displaystyle\int_{\mathbb{R}^N} \frac{L(y)U_n^{-q}(y)}{|x-y|^{N-\beta}} dy & \text{for all } x \in \overline{B}_n. \end{cases}$$

Since $(U_n, V_n) \in \mathcal{A}_n$, we can apply the Lebesgue dominated convergence theorem to obtain that (U, V) is a continuous solution of Eq (1.1). Moreover,

$$\begin{cases} U(x) = \lim_{n\to\infty} U_n(x) \simeq (1+|x|)^{-\kappa_1} & \text{for all } x \in \mathbb{R}^N \\ V(x) = \lim_{n\to\infty} V_n(x) \simeq (1+|x|)^{-\kappa_2} & \text{for all } x \in \mathbb{R}^N. \end{cases}$$

Finally, from Lemma 2.1 we deduce $(U, V) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$. This concludes the proof in (i).
(ii) Assume $0 < p < \frac{\gamma-N}{N-\beta}$, $0 < q < \frac{\gamma-N}{N-\alpha}$, and $pq < 1$.

For $n \geq 1$, we define the closed and convex subset $\mathcal{B}_n \subset C(\overline{B}_n) \times C(\overline{B}_n)$ by

$$\mathcal{B}_n = \left\{ (u, v) \in C(\overline{B}_n) \times C(\overline{B}_n) : \begin{array}{l} m_1(1+|x|)^{\alpha-N} \leq u(x) \leq M_1(1+|x|)^{\alpha-N} \\ m_2(1+|x|)^{\beta-N} \leq v(x) \leq M_2(1+|x|)^{\beta-N} \end{array} \text{ in } \overline{B}_n \right\}, \tag{3.10}$$

where $0 < m_1 < M_1$, $0 < m_2 < M_2$ are constants depending on $\alpha, \beta, \gamma, p, q$, and N.

For all $(u, v) \in \mathcal{B}_n$, we define

$$J_n(u, v)(x) = \Big(\int_{B_n} \frac{K(y)v(y)^{-p}}{|x-y|^{N-\alpha}} dy, \int_{B_n} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\beta}} dy \Big) \quad \text{for all } x \in \overline{B}_n.$$

By Lemma 2.2 (ii) applied twice for $\sigma = \gamma - p(N-\beta) > N$ and for $\sigma = \gamma - q(N-\alpha) > N$, respectively, there exist $c_2 > c_1 > 0$ independent of $n \geq 1$ such that

$$c_1(1+|x|)^{\alpha-N} \leq \int_{B_n} \frac{K(y)dy}{(1+|y|)^{-p(N-\beta)}|x-y|^{N-\alpha}} \leq c_2(1+|x|)^{\alpha-N} \quad \text{for all } x \in B_n, \tag{3.11}$$

and

$$c_1(1+|x|)^{\beta-N} \leq \int_{B_n} \frac{L(y)dy}{(1+|y|)^{-q(N-\alpha)}|x-y|^{N-\beta}} \leq c_2(1+|x|)^{\beta-N} \quad \text{for all } x \in B_n. \tag{3.12}$$

We choose m_1, m_2, M_1, and M_2 as given by Eq (3.8) with new constants c_1, c_2 from Eqs (3.11) and (3.12). Then, Lemma 3.1 holds, and we deduce $J_n(\mathcal{B}_n) \subset \mathcal{B}_n$.

Next, we select $s > 1$ such that

$$\alpha - 1 < \frac{N}{s} < \alpha < N < \gamma - p(N-\beta) \quad \text{and} \quad \beta - 1 < \frac{N}{s} < \beta < N < \gamma - q(N-\alpha).$$

Let $\mu = \alpha - \frac{N}{s} \in (0,1)$ and $\nu = \beta - \frac{N}{s} \in (0,1)$. By Lemma 2.1, we obtain

$$J_n : \mathcal{B}_n \subset L^s(B_n) \times L^s(B_n) \to C^{0,\mu}(B_n) \times C^{0,\nu}(B_n) \quad \text{is continuous}$$

and $J_n(\mathcal{B}_n) \subset \mathcal{B}_n$.

Hence, we can use the Schauder fixed-point theorem for J_n, which implies the existence of $(u_n, v_n) \in \mathcal{B}_n$ such that

$$(u_n, v_n) = \Big(\int_{B_n} \frac{K(y)v_n(y)^{-p}}{|x-y|^{N-\alpha}} dy, \int_{B_n} \frac{L(y)u_n(y)^{-q}}{|x-y|^{N-\beta}} dy \Big) \quad \text{for all } x \in \overline{B}_n.$$

As in part (i) above, the diagonal sequence $\{(U_n, V_n)\} = \{(u_n^n, v_n^n)\}_{n\geq 1}$ converges to a certain $(U, V) \in C(\mathbb{R}^N) \times C(\mathbb{R}^N)$ which fulfills

$$\begin{cases} U(x) = \displaystyle\int_{\mathbb{R}^N} \frac{K(y)V^{-p}(y)}{|x-y|^{N-\alpha}} dy & \text{for all } x \in \overline{B}_n, \\ V(x) = \displaystyle\int_{\mathbb{R}^N} \frac{L(y)U^{-q}(y)}{|x-y|^{N-\beta}} dy & \text{for all } x \in \overline{B}_n, \end{cases}$$

