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Research article

Dynamic analysis of one-unit repairable systems with imperfect repairs

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Abstract: This paper conducts a dynamic analysis of a one-unit repairable system with two different failure modes and imperfect repairs. Compared to classical models with single failure modes and perfect repair mechanisms, the proposed system more accurately reflects real-world maintenance scenarios. By employing the C_0-semigroup theory of linear operators, we prove the existence and uniqueness of a non-negative time-dependent solution (T-DS) for the system. Furthermore, we investigate the asymptotic behavior of the T-DS, demonstrate its exponential convergence to the steady-state solution (S-SS), and also derive explicit asymptotic expressions for the T-DS. Numerical examples illustrate how key parameters influence transient reliability metrics and their convergence characteristics. This study offers theoretical insights into the dynamics of repairable systems with complex failures and imperfect repair mechanisms, while providing practical guidelines for optimizing system design and reliability assessment.

Keywords: one-unit repairable system; imperfect repairs; C_0-semigroup; well-posedness; exponential stability

1. Introduction

The one-unit repairable system, consisting of a single component that can be repaired upon failure, plays a crucial role in determining the overall performance and efficiency of a larger system. Therefore, authors such as Nakagawa and Osaki [1], Al-Ali and Murari [2], Wang and Chiu [3], Garg et al. [4], Kadyan [5], Du et al. [6], El-Sherbeny and Hussien [7], Li et al. [8], and Shekhar et al. [9] have investigated the reliability models of one-unit systems under various assumptions about failure and repair policies. In most reliability analyses, the assumption is that the system is either perfect or failed. Another possible state included in general repair models is “better than old but worse than new” (BTOWTN). In view of this, Nikolov [10] considered a repairable system subject to multiple failure modes with imperfect repairs. The model presented in [10] utilized the supplementary variable technique and was formulated in terms of integro-differential equations, and under the following assumption:

(1) The system has a unique nonnegative dynamic solution.

(2) As $t \to \infty$, the dynamic solution converges to its S-SS.

The author derived the Laplace transform of the T-DS and expressions for the S-SS, focusing only on the steady-state scenario of the model. However, the validity of these assumptions remains unverified, posing significant challenges in proving their accuracy (see [11]). This paper aims to fill this gap. Since the S-SS is inherently linked to the T-DS, which clearly reflects the system's operational trends, dynamic analysis of the model is essential.

According to Nikolov [10], the one-unit repairable system with two different failure modes and imperfect repairs can be described by:

$$\begin{aligned}
&\frac{d\pi_1(t)}{dt} = -\lambda_1\pi_1(t) + \int_0^\infty \pi_{1,2}(t,x)h_{1,2}(x)\mathrm{d}x + \int_0^\infty \pi_{2,rpl}(t,x)h_2(x)\mathrm{d}x,\\
&\frac{d\pi_2(t)}{dt} = -\lambda_2\pi_2(t) + \int_0^\infty \pi_{1,1}(t,x)h_{1,1}(x)\mathrm{d}x,\\
&\frac{\partial\pi_{1,1}(t,x)}{\partial t} + \frac{\partial\pi_{1,1}(t,x)}{\partial x} = -h_{1,1}(x)\pi_{1,1}(t,x),\\
&\frac{\partial\pi_{1,2}(t,x)}{\partial t} + \frac{\partial\pi_{1,2}(t,x)}{\partial x} = -h_{1,2}(x)\pi_{1,2}(t,x),\\
&\frac{\partial\pi_{2,rpl}(t,x)}{\partial t} + \frac{\partial\pi_{2,rpl}(t,x)}{\partial x} = -h_2(x)\pi_{2,rpl}(t,x),
\end{aligned} \tag{1.1}$$

with the boundary conditions:

$$\begin{aligned}
\pi_{1,1}(t,0) &= \lambda_1\alpha_1\pi_1(t),\\
\pi_{1,2}(t,0) &= \lambda_1\alpha_2\pi_1(t),\\
\pi_{2,rpl}(t,0) &= \lambda_2\pi_2(t),
\end{aligned} \tag{1.2}$$

and the initial conditions:

$$\pi_1(0) = 1, \pi_2(0) = 0, \pi_{1,1}(0,x) = 0, \pi_{1,2}(0,x) = 0, \pi_{2,rpl}(0,x) = 0, \tag{1.3}$$

where $(x,t) \in [0,\infty) \times [0,\infty)$;

$\pi_1(t) = p\,\{$at time $t,$ the system is operating as "new" $\}$;
$\pi_{1,m}(x,t) = p\,\{$at time $t,$ the system is under repair after the first failure of type m $(m = 1,2)$, with x representing the elapsed repair time$\}$;
$\pi_2(t) = p\,\{$at time $t,$ the system is operating after the first failure as "BTOWTN" $\}$
$\pi_{2,rpl}(x,t) =$
$p\,\{$at time $t,$ the system is under repair after the second failure, and the elapsed repair time is $x\}$;
$h_{1,m}(x)$ represents the repair rate of a component of Type m and satisfies $h_{1,m}(x) \geqslant 0$, $\int_0^\infty h_{1,m}(x)\mathrm{d}x = \infty$, $m = 1,2$.
$h_2(x)$ represents the replacement rate after the second failure, and $h_2(x) \geqslant 0$, $\int_0^\infty h_2(x)\mathrm{d}x = \infty$.

The system is initiated at time $t = 0$, with two potential failure modes occurring with probabilities α_1 and α_2, respectively, where $\alpha_1 + \alpha_2 = 1$. The time to the first failure follows an exponential distribution.

The parameter is denoted by λ_1. Upon failure, the system is immediately repaired: if failure mode $m = 1$ occurs, an imperfect repair is performed, after which the system resumes operation with an increased failure rate $\lambda_2 > \lambda_1$, remaining susceptible to both failure modes; if the fatal failure $m = 2$ occurs, the system is replaced with a new one. The system operates for a maximum of two cycles: replacement is enforced either after the second failure (regardless of its mode) or immediately if $m = 2$ occurs in the first cycle.

In this paper, we conduct a dynamic analysis of the above system, employing concepts derived from Gupur [12] and Kasim and Yumaier [13]. The structure of the remainder of this work is as follows: In Section 2, we prove the system's well-posedness by demonstrating that the underlying operator generates a contraction C_0-semigroup, thereby ensuring the existence of a unique positive T-DS. The asymptotic behavior and asymptotic expressions of the T-DS for the system are examined in Sections 3 and 4, respectively. Lastly, Section 5 presents numerical illustrations that explain how various factors affect the system.

2. Well-posedness of the system

In this section, we prove the well-posedness of the system. To do this, we need to transform the system given by Eqs (1.1)–(1.3) into an abstract Cauchy problem (ACP).

Let

$$\mathbb{X} = \left\{ \mathbf{\Pi} \in \mathbb{R}^2 \times (L^1[0,\infty))^3 \;\middle|\; \|\Pi\| = |\pi_1| + |\pi_2| + \|\pi_{1,1}\|_{L^1[0,\infty)} + \|\pi_{1,2}\|_{L^1[0,\infty)} + \|\pi_{2,rpl}\|_{L^1[0,\infty)} < \infty \right\},$$

as a state space. Obviously, $\mathbb{X}$ is a Banach space. Now we define operators and their domain as follows:

$$\mathbb{A}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_{1,1}(x) \\ \pi_{1,2}(x) \\ \pi_{2,rpl}(x) \end{pmatrix} = \begin{pmatrix} -\lambda_1\pi_1 \\ -\lambda_2\pi_2 \\ -\pi'_{1,1}(x) - h_{1,1}(x) \\ -\pi'_{1,2}(x) - h_{1,2}(x) \\ -\pi'_{2,rpl}(x) - h_2(x) \end{pmatrix},$$

$$D(\mathbb{A}) = \left\{ \mathbf{\Pi} \in \mathbb{X} \;\middle|\; \begin{array}{l} \frac{d\pi_{1,1}(x)}{dx} \in L^1[0,\infty),\ \frac{d\pi_{1,2}(x)}{dx} \in L^1[0,\infty),\ \frac{d\pi_{2,rpl}(x)}{dx} \in L^1[0,\infty), \\ \pi_{1,1}(x)\ ,\pi_{1,2}(x)\ ,\pi_{2,rpl}(x) \text{ are absolutely continuous} \\ \text{and } \mathbf{\Pi}(0) = \int_0^\infty \Upsilon\mathbf{\Pi}(x)\,dx \end{array} \right\},$$

where

$$\Upsilon = \begin{pmatrix} e^{-x} & 0 & 0 & 0 & 0 \\ 0 & e^{-x} & 0 & 0 & 0 \\ \lambda_1\alpha_1 e^{-x} & 0 & 0 & 0 & 0 \\ \lambda_1\alpha_2 e^{-x} & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 e^{-x} & 0 & 0 & 0 \end{pmatrix}.$$

$$\mathbb{E}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_{1,1}(x) \\ \pi_{1,2}(x) \\ \pi_{2,rpl}(x) \end{pmatrix} = \begin{pmatrix} \int_0^\infty \pi_{1,2}(x)h_{1,2}(x)dx + \int_0^\infty \pi_{2,rpl}(x)h_2(x)dx \\ \int_0^\infty \pi_{1,1}(x)h_{1,1}(x)dx \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad D(\mathbb{E}) = \mathbb{X}.$$

The system (1.1)–(1.3) can thus be represented as the ACP on $\mathbb{X}$ shown below.

$$\begin{cases} \frac{\mathrm{d}\mathbf{\Pi}(t)}{\mathrm{d}t} = (\mathbb{A} + \mathbb{E})\mathbf{\Pi}(t), & t \in (0, \infty), \\ \mathbf{\Pi}(0) = (1, 0, 0, 0, 0)^T. \end{cases} \tag{2.1}$$

Now, we present the following result.

Theorem 2.1. If $\mathbb{M} = \sup_{x\in[0,\infty)} \{h_{1,1}(x), h_{1,2}(x), h_2(x)\} < \infty$, then $\mathbb{A} + \mathbb{E}$ generates a positive contraction C_0-semigroup $\mathbb{T}(t)$.

See the Appendix for a detailed proof. It is straightforward to confirm that $\mathbb{X}^*$, the dual space of $\mathbb{X}$, is as follows:

$$\mathbb{X}^* = \left\{ \mathbf{\Pi}^* \left| \begin{array}{l} \mathbf{\Pi}^*(\boldsymbol{x}) = (\pi_1^*, \pi_2^*, \pi_{1,1}^*(x), \pi_{1,2}^*(x), \pi_{2,rpl}^*(x))^T \\ \|\mathbf{\Pi}^*\| = \sup\{|\pi_1^*|, |\pi_2^*|, \|\pi_{1,1}^*\|_{L^\infty[0,\infty)}, \|\pi_{1,2}^*\|_{L^\infty[0,\infty)}, \|\pi_{2,rpl}^*\|_{L^\infty[0,\infty)}\} < \infty \end{array} \right. \right\}.$$

Obviously $\mathbb{X}^*$ is a Banach space. We define the subset in $\mathbb{X}$ as

$$Y = \left\{ \mathbf{\Pi} \in \mathbb{X} \left| \begin{array}{l} \mathbf{\Pi}(\boldsymbol{x}) = (\pi_1, \pi_2, \pi_{1,1}(x), \pi_{1,2}(x), \pi_{2,rpl}(x)) \\ \pi_1 \geqslant 0, \pi_2 \geqslant 0, \pi_{1,1}(x) \geqslant 0, \pi_{1,2}(x) \geqslant 0, \pi_{2,rpl}(x) \geqslant 0, \forall x \in [0, \infty) \end{array} \right. \right\}.$$

Then $T(t)Y \subset Y$ is guaranteed by Theorem 2.1. For $\mathbf{\Pi} \in D(\mathbb{A}) \cap Y$, we choose $\Xi^*(x) = \|\mathbf{\Pi}\|(1, 1, 1, 1, 1)^T$, then $q^* \in \mathbb{X}^*$ and

$$\begin{aligned} \langle(\mathbb{A} + \mathbb{E})\mathbf{\Pi}, \Xi^*\rangle = &\left\{-\lambda_1\pi_1 + \int_0^\infty \pi_{1,2}(x)h_{1,2}(x)\mathrm{d}x + \int_0^\infty \pi_{2,rpl}(x)h_2(x)\mathrm{d}x\right\}\|\mathbf{\Pi}\| \\ &+ \left\{-\lambda_2\pi_2 + \int_0^\infty \pi_{1,1}(x)h_{1,1}(x)\mathrm{d}x\right\}\|\mathbf{\Pi}\| \\ &+ \int_0^\infty \left\{-\pi_{1,1}(x)h_{1,1}(x) - \frac{\mathrm{d}\pi_{1,1}(x)}{\mathrm{d}x}\right\}\|\mathbf{\Pi}\|\mathrm{d}x \\ &+ \int_0^\infty \left\{-\pi_{1,2}(x)h_{1,2}(x) - \frac{\mathrm{d}\pi_{1,2}(x)}{\mathrm{d}x}\right\}\|\mathbf{\Pi}\|\mathrm{d}x \\ &+ \int_0^\infty \left\{-\pi_{2,rpl}(x)h_2(x) - \frac{\mathrm{d}\pi_{2,rpl}(x)}{\mathrm{d}x}\right\}\|\mathbf{\Pi}\|\mathrm{d}x \\ = &- \lambda_1\pi_1\|\mathbf{\Pi}\| - \lambda_2\pi_2\|\mathbf{\Pi}\| + \lambda_1\pi_1\alpha_1\|\mathbf{\Pi}\| + \lambda_1\pi_1\alpha_2\|\mathbf{\Pi}\| + \lambda_2\pi_2\|\mathbf{\Pi}\| \\ = &0. \end{aligned}$$

This shows that the $\mathbb{A} + \mathbb{E}$ is conservative with respect to set

$$\Theta(\mathbf{\Pi}) = \left\{\Xi^* \in X^* \left| \langle\mathbf{\Pi}, \Xi^*\rangle = \|\mathbf{\Pi}\|^2 = \||\Xi^*\||^2 \right.\right\}.$$

Since $\mathbf{\Pi}(0) \in D(\mathbb{A}^2) \cap Y$, applying the Fattorini theorem [14] yields.

Theorem 2.2. $\|\mathbb{T}(t)\mathbf{\Pi}(0)\| = \|\mathbf{\Pi}(0)\|, \forall t \in [0, \infty)$, that is, $\mathbb{T}(t)$ is isometric for $\mathbf{\Pi}(0)$ in Eq (2.1).

The desired result of this section is obtained from Theorems 2.1 and 2.2.

Theorem 2.3. If $M = \sup_{x\in[0,\infty)} \{h_{1.1}(x), h_{1.2}(x), h_2(x)\} < \infty$, then system (2.1) has a unique positive T-DS $\mathbf{\Pi}(x, t)$ satisfying $\|\mathbf{\Pi}(\cdot, t)\| = 1, \forall t \in [0, \infty)$.

