



Research article

Global well-posedness of strong solution to the kinetic Cucker-Smale model with external potential field

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Abstract: In this paper, we studied the initial problem of the kinetic Cucker-Smale model with noise, driven by pairwise alignment interactions under the influence of external potential field. Without the general compact support or smallness assumption on the initial datum, we established the global existence of strong solution. The proof was based on weighted energy estimates.

Keywords: kinetic Cucker-Smale model with noise; external potential field; global well-posedness

Mathematics Subject Classification: 35Q84, 35D35, 35B30, 35Q92

1. Introduction

This paper is concerned with the following kinetic Cucker-Smale model with external potential field:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x U(x) \cdot \nabla_v f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v f, \\ f(t, x, v)|_{t=0} = f_0(x, v). \end{cases} \quad (1.1)$$

Here, $f = f(t, x, v)$ is the particle distribution function in space $(x, v) \in \Omega = \mathbb{R}^d \times \mathbb{R}^d$, at time $t \geq 0$. The function $U(x) = \frac{1}{2}|x|^2$ represents the harmonic potential field. The constant $\sigma > 0$ represents the noise strength. The alignment force $L[f]$ is expressed in the form:

$$L[f](t, x, v) = \int_{\Omega} \varphi(|x - y|) f(t, y, v^*) (v^* - v) dy dv^*.$$

The interaction kernel function $\varphi(|x - y|)$ is a positive nonincreasing C^2 function. Without loss of generality, we assume that

$$\max\{|\varphi|, |\varphi'|, |\varphi''|\} \leq 1.$$

The system (1.1) arises as a mean-field kinetic description of the following stochastic Cucker-Smale model with external potential field:

$$\begin{cases} dx_i = v_i dt, \\ dv_i = \frac{1}{N} \sum_{j=1}^N \varphi(|x_i - x_j|)(v_j - v_i) dt - \nabla_x U(x_i) dt + \sigma dW_t^{(i)}, \quad i = 1, \dots, N, \end{cases} \quad (1.2)$$

where the deterministic system was studied by [1]. Here, $(x_i(t), v_i(t))$ are the position and velocity pair of i^{th} -particle, $W_t^{(i)}$ denote independent Wiener processes, and σ is the magnitude of the noise. The communication weight function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ satisfies some symmetry conditions.

The particle Cucker-Smale model was originally proposed to understand the flocking phenomena in bird populations by Cucker and Smale [2, 3]. Under the “molecular chaos” assumption, Ha and Tadmor [4] derived the kinetic Cucker-Smale model formally from the particle Cucker-Smale model using the BBGKY hierarchy method, e.g., [5–8]. For large-scale particle systems, Ha and Liu [9] rigorously justified the mean-field limit from the multi-particle Cucker-Smale model to the kinetic Cucker-Smale model, utilizing tools such as measure-valued solutions and the Kantorovich-Rubinstein distance. Furthermore, Carrillo et al. [10] proved that the solutions approached exponentially fast in velocity to the mean velocity of the initial condition, while in space they converged to a translational flocking solution.

The Cucker-Smale model has been extended to various complexities, including the presence of different network structures [11], communication mechanism [12], self-propulsion and friction forces [13], and external fields such as fluid field [14], temperature field [15, 16], potential fields [1], etc. These extensions significantly influence the dynamics of the system, leading to behaviors that are markedly different from the original model. Moreover, the connection between the kinetic Cucker-Smale model and the Euler-alignment system has been rigorously explored in recent literature. For the Euler-alignment system with pressure effects, Karper et al. [17] rigorously justified the hydrodynamic limit of the kinetic Cucker-Smale flocking model. Furthermore, Poyato and Soler [18] provided detailed analysis of a compressible Euler-type equation with singular commutator, which is derived from a hyperbolic limit of the kinetic Cucker-Smale model. In the pressure-less case, the derivation from the kinetic Cucker-Smale model to the nonlocal Euler-alignment system was established by [19]. Recently, Fabisiak and Peszek [20] rigorously derived the macroscopic fractional Euler-alignment system from the kinetic Cucker-Smale equation without performing any hydrodynamic limit.

The well-posedness of solution is a fundamental concept in the theory of partial differential equations. Previous works [21–23] have established the well-posedness of weak and strong solution to the kinetic Cucker-Smale model without external potentials. Recently, Jin [24] developed a unified framework to establish the well-posedness of the model with or without noise. In this paper, the global well-posedness of the noisy version of kinetic Cucker-Smale model with harmonic potential field is studied. We prove the global nonnegativity, existence, and uniqueness of the strong solution for the system (1.1). Our approach is based on a combination of weighted Sobolev spaces and approximation schemes, which have been shown to be effective in dealing with inherent nonlinear and nonlocal problems in [25–27].

Notation: We denote the usual L^p norms on Ω by $\|f(t)\|_{L^p} := \|f(t)\|_{L^p(\Omega)}$, and the i^{th} element $L[f]_i$

of the vector $L[f]$ by

$$L[f]_i = \int_{\Omega} \varphi(|x - y|) f(t, y, v_i^*) (v_i^* - v_i) \, dy \, dv^*.$$

Then, we construct three special weighted Sobolev spaces with power $\omega(x, v) = (1 + x^2 + v^2)^{\frac{1}{2}}$:

$$\begin{aligned} L_{\omega}^1(\Omega) &:= \{f(t, x, v) : \|f\|_{L_{\omega}^1(\Omega)} < \infty\}, \\ H_{\omega}^1(\Omega) &:= \{f(t, x, v) : \|f\|_{H_{\omega}^1(\Omega)} < \infty\}, \\ X(\Omega) &:= \{f(t, x, v) : \|f\|_{X(\Omega)} < \infty\}, \\ \|f\|_{H_{\omega}^1} &:= \|f\|_{L_{\omega}^2} + \|\nabla_x f\|_{L_{\omega}^2} + \|\nabla_v f\|_{L_{\omega}^2}, \\ \|f\|_X &:= \|\omega \nabla_v f\|_{L_{\omega}^2} + \|\omega \nabla_x f\|_{L_{\omega}^2} + \|\omega^2 f\|_{L_{\omega}^2}, \end{aligned}$$

where

$$\|f\|_{L_{\omega}^1} := \|f\|_{L_{\omega}^1(\Omega)} = \int_{\Omega} \omega f(t, x, v) \, dx \, dv, \quad \|f\|_{L_{\omega}^2} := \|f\|_{L_{\omega}^2(\Omega)} = \left(\int_{\Omega} \omega^2 f^2(t, x, v) \, dx \, dv \right)^{\frac{1}{2}}.$$

In the rest of the paper, we denote C_d and $C_{d,\sigma}$ as positive constants while subscripts are used to indicate specific dependencies of such constants.

Definition 1.1. Let $0 \leq f(t, x, v) \in C([0, \infty); L_{\omega}^1(\Omega))$. The function $f(t, x, v)$ is a weak solution to system (1.1) if

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U(x) \cdot \nabla_v f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v f, \quad \text{in } \mathcal{D}'([0, +\infty) \times \Omega).$$

We say $f(t, x, v)$ is a strong solution if $f(t, x, v)$ is a weak solution and $f(t, x, v) \in C([0, \infty); H_{\omega}^1(\Omega))$.

Now, our main results are stated as follows.

Theorem 1.1. Assume initial datum $f_0(x, v) \in X(\Omega) \cap L_{\omega}^1(\Omega)$. Then, the system (1.1) admits a unique strong solution in sense of Definition 1.1.

Remark 1.1. In this paper, we consider a special potential function $U = \frac{1}{2}|x|^2$. In fact, similar to the study in [1], the potential U can be extended to a more general case: assume that the potential function U satisfies the following conditions:

$$\frac{a}{2}|x|^2 \leq U(x) \leq \frac{A}{2}|x|^2, \quad a|x| \leq |\nabla U(x)| \leq A|x|, \quad \forall x \in \mathbb{R}^3, \quad 0 < a \leq A. \quad (1.3)$$

One can obtain the existence and uniqueness of a strong solution to the system (1.1) with (1.3).

The existence of a strong solution is constructed in the weighted Sobolev space $H_{\omega}^1(\Omega)$. However, to find the Cauchy sequences in this space, we need to give extra estimates for terms such as $\|\omega \nabla_v f_n(t)\|_{L_{\omega}^2}^2$ and $\|\omega \nabla_x f_n(t)\|_{L_{\omega}^2}^2$. For this purpose, we construct the weighted Sobolev space $X(\Omega)$ and establish the a priori estimate for the preparation; see Proposition 3.1.

The rest of this paper is organized as follows. In Section 2, we establish a priori estimates of the system (1.1) by taking advantage of three special weighted Sobolev spaces. In Section 3, we first prove the local existence and uniqueness of the strong solution to system (1.1) by an iteration scheme and extend the local existence to the global one.