Since

$$\begin{cases} U(x) = \lim_{n\to\infty} U_n(x) \simeq (1+|x|)^{\alpha-N} & \text{for all } x \in \mathbb{R}^N \\ V(x) = \lim_{n\to\infty} V_n(x) \simeq (1+|x|)^{\beta-N} & \text{for all } x \in \mathbb{R}^N, \end{cases}$$

we can apply Lemma 2.1, and we deduce $(U, V) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$. This finishes the proof of our theorem.

3.4. Proof of Theorem 1.5

Assume $0 < p < \frac{\gamma-N}{N-\beta}$ and $0 < q < \frac{\gamma-N}{N-\alpha}$. The existence of a positive solution to Eq (1.1) in this range of parameters was obtained in Theorem 1.4. We next discuss the uniqueness.

Let (u, v) be a positive solution of Eq (1.1). We note that if $|x| > 1 \geq |y|$, then $|x - y| \leq |x| + |y| \leq 2|x|$, and we combine this fact with $Kv^{-p} \in L^1_{loc}(\mathbb{R}^N)$ to obtain

$$u(x) \geq \int\limits_{|y|<1} \frac{K(y)v(y)^{-p}}{|x-y|^{N-\alpha}} dy \geq (2|x|)^{\alpha-N} \int\limits_{|y|<1} K(y)v(y)^{-p} dy$$

$$\geq C|x|^{\alpha-N} \geq C(1 + |x|)^{\alpha-N}.$$

This means that $u(x) \geq C(1 + |x|)^{\alpha-N}$ for $|x| > 1$.

If $|x|, |y| \leq 1$, then $|x - y| \leq 2$ and also $Kv^{-p} \in L^1(B_1)$ yield

$$u(x) \geq \int\limits_{|y|<1} \frac{K(y)v(y)^{-p}}{|x-y|^{N-\alpha}} dy \geq 2^{\alpha-N} \int\limits_{|y|<1} K(y)v(y)^{-p} dy \geq C \geq C(1 + |x|)^{\alpha-N}.$$

We have shown that

$$u(x) \geq C(1 + |x|)^{\alpha-N} \qquad \text{for all } x \in \mathbb{R}^N. \tag{3.13}$$

In the same way, we obtain

$$v(x) \geq C(1 + |x|)^{\beta-N} \quad \text{for all } x \in \mathbb{R}^N. \tag{3.14}$$

From Eqs (1.1) and (3.14), we estimate

$$u(x) \leq \int\limits_{\mathbb{R}^N} \frac{K(y)(1 + |y|)^{-p(\beta-N)}}{|x-y|^{N-\alpha}} dy$$

$$\leq C \int\limits_{\mathbb{R}^N} \frac{dy}{(1 + |y|)^{\gamma-p(N-\beta)}|x-y|^{N-\alpha}} dy.$$

We can now apply Eq (2.2) with $\sigma = \gamma - p(N - \beta) > N$ to obtain $u(x) \leq C(1 + |x|)^{\alpha-N}$ in $\mathbb{R}^N$, and similarly $v(x) \leq C(1 + |x|)^{\beta-N}$ in $\mathbb{R}^N$. Combining these two estimates with Eqs (3.13) and (3.14), we deduce

$$\begin{cases} u(x) \simeq (1 + |x|)^{\alpha-N} & \text{for all } x \in \mathbb{R}^N, \\ v(x) \simeq (1 + |x|)^{\beta-N} & \text{for all } x \in \mathbb{R}^N. \end{cases} \tag{3.15}$$

Now let (u_1, v_1) and (u_2, v_2) be two positive solutions of Eq (1.1). From Eq (3.15), we have

$$u_1(x) \simeq u_2(x) \simeq (1 + |x|)^{\alpha-N},$$

and so, we can find $C > c > 0$ such that

$$cu_1 \leq u_2 \leq Cu_1 \quad \text{in } \mathbb{R}^N.$$

Then, we define

$$M := \inf\{A > 1 : Au_1 \geq u_2 \text{ in } \mathbb{R}^N\} \geq 1. \tag{3.16}$$

Clearly, $Mu_1 \geq u_2$. Assume by contradiction that $M > 1$. Then, $u_2 \leq Mu_1$ implies that for all $x \in \mathbb{R}^N$, we have

$$\begin{aligned} v_2(x) &= \int_{\mathbb{R}^N} \frac{L(y)u_2(y)^{-q}}{|x-y|^{N-\beta}}dy \\ &\geq M^{-q} \int_{\mathbb{R}^N} \frac{L(y)u_1(y)^{-q}}{|x-y|^{N-\beta}}dy \\ &= M^{-q}v_1(x). \end{aligned}$$