Proof. According to Theorem 2.1 and [11, Theorem 1.81], system (2.1) has a unique positive T-DS $\mathbf{\Pi}(x,t)$ that can be represented as $\mathbf{\Pi}(x,t) = \mathbb{T}(t)\mathbf{\Pi}(0), t \in [0,\infty)$. Combining this with Theorem 2.2, we obtain

$$||\mathbf{\Pi}(\cdot,t)|| = ||\mathbb{T}(t)\mathbf{\Pi}(0)|| = ||\mathbf{\Pi}(0)|| = 1, \forall t \in [0,\infty).$$

3. Asymptotic behavior of the T-DS of Eq (2.1)

The Appendix's proof of Theorem 3.1 shows that the $\mathbb{A}$ results in the positive contraction C_0−semigroup $\mathbb{S}(t)$. We first demonstrate that S(t) is a quasi-compact operator (QCO), and then use $\mathbb{E}$'s compactness to prove that $\mathbb{T}(t)$ is also a QCO. Next, we show that 0 is an eigenvalue of $\mathbb{A}+\mathbb{E}$ and $(\mathbb{A}+\mathbb{E})^*$ with geometric multiplicity 1. Using [11, Theorem 1.90], we derive that $\mathbb{T}(t)$ converges exponentially to a projection operator ***Pr***, which we then state explicitly. Finally, we find that the T-DS of system (2.1) converges exponentially to its S-SS.

Lemma 3.1. If $\mathbf{\Pi}(x,t) = (\boldsymbol{S}\phi)(x)$ is a solution to the system

$$\begin{cases} \frac{\mathrm{d}\mathbf{\Pi}(t)}{\mathrm{dt}} = \mathbb{A}\mathbf{\Pi}(t), \quad t \in (0,\infty), \\ \mathbf{\Pi}(0) = \phi \in D(\mathbb{A}). \end{cases} \tag{3.1}$$

Then, it follows that

$$\mathbf{\Pi}(x,t) = (\mathbb{S}(t)\phi)(x) = \begin{cases} \begin{pmatrix} \phi_1 \mathrm{e}^{-\lambda_1 t} \\ \phi_2 \mathrm{e}^{-\lambda_2 t} \\ \pi_{1,1}(0,t-x)\mathrm{e}^{-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau} \\ \pi_{1,2}(0,t-x)\mathrm{e}^{-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau} \\ \pi_{2,rpl}(0,t-x)\mathrm{e}^{-\int_0^x h_{2}(\tau)\mathrm{d}\tau} \end{pmatrix}, when \quad x < t, \\ \begin{pmatrix} \phi_1 \mathrm{e}^{-\lambda_1 t} \\ \phi_2 \mathrm{e}^{-\lambda_2 t} \\ \phi_3(x-t)\mathrm{e}^{-\int_{x-t}^x h_{1,1}(\tau)\mathrm{d}\tau} \\ \phi_4(x-t)\mathrm{e}^{-\int_{x-t}^x h_{1,2}(\tau)\mathrm{d}\tau} \\ \phi_5(x-t)\mathrm{e}^{-\int_{x-t}^x h_{2}(\tau)\mathrm{d}\tau} \end{pmatrix}, when \quad x \geqslant t, \end{cases}$$

where $\pi_{1,1}(0,t-x), \pi_{1,2}(0,t-x), \pi_{2,rpl}(0,t-x)$ is given by Eq (1.2).

Proof. Given that $\mathbf{\Pi}(x,t)$=$(\boldsymbol{S}\phi)(x)$ is a solution to the system (3.1), $\mathbf{\Pi}(x,t)$ satisfies

$$\frac{\mathrm{d}\pi_1(t)}{dt} = -\lambda_1\pi_1(t), \tag{3.2a}$$

$$\frac{\mathrm{d}\pi_2(t)}{dt} = -\lambda_2\pi_2(t), \tag{3.2b}$$

$$\frac{\partial\pi_{1,1}(x,t)}{\partial t} + \frac{\partial\pi_{1,1}(x,t)}{\partial x} = -h_{1,1}(x)\pi_{1,1}(x,t), \tag{3.2c}$$

$$\frac{\partial\pi_{1,2}(x,t)}{\partial t} + \frac{\partial\pi_{1,2}(x,t)}{\partial x} = -h_{1,2}(x)\pi_{1,2}(x,t), \tag{3.2d}$$

$$\frac{\partial \pi_{2,rpl}(x,t)}{\partial t} + \frac{\partial \pi_{2,rpl}(x,t)}{\partial x} = -h_2(x)\pi_{2,rpl}(x,t), \tag{3.2e}$$
$$\pi_{1,1}(0,t) = \lambda_1\alpha_1\pi_1(t), \tag{3.2f}$$
$$\pi_{1,2}(0,t) = \lambda_1\alpha_2\pi_1(t), \tag{3.2g}$$
$$\pi_{2,rpl}(0,t) = \lambda_2\pi_2(t), \tag{3.2h}$$
$$\pi_1(0) = \phi_1, \pi_2(0) = \phi_2, \pi_{1,1}(0,x) = \phi_3(x), \pi_{1,2}(0,x) = \phi_4(x), \pi_{2,rpl}(0,x) = \phi_5(x). \tag{3.2i}$$

Take $\xi = x - t$ and $\Psi_1(t) = \pi_{1,1}(\xi + t, t), \Psi_2(t) = \pi_{1,2}(\xi + t, t), \Psi_3(t) = \pi_{2,rpl}(\xi + t, t)$; then Eqs (3.2c)–(3.2e) give

$$\frac{d\Psi_1(t)}{dt} = -h_{1,1}(\xi + t)\Psi_1(t), \tag{3.3a}$$
$$\frac{d\Psi_2(t)}{dt} = -h_{1,2}(\xi + t)\Psi_2(t), \tag{3.3b}$$
$$\frac{d\Psi_3(t)}{dt} = -h_2(\xi + t)\Psi_3(t). \tag{3.3c}$$

If $\xi < 0$ (i.e., $x < t$), then by integrating Eqs (3.3a)–(3.3c) from $-\xi$ to t separately, we have

$$\begin{aligned}\pi_{1,1}(x,t) &= \Psi_1(t) = \Psi_1(-\xi)e^{-\int_{-\xi}^{t} h_{1,1}(\xi+y)dy} \\ &\overset{\tau=\xi+y}{=\!=\!=} \pi_{1,1}(0, t - x)e^{-\int_0^x h_{1,1}(\tau)d\tau},\end{aligned} \tag{3.4a}$$
$$\begin{aligned}\pi_{1,2}(x,t) &= \Psi_2(t) = \Psi_2(-\xi)e^{-\int_{-\xi}^{t} h_{1,2}(\xi+y)dy} \\ &\overset{\tau=\xi+y}{=\!=\!=} \pi_{1,2}(0, t - x)e^{-\int_0^x h_{1,2}(\tau)d\tau},\end{aligned} \tag{3.4b}$$
$$\begin{aligned}\pi_{1,1}(x,t) &= \Psi_3(t) = \Psi_3(-\xi)e^{-\int_{-\xi}^{t} h_2(\xi+y)dy} \\ &\overset{\tau=\xi+y}{=\!=\!=} \pi_{2,rpl}(0, t - x)e^{-\int_0^x h_2(\tau)d\tau}.\end{aligned} \tag{3.4c}$$

Combining Eqs (3.2a) and (3.2b) with Eq (3.2i), we obtain

$$\pi_1(t) = \phi_1 e^{-\lambda_1 t}, \tag{3.5a}$$
$$\pi_2(t) = \phi_2 e^{-\lambda_2 t}. \tag{3.5b}$$

If $\xi \geqslant 0$ (i.e., $x \geqslant t$), then integrating Eqs (3.3a)–(3.3c) from 0 to t, we deduce

$$\begin{aligned}\pi_{1,1}(x,t) &= \Psi_1(t) = \Psi_1(0)e^{-\int_0^t h_{1,1}(\xi+\tau)d\tau} = \phi_3(x-t)e^{-\int_0^t h_{1,1}(\xi+\tau)d\tau} \\ &\overset{\sigma=\xi+\tau}{=\!=\!=} \phi_3(x-t)e^{-\int_{\xi}^{\xi+t} h_{1,1}(\sigma)d\sigma} = \phi_3(x-t)e^{-\int_{x-t}^{x} h_{1,1}(\tau)d\tau},\end{aligned} \tag{3.6a}$$
$$\begin{aligned}\pi_{1,2}(x,t) &= \Psi_2(t) = \Psi_2(0)e^{-\int_0^t h_{1,2}(\xi+\tau)d\tau} = \phi_4(x-t)e^{-\int_0^t h_{1,2}(\xi+\tau)d\tau} \\ &\overset{\sigma=\xi+\tau}{=\!=\!=} \phi_4(x-t)e^{-\int_{\xi}^{\xi+t} h_{1,2}(\sigma)d\sigma} = \phi_4(x-t)e^{-\int_{x-t}^{x} h_{1,2}(\tau)d\tau},\end{aligned} \tag{3.6b}$$
$$\begin{aligned}\pi_{2,rpl}(x,t) &= \Psi_3(t) = \Psi_3(0)e^{-\int_0^t h_2(\xi+\tau)d\tau} = \phi_5(x-t)e^{-\int_0^t h_2(\xi+\tau)d\tau} \\ &\overset{\sigma=\xi+\tau}{=\!=\!=} \phi_5(x-t)e^{-\int_{\xi}^{\xi+t} h_2(\sigma)d\sigma} = \phi_5(x-t)e^{-\int_{x-t}^{x} h_2(\tau)d\tau}.\end{aligned} \tag{3.6c}$$

Eqs (3.4)–(3.6) complete the proof.

In the subsequent analysis, we aim to demonstrate that the $\mathbb{S}(t)$ is QCO on $\mathbb{X}$. So, we define:

$$(\mathbb{V}(t)\mathbf{\Pi})(x) = \begin{cases} 0, & x \in [0, t), \\ (\mathbb{S}(t)\mathbf{\Pi})(x), & x \in [t, \infty). \end{cases}$$

$$(\mathbb{W}(t)\mathbf{\Pi})(x) = \begin{cases} (\mathbb{S}(t)\mathbf{\Pi})(x), & x \in [0, t), \\ 0, & x \in [t, \infty). \end{cases}$$

Clearly,

$$\mathbb{S}(t)\mathbf{\Pi} = \mathbb{V}(t)\mathbf{\Pi} + \mathbb{W}(t)\mathbf{\Pi}, \quad \forall\ \mathbf{\Pi} \in \mathbb{X}.$$

Referring to the work in [11, Corollary 1.37], we can readily derive the following result.

Lemma 3.2. A bounded subset F of $\mathbb{X}$ is said to be relatively compact if and only if it satisfies the following two conditions:

(1) $\lim\limits_{h\to\infty} \sum\limits_{n=3}^{5} \int_0^\infty |f_n(x + h) - f_n(x)|\mathrm{d}x = 0$, uniformly for $f = (f_1, f_2, f_3, f_4, f_5) \in F$;

(2) $\lim\limits_{h\to\infty} \sum\limits_{n=3}^{5} \int_h^\infty |f_n(x)|\mathrm{d}x = 0$, uniformly for $f = (f_1, f_2, f_3, f_4, f_5) \in F$.

Theorem 3.1. Assume that $h_j(x)$ $(j = 1.1, 1.2, 2)$ are Lipschitz continuous such that

$$0 < \underline{h_j} \leqslant h_j(x) \leqslant \overline{h_j} < \infty,$$

where $\underline{h_j}, \overline{h_j}$ are strictly positive constants. Then $\mathbb{W}(t)$ is a compact operator on $\mathbb{X}$.

Proof. To demonstrate the necessary result, Condition (1) in Lemma 3.2 is sufficient. Let $\mathbf{\Pi}(x, t) = (\boldsymbol{S}(t)\phi)(x), x \in [0, t)$ for bounded $\phi \in \mathbb{X}$. Then $\mathbf{\Pi}(x, t)$ is a generalized solution of Eq (3.1). By Lemma 3.1, we have, for $x \in [0, t)$, $h \in (0, t]$, $x + h \in [0, t)$,