2. A priori estimates

This section is devoted to the a priori estimates for the system (1.1). Directly integrating system (1.1) over $[0, t] \times \Omega$ gives that any smooth solution of it satisfies the following conservation law: $\int_{\Omega} f_0(x, v) dx dv = \int_{\Omega} f(t, x, v) dx dv =: \int_{\Omega} f(t) dx dv$. Therefore, w.o.l.g., we assume $\int_{\Omega} f_0(x, v) dx dv = 1$ in the rest of paper. We first give the following a priori estimates for $\|f(t)\|_{L^1_{\omega}}$ and $\|f(t)\|_{L^2_{\omega}}$, which are better integrabilities of f with large v and x .

Lemma 2.1. *Assume the function $f(t, x, v)$ is a smooth solution to system (1.1) with initial datum $0 \leq f_0 \in X(\Omega) \cap L^1_{\omega}(\Omega)$ and $\int_{\Omega} f_0(x, v) dx dv = 1$. Then, for $\forall t \geq 0$, we have*

- (1) $\int_{\Omega} f(t) dx dv = 1$, $\|f(t)\|_{L^1} = 1$, and $f(t) \geq 0$;
- (2) $\|f(t)\|_{L^1_{\omega}} \leq C(t)$;
- (3) $\|f(t)\|_{L^2_{\omega}}^2 + 2\sigma \int_0^t \|\nabla_v f(\tau)\|_{L^2_{\omega}}^2 d\tau \leq C \exp\left(\int_0^t C(\tau) d\tau\right)$;

where C and $C(t)$ are positive constant and positive continuous functions of t both depending on σ, d and the weighted norms of the initial datum f_0 .

Proof. (1) We multiply system (1.1) by $\text{sgn}(f)$ and integrate it over Ω to obtain

$$\frac{d}{dt} \|f(t)\|_{L^1} = 0,$$

which implies $\|f(t)\|_{L^1} = \|f_0\|_{L^1} = 1$, $\forall t \geq 0$. Note that

$$\int_{\Omega} f(t) dx dv = \int_{\Omega} f(t) \mathbf{1}_{[f(t) \geq 0]} dx dv + \int_{\Omega} f(t) \mathbf{1}_{[f(t) < 0]} dx dv = 1, \quad (2.1)$$

$$\|f(t)\|_{L^1} = \int_{\Omega} f(t) \mathbf{1}_{[f(t) \geq 0]} dx dv - \int_{\Omega} f(t) \mathbf{1}_{[f(t) < 0]} dx dv = 1. \quad (2.2)$$

We subtract system (2.1) from (2.2) to obtain $\int_{\Omega} f(t) \mathbf{1}_{[f(t) < 0]} dx dv = 0$, which gives $f(t) \geq 0$, $\forall t \geq 0$.

(2) Multiplying system (1.1) by ω and integrating by parts over Ω leads to

$$\begin{aligned} \frac{d}{dt} \|\omega f(t)\|_{L^1} &= - \int_{\Omega} \omega v \cdot \nabla_x f(t, x, v) dx dv + \int_{\Omega} \omega x \cdot \nabla_v f(t, x, v) dx dv \\ &\quad - \int_{\Omega} \omega \nabla_v \cdot (L[f] f(t, x, v)) dx dv + \sigma \int_{\Omega} \omega \Delta_v f(t, x, v) dx dv \\ &= \int_{\Omega} f(t, x, v) \nabla_x \cdot (v \omega) dx dv - \int_{\Omega} f(t, x, v) \nabla_v \cdot (x \omega) dx dv \\ &\quad + \int_{\Omega} f(t, x, v) L[f] \cdot \nabla_v \omega dx dv + \sigma \int_{\Omega} f(t, x, v) \Delta_v \omega dx dv \\ &= \int_{\Omega} f(t, x, v) L[f] \cdot \frac{v}{\omega} dx dv + \sigma \int_{\Omega} f(t, x, v) \Delta_v \omega dx dv \\ &\leq C_{d, \sigma} \|\omega f(t)\|_{L^1}. \end{aligned}$$

By applying Grönwall's lemma, we obtain

$$\|\omega f(t)\|_{L^1} \leq \|\omega f_0\|_{L^1} e^{C_{d, \sigma} t} =: C(t),$$

where $C(t)$ is the function of t depending on d, σ , and $\|f_0\|_{L^1_\omega}$.

(3) We multiply system (1.1) by $2\omega^2 f$ to obtain

$$\begin{aligned} \frac{d}{dt} (\omega^2 f^2) = & -\omega^2 v \cdot \nabla_x (f^2) + \omega^2 x \cdot \nabla_v (f^2) - 2(\nabla_v \cdot L[f])\omega^2 f^2 \\ & - \omega^2 L[f] \cdot \nabla_v (f^2) + 2\sigma\omega^2 f \Delta_v f. \end{aligned} \quad (2.3)$$

Then, we integrate system (2.3) by parts over Ω to obtain

$$\begin{aligned} & \frac{d}{dt} \|f(t)\|_{L^2_\omega}^2 \\ = & \int_\Omega \nabla_x \cdot (\omega^2 v) f^2(t, x, v) dx dv - \int_\Omega \nabla_v \cdot (\omega^2 x) f^2(t, x, v) dx dv \\ & - 2 \int_\Omega (\nabla_v \cdot L[f]) \omega^2 f^2(t, x, v) dx dv + \int_\Omega \nabla_v \cdot (\omega^2 L[f]) f^2(t, x, v) dx dv \\ & + 2\sigma \int_\Omega \omega^2 f(t, x, v) \Delta_v f(t, x, v) dx dv \\ = & 2 \int_\Omega x \cdot v f^2(t, x, v) dx dv - 2 \int_\Omega x \cdot v f^2(t, x, v) dx dv \\ & - d \int_\Omega \left(\int_\Omega \varphi(|x-y|) f(t, y, v^*) dy dv^* \right) \omega^2 f^2(t, x, v) dx dv \\ & + 2 \int_\Omega v \cdot L[f] f^2(t, x, v) dx dv - 2\sigma \int_\Omega \nabla_v f(t, x, v) \cdot \nabla_v (\omega^2 f(t, x, v)) dx dv \\ \leq & C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1}) \|f(t)\|_{L^2_\omega}^2 - 2\sigma \|\nabla_v f(t)\|_{L^2_\omega}^2, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} & - 2\sigma \int_\Omega \nabla_v f(t, x, v) \cdot \nabla_v (\omega^2 f(t, x, v)) dx dv \\ = & - 2\sigma \int_\Omega \nabla_v f(t, x, v) \cdot (\nabla_v \omega^2 f(t, x, v)) dx dv - 2\sigma \int_\Omega \nabla_v f(t, x, v) \cdot (\omega^2 \nabla_v f(t, x, v)) dx dv \\ = & - 2\sigma \int_\Omega \nabla_v f(t, x, v) \cdot (\nabla_v \omega^2 f(t, x, v)) dx dv - 2\sigma \|\nabla_v f(t)\|_{L^2_\omega}^2 \\ = & - \sigma \int_\Omega \nabla_v |f(t, x, v)|^2 \cdot \nabla_v \omega^2 dx dv - 2\sigma \|\nabla_v f(t)\|_{L^2_\omega}^2 \\ = & \sigma \int_\Omega |f(t, x, v)|^2 \Delta_v \omega^2 dx dv - 2\sigma \|\nabla_v f(t)\|_{L^2_\omega}^2. \end{aligned}$$

Applying Grönwall's lemma to system (2.4), we obtain

$$\|f(t)\|_{L^2_\omega}^2 + 2\sigma \int_0^t \|\nabla_v f(\tau)\|_{L^2_\omega}^2 d\tau \leq \|f_0\|_{L^2_\omega}^2 \exp\left(\int_0^t C(\tau) d\tau\right),$$

where $C(t)$ is the function of t depending on d, σ , and $\|f_0\|_{L^1_\omega}$.

Now we derive the estimates in the weighted Sobolev spaces. The first weighted space $H^1_\omega(\Omega)$ is prepared for the strong solution, while the second space $X(\Omega)$ is constructed to estimate the approximate solutions.

Proposition 2.1. Assume the function $f(t, x, v)$ is a smooth solution to system (1.1) with initial datum satisfying the condition of Lemma 2.1. Then, for $\forall t \geq 0$, we have

- (1) $\|f(t)\|_{H_\omega^1}^2 + \|\omega f(t)\|_{L_\omega^2}^2 \leq C \exp\left(\int_0^t C(\tau) d\tau\right);$
 (2) $\|f(t)\|_X^2 + \sigma \int_0^t \|\nabla_v f(\tau)\|_X^2 d\tau \leq C \exp\left(\int_0^t C(\tau) d\tau\right);$

where C and $C(t)$ are positive constant and positive continuous functions of t both depending on σ, d and the weighted norms of the initial datum f_0 .