Hence, $v_2 \geq M^{-q}v_1$ in $\mathbb{R}^N$, which implies

$$v_2^{-p} \leq M^{pq}v_1^{-p} \quad \text{in } \mathbb{R}^N.$$

Together with the previous estimates, we obtain

$$\begin{aligned} u_2(x) &= \int_{\mathbb{R}^N} \frac{K(y)v_2(y)^{-p}}{|x-y|^{N-\alpha}}dy \\ &\leq M^{pq} \int_{\mathbb{R}^N} \frac{K(y)v_1(y)^{-p}}{|x-y|^{N-\alpha}}dy \\ &= M^{pq}u_1(x) \quad \text{for all } x \in \mathbb{R}^N. \end{aligned}$$

Thus, $M^{pq}u_1 \geq u_2$ in $\mathbb{R}^N$.

Since $M > 1$ and $0 < pq < 1$, we have $M > M^{pq} > 1$, and this is a contradiction of the fact that M is the infimum value defined in Eq (3.16). Hence, $M = 1$ and $u_1 \geq u_2$. Swapping u_1 with u_2 in the above argument, we deduce $u_2 \geq u_1$, and thus $u_1 \equiv u_2$. From Eq (1.1), we also have $v_1 \equiv v_2$. This concludes the proof of the uniqueness of a positive solution to Eq (1.1).

3.5. Proof of Theorem 1.6

First, we start with the proof of the finite total mass solution.

(i) Recall that in the proof of Theorem 1.5, we established that (u, v) satisfies:

$$u(x) \geq C(1 + |x|)^{\alpha-N} \quad \text{and} \quad v(x) \geq C(1 + |x|)^{\beta-N} \tag{3.17}$$

in $\mathbb{R}^N$, for some $c > 0$. Using Eq (3.17) and $0 < p < \frac{\gamma-N}{N-\beta}$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)v(x)^{-p}dx &\leq C \int_{\mathbb{R}^N} (1 + |x|)^{-\gamma+p(N-\beta)}dx \\ &\leq C \int_0^\infty (1 + t)^{N-\gamma+p(N-\beta)-1}dt \\ &< \infty. \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N} L(x)u(x)^{-q}dx \le C\int_0^\infty (1+t)^{N-\gamma+q(N-\alpha)-1}dt < \infty.$$

Hence, (u, v) has finite total mass.

Next, in order to establish Eq (1.9), we note that

$$|x|^{N-\alpha}u(x) = \int_{\mathbb{R}^N} K(y)v(y)^{-p}(\frac{|x|}{|x-y|})^{N-\alpha}dy \quad \text{for all } x \in \mathbb{R}^N.$$

Using Fatou's lemma, we infer that

$$\begin{aligned} \liminf_{|x|\to\infty} |x|^{N-\alpha}u(x) &\ge \int_{\mathbb{R}^N} K(y)v(y)^{-p} \liminf_{|x|\to\infty} (\frac{|x|}{|x-y|})^{N-\alpha}dy \\ &= \int_{\mathbb{R}^N} K(y)v(y)^{-p}dy. \end{aligned} \tag{3.18}$$

For the converse inequality, take $\varepsilon \in (0, 1)$ small and write

$$\int_{\mathbb{R}^N} K(y)v(y)^{-p}(\frac{|x|}{|x-y|})^{N-\alpha}dy = S_1 + S_2 + S_3. \tag{3.19}$$

where

$$S_1 = \int_{|y|\le\varepsilon|x|} K(y)v(y)^{-p}(\frac{|x|}{|x-y|})^{N-\alpha}dy,$$

$$S_2 = \int_{\varepsilon|x|<|y|\le 2|x|} K(y)v(y)^{-p}(\frac{|x|}{|x-y|})^{N-\alpha}dy,$$

$$S_3 = \int_{|y|>2|x|} K(y)v(y)^{-p}(\frac{|x|}{|x-y|})^{N-\alpha}dy.$$

We note that $|y| \le \varepsilon|x|$ implies $|x-y| \ge |x| - |y| \ge (1-\varepsilon)|x|$, so $\frac{|x|}{|x-y|} \le \frac{1}{1-\varepsilon}$, and thus

$$S_1 \le \frac{1}{(1-\varepsilon)^{N-\alpha}} \int_{\mathbb{R}^N} K(y)v(y)^{-p}dy. \tag{3.20}$$

Next, using Eq (3.17) and the estimate on $K(y)$, we have

$$\begin{aligned} S_2 &\le C \int_{\varepsilon|x|<|y|\le 2|x|} (1+|y|)^{-\gamma+p(N-\beta)}(\frac{|x|}{|x-y|})^{N-\alpha}dy, \\ &\le C(1+\varepsilon|x|)^{-\gamma+p(N-\beta)}|x|^{N-\alpha} \int_{\varepsilon|x|<|y|\le 2|x|} \frac{dy}{|x-y|^{N-\alpha}}, \\ &\le C(1+\varepsilon|x|)^{-\gamma+p(N-\beta)}|x|^{N-\alpha} \int_{|x-y|\le 3|x|} \frac{dy}{|x-y|^{N-\alpha}}, \\ &= C(1+\varepsilon|x|)^{-\gamma+p(N-\beta)}|x|^{N}. \end{aligned} \tag{3.21}$$

Since $p < \frac{\gamma-N}{N-\beta}$, we have $S_2 \to 0$ as $|x| \to \infty$.