$$\begin{aligned}
&\int_0^t |\pi_{1,1}(x + h, t) - \pi_{1,1}(x, t)|\mathrm{d}x + \int_0^t |\pi_{1,2}(x + h, t) - \pi_{1,2}(x, t)|\mathrm{d}x \\
&\quad + \int_0^t |\pi_{2,rpl}(x + h, t) - \pi_{2,rpl}(x, t)|\mathrm{d}x \\
&= \int_0^t |\pi_{1,1}(0, t - x - h)\mathrm{e}^{-\int_0^{x+h} h_{1,1}(\tau)\mathrm{d}\tau} - \pi_{1,1}(0, t - x - h)\mathrm{e}^{-\int_0^{x} h_{1,1}(\tau)\mathrm{d}\tau} \\
&\quad + \pi_{1,1}(0, t - x - h)\mathrm{e}^{-\int_0^{x} h_{1,1}(\tau)\mathrm{d}\tau} - \pi_{1,1}(0, t - x)\mathrm{e}^{-\int_0^{x} h_{1,1}(\tau)\mathrm{d}\tau}|\mathrm{d}x \\
&\quad + \int_0^t |\pi_{1,2}(0, t - x - h)\mathrm{e}^{-\int_0^{x+h} h_{1,2}(\tau)\mathrm{d}\tau} - \pi_{1,2}(0, t - x - h)\mathrm{e}^{-\int_0^{x} h_{1,2}(\tau)\mathrm{d}\tau} \\
&\quad + \pi_{1,2}(0, t - x - h)\mathrm{e}^{-\int_0^{x} h_{1,2}(\tau)\mathrm{d}\tau} - \pi_{1,2}(0, t - x)\mathrm{e}^{-\int_0^{x} h_{1,2}(\tau)\mathrm{d}\tau}|\mathrm{d}x \\
&\quad + \int_0^t |\pi_{2,rpl}(0, t - x - h)\mathrm{e}^{-\int_0^{x+h} h_{2}(\tau)\mathrm{d}\tau} - \pi_{2,rpl}(0, t - x - h)\mathrm{e}^{-\int_0^{x} h_{2}(\tau)\mathrm{d}\tau} \\
&\quad + \pi_{2,rpl}(0, t - x - h)\mathrm{e}^{-\int_0^{x} h_{2}(\tau)\mathrm{d}\tau} - \pi_{2,rpl}(0, t - x)\mathrm{e}^{-\int_0^{x} h_{2}(\tau)\mathrm{d}\tau}|\mathrm{d}x \\
&\leqslant \int_0^t |\pi_1(t - x - h)\lambda_1\alpha_1||\mathrm{e}^{-\int_0^{x+h} h_{1,1}(\tau)\mathrm{d}\tau} - \mathrm{e}^{-\int_0^{x} h_{1,1}(\tau)\mathrm{d}\tau}|\mathrm{d}x \\
&\quad + \int_0^t |\lambda_1\alpha_1\pi_1(t - x - h) - \lambda_1\alpha_1\pi_1(t - x)|\mathrm{e}^{-\int_0^{x} h_{1,1}(\tau)\mathrm{d}\tau}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t |\pi_1(t-x-h)\lambda_1\alpha_2||e^{-\int_0^{x+h} h_{1,2}(\tau)d\tau}-e^{-\int_0^x h_{1,2}(\tau)d\tau}|dx\\
&+\int_0^t |\pi_{1,2}(0,t-x-h)-\pi_{1,2}(0,t-x)|e^{-\int_0^x h_{1,2}(\tau)d\tau}dx\\
&+\int_0^t |\pi_2(t-x-h)\lambda_2||e^{-\int_0^{x+h} h_2(\tau)d\tau}-e^{-\int_0^x h_2(\tau)d\tau}|dx\\
&+\int_0^t |\lambda_2\pi_2(t-x-h)-\lambda_2\pi_2(t-x)|e^{-\int_0^x h_2(\tau)d\tau}dx\\
\leqslant&\int_0^t \lambda_1\alpha_1\|S(t-x-h)\phi(\cdot)\|_X|e^{-\int_0^{x+h} h_{1,1}\tau d\tau}-e^{-\int_0^x h_{1,1}\tau d\tau}|dx\\
&+\int_0^t \lambda_1\alpha_1|\phi_1 e^{-\lambda_1(t-x-h)}-\phi_1 e^{-\lambda_1(t-x)}|e^{-\int_0^x h_{1,1}(\tau)d\tau}dx\\
&+\int_0^t \lambda_1\alpha_2\|S(t-x-h)\phi(\cdot)\|_X|e^{-\int_0^{x+h} h_{1,2}(\tau)d\tau}-e^{-\int_0^x h_{1,2}(\tau)d\tau}|dx\\
&+\int_0^t \lambda_1\alpha_2|\phi_1 e^{-\lambda_1(t-x-h)}-\phi_1 e^{-\lambda_1(t-x)}|e^{-\int_0^x h_{1,2}(\tau)d\tau}dx\\
&+\int_0^t \lambda_2\|S(t-x-h)\phi(\cdot)\|_X|e^{-\int_0^{x+h} h_2(\tau)d\tau}-e^{-\int_0^x h_2(\tau)d\tau}|dx\\
&+\int_0^t \lambda_2|\phi_2 e^{-\lambda_2(t-x-h)}-\phi_2 e^{-\lambda_2(t-x)}|e^{-\int_0^x h_2(\tau)d\tau}dx\\
\leqslant&\lambda_1\alpha_1\|\phi\|\int_0^t |e^{-\int_0^{x+h} h_{1,1}(\tau)d\tau}-e^{-\int_0^x h_{1,1}(\tau)d\tau}|dx\\
&+\lambda_1\alpha_1\|\phi\|\int_0^t |e^{-\lambda_1(t-x-h)-e^{-\lambda_1(t-x)}}|dx\\
&+\lambda_1\alpha_2\|\phi\|\int_0^t |e^{-\int_0^{x+h} h_{1,2}(\tau)d\tau}-e^{-\int_0^x h_{1,2}(\tau)d\tau}|dx\\
&+\lambda_1\alpha_2\|\phi\|\int_0^t |e^{-\lambda_1(t-x-h)-e^{-\lambda_1(t-x)}}|dx\\
&+\lambda_2\|\phi\|\int_0^t |e^{-\int_0^{x+h} h_2(\tau)d\tau}-e^{-\int_0^x h_2(\tau)d\tau}|dx\\
&+\lambda_2\|\phi\|\int_0^t |e^{-\lambda_2(t-x-h)-e^{-\lambda_2(t-x)}}|dx\to 0,\quad \text{as } |h|\to 0,\ \text{uniformly for } \phi.
\end{aligned}
$$

If $h \in [0,t)$, $x+h \in [0,h)$, a similar argument leads to the same conclusion and completes the proof.

Theorem 3.2. By the same conditions in Theorem 3.1, we have

$$\|\mathbb{V}(t)\phi\|_X \leqslant e^{-\min\{\lambda_1,\lambda_2,\underline{h_{1,1}},\underline{h_{1,2}},\underline{h_2}\}t}\|\phi\|_{\mathbb{X}}, \forall\phi \in \quad \mathbb{X}.$$

Proof. From the definition of $\mathbb{V}(t)$, for any $\phi \in X$, we estimate

$$\|\mathbb{V}(t)\phi(\cdot)\| = |\phi_1 e^{-\lambda_1 t}| + |\phi_2 e^{-\lambda_2 t}| + \int_t^\infty |\phi_3(x-t)e^{-\int_{x-t}^x h_{1,1}(\tau)d\tau}|dx$$

$$
\begin{aligned}
&+ \int_t^{\infty} |\phi_4(x-t)e^{-\int_{x-t}^{x} h_{1,2}(\tau)d\tau}|dx \\
&+ \int_t^{\infty} |\phi_5(x-t)e^{-\int_{x-t}^{x} h_2(\tau)d\tau}|dx \\
\leqslant\ & |\phi_1|e^{-\lambda_1 t} + |\phi_2|e^{-\lambda_2 t} + \sup_{x\in[0,\infty)} |e^{-\int_{x-t}^{x} h_{1,1}(\tau)d\tau}| \int_t^{\infty} |\phi_3(x-t)|dx \\
&+ \sup_{x\in[0,\infty)} |e^{-\int_{x-t}^{x} h_{1,2}(\tau)d\tau}| \int_t^{\infty} |\phi_4(x-t)|dx \\
&+ \sup_{x\in[0,\infty)} |e^{-\int_{x-t}^{x} h_2(\tau)d\tau}| \int_t^{\infty} |\phi_5(x-t)|dx \\
\leqslant\ & |\phi_1|e^{-\lambda_1 t} + |\phi_2|e^{-\lambda_2 t} + e^{-\underline{h_{1,1}}t} \int_t^{\infty} |\phi_3(x-t)|dx \\
&+ e^{-\underline{h_{1,2}}t} \int_t^{\infty} |\phi_4(x-t)|dx + e^{-\underline{h_2}t} \int_t^{\infty} |\phi_5(x-t)|dx \\
\overset{y=x-t}{=\!=\!=}\ & |\phi_1|e^{-\lambda_1 t} + |\phi_2|e^{-\lambda_2 t} + e^{-\underline{h_{1,1}}t} \int_0^{\infty} |\phi_3(y)|dy \\
&+ e^{-\underline{h_{1,2}}t} \int_0^{\infty} |\phi_4(y)|dy + e^{-\underline{h_2}t} \int_0^{\infty} |\phi_5(y)|dy \\
\leqslant\ & e^{-\min\{\lambda_1,\lambda_2,\underline{h_{1,1}},\underline{h_{1,2}},\underline{h_2}\}t} \Big\{ |\phi_1| + |\phi_2| + \int_0^{\infty} |\phi_3(y)|dy \\
&+ \int_0^{\infty} |\phi_4(y)|dy + \int_0^{\infty} |\phi_5(y)|dy \Big\} \\
=\ & e^{-\min\{\lambda_1,\lambda_2,\underline{h_{1,1}},\underline{h_{1,2}},\underline{h_2}\}t} \Big\{ |\phi_1| + |\phi_2| + \|\phi_3\|_{L^1[0,\infty)} \\
&+ \|\phi_4\|_{L^1[0,\infty)} + \|\phi_5\|_{L^1[0,\infty)} \Big\} \\
=\ & e^{-\min\{\lambda_1,\lambda_2,\underline{h_{1,1}},\underline{h_{1,2}},\underline{h_2}\}t} \|\phi\|_{\mathbb{X}}.
\end{aligned}
$$

From Theorems 3.1 and 3.2, we have

$$
\|\mathbb{S}(t) - \mathbb{W}(t)\| = \|\mathbb{V}(t)\| \leqslant e^{-\min\{\lambda_1,\lambda_2,\underline{h_{1,1}},\underline{h_{1,2}},\underline{h_2}\}t} \to 0, \quad t \to \infty.
$$

which, together with the definition of a QCO, yields the following result.

Theorem 3.3. $\mathbb{S}(t)$ is a QCO on $\mathbb{X}$ with the same conditions as in Theorem 3.1.

Because $\mathbb{E}$ is a compact operator on $\mathbb{X}$, we obtain the following result, according to Theorem 3.3 and Nagel [15, Prop. 2.9].

Corollary 3.1. $\mathbb{T}(t)$ is a QCO on $\mathbb{X}$. The spectral properties of $\mathbb{A} + \mathbb{E}$ are examined below.

Lemma 3.3. If $0 \leqslant \underline{h} \leqslant h_j(x) \leqslant \overline{h} < \infty$, then $\mathbb{A} + \mathbb{E}$ has at most finite eigenvalues in $\{\eta \in \mathbb{C} | -\underline{h_j} < \mathbb{R}\eta \leqslant 0\}$, each with a geometric multiplicity of one, and 0 is a strictly dominant eigenvalue.

Proof. Considering $(\mathbb{A} + \mathbb{E})\mathbf{\Pi} = \eta\mathbf{\Pi}$, i.e.,

$$
(\lambda_1 + \eta)\pi_1 = \int_0^{\infty} \pi_{1,2}(x)h_{1,2}(x)dx + \int_0^{\infty} \pi_{2,rpl}(x)h_2(x)dx, \tag{3.7a}
$$

$$(\lambda_2+\eta)\pi_2 = \int_0^\infty \pi_{1,1}(x)h_{1,1}(x)\mathrm{d}x, \tag{3.7b}$$

$$\frac{\mathrm{d}\pi_{1,1}(x)}{\mathrm{d}x} = -(\eta+h_{1,1}(x))\pi_{1,1}(x), \tag{3.7c}$$

$$\frac{\mathrm{d}\pi_{1,2}(x)}{\mathrm{d}x} = -(\eta+h_{1,2}(x))\pi_{1,2}(x), \tag{3.7d}$$

$$\frac{\mathrm{d}\pi_{2,rpl}(x)}{\mathrm{d}x} = -(\eta+h_2(x))\pi_{2,rpl}(x), \tag{3.7e}$$

$$\pi_{1,1}(0) = \pi_1\lambda_1\alpha_1, \tag{3.7f}$$

$$\pi_{1,2}(0) = \pi_1\lambda_1\alpha_2, \tag{3.7g}$$

$$\pi_{2,rpl}(0) = \pi_2\lambda_2. \tag{3.7h}$$

By calculating Eqs (3.7c)–(3.7e), and combine Eqs (3.7f)–(3.7h), we have

$$\pi_{1,1}(x) = \pi_1\lambda_1\alpha_1 \mathrm{e}^{-\eta x-\int_0^x h_{1,1}(x)\mathrm{d}x}, \tag{3.8a}$$

$$\pi_{1,2}(x) = \pi_1\lambda_1\alpha_2 \mathrm{e}^{-\eta x-\int_0^x h_{1,2}(x)\mathrm{d}x}, \tag{3.8b}$$

$$\pi_{2,rpl}(x) = \pi_2\lambda_2 \mathrm{e}^{-\eta x-\int_0^x h_2(x)\mathrm{d}x}. \tag{3.8c}$$

By inserting Eqs (3.8a)–(3.8c) into Eqs (3.7a) and (3.7b), we obtain

$$I(\eta)\pi_1 = I(\eta)\pi_2 = 0, \tag{3.9}$$

where

$$\begin{aligned} I(\eta) =& \lambda_1\lambda_2 + (\lambda_1+\lambda_2)\eta + \eta^2 - \lambda_1\alpha_2\eta\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\eta x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\mathrm{d}x \\ & - \lambda_1\lambda_2\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\eta x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\mathrm{d}x \\ & - \lambda_1\lambda_2\alpha_1\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\eta x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\mathrm{d}x \int_0^\infty h_2(x)\mathrm{e}^{-\eta x-\int_0^x h_2(\tau)\mathrm{d}\tau}\mathrm{d}x. \end{aligned}$$

If $\pi_1 = \pi_2 = 0$, then Eqs (3.8a)–(3.8c) imply $\pi_{1,1}(x) = \pi_{1,2}(x) = \pi_2(x) = 0$. That is, $\Pi(x) = (0,0,0,0,0)$. Hence, η is not an eigenvalue of $\mathbb{A}+\mathbb{E}$.

If $\pi_1 \neq 0, \pi_2 \neq 0$, then Eq (3.9) gives

$$I(\eta) = 0, \tag{3.10}$$

thus,

$$I(\eta) = 0 \Leftrightarrow \pi_1 \neq 0, \pi_2 \neq 0. \tag{3.11}$$

Using Eqs (3.8a)–(3.8c), we can estimate

$$\begin{aligned} \|\mathbf{\Pi}\| =& |\pi_1| + |\pi_2| + \|\pi_{1,1}\|_{L^1[0,\infty)} + \|\pi_{1,2}\|_{L^1[0,\infty)} + \|\pi_{2,rpl}\|_{L^1[0,\infty)} \\ =& |\pi_1| + |\pi_2| + |\pi_1|\lambda_1\alpha_1\int_0^\infty \mathrm{e}^{-\eta x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\mathrm{d}x \\ & + |\pi_1|\lambda_1\alpha_2\int_0^\infty \mathrm{e}^{-\eta x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\mathrm{d}x + |\pi_2|\lambda_2\int_0^\infty \mathrm{e}^{-\eta x-\int_0^x h_2(\tau)\mathrm{d}\tau}\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&=|\pi_1|\Big\{1+\lambda_1\alpha_1\int_0^\infty \mathrm{e}^{-\eta x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\mathrm{d}x+\lambda_1\alpha_2\int_0^\infty \mathrm{e}^{-\eta x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\mathrm{d}x\Big\}\\
&\quad+|\pi_2|\Big\{1+\lambda_2\int_0^\infty \mathrm{e}^{-\eta x-\int_0^x h_2(\tau)\tau}\mathrm{d}x\Big\}\\
&\leqslant|\pi_1|\Big\{1+\lambda_1\alpha_1\int_0^\infty \mathrm{e}^{-(\mathbb{R}\eta+\underline{h_{1,1}})x}\mathrm{d}x+\lambda_1\alpha_2\int_0^\infty \mathrm{e}^{-(\mathbb{R}\eta+\underline{h_{1,2}})x}\mathrm{d}x\Big\}\\
&\quad+|\pi_2|\Big\{1+\lambda_2\int_0^\infty \mathrm{e}^{-(\mathbb{R}\eta+\underline{h_2})x}\mathrm{d}x\Big\}\\
&=|\pi_1|\Big\{1+\frac{\lambda_1\alpha_1}{\mathbb{R}\eta+\underline{h_{1,1}}}+\frac{\lambda_1\alpha_2}{\mathbb{R}\eta+\underline{h_{1,2}}}\Big\}\\
&\quad+|\pi_2|\Big\{1+\frac{\lambda_2}{\mathbb{R}\eta+\underline{h_2}}\Big\}.
\end{aligned}
\tag{3.12}
$$

By Eqs (3.11) and (3.12), it is straightforward to deduce that all zeros of $I(\eta)$ in

$$
\Omega=\{\eta\in\mathbb{C}\;\big|\;-\min\underline{h_j}<\mathbb{R}\eta\leqslant 0\},
$$

are eigenvalues of $\mathbb{A}+\mathbb{E}$. Since $I(\eta)$ is analytic in Ω, by applying the zero-point theorem for analytic functions, we infer that $I(\eta)$ possesses at most countably isolated zeros within Ω.