Proof. (1) Applying ∇_v to system (1.1) gives

$$\begin{aligned} \partial_t \nabla_v f = & -\nabla_x f - v \cdot \nabla_x (\nabla_v f) + x \cdot \nabla_v (\nabla_v f) - \nabla_v (\nabla_v f) \cdot L[f] \\ & + (d+1) \left(\int_\Omega \varphi(|x-y|) f(t, y, v^*) dy dv^* \right) \nabla_v f + \sigma \Delta_v (\nabla_v f). \end{aligned} \quad (2.5)$$

Multiplying system (2.5) by $2\omega^2 \nabla_v f$ leads to

$$\begin{aligned} \frac{d}{dt} (\omega^2 |\nabla_v f|^2) = & -2\omega^2 \nabla_v f \cdot \nabla_x f - \omega^2 v \cdot \nabla_x (|\nabla_v f|^2) \\ & + \omega^2 x \cdot \nabla_v (|\nabla_v f|^2) - \omega^2 \nabla_v (|\nabla_v f|^2) \cdot L[f] \\ & + 2(d+1) \left(\int_\Omega \varphi(|x-y|) f(t, y, v^*) dy dv^* \right) \omega^2 |\nabla_v f|^2 \\ & + 2\sigma \omega^2 \nabla_v (\Delta_v f) \cdot \nabla_v f. \end{aligned} \quad (2.6)$$

Then we integrate system (2.6) by parts over Ω to obtain

$$\begin{aligned} & \frac{d}{dt} \|\nabla_v f(t)\|_{L_\omega^2}^2 \\ = & -2 \int_\Omega \omega^2 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) dx dv \\ & + \int_\Omega 2x \cdot v |\nabla_v f(t, x, v)|^2 dx dv - \int_\Omega 2x \cdot v |\nabla_v f(t, x, v)|^2 dx dv \\ & + 2(d+1) \int_\Omega \left(\int_\Omega \varphi(|x-y|) f(t, y, v^*) dy dv^* \right) \omega^2 |\nabla_v f(t, x, v)|^2 dx dv \\ & - \int_\Omega \omega^2 \nabla_v (|\nabla_v f(t, x, v)|^2) \cdot L[f] dx dv \\ & + 2\sigma \int_\Omega \omega^2 \nabla_v (\Delta_v f(t, x, v)) \cdot \nabla_v f(t, x, v) dx dv \\ \leq & -2 \int_\Omega \omega^2 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) dx dv + C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1}) \|\nabla_v f(t)\|_{L_\omega^2}^2 \\ & - 2\sigma \|\nabla_v^2 f(t)\|_{L_\omega^2}^2. \end{aligned} \quad (2.7)$$

By applying ∇_x to system (1.1), we obtain

$$\begin{aligned} \nabla_x f_t = & \nabla_v f - v \cdot \nabla_x (\nabla_x f) + x \cdot \nabla_v (\nabla_x f) - \nabla_x (\nabla_v \cdot L[f]) f \\ & - \nabla_v \cdot L[f] \nabla_x f - \nabla_x (\nabla_v f \cdot L[f]) + \sigma \nabla_x (\Delta_v f). \end{aligned} \quad (2.8)$$

Multiplying system (2.8) by $2\omega^2 \nabla_x f$ gives

$$\begin{aligned} \frac{d}{dt} (\omega^2 |\nabla_x f|^2) &= 2\omega^2 \nabla_x f \cdot \nabla_v f - \omega^2 v \cdot \nabla_x (|\nabla_x f|^2) \\ &\quad + \omega^2 x \cdot \nabla_v (|\nabla_x f|^2) - \omega^2 \nabla_x (\nabla_v \cdot L[f]) \cdot \nabla_x (f^2) \\ &\quad - 2\omega^2 (\nabla_v \cdot L[f]) |\nabla_x f|^2 - \omega^2 \nabla_v (|\nabla_x f|^2) \cdot L[f] \\ &\quad - 2\omega^2 \nabla_x f \cdot (\nabla_x L[f] \cdot \nabla_v f) + 2\sigma \omega^2 \nabla_x f \cdot \nabla_x (\Delta_v f). \end{aligned} \quad (2.9)$$

Then, we integrate system (2.9) by parts over Ω to obtain

$$\begin{aligned} &\frac{d}{dt} \|\nabla_x f(t)\|_{L_\omega^2}^2 \\ &= 2 \int_{\Omega} \omega^2 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) dx dv + 2 \int_{\Omega} |\nabla_x f(t, x, v)|^2 \nabla_x \cdot (\omega^2 v) dx dv \\ &\quad - 2 \int_{\Omega} |\nabla_x f(t, x, v)|^2 \nabla_v \cdot (\omega^2 x) dx dv + \int_{\Omega} f^2(t, x, v) \nabla_x \cdot (\omega^2 \nabla_x (\nabla_v \cdot L[f])) dx dv \\ &\quad - 2 \int_{\Omega} \omega^2 (\nabla_v \cdot L[f]) |\nabla_x f|^2 dx dv + \int_{\Omega} |\nabla_x f(t, x, v)|^2 \nabla_v \cdot (\omega^2 L[f]) dx dv \\ &\quad - 2\sigma \sum_{i,j=1}^d \int_{\Omega} \partial_{v_i} (\omega^2 \partial_{x_j} f) (\partial_{v_i x_j}^2 f) dx dv - 2 \int_{\Omega} \omega^2 \nabla_x f(t, x, v) \cdot (\nabla_x L[f] \cdot \nabla_v f(t, x, v)) dx dv \\ &\leq 2 \int_{\Omega} \omega^2 \nabla_x f(t, x, v) \cdot \nabla_v f(t, x, v) dx dv \\ &\quad + C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1}) \|\nabla_x f(t)\|_{L_\omega^2}^2 + C_d \|f(t)\|_{L_\omega^2}^2 - 2\sigma \|\nabla_v \nabla_x f(t)\|_{L_\omega^2}^2 \\ &\quad - \underbrace{2 \int_{\Omega} \omega^2 \nabla_x f(t, x, v) \cdot (\nabla_x L[f] \cdot \nabla_v f(t, x, v)) dx dv}_A. \end{aligned} \quad (2.10)$$

Now we estimate the last term A on the right hand side of system (2.10).

$$\begin{aligned} A &= -2 \sum_{i,j=1}^d \int_{\Omega} \omega^2 \partial_{v_i} f(t, x, v) \partial_{x_j} f(t, x, v) \partial_{x_j} L[f]_i dx dv \\ &= 2 \sum_{i,j=1}^d \int_{\Omega} f(t, x, v) \partial_{v_i} (\omega^2 \partial_{x_j} f(t, x, v) \partial_{x_j} L[f]_i) dx dv \\ &= 4 \sum_{i,j=1}^d \int_{\Omega} f(t, x, v) v_i \partial_{x_j} f(t, x, v) \partial_{x_j} L[f]_i dx dv \\ &\quad + 2 \sum_{i,j=1}^d \int_{\Omega} f(t, x, v) \omega^2 \partial_{v_i x_j}^2 f(t, x, v) \partial_{x_j} L[f]_i dx dv \\ &\quad + 2 \sum_{i,j=1}^d \int_{\Omega} f(t, x, v) \omega^2 \partial_{x_j} f(t, x, v) \partial_{v_i x_j}^2 L[f]_i dx dv \\ &\leq C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1} + \|\omega f(t)\|_{L^1}^2) (\|f(t)\|_{L_\omega^2}^2 + \|\omega f(t)\|_{L_\omega^2}^2) \\ &\quad + C_d \|\nabla_x f(t)\|_{L_\omega^2}^2 + \sigma \|\nabla_v \nabla_x f(t)\|_{L_\omega^2}^2, \end{aligned} \quad (2.11)$$

where we have used the σ -Young's inequality and the following facts:

$$\begin{aligned}\partial_{x_j} L[f]_i &= \int_{\Omega} \partial_{x_j} \varphi(|x-y|) f(t, y, v_i^*) (v_i^* - v_i) \, dy \, dv^* \\ &\leq C(1+v^2)^{\frac{1}{2}} \int_{\Omega} f(t, y, v_i^*) (1+|v^*|^2)^{\frac{1}{2}} \, dy \, dv^* \\ &\leq C\omega \|\omega f\|_{L^1}\end{aligned}$$

and

$$\begin{aligned}\partial_{v_i x_j}^2 L[f]_i &= \int_{\Omega} \partial_{x_j} \varphi(|x-y|) f(t, y, v_i^*) \partial_{v_i} (v_i^* - v_i) \, dy \, dv^* \\ &= - \int_{\Omega} \partial_{x_j} \varphi(|x-y|) f(t, y, v_i^*) \, dy \, dv^* \\ &\leq C.\end{aligned}$$