In order to estimate S_3, we note that $|y| > 2|x|$ yields $|x-y| \geq |y| - |x| \geq |x|$, so $\frac{|x|}{|x-y|} \leq 1$, and then

$$S_3 \leq \int\limits_{|y|>2|x|} K(y)v(y)^{-p}dy \to 0 \quad \text{as} \quad |x| \to \infty. \tag{3.22}$$

Now, using Eqs (3.19)–(3.22), we deduce

$$\lim_{|x|\to\infty} \sup |x|^{N-\alpha} u(x) \leq \frac{1}{(1-\varepsilon)^{N-\alpha}} \int\limits_{\mathbb{R}^N} K(y)v(y)^{-p}dy.$$

Since $\varepsilon > 0$ was arbitrarily chosen, this yields

$$\lim_{|x|\to\infty} \sup |x|^{N-\alpha} u(x) \leq \int\limits_{\mathbb{R}^N} K(y)v(y)^{-p}dy. \tag{3.23}$$

From Eqs (3.18) and (3.23), we complete the proof of Eq (1.9). The proof of Eq (1.10) follows similarly.

(ii) Assume, without loss of the generality, that $p = \frac{\gamma-N}{N-\beta}$, $q > \frac{\gamma-N}{N-\alpha}$, and let $\varepsilon > 0$ be such that

$$q > \frac{\gamma - N + \varepsilon}{N - \alpha}. \tag{3.24}$$

Using Eq (3.17) and Lemma 2.2 (iii), we find:

$$\begin{aligned} u(x) &\leq \int\limits_{\mathbb{R}^N} \frac{K(y)v(y)^{-p}}{|x-y|^{N-\alpha}}dy, \\ &\leq C \int\limits_{\mathbb{R}^N} \frac{dy}{(1+|y|)^{\gamma-p(N-\beta)}|x-y|^{N-\alpha}}dy, \\ &\leq C(1+|x|)^{\alpha-N}\log(e+|x|) \quad \text{in} \quad \mathbb{R}^n. \end{aligned}$$

This last inequality and Eq (3.24) yields,

$$\begin{aligned} \int\limits_{\mathbb{R}^N} L(x)u(x)^{-q}dy &\geq C \int\limits_{\mathbb{R}^N} (1+|x|)^{-\gamma+q(N-\alpha)} \log^{-q}(e+|x|)dx, \\ &= C \int\limits_0^\infty t^{N-1}(1+t)^{-\gamma+q(N-\alpha)} \log^{-q}(e+t)dt, \\ &\geq C \int\limits_1^\infty t^{N-\gamma+q(N-\alpha)-1} \log^{-q}(e+t)dt, \\ &\geq C \int\limits_1^\infty t^{\varepsilon-1} \log^{-q}(e+t)dt, \\ &= \infty. \end{aligned}$$

Hence, (u, v) has infinite total mass. This concludes our proof.

4. Further extensions

In this section we explain how our approach can be used to investigate the existence of solutions to the system

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{K(y)v^p(y)}{|x-y|^{N-\alpha}}dy & \text{for all } x \in \mathbb{R}^N, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{L(y)u^{-q}(y)}{|x-y|^{N-\beta}}dy & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $p, q > 0$, $\alpha, \beta \in (0, N)$, and $K, L : \mathbb{R}^N \to (0, \infty)$ are continuous functions which satisfy Eq (1.2). Our main result in this section is stated below.

Theorem 4.1. (Existence)

Assume $\max\{\alpha, \beta\} < \gamma$.

(i) *If* $\gamma > N$, $0 < q < \frac{\gamma-N}{N-\alpha}$, *and* $pq < 1$, *then the integral system* (4.1) *has a solution* (u, v) *such that*

$$\begin{cases} u(x) \simeq (1+|x|)^{\alpha-N}, \\ v(x) \simeq (1+|x|)^{\beta-N}. \end{cases}$$

Moreover, $(u, v) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$ *for some* $\mu, \nu \in (0, 1)$.

(ii) *If* $\max\{\alpha, \beta\} < \gamma < N$, $0 < p < \frac{N-\gamma}{\gamma-\beta}$, *and* $0 < q < \frac{\gamma-\beta}{\gamma-\alpha}$, *then the integral system* (4.1) *has a solution* (u, v) *such that*

$$\begin{cases} u(x) \simeq (1+|x|)^{-\frac{\gamma-\alpha+p(\gamma-\beta)}{1+pq}}, \\ v(x) \simeq (1+|x|)^{-\frac{\gamma-\beta-q(\gamma-\alpha)}{1+pq}}. \end{cases}$$

Moreover, $(u, v) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$ *for some* $\mu, \nu \in (0, 1)$.