In the subsequent analysis, we aim to confirm the aforementioned findings. Suppose $I(\eta)$ has infinitely many zeros within Ω, denoted as $\eta_l=\nu_l+\xi_l\in\Omega$, where $\kappa_l\in(-\underline{h_i},0]$ and $\xi_l\in\mathbb{R}$. By the Bolzano-Weierstrass theorem, we can assert that there exists a convergent subsequence among these zeros. Without loss of generality, let us consider the subsequence $\eta_k=\nu_k+\mathrm{i}\xi_k$ such that

$$
\lim_{k\to\infty}\nu_k=\nu\in(-\underline{h_i},0],\quad \lim_{k\to\infty}|\beta_k|=\infty,\quad I(\eta_k)=0\quad \forall k\geq 1.
$$

By substituting $\eta_k=\nu_k+\mathrm{i}\xi_k$ into Eq (3.10), we derive

$$
\begin{aligned}
&-\lambda_1\lambda_2-(\lambda_1+\lambda_2)\eta-\eta^2+\lambda_1\alpha_2\eta\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\eta x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\mathrm{d}x\\
&+\lambda_1\lambda_2\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\eta x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\mathrm{d}x\\
&+\lambda_1\lambda_2\alpha_1\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\eta x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\mathrm{d}x\int_0^\infty h_2(x)\mathrm{e}^{-\eta x-\int_0^x h_2(\tau)\mathrm{d}\tau}\mathrm{d}x=0\\
\Longrightarrow\\
&-\lambda_1\lambda_2-(\lambda_1+\lambda_2)\nu_k-\mathrm{i}(\lambda_1+\lambda_2)\xi_k-\nu_k^2-\xi_k^2-2\mathrm{i}\nu_k\xi_k\\
&+\lambda_1\lambda_2\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}[\cos(\xi_k x)-\mathrm{i}\sin(\xi_k x)]\mathrm{d}x\\
&+\lambda_1\alpha_2(\nu_k+\mathrm{i}\xi_k)\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^\infty h_{1,2}(\tau)\mathrm{d}\tau}[\cos(\xi_k x)-\mathrm{i}\sin(\xi_k x)]\mathrm{d}x\\
&+\lambda_1\lambda_2\alpha_1\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\nu_k x-\int_0^\infty h_{1,1}(\tau)\mathrm{d}\tau}[\cos(\xi_k x)-\mathrm{i}\sin(\xi_k x)]\mathrm{d}x\\
&\times\int_0^\infty h_2(x)\mathrm{e}^{-\nu_k x-\int_0^\infty h_2(\tau)\mathrm{d}\tau}[\cos(\xi_k x)-\mathrm{i}\sin(\xi_k x)]\mathrm{d}x=0
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow&\\
&-\lambda_1\lambda_2-(\lambda_1+\lambda_2)\nu_k-\nu_k^2+\xi_k^2\\
&+\lambda_1\lambda_2\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x\\
&+\lambda_1\alpha_2\nu_k\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x\\
&+\lambda_1\alpha_2\xi_k\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x\\
&+\lambda_1\lambda_2\alpha_1\Big[\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x\\
&\times\int_0^\infty h_2(x)\mathrm{e}^{-\nu_k x-\int_0^x h_2(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x\\
&-\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x\\
&\times\int_0^\infty h_2(x)\mathrm{e}^{-\nu_k x-\int_0^x h_2(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x\Big]=0.
\end{aligned}\tag{3.13}
$$

$$
\begin{aligned}
&-(\lambda_1+\lambda_2)\xi_k-2\nu_k\xi_k-\lambda_1\lambda_2\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x\\
&+\lambda_1\alpha_2\xi_k\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x\\
&-\lambda_1\alpha_2\nu_k\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,2}(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x\\
&-\lambda_1\lambda_2\alpha_1\Big[\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x\\
&\times\int_0^\infty h_2(x)\mathrm{e}^{-\nu_k x-\int_0^x h_2(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x\\
&+\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\nu_k x-\int_0^x h_{1,1}(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x\\
&\times\int_0^\infty h_2(x)\mathrm{e}^{-\nu_k x-\int_0^x h_2(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x=0.
\end{aligned}\tag{3.14}
$$

Given that $\eta_k\in\Omega$, and by applying the Riemann-Lebesgue theorem, we deduce

$$\lim_{k\to\infty}\int_0^\infty h_j(x)\mathrm{e}^{-\nu_k x-\int_0^x h_j(\tau)\mathrm{d}\tau}\cos(\xi_k x)\mathrm{d}x=0,\tag{3.15}$$

$$\lim_{k\to\infty}\int_0^\infty h_j(x)\mathrm{e}^{-\nu_k x-\int_0^x h_j(\tau)\mathrm{d}\tau}\sin(\xi_k x)\mathrm{d}x=0.\tag{3.16}$$

From Eqs (3.13), (3.15), (3.16) and taking the limit as $k\to\infty$ in Eq (3.13), we arrive at a contradiction where $\infty=0$. This contradiction implies that $I(\eta)$ can have at most a finite number of zeros in Ω. In other words, the operator $\mathbb{A}+\mathbb{E}$ has at most a finite number of eigenvalues in Ω. Furthermore, based on Eqs (3.8a)–(3.8c), the eigenvectors associated with η span a one-dimensional linear space. This indicates that the geometric multiplicity of η is one.

Remark 3.1. It can easily be seen that $I(0) = 0$. Consequently, 0 is an eigenvalue of the operator $\mathbb{A} + \mathbb{E}$ with a geometric multiplicity of one. Given that $\mathbb{A} + \mathbb{E}$ has a finite number of eigenvalues, and all non-zero eigenvalues have strictly negative real part in Ω. it follows that 0 is a strictly dominant eigenvalue of $\mathbb{A} + \mathbb{E}$.

Lemma 3.4. The adjoint operator $(\mathbb{A} + \mathbb{E})^*$ is given as

$$(\mathbb{A} + \mathbb{E})^* \mathbf{\Pi}^* = (\mathbb{G} + \mathbb{F})\mathbf{\Pi}^*, \quad \mathbf{\Pi}^* \in D((\mathbb{A} + \mathbb{E})^*) = D(\mathbb{G}),$$

where

$$\mathbb{G}\mathbf{\Pi}^* = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}x} - h_{1,1}(x) & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}x} - h_{1,2}(x) & 0 \\ 0 & 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}x} - h_2(x) \end{pmatrix} \begin{pmatrix} \pi_1^* \\ \pi_2^* \\ \pi_{1,1}^*(x) \\ \pi_{1,2}^*(x) \\ \pi_{2,rpl}^*(x) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & h_{1,1}(x) & 0 & 0 & 0 \\ h_{1,2}(x) & 0 & 0 & 0 & 0 \\ h_2(x) & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1^* \\ \pi_2^* \\ \pi_{1,1}^*(0) \\ \pi_{1,2}^*(0) \\ \pi_{2,rpl}^*(0) \end{pmatrix},$$

$$\mathbb{F}\mathbf{\Pi}^* = \begin{pmatrix} 0 & 0 & \lambda_1\alpha_1 & \lambda_1\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1^* \\ \pi_2^* \\ \pi_{1,1}^*(0) \\ \pi_{1,2}^*(0) \\ \pi_{2,rpl}^*(0) \end{pmatrix},$$

$$D(\mathbb{G}) = \left\{ \mathbf{\Pi}^* \in X \,\middle|\, \frac{\mathrm{d}\mathbf{\Pi}^*(x)}{\mathrm{d}x} \text{ exists and } \pi_{1,1}^*(\infty) = \pi_{1,2}^*(\infty) = \pi_{2,rpl}^*(\infty) = \varepsilon \right\}.$$

and ε in $D(\mathbb{G})$ is a constant that is independent of state.

Lemma 3.5. Suppose that for $i = 1.1,\ 1.2,\ 2$, the function $h_i(x)$ satisfies $0 \leqslant \underline{h_i} \leqslant h_i(x) \leqslant \overline{h_i} < \infty$. the adjoint operator $(\mathbb{A} + \mathbb{E})^*$ has at most a finite number of eigenvalues within the region $\{\eta \in \mathbb{C} | -\min \underline{h_i} < \mathbb{R}\eta \leqslant 0\}$. Furthermore, if the geometric multiplicity of each eigenvalue is one, then 0 is a strictly dominant eigenvalue.

Proof. Consider $(\mathbb{A} + \mathbb{E})^* \mathbf{\Pi}^* = \eta \mathbf{\Pi}^*$, i.e.,

$$\eta\pi_1^* = -\lambda_1\pi_1^* + \lambda_1\alpha_1\pi_{1.1}^*(0) + \lambda_1\alpha_2\pi_{1.2}^*(0), \tag{3.17a}$$

$$\eta\pi_2^* = -\lambda_2\pi_2^* + \lambda_2\pi_{2,rpl}^*(0), \tag{3.17b}$$

$$\frac{\mathrm{d}\pi_{1.1}^*(x)}{\mathrm{d}x} = (\eta + h_{1,1}(x))\pi_{1.1}^*(x) - h_{1,1}(x)\pi_2^*, \tag{3.17c}$$

$$\frac{\mathrm{d}\pi_{1.2}^*(x)}{\mathrm{d}x} = (\eta + h_{1,2}(x))\pi_{1.2}^*(x) - h_{1,2}(x)\pi_1^*, \tag{3.17d}$$

$$\frac{\mathrm{d}\pi_{2,rpl}^*(x)}{\mathrm{d}x} = (\eta + h_2(x))\pi_{2,rpl}^*(x) - h_2(x)\pi_1^*, \tag{3.17e}$$

$$\pi_j^*(\infty) = \varepsilon. \; (j = 1.1, 1.2, 2, rpl). \tag{3.17f}$$

By solving Eqs (3.17c)–(3.17e), we deduce

$$\pi_{1.1}^*(x) = \pi_{1.1}^*(0)e^{\eta x + \int_0^x h_{1,1}(\tau)d\tau} - e^{\eta x + \int_0^x h_{1,1}(\tau)d\tau} \int_0^x h_{1,1}(x)\pi_2^* e^{-\eta x - \int_0^x h_{1,1}(\tau)d\tau}dx, \tag{3.18a}$$

$$\pi_{1.2}^*(x) = \pi_{1.2}^*(0)e^{\eta x + \int_0^x h_{1,2}(\tau)d\tau} - e^{\eta x + \int_0^x h_{1,2}(\tau)d\tau} \int_0^x h_{1,2}(x)\pi_1^* e^{-\eta x - \int_0^x h_{1,2}(\tau)d\tau}dx, \tag{3.18b}$$

$$\pi_{2,rpl}^*(x) = \pi_{2,rpl}^*(0)e^{\eta x + \int_0^x h_2(\tau)d\tau} - e^{\eta x + \int_0^x h_2(\tau)d\tau} \int_0^x h_2(x)\pi_1^* e^{-\eta x - \int_0^x h_2(\tau)d\tau}dx. \tag{3.18c}$$

Multiplying both sides of $e^{-\eta x - \int_0^x h_i(\tau)d\tau}$, $i = \{(1.1), (1, 2), (2, rpl)\}$ by Eqs (3.18a)–(3.18c), taking the limit as x approaches infinity, and utilizing Eq (3.17f), we derive

$$\pi_{1.1}^*(0) = \pi_2^* \int_0^\infty h_{1,1}(x)e^{-\eta x - \int_0^x h_{1,1}(\tau)d\tau}dx, \tag{3.19a}$$

$$\pi_{1.2}^*(0) = \pi_1^* \int_0^\infty h_{1,2}(x)e^{-\eta x - \int_0^x h_{1,2}(\tau)d\tau}dx, \tag{3.19b}$$

$$\pi_{2,rpl}^*(0) = \pi_1^* \int_0^\infty h_2(x)e^{-\eta x - \int_0^x h_2(\tau)d\tau}dx. \tag{3.19c}$$

By substituting Eqs (3.19a)–(3.19c) into Eqs (3.17a) and (3.17b), we obtain

$$\begin{aligned}
&(\eta + \lambda_1 + \lambda_1\alpha_2 \int_0^\infty h_{1,2}(x)e^{-\eta x - \int_0^x h_{1,2}(\tau)d\tau}dx)\pi_1^* \\
&- \lambda_1\alpha_1 \int_0^\infty h_{1,1}(x)e^{-\eta x - \int_0^x h_2(\tau)d\tau}dx \times \pi_2^* = 0, \\
&\lambda_2 \int_0^\infty h_2(x)e^{-\eta x - \int_0^x h_2(\tau)d\tau}dx \times \pi_1^* - (\eta + \lambda_2)\pi_2^* = 0, \\
\Longrightarrow& \\
&\Big[\lambda_1\lambda_2 + (\lambda_1 + \lambda_2)\eta + \eta^2 - \lambda_1\alpha_2\eta \int_0^\infty h_{1,2}(x)e^{-\eta x - \int_0^x h_{1,2}(\tau)d\tau}dx \\
&- \lambda_1\lambda_2\alpha_2 \int_0^\infty h_{1,2}(x)e^{-\eta x - \int_0^x h_{1,2}(\tau)d\tau}dx - \lambda_1\lambda_2\alpha_1 \\
&\times \int_0^\infty h_{1,1}(x)e^{-\eta x - \int_0^x h_{1,1}(\tau)d\tau}dx \int_0^\infty h_2(x)e^{-\eta x - \int_0^x h_2(\tau)d\tau}dx\Big]\pi_i^* = 0, \quad i = 1, 2.
\end{aligned} \tag{3.20}$$

If $\pi_i^* = 0$ ($i = 1,\ 2$), Eqs (3.18a)–(3.18c) mean $\pi_j^*(x) = 0$ ($j = 1.1,\ 1.2,\ 2.\ rpl$). Evidently, $\mathbf{\Pi}^*(x) = (0, 0, 0, 0, 0)$, which shows that η is not an eigenvalue of $(\mathbb{A} + \mathbb{E})^*$.