Combining system (2.11) with (2.10) yields

$$\begin{aligned}&\frac{d}{dt} \|\nabla_x f(t)\|_{L_{\omega}^2}^2 \\ &\leq 2 \int_{\Omega} \omega^2 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) \, dx \, dv + C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1}) \|\nabla_x f(t)\|_{L_{\omega}^2}^2 \\ &\quad - \sigma \|\nabla_v \nabla_x f(t)\|_{L_{\omega}^2}^2 + C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1} + \|\omega f(t)\|_{L^1}^2) (\|f(t)\|_{L_{\omega}^2}^2 + \|\omega f(t)\|_{L_{\omega}^2}^2).\end{aligned}\tag{2.12}$$

Due to the term $\|\omega f(t)\|_{L_{\omega}^2}^2$ which appeared in system (2.12), we need to analyze it in detail to close the a priori estimate in H_{ω}^1 . Similarly to the estimate of system (2.4), we multiply system (1.1) by $2\omega^4 f$ and integrate it over Ω to have

$$\frac{d}{dt} \|\omega f(t)\|_{L_{\omega}^2}^2 + 2\sigma \|\omega \nabla_v f(t)\|_{L_{\omega}^2}^2 \leq C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1}) \|\omega f(t)\|_{L_{\omega}^2}^2.\tag{2.13}$$

Adding up systems (2.4), (2.7), (2.12), and (2.13), we can get

$$\begin{aligned}&\frac{d}{dt} (\|f(t)\|_{H_{\omega}^1}^2 + \|\omega f(t)\|_{L_{\omega}^2}^2) + \sigma (\|\nabla_v f(t)\|_{H_{\omega}^1}^2 + \|\omega \nabla_v f(t)\|_{L_{\omega}^2}^2) \\ &\leq C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1} + \|\omega f(t)\|_{L^1}^2) (\|f(t)\|_{H_{\omega}^1}^2 + \|\omega f(t)\|_{L_{\omega}^2}^2).\end{aligned}$$

By Grönwall's lemma and the estimate in Lemma 2.1, we obtain

$$\begin{aligned}&\|f(t)\|_{H_{\omega}^1}^2 + \|\omega f(t)\|_{L_{\omega}^2}^2 + \sigma \int_0^t (\|\nabla_v f(\tau)\|_{H_{\omega}^1}^2 + \|\omega \nabla_v f(\tau)\|_{L_{\omega}^2}^2) \, d\tau \\ &\leq (\|f_0\|_{H_{\omega}^1}^2 + \|\omega f_0\|_{L_{\omega}^2}^2) \exp\left(\int_0^t C(\tau) \, d\tau\right),\end{aligned}$$

where $C(t)$ depends on d , σ , and $\|f_0\|_{L_{\omega}^1}$.

(2) We estimate the first term of $\|f(t)\|_X^2$. Multiplying system (2.5) by $2\omega^4 \nabla_v f$ gives

$$\begin{aligned} \frac{d}{dt} (\omega^4 |\nabla_v f|^2) &= -2\omega^4 \nabla_v f \cdot \nabla_x f - \omega^4 v \cdot \nabla_x (|\nabla_v f|^2) \\ &\quad + \omega^4 x \cdot \nabla_v (|\nabla_v f|^2) - \omega^4 \nabla_v (|\nabla_v f|^2) \cdot L[f] \\ &\quad + 2(d+1) \left(\int_{\Omega} \varphi(|x-y|) f(t, y, v^*) dy dv^* \right) \omega^4 |\nabla_v f|^2 \\ &\quad + 2\sigma \omega^4 \nabla_v (\Delta_v f) \cdot \nabla_v f =: \sum_{k=1}^6 I_k. \end{aligned}$$

Integrating it over Ω , we can get

$$\int_{\Omega} I_1 dx dv = -2 \int_{\Omega} \omega^4 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) dx dv;$$

$$\int_{\Omega} I_2 dx dv = \int_{\Omega} |\nabla_v f(t, x, v)|^2 \nabla_x \cdot (\omega^4 v) dx dv = 4 \int_{\Omega} |\nabla_v f(t, x, v)|^2 \omega^2 x \cdot v dx dv;$$

$$\int_{\Omega} I_3 dx dv = - \int_{\Omega} |\nabla_v f(t, x, v)|^2 \nabla_v \cdot (\omega^4 x) dx dv = -4 \int_{\Omega} |\nabla_v f(t, x, v)|^2 \omega^2 x \cdot v dx dv;$$

$$\begin{aligned} \int_{\Omega} I_4 dx dv &= \int_{\Omega} |\nabla_v f(t, x, v)|^2 \nabla_v \cdot (\omega^4 L[f]) dx dv \\ &= 4 \int_{\Omega} |\nabla_v f(t, x, v)|^2 \omega^2 v \cdot L[f] dx dv + \int_{\Omega} |\nabla_v f(t, x, v)|^2 \omega^4 (\nabla_v \cdot L[f]) dx dv \\ &\leq C_d (1 + \|\omega f(t)\|_{L^1}) \|\omega \nabla_v f(t)\|_{L_{\omega}^2}^2; \end{aligned}$$

$$\int_{\Omega} I_5 dx dv \leq C_d \|\omega \nabla_v f(t)\|_{L_{\omega}^2}^2;$$

$$\begin{aligned} \int_{\Omega} I_6 dx dv &= -2\sigma \sum_{i,j=1}^d \int_{\Omega} \partial_{v_i v_j}^2 f \partial_{v_i} (\omega^4 \partial_{v_j} f) dx dv \\ &= -2\sigma \sum_{i,j=1}^d \int_{\Omega} \partial_{v_i v_j}^2 f \partial_{v_i} \omega^4 \partial_{v_j} f dx dv - 2\sigma \sum_{i,j=1}^d \int_{\Omega} \partial_{v_i v_j}^2 f \omega^4 \partial_{v_i v_j}^2 f dx dv \\ &= -\sigma \sum_{i,j=1}^d \int_{\Omega} \partial_{v_i} |\partial_{v_j} f|^2 \partial_{v_i} \omega^4 dx dv - 2\sigma \|\omega \nabla_v^2 f(t)\|_{L_{\omega}^2}^2 \\ &= \sigma \sum_{i,j=1}^d \int_{\Omega} |\partial_{v_j} f|^2 (\partial_{v_i v_i}^2 \omega^4) dx dv - 2\sigma \|\omega \nabla_v^2 f(t)\|_{L_{\omega}^2}^2 \end{aligned}$$

$$\leq -2\sigma \|\omega \nabla_v^2 f(t)\|_{L_\omega^2}^2 + C_{d,\sigma} \|\omega \nabla_v f(t)\|_{L_\omega^2}^2.$$

Adding up the above estimates leads to

$$\begin{aligned} & \frac{d}{dt} \|\omega \nabla_v f(t)\|_{L_\omega^2}^2 + 2\sigma \|\omega \nabla_v^2 f(t)\|_{L_\omega^2}^2 \\ & \leq -2 \int_{\Omega} \omega^4 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) dx dv + C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1}) \|\omega \nabla_v f(t)\|_{L_\omega^2}^2. \end{aligned} \quad (2.14)$$

Then, we estimate the second term of $\|f(t)\|_X^2$. Multiplying system (2.8) by $2\omega^4 \nabla_x f$ leads to

$$\begin{aligned} \frac{d}{dt} (\omega^4 |\nabla_x f|^2) &= 2\omega^4 \nabla_v f \cdot \nabla_x f - \omega^4 v \cdot \nabla_x (|\nabla_x f|^2) \\ &+ \omega^4 x \cdot \nabla_v (|\nabla_x f|^2) - \omega^4 \nabla_v (|\nabla_x f|^2) \cdot L[f] \\ &- \omega^4 \nabla_x (\nabla_v \cdot L[f]) \cdot \nabla_x (f^2) - 2\omega^4 (\nabla_v \cdot L[f]) |\nabla_x f|^2 \\ &+ 2\sigma \omega^4 \nabla_x f \cdot \nabla_x (\Delta_v f) - 2\omega^4 \nabla_x f \cdot (\nabla_x L[f] \cdot \nabla_v f) =: \sum_{k=1}^8 J_k. \end{aligned}$$