Proof. (i) Assume $\gamma > N$, $0 < q < \frac{\gamma-N}{N-\alpha}$, and $pq < 1$.

For $n \geq 1$, we define the closed and convex subset $\mathcal{A}_n \subset C(\overline{B}_n) \times C(\overline{B}_n)$ by

$$\mathcal{A}_n = \left\{ (u, v) \in C(\overline{B}_n) \times C(\overline{B}_n) : \begin{matrix} m_1(1+|x|)^{\alpha-N} \leq u(x) \leq M_1(1+|x|)^{\alpha-N} \\ m_2(1+|x|)^{\beta-N} \leq v(x) \leq M_2(1+|x|)^{\beta-N} \end{matrix} \ \text{ in } \overline{B}_n \right\}, \tag{4.2}$$

where $0 < m_1 < M_1$, $0 < m_2 < M_2$ are constants depending on $\alpha, \beta, \gamma, p, q$, and N.

For all $(u, v) \in \mathcal{B}_n$ we define

$$J_n(u, v)(x) = \Big(\int_{B_n} \frac{K(y)v(y)^p}{|x-y|^{N-\alpha}}dy, \int_{B_n} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\beta}}dy \Big) \quad \text{for all } x \in \overline{B}_n.$$

By Lemma 2.2 (ii) applied twice for $\sigma = \gamma + p(N-\beta) > N$ and for $\sigma = \gamma - q(N-\alpha) > N$, respectively, there exist $c_2 > c_1 > 0$ independent of $n \geq 1$ such that

$$c_1(1+|x|)^{\alpha-N} \leq \int_{B_n} \frac{K(y)dy}{(1+|y|)^{p(N-\beta)}|x-y|^{N-\alpha}} \leq c_2(1+|x|)^{\alpha-N} \quad \text{for all } x \in B_n, \tag{4.3}$$

and

$$c_1(1+|x|)^{\beta-N} \leq \int_{B_n} \frac{L(y)dy}{(1+|y|)^{-q(N-\alpha)}|x-y|^{N-\beta}} \leq c_2(1+|x|)^{\beta-N} \quad \text{for all } x \in B_n. \tag{4.4}$$

Lemma 4.2. *Let*

$$m_1 = \left(\frac{c_2^{p^2q+pq}}{c_1^{p+1}}\right)^{\frac{1}{p^2q^2-1}}, \quad M_1 = \left(\frac{c_1^{p^2q+pq}}{c_2^{p+1}}\right)^{\frac{1}{p^2q^2-1}}, \quad m_2 = \left(\frac{c_2^{pq+q}}{c_1^{pq^2+1}}\right)^{\frac{1}{p^2q^2-1}}, \quad M_2 = \left(\frac{c_1^{pq+q}}{c_2^{pq^2+1}}\right)^{\frac{1}{p^2q^2-1}}. \tag{4.5}$$

where c_1, c_2 are defined in Eqs (4.3) *and* (4.4)*. If $pq < 1$, then $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$.*

Proof. Since $pq < 1$, it is obvious to see that $m_1 < M_1$ and $m_2 < M_2$. We also have

$$\begin{cases} m_1 = c_1 m_2^p, \\ M_1 = c_2 M_2^p, \\ m_2 = c_1 M_1^{-q}, \\ M_2 = c_2 m_1^{-q}. \end{cases} \tag{4.6}$$

To prove $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$, let $(u, v) \in \mathcal{A}_n$.
Since $v(x) \le M_2(1 + |x|)^{\beta-N}$ in B_n, by Eqs (4.3) and $(4.6)_2$ we have

$$\begin{aligned} \int_{B_n} \frac{K(y)v(y)^p}{|x-y|^{N-\alpha}} dy &\le M_2^p \int_{B_n} \frac{K(y)}{(1+|x|)^{p(N-\beta)}|x-y|^{N-\alpha}} dy \\ &\le M_2^p c_2 (1+|x|)^{\alpha-N} \\ &= M_1 (1+|x|)^{\alpha-N} \quad \text{in } B_n. \end{aligned}$$

Also, $v(x) \ge m_2(1 + |x|)^{\beta-N}$ in B_n, and by Eqs (4.3) and $(4.6)_1$ we have

$$\begin{aligned} \int_{B_n} \frac{K(y)v(y)^p}{|x-y|^{N-\alpha}} dy &\ge m_2^p \int_{B_n} \frac{K(y)}{(1+|x|)^{p(N-\beta)}|x-y|^{N-\alpha}} dy \\ &\ge m_2^p c_1 (1+|x|)^{\alpha-N} \\ &= m_1 (1+|x|)^{\alpha-N} \quad \text{in } B_n. \end{aligned}$$