If $\pi_i^* \neq 0$ ($i = 1,\ 2$), from Eq (3.20), we obtain

$$\begin{aligned}
B(\eta) =& \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)\eta + \eta^2 - \lambda_1\alpha_2\eta \int_0^\infty h_{1,2}(x)e^{-\eta x - \int_0^x h_{1,2}(\tau)d\tau}dx \\
&- \lambda_1\lambda_2\alpha_2 \int_0^\infty h_{1,2}(x)e^{-\eta x - \int_0^x h_{1,2}(\tau)d\tau}dx
\end{aligned}$$

$$- \lambda_1 \lambda_2 \alpha_1 \int_0^\infty h_{1,1}(x) e^{-\eta x - \int_0^x h_{1,1}(\tau) d\tau} dx \int_0^\infty h_2(x) e^{-\eta x - \int_0^x h_2(\tau) d\tau} dx = 0. \tag{3.21}$$

Hence, we can get $I(\eta) = B(\eta) = 0$. Therefore,

$$I(\eta) = 0 \Leftrightarrow \pi_i^* \neq 0, \ i = 1, 2. \tag{3.22}$$

According to Eqs (3.18a)–(3.18c), we have (assume $\mathbb{R}\eta + \underline{h_j}) > 0)$

$$\begin{aligned} \|\pi_{1.1}^*\|_{L^\infty[0,\infty)} &\leqslant |\pi_2^*| \sup_{x \in [0,\infty)} e^{\mathbb{R}\eta x + \int_0^x h_{1,1}(\tau) d\tau} \times \int_x^\infty h_{1,1}(\xi) e^{-\mathbb{R}\eta \xi - \int_0^\xi h_{1,1}(\tau) d\tau} d\xi \\ &\leqslant |\pi_2^*| \sup_{x \in [0,\infty)} \int_x^\infty \overline{h_{1,1}} e^{-\mathbb{R}\eta(\xi - x) - \underline{h_{1,1}}(\xi - x)} d\xi \\ &= |\pi_2^*| \frac{\overline{h_{1,1}}}{\mathbb{R}\eta + \underline{h_{1,1}}}. \end{aligned} \tag{3.23}$$

Similarly, we obtain

$$\|\pi_{1.2}^*\|_{L^\infty[0,\infty)} \leqslant |\pi_1^*| \frac{\overline{h_{1,2}}}{\mathbb{R}\eta + \underline{h_{1,2}}}. \tag{3.24}$$

$$\|\pi_{2,rpl}^*\|_{L^\infty[0,\infty)} \leqslant |\pi_1^*| \frac{\overline{h_2}}{\mathbb{R}\eta + \underline{h_2}}. \tag{3.25}$$

Combining $|\pi_i^*|$ $(i = 1, 2)$ with Eqs (3.23)–(3.25), we obtain

$$\begin{aligned} \|Q^*\| &= \sup \left\{ |\pi_1^*|, |\pi_2^*|, \cdots, \|\pi_{1.1}^*\|_{L^\infty[0,\infty)}, \|\pi_{1.2}^*\|_{L^\infty[0,\infty)}, \|\pi_{2,rpl}^*\|_{L^\infty[0,\infty)} \right\} \\ &= \sup \left\{ |\pi_1^*|, |\pi_2^*|, |\pi_2^*| \frac{\overline{h_{1,1}}}{\mathbb{R}\eta + \underline{h_{1,1}}}, |\pi_1^*| \frac{\overline{h_{1,2}}}{\mathbb{R}\eta + \underline{h_{1,2}}}, |\pi_1^*| \frac{\overline{h_2}}{\mathbb{R}\eta + \underline{h_2}} \right\}. \end{aligned} \tag{3.26}$$

Eqs (3.23)–(3.25) imply that all zeros of $B(\eta)$ in Ω are eigenvalues of $(\mathbb{A} + \mathbb{E})^*$. Since $B(\eta)$ is analytic in Ω, by the zero-point theorem for analytic functions, $B(\eta)$ has at most countably many isolated zeros in Ω. Given the similarity to Lemma 3.3, we can conclude that $B(\eta)$ has at most a finite number of zeros in Ω. In other words, $(\mathbb{A} + \mathbb{E})^*$ has at most a finite number of eigenvalues in Ω.

Remark 3.2. It is easy to verify that $B(0) = 0$ by Eq (3.21). Consequently, 0 is the eigenvalue of $(\mathbb{A} + \mathbb{E})^*$ with geometric multiplicity of one.

By combining Lemmas 3.3 and 3.5 with Theorem 2.3, we can determine that the algebraic multiplicity of 0 is 1 and $s(\mathbb{A} + \mathbb{E}) = 0$, i.e., the spectral bound of $\mathbb{A} + \mathbb{E}$ is zero. Thus, with the Corollary 4.1 in [11], we obtain the following:

Theorem 3.4. *If $h_i(x)$ $(i = 1.1, \ 1.2, \ 2)$ are Lipschitz continuous and satisfy*

$$0 < \underline{h_i} \leqslant h_i(x) \leqslant \overline{h_i} < \infty,$$

then, there exists a positive projection $\boldsymbol{Pr}$ *of rank one and appropriate constants* $\theta > 0$, $\mathcal{M} \geqslant 0$ *such that*

$$\|\mathbb{T}(t) - \boldsymbol{Pr}\| \leqslant \mathcal{M}\mathrm{e}^{-\theta t},$$

where $\boldsymbol{P_r} = \frac{1}{2\pi i}\int_{\bar{\Gamma}}(zI - \mathbb{A} - \mathbb{E})^{-1}\mathrm{d}z$ and $\bar{\Gamma}$ is a circle centered at 0 with sufficiently small radius.

The main results of this section will be presented in the following analysis, where we will discuss the growth bound of $\mathbb{T}(t)$ and determine the explicit form of $\boldsymbol{Pr}$.

Lemma 3.6. For $\gamma \in \rho(\mathbb{A} + \mathbb{E})$, yields

$$(\gamma I - \mathbb{A} - \mathbb{E})^{-1}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}, \quad \forall z \in \ X,$$

where

$$\begin{aligned} y_1 =&(\gamma + \lambda_2)\Big[\int_0^\infty h_{1.2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\int_0^x z_4(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x \\ &+ \int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_1\Big] \\ &+ \lambda_2\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\Big[\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi} \\ &\times\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_2\Big]\Big/F(\gamma), \end{aligned} \tag{3.27a}$$

$$\begin{aligned} y_2 =&\Big[\gamma + \lambda_1 - \lambda_1\alpha_2 h_{1.2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big] \\ &\times\Big[\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_2\Big] \\ &+ \lambda_1\alpha_1\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big[\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi} \\ &\times\int_0^x z_4(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + \int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi} \\ &\times\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_1\Big]\Big/F(\gamma), \end{aligned} \tag{3.27b}$$

$$\begin{aligned} y_3(x) =&\Big\{(\gamma + \lambda_2)\Big[\int_0^\infty h_{1.2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\int_0^x z_4(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x \\ &+ \int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_1\Big] \\ &+ \lambda_2\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\Big[\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi} \\ &\times\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_2\Big]\Big/F(\gamma)\Big\} \end{aligned}$$

$$\times\Big[\lambda_1\alpha_1 e^{-\gamma x-\int_0^x h_{1,1}(\xi)d\xi} + e^{-\gamma x-\int_0^x h_{1,1}(\xi)d\xi}\int_0^x z_3(\tau)e^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)d\xi}d\tau\Big], \tag{3.27c}$$

$$\begin{aligned} y_4(x) =&\Big\{(\gamma+\lambda_2)\Big[\int_0^\infty h_{1.2}(x)e^{-\gamma x-\int_0^x h_{1,2}(\xi)d\xi}\int_0^x z_4(\tau)e^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)d\xi}d\tau dx \\ &+\int_0^\infty h_2(x)e^{-\gamma x-\int_0^x h_2(\xi)d\xi}\int_0^x z_5(\tau)e^{\gamma\tau+\int_0^\tau h_2(\xi)d\xi}d\tau dx + z_1\Big] \\ &+\lambda_2\int_0^\infty h_2(x)e^{-\gamma x-\int_0^x h_2(\xi)d\xi}dx\Big[\int_0^\infty h_{1,1}(x)e^{-\gamma x-\int_0^x h_{1,1}(\xi)d\xi} \\ &\times\int_0^x z_3(\tau)e^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)d\xi}d\tau dx + z_2\Big]\Big/F(\gamma)\Big\} \\ &\times\Big[\lambda_1\alpha_2 e^{-\gamma x-\int_0^x h_{1,2}(\xi)d\xi} + e^{-\gamma x-\int_0^x h_{1,2}(\xi)d\xi}\int_0^x z_4(\tau)e^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)d\xi}d\tau\Big], \end{aligned} \tag{3.27d}$$

$$\begin{aligned} y_5(x) =&\Big\{(\gamma+\lambda_2)\Big[\int_0^\infty h_{1.2}(x)e^{-\gamma x-\int_0^x h_{1,2}(\xi)d\xi}\int_0^x z_4(\tau)e^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)d\xi}d\tau dx \\ &+\int_0^\infty h_2(x)e^{-\gamma x-\int_0^x h_2(\xi)d\xi}\int_0^x z_5(\tau)e^{\gamma\tau+\int_0^\tau h_2(\xi)d\xi}d\tau dx + z_1\Big] \\ &+\lambda_2\int_0^\infty h_2(x)e^{-\gamma x-\int_0^x h_2(\xi)d\xi}dx\Big[\int_0^\infty h_{1,1}(x)e^{-\gamma x-\int_0^x h_{1,1}(\xi)d\xi} \\ &\times\int_0^x z_3(\tau)e^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)d\xi}d\tau dx + z_2\Big]\Big/F(\gamma)\Big\} \\ &\times\lambda_2 e^{-\gamma x-\int_0^x h_2(\xi)d\xi} + e^{-\gamma x-\int_0^x h_2(\xi)d\xi}\int_0^x z_5(\tau)e^{\gamma\tau+\int_0^\tau h_2(\xi)d\xi}d\tau, \end{aligned} \tag{3.27e}$$

where

$$\begin{aligned} F(\gamma) =&(\gamma+\lambda_2)\Big[\gamma+\lambda_1-\lambda_1\alpha_2\int_0^\infty h_{1.2}(x)e^{-\gamma x-\int_0^x h_{1,2}(\xi)d\xi}dx\Big] \\ &-\alpha_1\lambda_1\lambda_2\int_0^\infty h_2(x)e^{-\gamma x-\int_0^x h_2(\xi)d\xi}dx\int_0^\infty h_{1,1}(x)e^{-\gamma x-\int_0^x h_{1,1}(\xi)d\xi}dx. \end{aligned}$$

Proof. Considering $(\gamma I-\mathbb{A}-\mathbb{E})y=z$, for $\forall z\in\ x$, i.e.,

$$(\gamma+\lambda_1)y_1 = \int_0^\infty h_{1,2}(x)y_4(x)dx + \int_0^\infty h_2(x)y_5(x)dx + z_1, \tag{3.28a}$$

$$(\gamma+\lambda_2)y_2 = \int_0^\infty h_{1,1}(x)y_3(x)dx + z_2, \tag{3.28b}$$

$$\frac{dy_3(x)}{dx} = -(\gamma+h_{1,1}(x))y_3(x) + z_3(x), \tag{3.28c}$$

$$\frac{dy_4(x)}{dx} = -(\gamma+h_{1,2}(x))y_4(x) + z_4(x), \tag{3.28d}$$

$$\frac{dy_5(x)}{dx} = -(\gamma+h_2(x))y_5(x) + z_5(x), \tag{3.28e}$$

$$y_3(0) = y_1\lambda_1\alpha_1, \tag{3.28f}$$

$$y_4(0) = y_1\lambda_1\alpha_2, \tag{3.28g}$$

$$y_5(0) = y_2\lambda_2. \tag{3.28h}$$

By solving Eqs (3.28c)–(3.28e), and using Eqs (3.28f)–(3.28h), we have

$$y_3(x) = y_1\lambda_1\alpha_1 \mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi} + \mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau, \tag{3.29a}$$

$$y_4(x) = y_1\lambda_1\alpha_2 \mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi} + \mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\int_0^x z_4(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau, \tag{3.29b}$$

$$y_5(x) = y_2\lambda_2 \mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi} + \mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau. \tag{3.29c}$$

Combining Eqs (3.29a)–(3.29c) with Eqs (3.28a) and (3.28b) yields

$$\begin{aligned}
(\gamma+\lambda_1)y_1 =& \int_0^\infty h_{1,2}(x)y_1\lambda_1\alpha_2\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x + \int_0^\infty h_{1,2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_4(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + \int_0^\infty h_2(x)y_2\lambda_2\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\\
&+\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_1\\
&\Big(\gamma+\lambda_1-\lambda_1\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big)y_1 - \Big(\lambda_2\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\Big)y_2\\
&=\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\gamma x-\int_0^x - \int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\int_0^\infty z_4(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x\\
&+\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_1,
\end{aligned} \tag{3.30a}$$

$$\begin{aligned}
(\gamma+\lambda_2)y_2 =& \int_0^\infty h_{1,1}(x)y_1\lambda_1\alpha_1\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x + \int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_2\\
&\Big(-\lambda_1\alpha_1\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big)y_1 + (\gamma+\lambda_2)y_2\\
&=\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x + z_2.
\end{aligned} \tag{3.30b}$$

Applying Cramer's rule to Eqs (3.30a) and (3.30b), we obtain Eqs (3.27a) and (3.27b), and substituting this into Eqs (3.29a)–(3.29c) separately, we obtain Eqs (3.27c)–(3.27e).