Similarly, we integrate it over Ω as follows:

$$\int_{\Omega} J_1 dx dv = 2 \int_{\Omega} \omega^4 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) dx dv;$$

$$\int_{\Omega} J_2 dx dv = 4 \int_{\Omega} |\nabla_x f(t, x, v)|^2 \omega^2 x \cdot v dx dv;$$

$$\int_{\Omega} J_3 dx dv = -4 \int_{\Omega} |\nabla_x f(t, x, v)|^2 \omega^2 x \cdot v dx dv;$$

$$\begin{aligned} \int_{\Omega} J_4 dx dv &= \int_{\Omega} |\nabla_x f(t, x, v)|^2 \nabla_v \cdot (\omega^4 L[f]) dx dv \\ &= 4 \int_{\Omega} |\nabla_x f(t, x, v)|^2 \omega^2 v \cdot L[f] dx dv + \int_{\Omega} (\nabla_v \cdot L[f]) |\nabla_x f(t, x, v)|^2 \omega^4 dx dv \\ &\leq C_d (1 + \|\omega f(t)\|_{L^1}) \|\omega \nabla_x f(t)\|_{L_\omega^2}^2; \end{aligned}$$

$$\begin{aligned} \int_{\Omega} J_5 dx dv &= \int_{\Omega} f^2(t, x, v) \nabla_x \cdot (\omega^4 \nabla_x (\nabla_v \cdot L[f])) dx dv \\ &= 4 \int_{\Omega} f^2(t, x, v) \omega^2 x \cdot \nabla_x (\nabla_v \cdot L[f]) dx dv + \int_{\Omega} f^2(t, x, v) \omega^4 \Delta_x (\nabla_v \cdot L[f]) dx dv \\ &\leq C_d \|\omega f(t)\|_{L_\omega^2}^2, \end{aligned}$$

where we use the fact $\max\{|\varphi|, |\varphi'|, |\varphi''|\} \leq 1$;

$$\int_{\Omega} J_6 dx dv \leq C_d \|\omega \nabla_x f(t)\|_{L_\omega^2}^2;$$

$$\begin{aligned}
\int_{\Omega} J_7 \, dx \, dv &= -2\sigma \sum_{i,j=1}^d \int_{\Omega} \partial_{v_i} (\omega^4 \partial_{x_j} f) (\partial_{v_i x_j}^2 f) \, dx \, dv \\
&= -2\sigma \sum_{i,j=1}^d \int_{\Omega} (\partial_{v_i} \omega^4 \partial_{x_j} f) (\partial_{v_i x_j}^2 f) \, dx \, dv - 2\sigma \sum_{i,j=1}^d \int_{\Omega} (\omega^4 \partial_{v_i x_j}^2 f) (\partial_{v_i x_j}^2 f) \, dx \, dv \\
&= -\sigma \sum_{i,j=1}^d \int_{\Omega} (\partial_{v_i} \omega^4) \partial_{v_i} |\partial_{x_j} f|^2 \, dx \, dv - 2\sigma \|\omega \nabla_v \nabla_x f(t)\|_{L_{\omega}^2}^2 \\
&= \sigma \sum_{i,j=1}^d \int_{\Omega} (\partial_{v_i v_i}^2 \omega^4) |\partial_{x_j} f|^2 \, dx \, dv - 2\sigma \|\omega \nabla_v \nabla_x f(t)\|_{L_{\omega}^2}^2 \\
&\leq -2\sigma \|\omega \nabla_v \nabla_x f(t)\|_{L_{\omega}^2}^2 + C_{d,\sigma} \|\omega \nabla_x f(t)\|_{L_{\omega}^2}^2.
\end{aligned}$$

Similar to the way of estimating the term A , we have

$$\int_{\Omega} J_8 \, dx \, dv \leq C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1} + \|\omega f(t)\|_{L^1}^2) (\|\omega^2 f(t)\|_{L_{\omega}^2}^2 + \|\omega \nabla_x f(t)\|_{L_{\omega}^2}^2) + \sigma \|\omega \nabla_v \nabla_x f(t)\|_{L_{\omega}^2}^2.$$

Adding up the above estimates for the integrals $\int_{\Omega} J_i \, dx \, dv$ ($i = 1, 2, \dots, 8$), we obtain

$$\begin{aligned}
&\frac{d}{dt} \|\omega \nabla_x f(t)\|_{L_{\omega}^2}^2 + \sigma \|\omega \nabla_x \nabla_v f(t)\|_{L_{\omega}^2}^2 \\
&\leq 2 \int_{\Omega} \omega^4 \nabla_v f(t, x, v) \cdot \nabla_x f(t, x, v) \, dx \, dv \\
&\quad + C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1} + \|\omega f(t)\|_{L^1}^2) (\|\omega \nabla_x f(t)\|_{L_{\omega}^2}^2 + \|\omega^2 f(t)\|_{L_{\omega}^2}^2).
\end{aligned} \tag{2.15}$$

Finally, we consider the last term of $\|f(t)\|_X^2$. Similar to the estimate of system (2.4), we multiply system (2.5) by $2\omega^6 f$ and integrate over Ω to have

$$\frac{d}{dt} \|\omega^2 f(t)\|_{L_{\omega}^2}^2 + 2\sigma \|\omega^2 \nabla_v f(t)\|_{L_{\omega}^2}^2 \leq C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1}) \|\omega^2 f(t)\|_{L_{\omega}^2}^2. \tag{2.16}$$

Adding up systems (2.14)–(2.16), one has

$$\frac{d}{dt} \|\omega^2 f(t)\|_X^2 + \sigma \|\nabla_v f(t)\|_X^2 \leq C_{d,\sigma} (1 + \|\omega f(t)\|_{L^1} + \|\omega f(t)\|_{L^1}^2) \|\omega^2 f(t)\|_X^2.$$

Applying Grönwall's lemma and the estimate $\|f(t)\|_{L_{\omega}^1}$ in Lemma 2.1, we obtain

$$\|f(t)\|_X^2 + \sigma \int_0^t \|\nabla_v f(\tau)\|_X^2 \, d\tau \leq \|f_0\|_X^2 \exp \left(\int_0^t C(\tau) \, d\tau \right),$$

where $C(t)$ depends on d, σ , and $\|f_0\|_{L_{\omega}^1}$. Therefore, we finish the proof.

3. Existence and uniqueness of strong solutions

In this section, we prove the local existence and uniqueness of the strong solution to system (1.1) by constructing a sequence of approximate solutions and extend the local existence to the global. First, let us recall a Grönwall-type lemma in [28].

Lemma 3.1. Let $T > 0$ and $(a_n)_{n \in \mathbb{N}}$ be the sequence of the nonnegative continuous functions on $[0, T]$. Assume that (a_n) satisfies

$$a_{n+1}(t) \leq A + B \int_0^t a_n(s) ds + C \int_0^t a_{n+1}(s) ds, \quad 0 \leq t \leq T,$$

where A, B , and C are nonnegative constants.

If $A = 0$, there exists a constant $K \geq 0$ depending on B, C such that

$$a_n(t) \leq \frac{K^n t^n}{n!}, \quad 0 \leq t \leq T, \quad n \in \mathbb{N}.$$

If $A > 0$, there exists a constant $K \geq 0$ depending on A, B, C such that

$$a_n(t) \leq K \exp(Kt), \quad 0 \leq t \leq T, \quad n \in \mathbb{N}.$$

3.1. Uniform bound on approximate solutions.

To begin with, we construct the following iteration scheme on finite time $[0, T]$.

$$\begin{cases} \partial_t f_n + v \cdot \nabla_x f_n - x \cdot \nabla_v f_n + \nabla_v \cdot (L[f_{n-1}]f_n) = \sigma \Delta_v f_n, & \text{in } [0, T] \times \Omega, \\ f_n(t, x, v)|_{t=0} = f_0(x, v), & \text{in } \{t = 0\} \times \Omega, \end{cases} \quad (3.1)$$

for $n \geq 1$. We define a sequence of approximate solutions $\{f_n\}$ as the solution to the above iterative system (3.1) by induction.

◊ Initial step ($n = 1$): we set

$$f_0(t, x, v) := f_0(x, v).$$

With this, we solve the initial value problem for the Cucker-Smale model with external potential field and noise:

$$\partial_t f_1 + v \cdot \nabla_x f_1 - x \cdot \nabla_v f_1 + \nabla_v \cdot [L(f_0)f_1] = \sigma \Delta_v f_1, \quad (x, v) \in \Omega$$

subject to

$$f_1(0, x, v) = f_0(x, v).$$

◊ Inductive step: Suppose we have the sequence of smooth approximate solutions $\{f_k\}_{k=1}^n$. Then, we can solve the following model:

$$\partial_t f_{n+1} + v \cdot \nabla_x f_{n+1} - x \cdot \nabla_v f_{n+1} + \nabla_v \cdot [L(f_n)f_{n+1}] = \sigma \Delta_v f_{n+1}, \quad \text{in } [0, T] \times \Omega, \quad (3.2)$$

subject to initial datum:

$$f_{n+1}(0, x, v) = f_0(x, v).$$

Thus, we can construct the smooth function $\{f_{n+1}\}$ from $\{f_n\}$. The solvability of system (3.2) is similar to that in the appendix of reference [29]. Therefore, the sequence $\{f_n\}$ is well-defined.

Paralleling to the a priori estimate for the solution f in Section 2, we can establish the uniform energy estimates for the approximate sequence f_n in the weighted Sobolev spaces.