Similarly, $u(x) \le M_1(1 + |x|)^{\alpha-N}$ in B_n combined with Eqs (4.4) and $(4.6)_3$ yields

$$\begin{aligned} \int_{B_n} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\beta}} dy &\ge M_1^{-q} \int_{B_n} \frac{L(y)}{(1+|x|)^{-q(N-\alpha)}|x-y|^{N-\beta}} dy \\ &\ge M_1^{-q} c_1 (1+|x|)^{\alpha-N} \\ &= m_2 (1+|x|)^{\alpha-N} \quad \text{in } B_n. \end{aligned}$$

Finally, since $u(x) \ge m_1(1 + |x|)^{\alpha-N}$ in B_n, Eqs (4.4) and $(4.6)_4$ produces

$$\begin{aligned} \int_{B_n} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\beta}} dy &\le m_1^{-q} \int_{B_n} \frac{L(y)}{(1+|x|)^{-q(N-\alpha)}|x-y|^{N-\beta}} dy \\ &\le m_1^{-q} c_2 (1+|x|)^{\alpha-N} \\ &= M_2 (1+|x|)^{\alpha-N} \quad \text{in } B_n. \end{aligned}$$

Hence, $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$.

Let us recall that

$$\gamma + p(N-\beta) > N \quad \text{and} \quad \gamma - q(N-\alpha) > N.$$

Next, we select $s > 1$ such that

$$\alpha - 1 < \frac{N}{s} < \alpha < N < \gamma + p(N-\beta) \quad \text{and} \quad \beta - 1 < \frac{N}{s} < \beta < N < \gamma - q(N-\alpha).$$

Let $\mu = \alpha - \frac{N}{s} \in (0,1)$ and $\nu = \beta - \frac{N}{s} \in (0,1)$. By Lemma 2.1, we obtain that

$$J_n = J_{\alpha,\beta,n} : \mathcal{A}_n \subset L^s(B_n) \times L^s(B_n) \to C^{0,\mu}(B_n) \times C^{0,\nu}(B_n) \quad \text{is continuous}$$

and $J_n(\mathcal{A}_n) \subset \mathcal{A}_n$.

Hence, we can use the Schauder fixed-point theorem for J_n, which implies the existence of $(u_n, v_n) \in \mathcal{A}_n$ such that

$$(u_n, v_n) = \Big(\int_{B_n} \frac{K(y) v_n(y)^p}{|x-y|^{N-\alpha}} dy, \int_{B_n} \frac{L(y) u_n(y)^{-q}}{|x-y|^{N-\beta}} dy \Big) \quad \text{for all } x \in \overline{B}_n.$$

We next argue as follows.

- Since $\{(u_n, v_n)\}$ is bounded in $C^{0,\mu}(\overline{B}_1) \times C^{0,\nu}(\overline{B}_1)$. Since the embedding $C^{0,\mu}(\overline{B}_1) \times C^{0,\nu}(\overline{B}_1) \hookrightarrow C(\overline{B}_1) \times C(\overline{B}_1)$ is compact, there exists a subsequence $\{(u_n^1, v_n^1)\}_{n\geq 1}$ of $\{(u_n, v_n)\}_{n\geq 1}$ which converges in $C(\overline{B}_1) \times C(\overline{B}_1)$.

- Since $\{(u_n^1, v_n^1)\}_{n\geq 2}$ is bounded in $C^{0,\mu}(\overline{B}_2) \times C^{0,\nu}(\overline{B}_2)$ and the embedding $C^{0,\mu}(\overline{B}_2) \times C^{0,\nu}(\overline{B}_2) \hookrightarrow C(\overline{B}_2) \times C(\overline{B}_2)$ is compact, there exists a subsequence $\{(u_n^2, v_n^2)\}_{n\geq 2}$ of $\{(u_n^1, v_n^1)\}_{n\geq 2}$ which converges in $C(\overline{B}_2) \times C(\overline{B}_2)$.

- Since $\{(u_n^2, v_n^2)\}_{n\geq 3}$ is bounded in $C^{0,\mu}(\overline{B}_3) \times C^{0,\nu}(\overline{B}_3)$ and the embedding $C^{0,\mu}(\overline{B}_3) \times C^{0,\nu}(\overline{B}_3) \hookrightarrow C(\overline{B}_3) \times C(\overline{B}_3)$ is compact, we can find a subsequence $\{(u_n^3, v_n^3)\}_{n\geq 3}$ of $\{(u_n^2, v_n^2)\}_{n\geq 3}$ which converges in $C(\overline{B}_3) \times C(\overline{B}_3)$.

- Inductively, we deduce that, for all $k \geq 1$, there exists a subsequence $\{(u_n^k, v_n^k)\}_{n\geq k}$ of $\{(u_n^{k-1}, v_n^{k-1})\}_{n\geq k}$ which converges in $C(\overline{B}_k) \times C(\overline{B}_k)$.