Theorem 3.5. *Under conditions of Theorem 3.4, the T-DS of system* (2.4) *converges exponentially to its S-SS, i.e.*,

$$\|\mathbf{\Pi}(\cdot.t) - \mathbf{\Pi}(\cdot)\| \leqslant \mathcal{M}\mathrm{e}^{-\theta t}, \quad t > 0.$$

Proof. Theorems 3.1 and 3.2 imply

$$\ln\|\mathbb{S}(t) - \mathbb{W}(t)\| = \ln\|\mathbb{V}(t)\| \leqslant -\min\{\lambda_1, \lambda_2, \underline{h}\}t$$

$$\Rightarrow$$
$$\frac{\ln\|\mathbb{S}(t)-\mathbb{W}(t)\|}{t}\leqslant-\min\{\lambda_1,\lambda_2,\underline{h}\}.$$

Combining this with Engel et al. [16, Prop. 2.10], we deduce that $w_{ess}(\mathbb{S}(t))$ (*i.e.*, $w_{ess}(\mathbb{A})$), the essential growth bound of $\mathbb{S}(t)$ (*i.e.*, $\mathbb{A}$), satisfies

$$w_{ess}(\mathbb{S}(t))\leqslant-\min\{\lambda_1,\lambda_2,\underline{h}\}.$$

Since $\mathbb{E}:\mathbb{R}^4\to\mathbb{R}^4$ is a compact operator, we derive by [16, Prop. 2.12] that

$$w_{ess}(\mathbb{A}+\mathbb{E})=w_{ess}(\mathbb{T}(t))=w_{ess}(\mathbb{S}(t))\leqslant-\min\{\lambda_1,\lambda_2,\underline{h}\}.$$

We get that 0 is a pole of $(\gamma I-\mathbb{A}-\mathbb{E})^{-1}$ of order 1, using the above result together with Engel et al. [16, Coro. 2.11] and Theorem 3.1. Therefore, by the residue theorem, we have

$$\begin{pmatrix}z_1\\z_2\\z_3\\z_4\\z_5\end{pmatrix}=\lim_{\gamma\to0}\gamma(\gamma I-\mathbb{A}-\mathbb{E})^{-1}\begin{pmatrix}z_1\\z_2\\z_3(x)\\z_4(x)\\z_5(x)\end{pmatrix}=\begin{pmatrix}\lim\limits_{\gamma\to0}\gamma y_1\\\lim\limits_{\gamma\to0}\gamma y_2\\\lim\limits_{\gamma\to0}\gamma y_3(x)\\\lim\limits_{\gamma\to0}\gamma y_4(x)\\\lim\limits_{\gamma\to0}\gamma y_5(x)\end{pmatrix}.$$

Determining the projection operator is now possible by calculating the above limit. Given that

$$\int_0^\infty h_i(x)\mathrm{e}^{-\int_0^x h_i(\xi)\mathrm{d}\xi}\mathrm{d}x=-\mathrm{e}^{-\int_0^x h_i(\xi)\mathrm{d}\xi}\,|_0^\infty=1,\quad(i=1.1,\ 1,2,\ 2),$$

and we obtain the following using L'Hospital's rule:

$$\begin{aligned}\lim_{\gamma\to0}\gamma\Big/F(\gamma)=&\lim_{\gamma\to0}1\Big/\Big[\gamma+\lambda_1-\lambda_1\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\\&+(\gamma+\lambda_2)\Big\{1+\lambda_1\alpha_2\int_0^\infty xh_{1,2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\}\\&+\alpha_1\lambda_1\lambda_2\int_0^\infty xh_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\\&+\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\int_0^\infty xh_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big]\\=&1\Big/\Big[\lambda_1-\lambda_1\alpha_2\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\\&+\lambda_2\Big\{1+\lambda_1\alpha_2\int_0^\infty xh_{1,2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\}\\&+\alpha_1\lambda_1\lambda_2\Big\{\int_0^\infty xh_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\\&+\int_0^\infty h_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\int_0^\infty xh_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\}\Big]\end{aligned}$$

$$
\begin{aligned}
=&1\Big/\Big[\lambda_1-\lambda_1\alpha_2+\lambda_2\Big\{1+\lambda_1\alpha_2\int_0^\infty xh_{1,2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\}\\
&+\alpha_1\lambda_1\lambda_2\Big\{\int_0^\infty xh_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x+\int_0^\infty xh_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\}\Big]\\
=&1\Big/\Big[\lambda_1\alpha_1+\lambda_2+\lambda_1\lambda_2\alpha_2\int_0^\infty xh_{1,2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\\
&+\alpha_1\lambda_1\lambda_2\Big\{\int_0^\infty xh_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x+\int_0^\infty xh_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\}\Big]\\
=&\frac{1}{H}.
\end{aligned}
$$

The Fubini theorem gives

$$
\begin{aligned}
&\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\int_0^x z_4(\eta)\mathrm{e}^{\int_0^\eta h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\eta\mathrm{d}x\\
=&\int_0^\infty z_4(\eta)\mathrm{e}^{\int_0^\eta h_{1,2}(\xi)\mathrm{d}\xi}\int_\eta^\infty h_{1,2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\mathrm{d}\eta\\
=&\int_0^\infty z_4(\eta)\mathrm{d}\eta=\int_0^\infty z_4(x)\mathrm{d}x.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\int_0^x z_3(\eta)\mathrm{e}^{\int_0^\eta h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\eta\mathrm{d}x=\int_0^\infty z_3(x)\mathrm{d}x,\\
&\int_0^\infty h_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\eta)\mathrm{e}^{\int_0^\eta h_2(\xi)\mathrm{d}\xi}\mathrm{d}\eta\mathrm{d}x=\int_0^\infty z_5(x)\mathrm{d}x.
\end{aligned}
$$

Using this and $z_1+z_2+\sum_{i=3}^{5}\int_0^\infty z_i(x)\mathrm{d}x=1$ we derive

$$
\begin{aligned}
\lim_{\gamma\to 0}\gamma y_1=&+\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+z_1\Big]\\
&+\lambda_2\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\Big[\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+z_2\Big]\Big\}\Big/H\\
=&\Big[\lambda_2\Big\{\int_0^\infty h_{1,2}(x)\mathrm{e}^{-int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\int_0^x z_4(\eta)\mathrm{e}^{\int_0^\eta h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\eta\mathrm{d}x\\
&+\int_0^\infty h_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\int_0^x z_5(\eta)\mathrm{e}^{\int_0^\eta h_2(\xi)\mathrm{d}\xi}\mathrm{d}\eta\mathrm{d}x+z_1\Big\}\\
&+\lambda_2\int_0^\infty h_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\{\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_3(\eta)\mathrm{e}^{\int_0^\eta h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\eta\mathrm{d}x+z_2\Big\}\Big]\Big/H
\end{aligned}
$$

$$
\begin{aligned}
&=\lambda_2\Big\{\int_0^\infty z_4(x)\mathrm{d}x+\int_0^\infty z_5(x)\mathrm{d}x+z_1\Big\}+\lambda_2\Big\{\int_0^\infty z_3(x)\mathrm{d}x+z_2\Big\}\Big/H\\
&=\lambda_2\Big\{z_1+z_2+\int_0^\infty z_3(x)\mathrm{d}x+\int_0^\infty z_4(x)\mathrm{d}x+\int_0^\infty z_5(x)\mathrm{d}x\Big\}\Big/H\\
&=\frac{\lambda_2}{H}=\pi_1, \qquad (3.31a)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\gamma\to 0}\gamma y_2 =&\times\Big\{\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\int_0^x z_3(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+z_2\Big\}\\
&+\lambda_1\alpha_1\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\{\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\gamma x-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_4(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+\int_0^\infty h_2(x)\mathrm{e}^{-\gamma x-\int_0^x h_2(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_5(\tau)\mathrm{e}^{\gamma\tau+\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+z_1\Big\}\Big/H\\
=&\Big\{\lambda_1-\lambda_1\alpha_2\int_0^\infty h_{1.2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\}\\
&\times\Big\{\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\int_0^x z_3(\tau)\mathrm{e}^{\int_0^\tau h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+z_2\Big\}\\
&+\lambda_1\alpha_1\int_0^\infty h_{1,1}(x)\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\mathrm{d}x\Big\{\int_0^\infty h_{1,2}(x)\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_4(\tau)\mathrm{e}^{\int_0^\tau h_{1,2}(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+\int_0^\infty h_2(x)\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\\
&\times\int_0^x z_5(\tau)\mathrm{e}^{\int_0^\tau h_2(\xi)\mathrm{d}\xi}\mathrm{d}\tau\mathrm{d}x+z_1\Big\}\Big/H\\
=&(\lambda_1-\lambda_1\alpha_2)\Big\{\int_0^\infty z_3(\tau)\mathrm{d}\tau+z_2\Big\}\\
&+\lambda_1\alpha_1\Big\{\int_0^\infty z_4(\tau)\mathrm{d}\tau+\int_0^\infty z_5(\tau)\mathrm{d}\tau+z_1\Big\}\Big/H\\
=&\lambda_1\alpha_1\Big\{z_1+z_2+\int_0^\infty z_3(x)\mathrm{d}x+\int_0^\infty z_4(x)\mathrm{d}x+\int_0^\infty z_5(x)\mathrm{d}x\Big\}\Big/H\\
=&\frac{\lambda_1\alpha_1}{H}=\pi_2, \qquad (3.31b)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\gamma\to 0}\gamma y_3(x) =&\lambda_1\alpha_1\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}\lim_{\gamma\to 0}\gamma y_1\\
=&\frac{\lambda_2}{H}\times\lambda_1\alpha_1\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}=\frac{\lambda_1\lambda_2\alpha_1\mathrm{e}^{-\int_0^x h_{1,1}(\xi)\mathrm{d}\xi}}{H}=\pi_{1,1}(x), \qquad (3.31c)\\
\lim_{\gamma\to 0}\gamma y_4(x) =&\lambda_1\alpha_2\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}\lim_{\gamma\to 0}\gamma y_1\\
=&\frac{\lambda_2}{H}\times\lambda_1\alpha_2\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}=\frac{\lambda_1\lambda_2\alpha_2\mathrm{e}^{-\int_0^x h_{1,2}(\xi)\mathrm{d}\xi}}{H}=\pi_{1,2}(x), \qquad (3.31d)\\
\lim_{\gamma\to 0}\gamma y_5(x) =&\lambda_2\mathrm{e}^{-\int_0^x h_2(\xi)\mathrm{d}\xi}\lim_{\gamma\to 0}\gamma y_2
\end{aligned}
$$

$$=\frac{\lambda_1\alpha_1}{H}\times\lambda_2 e^{-\int_0^x h_2(\xi)d\xi}=\frac{\lambda_1\lambda_2\alpha_1 e^{-\int_0^x h_2(\xi)d\xi}}{H}=\pi_{2,rpl}(x). \tag{3.31e}$$

Combining Eqs (3.31a)–(3.31e) with Theorem 3.5, we obtain

$$\boldsymbol{Pr}\boldsymbol{\Pi}(0)=\boldsymbol{\Pi}(x). \tag{3.32}$$

From Theorem 2.3, Eq (3.32), and Theorem 3.5, it follows that

$$\|\boldsymbol{\Pi}(\cdot,t)-\boldsymbol{\Pi}(\cdot)\|=\|T(t)\boldsymbol{\Pi}(0)-\boldsymbol{Pr}\boldsymbol{\Pi}(0)\|\leqslant\|T(t)-\boldsymbol{Pr}\|\|\boldsymbol{\Pi}(0)\|\leqslant\mathcal{M}e^{-\theta t}\|\boldsymbol{\Pi}(0)\|=\mathcal{M}e^{-\theta t},\quad t\geqslant 0.$$

4. Asymptotic expression of the T-DS of Eq (2.1)

The algebraic multiplicity of all eigenvalues of $\mathbb{A}+\mathbb{E}$ in Ω is 1, as can easily be demonstrated using the same method as in [12]. Without loss of generality, suppose

$$\eta_l\in\Xi=\{\eta\in\mathbb{C}\mid-\min\underline{h_i}<\Re\eta\leq 0\},\ l=0,1,\cdots,q,$$
$$-\min\underline{h_i}<\eta_q<\eta_{q-1}<\cdots<\eta_1<\eta_0=0,$$

are $q+1$ real eigenvalues of $\mathbb{A}+\mathbb{E}$. Thus, by combining Theorem 2.1 with [11, Thm.1.89] (see also [15]), we deduce

$$\boldsymbol{\Pi}(x,t)=\mathbb{T}(t)\boldsymbol{\Pi}(0)=\sum_{l=0}^{q}\mathbb{T}_l(t)\boldsymbol{\Pi}(0)+R_q(t)\boldsymbol{\Pi}(0), \tag{4.1a}$$
$$\mathbb{T}_l(t)\boldsymbol{\Pi}(0)=e^{\eta_l t}\boldsymbol{Pr}_l\boldsymbol{\Pi}(0),\ \ l=0,1,\cdots,q, \tag{4.1b}$$
$$\boldsymbol{Pr}_l\boldsymbol{\Pi}(0)=\frac{1}{2\pi i}\int_{\Gamma_l}(\eta I-\mathbb{A}-\mathbb{E})^{-1}\boldsymbol{\Pi}(0)d\eta,\ \ l=0,1,\cdots,s, \tag{4.1c}$$
$$\|R_q(t)\|\leq\mathcal{M}e^{-\theta t},\mathcal{M}>0,\theta>0, \tag{4.1d}$$

where η_l ($l=0,1,\cdots,q$) is an order-one pole of $(\eta I-\mathbb{A}-\mathbb{E})^{-1}$ due to its algebraic multiplicity of one. By the residual theorem, we have

$$\boldsymbol{Pr}_l\boldsymbol{\Pi}(0)=\lim_{\eta\to\eta_l}\eta(\eta I-\mathbb{A}-\mathbb{E})^{-1}\boldsymbol{\Pi}(0), \tag{4.2}$$

and

$$(\eta I-\mathbb{A}-\mathbb{E})^{-1}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{pmatrix}=\begin{pmatrix}\Pi_1\\\Pi_2\\\Pi_3\\\Pi_4\\\Pi_5\end{pmatrix},\quad\forall y\in X. \tag{4.3}$$

By Eqs (4.2) and (4.3), we can determine

$$\boldsymbol{Pr}_0\Pi(0)=\langle\Pi(0),\boldsymbol{\Pi}^*\rangle\widetilde{\Pi}(x),$$

here $\widetilde{\mathbf{\Pi}}(x)$ and $\mathbf{\Pi}^*$ satisfy $(\mathbb{A}+\mathbb{E})\widetilde{\mathbf{\Pi}}(x) = 0$, $(\mathbb{A}+\mathbb{E})^*\mathbf{\Pi}^* = 0, \langle\widetilde{\mathbf{\Pi}}, \mathbf{\Pi}^*\rangle = 1$. Finally, we deduce the following main results.

Theorem 4.1. If the condition of Theorem 3.4 holds, then the TDS of Eq (2.1) can be written as

$$\begin{aligned}&\mathbf{\Pi}(x,t) = \langle\mathbf{\Pi}(0), \mathbb{Q}^*\rangle\widetilde{\mathbf{\Pi}}(x) + \sum_{l=1}^{q} \mathrm{e}^{\eta_l t} \lim_{\eta\to\eta_l} \eta(\eta I - \mathbb{A} - \mathbb{E})^{-1}\mathbf{\Pi}(0) + R_q(t)\mathbf{\Pi}(0),\\ &\|R_q(t)\| \leqslant \mathcal{M}\mathrm{e}^{-\theta t}, \mathcal{M} > 0, \theta > 0,\end{aligned}$$

where $\eta_l(l = 1, \cdots, q)$ are isolated eigenvalues of $\mathbb{A}+\mathbb{E}$ in Ω.