Lemma 3.2. Let $T_1 > 0$. Assume the function $f_n(t, x, v)$ is a smooth solution to system (3.1) with the initial datum satisfying $f_0 \in C^\infty(\Omega) \cap X(\Omega) \cap L_\omega^1(\Omega)$ and $\int_\Omega f_0(x, v) dx dv = 1$. Then, for $\forall t \leq T_1$, we have

- (1) $\int_\Omega f_n(t) dx dv = 1$, $\|f_n(t)\|_{L^1} = 1$, and $f_n(t) \geq 0$;
- (2) $\|\omega f_n(t)\|_{L^1} \leq C$;
- (3) $\|f_n(t)\|_{L_\omega^2} \leq C$;

where C denotes the positive constant only depending on σ, d, T_1 , and the weighted norms of the initial datum f_0 .

Proof. (1) The results are obvious.

(2) We multiply system (3.1) by ω and integrate it over Ω to obtain

$$\begin{aligned}
 & \frac{d}{dt} \|\omega f_n(t)\|_{L^1} \\
 &= - \int_\Omega \omega v \cdot \nabla_x f_n(t, x, v) dx dv + \int_\Omega \omega x \cdot \nabla_v f_n(t, x, v) dx dv \\
 & \quad - \int_\Omega \omega \nabla_v \cdot (L[f_{n-1}] f_n(t, x, v)) dx dv + \sigma \int_\Omega \omega \Delta_v f_n(t, x, v) dx dv \\
 &= \int_\Omega f_n(t, x, v) L[f_{n-1}] \cdot \nabla_v \omega dx dv + \sigma \int_\Omega f_n(t, x, v) \Delta_v \omega dx dv \\
 &\leq C_d \|\omega f_{n-1}(t)\|_{L^1} + C_{d,\sigma} \|\omega f_n(t)\|_{L^1}.
 \end{aligned} \tag{3.3}$$

Then we integrate system (3.3) over $[0, t]$ to obtain

$$\|\omega f_n(t)\|_{L^1} \leq \|\omega f_0\|_{L^1} + C_d \int_0^t \|\omega f_{n-1}(\tau)\|_{L^1} d\tau + C_{d,\sigma} \int_0^t \|\omega f_n(\tau)\|_{L^1} d\tau. \tag{3.4}$$

Applying Lemma 3.1 to system (3.4), one gets

$$\|\omega f_n(t)\|_{L^1} \leq C \exp(Ct), \quad n \in \mathbb{N},$$

where the positive constant C depends on $\|f_0\|_{L_\omega^1}$, d , and σ . Then, for any given $T_1 > 0$, there exists a positive C depending σ, d, T_1 , and the weighted norms of the initial datum f_0 such that for all $n \geq 1$,

$$\|\omega f_n(t)\|_{L^1(\Omega)} \leq C, \quad \forall 0 \leq t \leq T_1.$$

(3) Following the way of the computation for equations (2.3) and (2.4), we can conclude that $\|f_n(t)\|_{L_\omega^2} \leq C$.

Proposition 3.1. Let $T_1 > 0$. Assume the function $f_n(t, x, v)$ is a smooth solution to system (3.1) with initial datum satisfying the condition of Lemma 3.2. Then, for $\forall t \leq T_1$, we have

- (1) $\left(\|f_n(t)\|_{H_\omega^1(\Omega)}^2 + \|\omega f_n(t)\|_{L_\omega^2}^2 \right) \leq C$;
- (2) $\|f_n(t)\|_X^2 \leq C$;

where C denotes positive constant only depending on σ, d, T_1 , and the weighted norms of the initial datum f_0 .

Proof. Following the similar proof of Proposition 2.1 in Section 2, we can obtain the parallel results.

3.2. Convergence of approximate solutions.

Next, we show that the approximate solution f_n is the Cauchy sequence in $C([0, T]; L^1_\omega(\Omega) \cap H^1_\omega(\Omega))$, where $0 < T < 1$. Setting $h_n := f_{n+1} - f_n$, it follows from system (3.1) that

$$\partial_t h_n = -v \cdot \nabla_x h_n + x \cdot \nabla_v h_n - \nabla_v \cdot (L[f_n]h_n) - \nabla_v \cdot (L[h_{n-1}]f_n) + \sigma \Delta_v h_n. \quad (3.5)$$

It is obvious to see that

$$h_n(0, x, v) = f_{n+1}(0, x, v) - f_n(0, x, v) = f_0(x, v) - f_0(x, v) \equiv 0.$$

Proposition 3.2. Assume that initial datum $f_0(x, v) \in C^\infty(\Omega) \cap X(\Omega) \cap L^1_\omega(\Omega)$, $\|f_0\|_{L^1(\Omega)} = 1$, and $f_0 \geq 0$. For any given positive small time $T < 1$, we have

$$\sup_{0 \leq t \leq T} \|h_n(t)\|_{L^1_\omega} \leq CT^n,$$

where C denotes the positive constant only depending on σ, d , and the weighted norms of the initial datum f_0 .

Furthermore, there exists a limit function $f(t, x, v) \in C([0, T]; L^1_\omega(\Omega))$ such that

$$\sup_{0 \leq t \leq T} \|f_n(t) - f(t)\|_{L^1_\omega} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proof. For arbitrary $T_1 > 0$ and $t \in [0, T_1]$, multiplying system (3.5) by $\omega \operatorname{sgn}(h_n)$ and integrating it over Ω by parts lead to

$$\begin{aligned} & \frac{d}{dt} \|\omega h_n(t)\|_{L^1} \\ &= - \int_{\Omega} \omega \operatorname{sgn}(h_n) v \cdot \nabla_x h_n \, dx \, dv + \int_{\Omega} \omega \operatorname{sgn}(h_n) x \cdot \nabla_v h_n \, dx \, dv \\ & \quad - \int_{\Omega} \omega \operatorname{sgn}(h_n) \nabla_v \cdot (L[f_n]h_n) \, dx \, dv - \int_{\Omega} \omega \operatorname{sgn}(h_n) \nabla_v \cdot (L[h_{n-1}]f_n) \, dx \, dv \\ & \quad + \sigma \int_{\Omega} \omega \operatorname{sgn}(h_n) \Delta_v h_n \, dx \, dv \\ &= \int_{\Omega} |h_n| L[f_n] \cdot \frac{v}{\omega} \, dx \, dv + \int_{\Omega} \operatorname{sgn}(h_n) f_n L[h_{n-1}] \cdot \frac{v}{\omega} \, dx \, dv + \sigma \int_{\Omega} \Delta_v \omega \operatorname{sgn}(h_n) h_n \, dx \, dv \\ &\leq C(\|\omega h_n(t)\|_{L^1} + \|\omega h_{n-1}(t)\|_{L^1}), \end{aligned} \quad (3.6)$$

where C depends on σ, d , and the weighed norms of the initial datum f_0 and T_1 , since we used the conclusion of Lemma 3.2. Then, we integrate system (3.6) to obtain

$$\|\omega h_n(t)\|_{L^1} \leq C \int_0^t \|\omega h_{n-1}(\tau)\|_{L^1} \, d\tau + C \int_0^t \|\omega h_n(\tau)\|_{L^1} \, d\tau.$$

Using Lemma 3.1, we can derive that

$$\|\omega h_n(t)\|_{L^1} \leq \frac{C^n t^n}{n!}, \quad n \in \mathbb{N}.$$

Then, for any given small time $0 < T < 1$, we have

$$\sup_{0 \leq t \leq T} \|h_n(t)\|_{L_\omega^1} \leq CT^n, \quad (3.7)$$

where C denotes the positive constant only depending on σ, d , and the weighted norms of the initial datum f_0 . This means that f_n is the Cauchy sequence in $C([0, T], L_\omega^1(\Omega))$. Moreover, there exists a limit function $f(t, x, v) \in C([0, T], L_\omega^1(\Omega))$ such that

$$\sup_{0 \leq t \leq T} \|f_n(t) - f(t)\|_{L_\omega^1} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proposition 3.3. *Assume that the initial datum $f_0(x, v)$ satisfies the condition of Proposition 3.2. For any given positive small time $T < 1$, we have*

$$(1) \sup_{0 \leq t \leq T} \|h_n(t)\|_{L_\omega^2}^2 + \int_0^T \|\nabla_v h_n(\tau)\|_{L_\omega^2}^2 d\tau \leq CT^n; \quad (3.8)$$

$$(2) \sup_{0 \leq t \leq T} \|\omega h_n(t)\|_{L_\omega^2}^2 + \int_0^T \|\omega \nabla_v h_n(\tau)\|_{L_\omega^2}^2 d\tau \leq CT^n; \quad (3.9)$$

$$(3) \sup_{0 \leq t \leq T} \|h_n(t)\|_{H_\omega^1}^2 + \int_0^T \|\nabla_v h_n(\tau)\|_{H_\omega^1}^2 d\tau \leq CT^n; \quad (3.10)$$

where C denotes the positive constant only depending on σ, d , and the weighted norms of the initial datum f_0 . Furthermore, there exists a limit function $f(t, x, v) \in C([0, T]; H_\omega^1(\Omega))$ such that

$$\sup_{0 \leq t \leq T} \|f_n(t) - f(t)\|_{H_\omega^1(\Omega)} + \|\nabla_v f_n(t) - \nabla_v f(t)\|_{L^2(0, T; H_\omega^1(\Omega))} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proof. (1) Multiplying system (3.5) by $2\omega^2 h_n$ leads to

$$\begin{aligned} \frac{d}{dt} (\omega^2 h_n^2) &= -\omega^2 v \cdot \nabla_x (h_n^2) + \omega^2 x \cdot \nabla_v (h_n^2) \\ &\quad - \omega^2 \nabla_v (h_n^2) \cdot L[f_n] - 2\omega^2 h_n^2 \nabla_v \cdot L[f_n] \\ &\quad - 2\omega^2 \nabla_v \cdot L[h_{n-1}] h_n f_n - 2\omega^2 h_n \nabla_v f_n \cdot L[h_{n-1}] \\ &\quad + 2\sigma \omega^2 h_n \Delta_v h_n = \sum_{i=1}^7 K_i. \end{aligned} \quad (3.11)$$