Let $\{(U_n, V_n)\} = \{(u_n^n, v_n^n)\}_{n\geq 1}$ which converges to a certain $(U, V) \in C(\mathbb{R}^N) \times C(\mathbb{R}^N)$ and fulfills

$$\begin{cases} U_n(x) = \displaystyle\int_{\mathbb{R}^N} \frac{K(y) V_n^p(y)}{|x-y|^{N-\alpha}} dy & \text{for all } x \in \overline{B}_n, \\ V_n(x) = \displaystyle\int_{\mathbb{R}^N} \frac{L(y) U_n^{-q}(y)}{|x-y|^{N-\beta}} dy & \text{for all } x \in \overline{B}_n. \end{cases}$$

Since $(U_n, V_n) \in \mathcal{A}_n$, we can apply the Lebesgue dominated convergence theorem to obtain that (U, V) is a continuous solution of Eq (4.1). Moreover,

$$\begin{cases} U(x) = \lim_{n\to\infty} U_n(x) \simeq (1+|x|)^{\alpha-N} & \text{for all } x \in \mathbb{R}^N \\ V(x) = \lim_{n\to\infty} V_n(x) \simeq (1+|x|)^{\beta-N} & \text{for all } x \in \mathbb{R}^N. \end{cases}$$

Finally, from Lemma 2.1 we deduce $(U, V) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$. This concludes the proof in (i).

(ii) Assume $\max\{\alpha,\beta\} < \gamma < N$, $0 < p < \frac{N-\gamma}{\gamma-\beta}$, and $0 < q < \frac{\gamma-\beta}{\gamma-\alpha}$. Let

$$\kappa_1 = \frac{\gamma - \alpha + p(\gamma-\beta)}{1+pq} > 0 \quad \text{and} \quad \kappa_2 = \frac{\gamma-\beta-q(\gamma-\alpha)}{1+pq} > 0. \tag{4.7}$$

For $n \geq 1$, we define the closed and convex set $\mathcal{B}_n \subset C(\overline{B}_n) \times C(\overline{B}_n)$ by

$$\mathcal{B}_n = \left\{ (u,v) \in C(\overline{B}_n) \times C(\overline{B}_n) : \begin{array}{l} m_1(1+|x|)^{-\kappa_1} \leq u(x) \leq M_1(1+|x|)^{-\kappa_1} \\ m_2(1+|x|)^{-\kappa_2} \leq v(x) \leq M_2(1+|x|)^{-\kappa_2} \end{array} \text{ in } \overline{B}_n \right\}, \tag{4.8}$$

where $0 < m_1 < M_1$, $0 < m_2 < M_2$ are constants depending on $\alpha, \beta, \gamma, p, q$, and N that will be chosen in Lemma 4.2 below.

For all $u, v \in \mathcal{B}_n$, we define

$$J_n(u,v)(x) = \Big(\int_{B_n} \frac{K(y)v(y)^p}{|x-y|^{N-\alpha}} dy, \int_{B_n} \frac{L(y)u(y)^{-q}}{|x-y|^{N-\beta}} dy \Big) \quad \text{for all } x \in \overline{B}_n.$$

Since $0 < q < \frac{\gamma-\beta}{\gamma-\alpha}$, we can check

$$\begin{aligned} \gamma - \kappa_1 q &= \gamma - \frac{(\gamma-\alpha)q + pq(\gamma-\beta)}{1+pq} \\ &> \gamma - \frac{(\gamma-\beta) + pq(\gamma-\beta)}{1+pq} \\ &> \beta. \end{aligned}$$

Also, by $N > \gamma$, we deduce $\gamma - \kappa_1 q < N$.

Next, since $0 < p < \frac{N-\gamma}{\gamma-\beta}$, we obtain

$$\begin{aligned} \gamma + \kappa_2 p &= \gamma + \frac{(\gamma-\beta)p - pq(\gamma-\alpha)}{1+pq}, \\ &< \gamma + \frac{(N-\gamma) + pq(\alpha-\gamma)}{1+pq}, \\ &< \gamma + \frac{(N-\gamma) + pq(N-\gamma)}{1+pq}, \\ &< N. \end{aligned}$$

Also, by $\max\{\alpha,\beta\} < \gamma < N$, it follows that $\gamma + \kappa_2 p > \alpha$.

Now we have $\beta < \gamma - \kappa_1 q < N$ and $\alpha < \gamma + \kappa_2 p < N$. Hence, by Eq (3.7), there exists $c_2 > c_1 > 0$ independent of $n \geq 1$ such that

$$c_1(1+|x|)^{-\kappa_1} \leq \int_{B_n} \frac{K(y)dy}{(1+|y|)^{p\kappa_2}|x-y|^{N-\alpha}} \leq c_2(1+|x|)^{-\kappa_1} \quad \text{for all } x \in B_n, \tag{4.9}$$

and

$$c_1(1+|x|)^{-\kappa_2} \le \int\limits_{B_n} \frac{L(y)dy}{(1+|y|)^{-q\kappa_1}|x-y|^{N-\beta}} \le c_2(1+|x|)^{-\kappa_2} \quad \text{for all } x \in B_n. \tag{4.10}$$

We choose m_1, m_2, M_1, and M_2 as given by Eq (4.5) with new constants c_1, c_2 from Eqs (4.9) and (4.10). Then, Lemma 4.2 holds, and we deduce $J_n(\mathcal{B}_n) \subset \mathcal{B}_n$.