5. Numerical results

In this section, we analyze the reliability indices of the system through specific examples and numerical analysis. The key indices under consideration include the instantaneous availability $A(t)$, failure frequency $m_f(t)$, renewal frequency $m_r(t)$, and reliability $R(t)$. These indices are defined as follows, based on [11],

$$\begin{aligned} A(t) =& \pi_1(t) + \pi_2(t),\\ m_f(t) =& \lambda_1\pi_1(t) + \lambda_2\pi_2(t),\\ m_r(t) =& \int_0^\infty \pi_{1,2}(x)h_{1,2}(x)\mathrm{d}x + \int_0^\infty \pi_{2,rpl}(x)h_2(x)\mathrm{d}x.\end{aligned}$$

According to Theorem 3.5, the system's reliability indices converge to steady-state values as time approaches infinity:

$$\lim_{t\to\infty} A(t) = A, \quad \lim_{t\to\infty} m_f(t) = M_f, \quad \lim_{t\to\infty} m_r(t) = M_r.$$

To validate the aforementioned results numerically, we assume that both repair and replacement times follow to an exponential distribution. Furthermore, we examine how variations in system parameters affect these reliability indices. The system parameters are initially set to the following values:

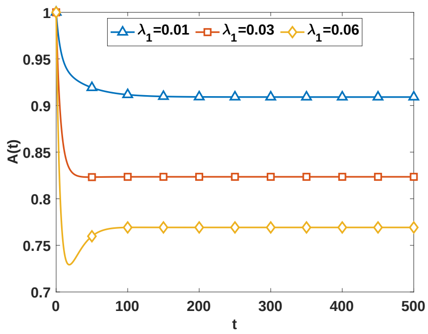
$$\lambda_1 = 0.01, \lambda_2 = 0.02, \alpha_1 = 0.5, \alpha_2 = 0.5, h_{1,1} = 0.1, h_{1,2} = 0.2, h_2 = 0.1.$$

By varying these parameters, we analyze their impact on the reliability indices and discuss the implications of these findings.

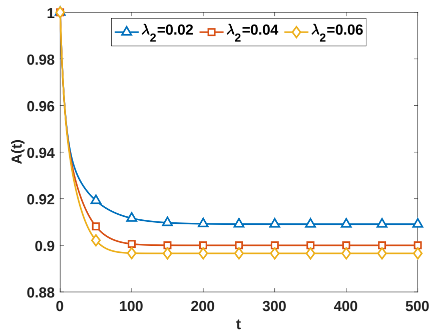
Figure 1 illustrates the impact of the parameters on $A(t)$ over time t. It is evident that $A(t)$ decreases rapidly with time and stabilizes at a constant value after extended operation.

Figure 2 describes the effect of on the instantaneous failure frequency $m_f(t)$ over time. Initially, $m_f(t)$ increases rapidly and later stabilizes to a constant value after a period of operation.

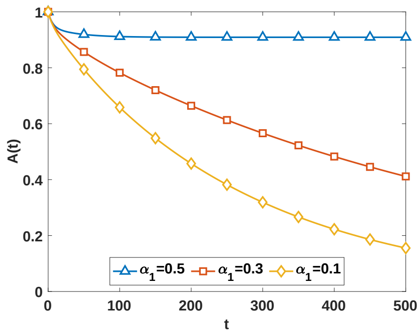
Figure 3 depicts the effect of the parameters on $m_r(t)$. Similarly, $m_r(t)$ exhibits a rapid decline and then converges to a fixed value after long-term operation.



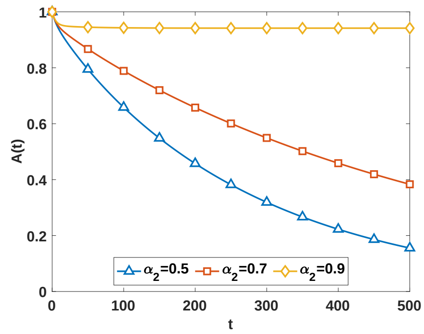
(a) $A(t)$ for different λ_1



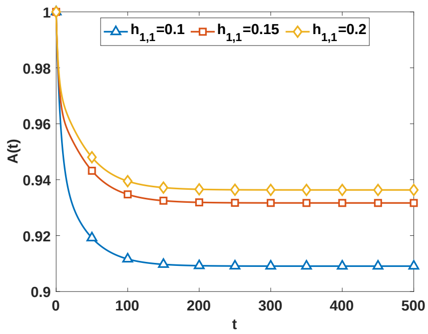
(b) $A(t)$ for λ_2



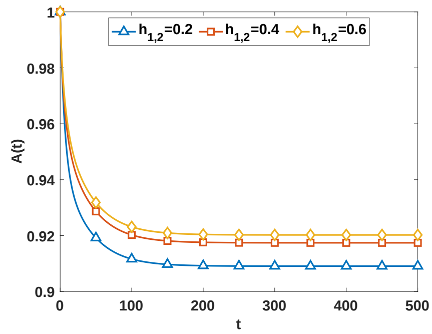
(c) $A(t)$ for α_1



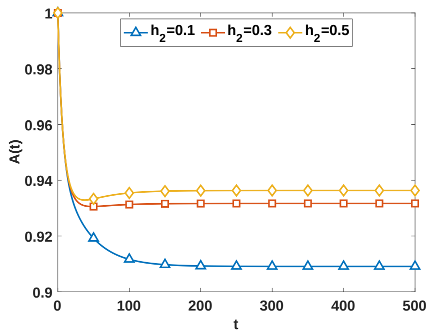
(d) $A(t)$ for α_2



(e) $A(t)$ for $h_{1,1}$

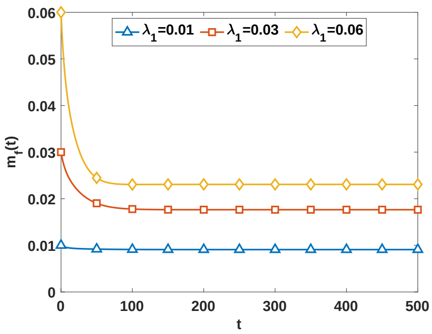


(f) $A(t)$ for $h_{1,2}$

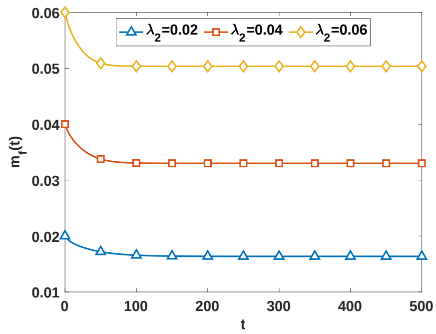


(g) $A(t)$ for h_2

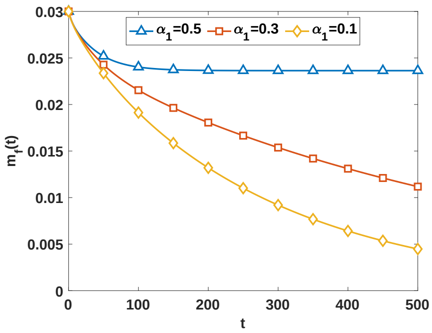
Figure 1. Effect of parameters on system availability.



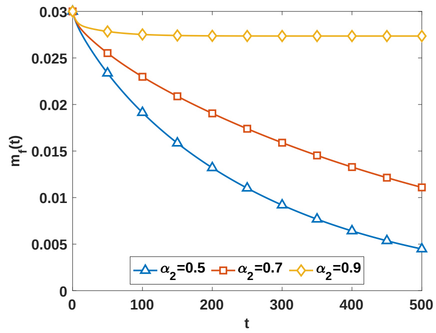
(a) $m_f(t)$ for λ_1



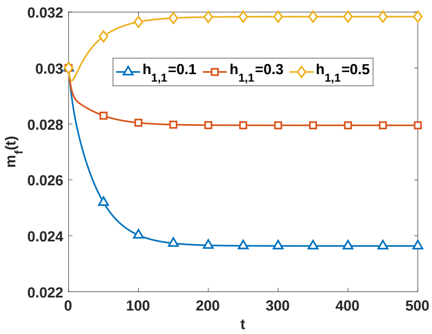
(b) $m_f(t)$ for λ_2



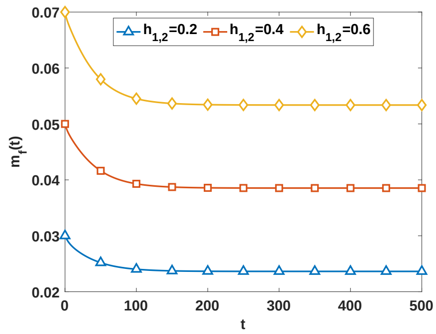
(c) $m_f(t)$ for α_1



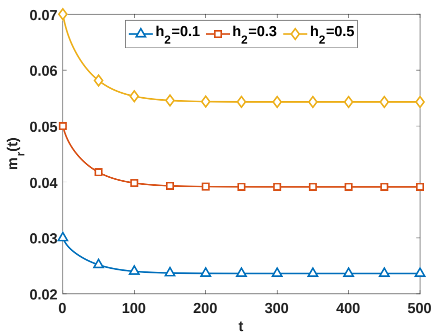
(d) $m_f(t)$ for α_2



(e) $m_f(t)$ for $h_{1,1}$

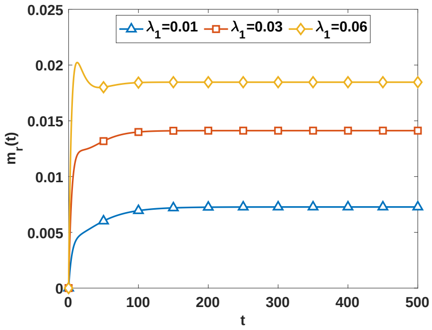


(f) $m_f(t)$ for $h_{1,2}$

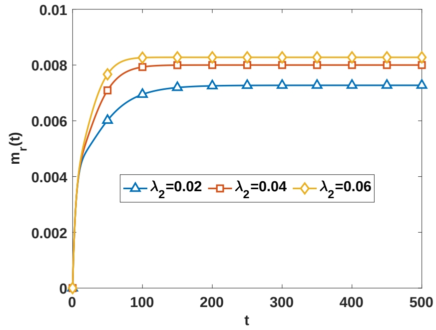


(g) $m_f(t)$ for h_2

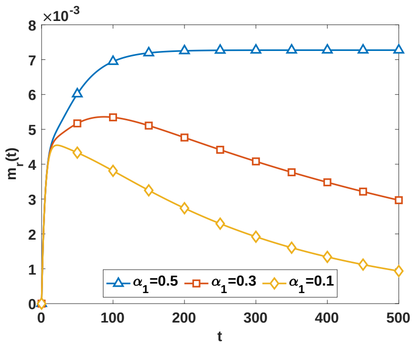
Figure 2. Effect of parameters on failure frequency.



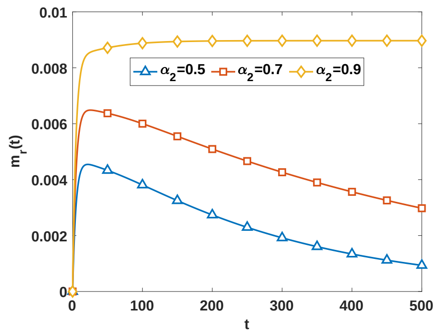
(a) $m_r(t)$ for λ_1



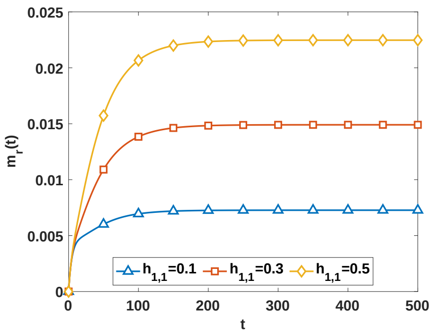
(b) $m_r(t)$ for λ_2



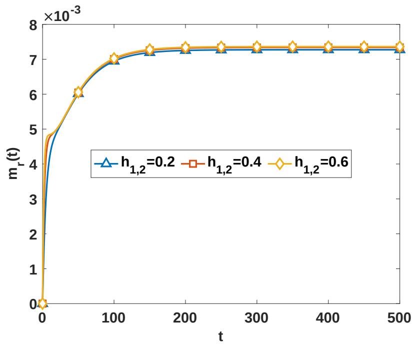
(c) $m_r(t)$ for α_1



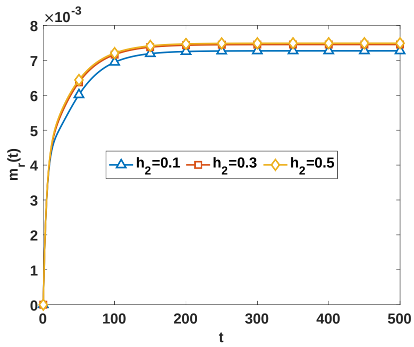
(d) $m_r(t)$ for α_2



(e) $m_r(t)$ for $h_{1,1}$



(f) $m_r(t)$ for $h_{1,2}$



(g) $m_r(t)$ for h_2

Figure 3. Effect of parameters on renewal frequency.

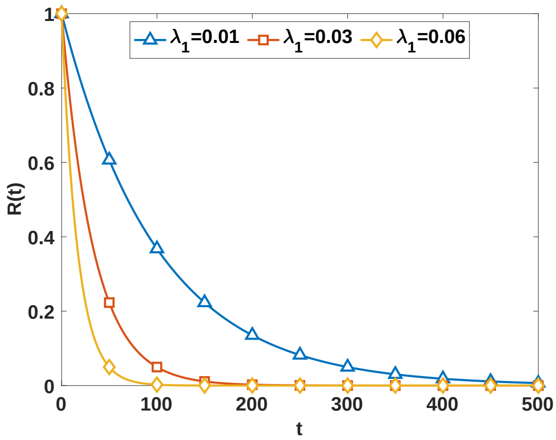


Figure 4. System reliability for λ_1.

To evaluate system reliability, we define failure states as absorbing states in the model. The reliability function is derived as follows:

$$R(t) = \mathrm{e}^{-\lambda_1 t}.$$

Figure 4 shows how the system reliability $R(t)$ changes over time for different values of λ_1. A higher λ_1 leads to a faster decline in $R(t)$, indicating accelerated system degradation. Conversely, a lower λ_1 results in slower reliability decay, reflecting improved system longevity. As expected, the reliability of the system tends towards zero as time goes to infinity.

In addition, the effect of different parameters $(\lambda_1, \lambda_2, \alpha_1, \alpha_2, h_{1,1}, h_{1,2}, h_2)$ on the transient reliability indices of the system is illustrated in these figures. Overall, changes in the parameters significantly alter their decay rate or steady-state values.

6. Conclusions

In this paper, we have studied a one-unit repairable system characterized by two distinct failure modes and subjected to imperfect repairs. We transformed the model into an ACP in Banach space and conducted a dynamic analysis using the operator semigroup theory of linear operators. Our analysis demonstrated that the system has unique nonnegative T-DS, which exponentially converges to its S-SS. Furthermore, we present the asymptotic expressions for the T-DS. Moreover, we analyzed the impact of each parameter on system reliability through numerical examples. These results are helpful for engineers to build systems that are more reliable, secure, and cost-effective.