Similar to the way to estimate system (2.3), we only need to estimate two extra terms $\int_\Omega K_5 dx dv$ and $\int_\Omega K_6 dx dv$. Note that

$$K_5 \leq C_d \|\omega h_{n-1}(t)\|_{L^1} \left(\|h_n(t)\|_{L_\omega^2}^2 + \|f_n(t)\|_{L_\omega^2}^2 \right),$$

$$\begin{aligned}
|K_6| &\leq \left| \int_{\Omega \times \Omega^*} \varphi(|x-y|) \omega^* h_{n-1}(t, y, v^*) \omega^3 h_n(t, x, v) \nabla_v f_n(t, x, v) dy dv^* dx dv \right| \\
&\leq \|\omega h_{n-1}\|_{L^1} \left| \int_{\Omega} \omega^3 h_n(t, x, v) \nabla_v f_n(t, x, v) dx dv \right| \\
&\leq \|\omega h_{n-1}\|_{L^1} \left| \int_{\Omega} f_n(t, x, v) \nabla_v (\omega^3 h_n(t, x, v)) dx dv \right| \\
&\leq C_{d,\sigma} \|\omega h_{n-1}(t)\|_{L^1} (\|h_n(t)\|_{L_\omega^2}^2 + \|f_n(t)\|_{L_\omega^2}^2 + \|\omega f_n(t)\|_{L_\omega^2}^2) + \sigma \|\nabla_v h_n(t)\|_{L_\omega^2}^2,
\end{aligned}$$

where $\omega^* = (1 + y^2 + v^{*2})^{\frac{1}{2}}$, and we have used the σ -Young's inequality.

Thus, by integrating system (3.11) over Ω and using the estimate of Lemma 3.2 and Proposition 3.1, we have

$$\begin{aligned}
&\frac{d}{dt} \|h_n(t)\|_{L_\omega^2}^2 + \sigma \|\nabla_v h_n(t)\|_{L_\omega^2}^2 \\
&\leq C_{d,\sigma} (1 + \|\omega f_n(t)\|_{L^1}) \|h_n(t)\|_{L_\omega^2}^2 + C_d \|\omega h_{n-1}(t)\|_{L^1} (\|h_n(t)\|_{L_\omega^2}^2 + \|f_n(t)\|_{L_\omega^2}^2) \\
&\quad + C_{d,\sigma} \|\omega h_{n-1}(t)\|_{L^1} (\|h_n(t)\|_{L_\omega^2}^2 + \|f_n(t)\|_{L_\omega^2}^2 + \|\omega f_n(t)\|_{L_\omega^2}^2) \\
&\leq C(\|h_n(t)\|_{L_\omega^2}^2 + \|h_{n-1}(t)\|_{L_\omega^1}).
\end{aligned} \tag{3.12}$$

Here, C depends on d, σ , and weighed norms of initial datum.

By applying Grönwall's lemma to system (3.12) on $0 < t < 1$ and using the conclusion of Proposition 3.2, we obtain

$$\begin{aligned}
&\|h_n(t)\|_{L_\omega^2}^2 + \sigma \int_0^t \|\nabla_v h_n(\tau)\|_{L_\omega^2}^2 d\tau \\
&\leq e^{Ct} \left(\|h_n(0)\|_{L_\omega^2}^2 + \int_0^t C \|h_{n-1}(\tau)\|_{L_\omega^1} d\tau \right) \\
&\leq e^{Ct} C t^{n-1} t.
\end{aligned} \tag{3.13}$$

Then for any given small time $T < 1$, we have

$$\sup_{0 \leq t \leq T} \left\{ \|h_n(t)\|_{L_\omega^2}^2 + \sigma \int_0^t \|\nabla_v h_n(\tau)\|_{L_\omega^2}^2 d\tau \right\} \leq CT^n. \tag{3.14}$$

Hence, we obtain the estimate system (3.8).

(2) Multiplying system (3.5) by $2\omega^4 h_n$ and integrating it over Ω leads to

$$\frac{d}{dt} \|\omega h_n(t)\|_{L_\omega^2}^2 + \sigma \|\omega \nabla_v h_n(t)\|_{L_\omega^2}^2 \leq C (\|h_{n-1}(t)\|_{L_\omega^1} + \|\omega h_n(t)\|_{L_\omega^2}^2),$$

which is similar in form to the result of system (3.12). Hence, we obtain estimate system (3.9).

(3) Applying ∇_v to system (3.5) gives

$$\begin{aligned}
 \partial_t \nabla_v h_n &= -\nabla_x h_n - v \cdot \nabla_x (\nabla_v h_n) + x \cdot \nabla_v (\nabla_v h_n) - \nabla_v (\nabla_v h_n \cdot L[f_n]) \\
 &\quad - (\nabla_v \cdot L[f_n]) \nabla_v h_n - (\nabla_v \cdot L[h_{n-1}]) \nabla_v f_n - \nabla_v (\nabla_v f_n \cdot L[h_{n-1}]) \\
 &\quad + \sigma \Delta_v (\nabla_v h_n) \\
 &= -\nabla_x h_n - v \cdot \nabla_x (\nabla_v h_n) + x \cdot \nabla_v (\nabla_v h_n) - \nabla_v (\nabla_v h_n) \cdot L[f_n] \\
 &\quad + (d+1) \left(\int_{\Omega} \varphi(|x-y|) f_n(t, y, v^*) dy dv^* \right) \nabla_v h_n + \sigma \Delta_v (\nabla_v h_n) \\
 &\quad - \underbrace{\nabla_v^2 f_n \cdot L[h_{n-1}]}_{L_1} + (d+1) \underbrace{\left(\int_{\Omega} \varphi(|x-y|) h_{n-1}(t, y, v^*) dy dv^* \right) \nabla_v f_n}_{L_2}.
 \end{aligned} \tag{3.15}$$

We multiply system (3.15) by $2\omega^2 \nabla_v h_n$ and integrate it over Ω . Comparing to system (2.7), we only need to estimate two extra terms $2 \int_{\Omega} \omega^2 \nabla_v h_n \cdot L_1 dx dv$ and $2 \int_{\Omega} \omega^2 \nabla_v h_n \cdot L_2 dx dv$. Note that

$$\begin{aligned}
 &\int_{\Omega} L_1 \cdot (2\omega^2 \nabla_v h_n) dx dv \\
 &= -2 \sum_{i,j=1}^d \int_{\Omega} \omega^2 \frac{\partial h_n(t, x, v)}{\partial v_i} \frac{\partial^2 f_n(t, x, v)}{\partial v_i \partial v_j} L[h_{n-1}]_j dx dv \\
 &= 2 \sum_{i,j=1}^d \int_{\Omega} \frac{\partial f_n(t, x, v)}{\partial v_i} \frac{\partial}{\partial v_j} \left(\omega^2 \frac{\partial h_n(t, x, v)}{\partial v_i} L[h_{n-1}]_j \right) dx dv \\
 &= 4 \int_{\Omega} (v \cdot L[h_{n-1}]) \nabla_v f_n(t, x, v) \cdot \nabla_v h_n(t, x, v) dx dv \\
 &\quad + 2 \int_{\Omega} \omega^2 \nabla_v f_n(t, x, v) \cdot (\nabla_v^2 h_n(t, x, v) \cdot L[h_{n-1}]) dx dv \\
 &\quad + 2 \int_{\Omega} \omega^2 (\nabla_v \cdot L[h_{n-1}]) \nabla_v f_n(t, x, v) \cdot \nabla_v h_n(t, x, v) dx dv \\
 &\leq C_{d,\sigma} (\|\omega h_{n-1}(t)\|_{L^1} + \|\omega h_{n-1}(t)\|_{L^1}^2) (\|\nabla_v f_n(t)\|_{L_{\omega}^2}^2 + \|\nabla_v h_n(t)\|_{L_{\omega}^2}^2 + \|\omega \nabla_v f_n(t)\|_{L_{\omega}^2}^2) \\
 &\quad + \frac{\sigma}{2} \|\nabla_v^2 h_n(t)\|_{L_{\omega}^2}^2,
 \end{aligned} \tag{3.16}$$

where we have used the σ -Young's inequality.