Thus, we can select $s > 1$ such that

$$\alpha - 1 < \frac{N}{s} < \alpha < \gamma + p\kappa_2 \quad \text{and} \quad \beta - 1 < \frac{N}{s} < \beta < \gamma - q\kappa_1.$$

Let $\mu = \alpha - \frac{N}{s} \in (0,1)$ and $\nu = \beta - \frac{N}{s} \in (0,1)$. By Lemma 2.1, we obtain that

$$J_n : \mathcal{B}_n \subset L^s(B_n) \times L^s(B_n) \to C^{0,\mu}(B_n) \times C^{0,\nu}(B_n) \quad \text{is continuous}$$

and $J_n(\mathcal{B}_n) \subset \mathcal{B}_n$.

Hence, we can use the Schauder fixed-point theorem for J_n, which implies the existence of $(u_n, v_n) \in \mathcal{B}_n$ such that

$$(u_n, v_n) = \Big(\int_{B_n} \frac{K(y)v_n(y)^p}{|x-y|^{N-\alpha}} dy, \int_{B_n} \frac{L(y)u_n(y)^{-q}}{|x-y|^{N-\beta}} dy \Big) \quad \text{for all } x \in \overline{B}_n.$$

As in part (i) above, the diagonal sequence $\{(U_n, V_n)\} = \{(u_n^n, v_n^n)\}_{n\ge1}$ converges to a certain $(U, V) \in C(\mathbb{R}^N) \times C(\mathbb{R}^N)$, which fulfills

$$\begin{cases} U(x) = \displaystyle\int\limits_{\mathbb{R}^N} \frac{K(y)V^p(y)}{|x-y|^{N-\alpha}} dy & \text{for all } x \in \overline{B}_n, \\ V(x) = \displaystyle\int\limits_{\mathbb{R}^N} \frac{L(y)U^{-q}(y)}{|x-y|^{N-\beta}} dy & \text{for all } x \in \overline{B}_n, \end{cases}$$

Since

$$\begin{cases} U(x) = \lim_{n\to\infty} U_n(x) \simeq (1+|x|)^{-\kappa_1} & \text{for all } x \in \mathbb{R}^N \\ V(x) = \lim_{n\to\infty} V_n(x) \simeq (1+|x|)^{-\kappa_2} & \text{for all } x \in \mathbb{R}^N, \end{cases}$$

we can apply Lemma 2.1 to deduce $(U, V) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$. This finishes the proof of our theorem.

With a similar approach, we can also obtain the existence of solutions to the system

$$\begin{cases} u(x) = \displaystyle\int\limits_{\mathbb{R}^N} \frac{K(y)v^p(y)}{|x-y|^{N-\alpha}} dy & \text{for all } x \in \mathbb{R}^N, \\ v(x) = \displaystyle\int\limits_{\mathbb{R}^N} \frac{L(y)u^q(y)}{|x-y|^{N-\beta}} dy & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{4.11}$$

where $p, q > 0$, $\alpha, \beta \in (0, N)$, and $K, L : \mathbb{R}^N \to (0, \infty)$ are continuous functions which satisfy Eq (1.2). Our main result regarding the system (4.11) reads as follows.

Theorem 4.3. (Existence)

Assume $\max\{\alpha, \beta\} < \gamma$.

(i) *If* $\max\{\alpha,\beta\} < \gamma < N$*,* $p > \frac{N-\gamma}{N-\beta}$*,* $q > \frac{N-\gamma}{N-\alpha}$*, and* $pq < 1$*, then the integral system* (4.11) *has a solution* (u, v) *such that*

$$\begin{cases} u(x) \simeq (1 + |x|)^{\alpha-N}, \\ v(x) \simeq (1 + |x|)^{\beta-N}. \end{cases}$$

Moreover, $(u, v) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$ *for some* $\mu, \nu \in (0, 1)$*.*

(ii) *If* $\gamma > N$ *and* $pq < 1$*, then the integral system* (4.11) *has a solution* (u, v) *such that*

$$\begin{cases} u(x) \simeq (1 + |x|)^{\alpha-N}, \\ v(x) \simeq (1 + |x|)^{\beta-N}. \end{cases}$$

Moreover, $(u, v) \in C^{0,\mu}(\mathbb{R}^N) \times C^{0,\nu}(\mathbb{R}^N)$ *for some* $\mu, \nu \in (0, 1)$*.*

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.s

Acknowledgements

The author would like to thank the anonymous a for the careful reading of the manuscript which led to the current form.

This publication has emanated from research conducted with the financial support of Taighde Éireann-Research Ireland under Grant number 18/CRT/6049.

Conflict of interest

The author declares there is no conflict of interest.

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