Our future research will extend the scope to include systems with multiple failure modes and imperfect repairs. This will allow us to investigate increased failure mode complexity and imperfect repair effects on system reliability and maintenance costs.

Author contributions

Conceptualization, H. W. and E. K.; methodology, H. W. and E. K.; validation, H. W., B. N. and C. L.; writing-original draft preparation, H. W., B. N. and C. L.; writing-review and editing, H. W. and E. K. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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A. Appendix

Proof of Theorem 3.1. We start by estimating $\|(\gamma I - \mathbb{A})^{-1}\|$. Given any arbitrary $\Phi \in \mathbb{X}$, and considering the equation $(\gamma I - \mathbb{A})\mathbf{\Pi} = \mathbf{\Phi}$, i.e.,

$$(\gamma + \lambda_1)\pi = \Phi_1, \tag{A.1}$$
$$(\gamma + \lambda_2)\pi = \Phi_2, \tag{A.2}$$
$$\frac{d\pi_{1,1}(x)}{dx} = -(\gamma + h_{1,1}(x))\pi_{1,1}(x) + \Phi_3(x), \tag{A.3}$$
$$\frac{d\pi_{1,2}(x)}{dx} = -(\gamma + h_{1,2}(x))\pi_{1,2}(x) + \Phi_4(x), \tag{A.4}$$
$$\frac{d\pi_{2,rpl}(x)}{dx} = -(\gamma + h_2(x))\pi_{2,rpl}(x) + \Phi_5(x), \tag{A.5}$$
$$\pi_{1,1}(0) = \pi_1\lambda_1\alpha_1, \tag{A.6}$$
$$\pi_{1,2}(0) = \pi_1\lambda_1\alpha_2, \tag{A.7}$$
$$\pi_{2,rpl}(0) = \pi_2\lambda_2. \tag{A.8}$$

By solving Eqs (A.1)–(A.5), and using Eqs (A.6)–(A.8), we have

$$\pi_1 = \frac{1}{\gamma + \lambda_1}\Phi_1, \tag{A.9}$$
$$\pi_2 = \frac{1}{\gamma + \lambda_2}\Phi_2, \tag{A.10}$$
$$\pi_{1,1}(x) = \lambda_1\alpha_1\pi_1 e^{-\gamma x - \int_0^x h_{1,1}(\tau)d\tau} + e^{-\gamma x - \int_0^x h_{1,1}(\tau)d\tau}\int_0^x \Phi_3(\xi)e^{\gamma\xi + \int_0^\xi h_{1,1}(\tau)d\tau}d\xi, \tag{A.11}$$
$$\pi_{1,2}(x) = \lambda_1\alpha_2\pi_1 e^{-\gamma x - \int_0^x h_{1,2}(\tau)d\tau} + e^{-\gamma x - \int_0^x h_{1,2}(\tau)d\tau}\int_0^x \Phi_4(\xi)e^{\gamma\xi + \int_0^\xi h_{1,2}(\tau)d\tau}d\xi, \tag{A.12}$$
$$\pi_{2,rpl}(x) = \lambda_2\pi_2 e^{-\gamma x - \int_0^x h_2(\tau)d\tau} + e^{-\gamma x - \int_0^x h_2(\tau)d\tau}\int_0^x \Phi_5(\xi)e^{\gamma\xi + \int_0^\xi h_2(\tau)d\tau}d\xi. \tag{A.13}$$

The following inequalities are used above:

$$e^{-\int_{\xi}^{x} h_j(\tau)d\tau} \leqslant 1, \quad x > \xi > 0, \quad j = 1.\ 1,\ 1.2,\ 2.$$

Now, if we combine Eqs (A.11)–(A.13) with Eqs (A.9) and (A.10) and use the Fubini theorem, we get

$$\begin{aligned}
\|\pi_{1,1}\|_{L^1[0,\infty)} &= \int_0^{\infty} |\pi_{1,1}(x)|dx \\
&\leqslant \frac{\lambda_1\alpha_1}{\gamma(\gamma+\lambda_1)}|\Phi_1| + \int_0^{\infty} e^{-\gamma x}\int_0^x |\Phi_3(\xi)|e^{\gamma\xi}d\xi dx \\
&= \frac{1}{\gamma}|a_1| + \int_0^{\infty} |\Phi_3(\xi)|e^{\gamma\xi}\int_{\xi}^{\infty} e^{-\gamma x}dxd\xi \\
&= \frac{1}{\gamma}|a_1| + \frac{1}{\gamma}\|\Phi_3\|_{L^1[0,\infty)}, && \text{(A.14)} \\
\|\pi_{1,2}\|_{L^1[0,\infty)} &\leqslant \frac{1}{\gamma}|a_2| + \frac{1}{\gamma}\|\Phi_4\|_{L^1[0,\infty)}, && \text{(A.15)} \\
\|\pi_{2,rpl}\|_{L^1[0,\infty)} &\leqslant \frac{\lambda_2}{\gamma(\gamma+\lambda_2)}|\Phi_2| + \frac{1}{\gamma}\|\Phi_5\|_{L^1[0,\infty)}. && \text{(A.16)}
\end{aligned}$$

Eqs (A.14)–(A.16) give

$$\begin{aligned}
\|\Pi\| =& \|\pi_1\| + \|\pi_2\| + \|\pi_{1,1}\|_{L^1[0,\infty)} + \|\pi_{1,2}\|_{L^1[0,\infty)} + \|\pi_{2,rpl}\|_{L^1[0,\infty)} \\
\leqslant& \frac{1}{\gamma+\lambda_1}|\Phi_1| + \frac{1}{\gamma+\lambda_2}|\Phi_2| + \frac{\lambda_1\alpha_1}{\gamma(\gamma+\lambda_1)}|\Phi_1| + \frac{1}{\gamma}\|\Phi_3\|_{L^1[0,\infty)} \\
&+ \frac{\lambda_1\alpha_2}{\gamma(\gamma+\lambda_1)}|\Phi_1| + \frac{1}{\gamma}\|\Phi_4\|_{L^1[0,\infty)} + \frac{\lambda_2}{\gamma(\gamma+\lambda_2)}|\Phi_1| + \frac{1}{\gamma}\|\Phi_5\|_{L^1[0,\infty)} \\
=& \frac{\gamma+\lambda_1(\alpha_1+\alpha_2)}{\gamma(\gamma+\lambda_1)}|\Phi_1| + \frac{\gamma+\lambda_2}{\gamma(\gamma+\lambda_2)}|\Phi_2| + \frac{1}{\gamma}\|\Phi_3\|_{L^1[0,\infty)} \\
&+ \frac{1}{\gamma}\|\Phi_4\|_{L^1[0,\infty)} + \frac{1}{\gamma}\|\Phi_5\|_{L^1[0,\infty)} \\
=& \frac{1}{\gamma}\Big\{|\Phi_1| + |\Phi_2| + \|\Phi_3\|_{L^1[0,\infty)} + \|\Phi_4\|_{L^1[0,\infty)} + \|\Phi_5\|_{L^1[0,\infty)}\Big\} \\
=& \frac{1}{\gamma}\|\Phi\|_{L^1[0,\infty)}. && \text{(A.17)}
\end{aligned}$$

Eq (A.17) shows that $(\gamma I - \mathbb{A})^{-1}$ exists, and

$$(\gamma I - \mathbb{A})^{-1} : X \to D(\mathbb{A}), \quad \|(\gamma I - \mathbb{A})^{-1}\| \leqslant \frac{1}{\gamma}, \quad \text{for} \quad \gamma > 0.$$

It is evident that $\overline{D(\mathbb{A})} = \mathbb{X}$, and the proof is similar to Guper [11], thus we omit the particular procedures. Based on the results above, linear boundedness of $\mathbb{E}$, the Hille-Yosida theorem and the perturbation theory [11, 15], we determine that $\mathbb{A} + \mathbb{E}$ generates a C_0-semigroup $T(t)$.

In the final step, we show that $\mathbb{A}+\mathbb{E}$ is a dispersive operator. Choose

$$\xi(x)=\left(\frac{[\pi_1^+]}{\pi_1},\frac{[\pi_2^+]}{\pi_2},\frac{[\pi_{1,1}(x)^+]}{\pi_{1,1}(x)},\frac{[\pi_{1,2}(x)^+]}{\pi_{1,2}(x)},\frac{[\pi_{2,rpl}(x)^+]}{\pi_{2,rpl}(x)}\right),$$

where

$$[\pi_i]^+=\begin{cases}\pi_i, & \pi_i>0,\\ 0, & \pi_i\leqslant 0.\end{cases}\quad i=1,2,$$

$$[\pi_j(x)]^+=\begin{cases}\pi_j(x), & \pi_j(x)>0,\\ 0, & \pi_j(x)\leqslant 0.\end{cases}\quad j=1.1,1.2,\ 2.\ rpl,$$

for $\mathbf{\Pi}\in D(\mathbb{A})$. Define $W_j=\{x\in[0,\infty)|\pi_j(x)>0\}$ and $\widetilde{W}_j=\{x\in[0,\infty)|\pi_j(x)\leqslant 0\}$, then we get

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}\pi_j(x)}{\mathrm{d}x}\frac{[\pi_j(x)]^+}{\pi_j(x)}\mathrm{d}x=&\int_{W_j}\frac{\mathrm{d}\pi_j(x)}{\mathrm{d}x}\frac{[\pi_j(x)]^+}{\pi_j(x)}\mathrm{d}x+\int_{\widetilde{W}_j}\frac{d\pi_j(x)}{\mathrm{d}x}\frac{[\pi_j(x)]^+}{\pi_j(x)}\mathrm{d}x\\ =&\int_{W_j}\frac{\mathrm{d}\pi_{1,1}(x)}{\mathrm{d}x}\frac{[\pi_j(x)]^+}{\pi_j(x)}\mathrm{d}x=\int_{W_j}\frac{\mathrm{d}\pi_j(x)}{\mathrm{d}x}\mathrm{d}x=\int_0^\infty\frac{\mathrm{d}[\pi_j(x)]^+}{\mathrm{d}x}\mathrm{d}x\\ =&[\pi_j(x)]^+\,|_0^\infty=-[\pi_j(0)]^+,\quad j=1.1,\ 1.2,\ 2.\ rpl.\end{aligned}\tag{A.18}$$

Thus, we have

$$\begin{aligned}\langle(\mathbb{A}+\mathbb{E})\mathbf{\Pi},\xi\rangle=&\Big\{-\lambda_1\pi_1+\int_0^\infty h_{1,2}(x)\pi_{1,2}\mathrm{d}x+\int_0^\infty h_2(x)\pi_{2,rpl}(x)\mathrm{d}x\Big\}\frac{[\pi_1]^+}{\pi_1}\\ &+\Big\{-\lambda_2\pi_2+\int_0^\infty h_{1,1}(x)\pi_{1,1}(x)\mathrm{d}x\Big\}\frac{[\pi_2]^+}{\pi_2}\\ &+\int_0^\infty\Big\{-\frac{\mathrm{d}\pi_{1,1}(x)}{\mathrm{d}x}-h_{1,1}(x)\pi_{1,1}(x)\Big\}\frac{[\pi_{1,1}(x)]^+}{\pi_{1,1}(x)}\mathrm{d}x\\ &+\int_0^\infty\Big\{-\frac{d\pi_{1,2}(x)}{\mathrm{d}x}-h_{1,2}(x)\pi_{1,2}(x)\Big\}\frac{[\pi_{1,2}(x)]^+}{\pi_{1,2}(x)}\mathrm{d}x\\ &+\int_0^\infty\Big\{-\frac{\mathrm{d}\pi_{2,rpl}(x)}{\mathrm{d}x}-h_2(x)\pi_{2,rpl}(x)\Big\}\frac{[\pi_{2,rpl}(x)]^+}{\pi_{2,rpl}(x)}\mathrm{d}x\\ \leqslant&-\lambda_1[\pi_1]^++\frac{[\pi_1]^+}{\pi_1}\int_0^\infty h_{1,2}(x)[\pi_{1,2}(x)]^+\mathrm{d}x+\frac{[\pi_1]^+}{\pi_1}\int_0^\infty h_2(x)[\pi_{2,rpl}(x)]^+\mathrm{d}x\\ &-\lambda_2[\pi_2]^++\frac{[\pi_2]^+}{\pi_2}\int_0^\infty h_{1,1}(x)[\pi_{1,1}(x)]^+\mathrm{d}x+[\pi_{1,1}(0)]^+\\ &-\int_0^\infty h_{1,1}(x)[\pi_{1,1}(x)]^+\mathrm{d}x+[\pi_{1,2}(0)]^+-\int_0^\infty h_{1,2}(x)[\pi_{1,2}(x)]^+\mathrm{d}x\\ &+[\pi_{2,rpl}(0)]^+-\int_0^\infty h_2(x)[\pi_{2,rpl}(x)]^+\mathrm{d}x\\ \leqslant&-\lambda_1[\pi_1]^+-\lambda_2[\pi_2]^++\frac{[\pi_1]^+}{\pi_1}\int_0^\infty h_{1,2}(x)[\pi_{1,2}(x)]^+\mathrm{d}x\\ &+\frac{[\pi_1]^+}{\pi_1}\int_0^\infty h_2(x)[\pi_{2,rpl}(x)]^+\mathrm{d}x+\frac{[\pi_2]^+}{\pi_2}\int_0^\infty h_{1,1}(x)[\pi_{1,1}(x)]^+\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\lambda_1\alpha_1[\pi_1]^+ + \lambda_1\alpha_2[\pi_1]^+ + \lambda_2[\pi_2]^+ - \int_0^\infty h_{1,1}(x)[\pi_{1,1}(x)]^+\mathrm{d}x \\
&- \int_0^\infty h_{1,2}(x)[\pi_{1,2}(x)]^+\mathrm{d}x - \int_0^\infty h_2(x)[\pi_{2,rpl}(x)]^+\mathrm{d}x \\
=&\Big(\frac{[\pi_1]^+}{\pi_1} - 1\Big)\int_0^\infty h_{1,2}(x)[\pi_{1,2}(x)]^+\mathrm{d}x \\
&+\Big(\frac{[\pi_1]^+}{\pi_1} - 1\Big)\int_0^\infty h_2(x)[\pi_{2,rpl}(x)]^+\mathrm{d}x \\
&+\Big(\frac{[\pi_2]^+}{\pi_2} - 1\Big)\int_0^\infty h_{1,1}(x)[\pi_{1,1}(x)]^+\mathrm{d}x \\
\leqslant& 0,
\end{aligned}
$$

which means that $\mathbb{A}+\mathbb{E}$ is a dispersive operator. Therefore, we can conclude $\mathbb{A}+\mathbb{E}$ generates a positive contraction C_0-semigroup $\mathbb{T}(t)$ by the Fillips theorem.