$$\int_{\Omega} L_2 \cdot (2\omega^2 \nabla_v h_n) dx dv \leq C_d \|\omega h_{n-1}(t)\|_{L^1} (\|\nabla_v f_n(t)\|_{L_{\omega}^2}^2 + \|\nabla_v h_n(t)\|_{L_{\omega}^2}^2). \tag{3.17}$$

Thus, similar to system (2.7), we use systems (3.16) and (3.17) to obtain

$$\begin{aligned}
 &\frac{d}{dt} \|\nabla_v h_n(t)\|_{L_{\omega}^2}^2 + \frac{\sigma}{2} \|\nabla_v^2 h_n(t)\|_{L_{\omega}^2}^2 \\
 &\leq -2 \int_{\Omega} \omega^2 \nabla_x h_n(t, x, v) \cdot \nabla_v h_n(t, x, v) dx dv \\
 &\quad + C_{d,\sigma} (1 + \|\omega f_n(t)\|_{L^1} + \|\omega h_{n-1}(t)\|_{L^1} + \|\omega h_{n-1}(t)\|_{L^1}^2) \|\nabla_v h_n(t)\|_{L_{\omega}^2}^2 \\
 &\quad + C_{d,\sigma} (\|\omega h_{n-1}(t)\|_{L^1} + \|\omega h_{n-1}(t)\|_{L^1}^2) (\|\nabla_v f_n(t)\|_{L_{\omega}^2}^2 + \|\omega \nabla_v f_n(t)\|_{L_{\omega}^2}^2).
 \end{aligned} \tag{3.18}$$

Applying ∇_x to system (3.5) gives

$$\begin{aligned} \partial_t \nabla_x h_n = & \nabla_v h_n - v \cdot \nabla_x (\nabla_x h_n) + x \cdot \nabla_v (\nabla_x h_n) - \nabla_x (\nabla_v \cdot L[f_n]) h_n \\ & - (\nabla_v \cdot L[f_n]) \nabla_x h_n - \nabla_x (\nabla_v h_n \cdot L[f_n]) + \sigma \Delta_v (\nabla_x h_n) \\ & - \underbrace{\nabla_x (\nabla_v \cdot L[h_{n-1}]) f_n}_{L_3} - \underbrace{(\nabla_v \cdot L[h_{n-1}]) \nabla_x f_n}_{L_4} - \underbrace{\nabla_x (\nabla_v f_n \cdot L[h_{n-1}])}_{L_5}. \end{aligned} \quad (3.19)$$

We multiply system (3.19) by $2\omega^2 \nabla_x h_n$ and integrate it over Ω . Comparing to system (2.9), we only estimate the extra terms $2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot L_3 \, dx \, dv$, $2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot L_4 \, dx \, dv$, and $2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot L_5 \, dx \, dv$.

$$\begin{aligned} 2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot L_3 \, dx \, dv & \leq C_d \|\omega h_{n-1}(t)\|_{L^1} \left(\|f_n(t)\|_{L^2_{\omega}}^2 + \|\nabla_x h_n(t)\|_{L^2_{\omega}}^2 \right), \\ 2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot L_4 \, dx \, dv & \leq C_d \|\omega h_{n-1}(t)\|_{L^1} \left(\|\nabla_x f_n(t)\|_{L^2_{\omega}}^2 + \|\nabla_x h_n(t)\|_{L^2_{\omega}}^2 \right), \end{aligned} \quad (3.20)$$

$$\begin{aligned} & 2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot L_5 \, dx \, dv \\ = & -2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot (\nabla_x \nabla_v f_n \cdot L[h_{n-1}]) \, dx \, dv - 2 \int_{\Omega} \omega^2 \nabla_x h_n \cdot (\nabla_x L[h_{n-1}] \cdot \nabla_v f_n) \, dx \, dv \\ = & :Q_1 + Q_2. \end{aligned}$$

Note that

$$\begin{aligned} Q_1 = & -2 \sum_{i,j=1}^d \int_{\Omega} \omega^2 \frac{\partial h_n(t, x, v)}{\partial x_i} \frac{\partial^2 f_n(t, x, v)}{\partial x_i \partial v_j} L[h_{n-1}]_j \, dx \, dv \\ = & 2 \sum_{i,j=1}^d \int_{\Omega} \frac{\partial f_n(t, x, v)}{\partial x_i} \frac{\partial}{\partial v_j} \left(\omega^2 \frac{\partial h_n(t, x, v)}{\partial x_i} L[h_{n-1}]_j \right) \, dx \, dv \\ = & 4 \int_{\Omega} (v \cdot L[h_{n-1}]) \nabla_x f_n(t, x, v) \cdot \nabla_x h_n(t, x, v) \, dx \, dv \\ & + 2 \int_{\Omega} \omega^2 \nabla_x f_n(t, x, v) \cdot (\nabla_x \nabla_v h_n(t, x, v) \cdot L[h_{n-1}]) \, dx \, dv \\ & + 2 \int_{\Omega} (\nabla_v \cdot L[h_{n-1}]) \omega^2 \nabla_x f_n(t, x, v) \cdot \nabla_x h_n(t, x, v) \, dx \, dv \\ \leq & C_{d,\sigma} (\|\omega h_{n-1}(t)\|_{L^1} + \|\omega h_{n-1}(t)\|_{L^1}^2) (\|\omega \nabla_x f_n(t)\|_{L^2_{\omega}}^2 + \|\nabla_x h_n(t)\|_{L^2_{\omega}}^2 + \|\nabla_x f_n(t)\|_{L^2_{\omega}}^2) \\ & + \frac{\sigma}{2} \|\nabla_v \nabla_x h_n(t)\|_{L^2_{\omega}}^2, \end{aligned} \quad (3.21)$$

where we have used the σ -Young's inequality.

$$\begin{aligned} Q_2 = & -2 \sum_{i,j=1}^d \int_{\Omega} \omega^2 \frac{\partial h_n(t, x, v)}{\partial x_i} \frac{\partial f_n(t, x, v)}{\partial v_j} \frac{\partial L[h_{n-1}]_j}{\partial x_i} \, dx \, dv \\ \leq & C_d \|\omega h_{n-1}(t)\|_{L^1} \left(\|\nabla_x h_n(t)\|_{L^2_{\omega}}^2 + \|\omega \nabla_v f_n(t)\|_{L^2_{\omega}}^2 \right). \end{aligned} \quad (3.22)$$

Thus, similar to system (2.12), we use systems (3.20)–(3.22) to obtain

$$\begin{aligned}
 & \frac{d}{dt} \|\nabla_x h_n(t)\|_{L_\omega^2}^2 + \frac{\sigma}{2} \|\nabla_v \nabla_x h_n(t)\|_{L_\omega^2}^2 \\
 & \leq 2 \int_{\Omega} \omega^2 \nabla_x h_n(t, x, v) \cdot \nabla_v h_n(t, x, v) \, dx \, dv \\
 & + C_{d,\sigma} ((1 + \|\omega f_n(t)\|_{L^1}) (\|\nabla_x h_n(t)\|_{L_\omega^2}^2 + \|h_n(t)\|_{L_\omega^2}^2) + \|\omega h_n(t)\|_{L_\omega^2}^2 \\
 & + \|\omega h_{n-1}(t)\|_{L^1} (\|f_n(t)\|_{L_\omega^2}^2 + \|\nabla_x f_n(t)\|_{L_\omega^2}^2 + \|\nabla_v f_n(t)\|_{L_\omega^2}^2 + \|\nabla_x h_n(t)\|_{L_\omega^2}^2) \\
 & + \|\omega h_{n-1}(t)\|_{L^1} (\|\omega \nabla_v f_n(t)\|_{L_\omega^2}^2 + \|\omega \nabla_x f_n(t)\|_{L_\omega^2}^2)).
 \end{aligned} \tag{3.23}$$

Adding up systems (3.12), (3.18), and (3.23), and applying Lemma 3.2-Proposition 3.2 and Proposition 3.3(2), we obtain

$$\frac{d}{dt} \|h_n(t)\|_{H_\omega^1}^2 + \frac{\sigma}{2} \|\nabla_v h_n(t)\|_{H_\omega^1}^2 \leq C \|h_n(t)\|_{H_\omega^1}^2 + C t^{n-1}.$$

Similar to systems (3.13) and (3.14), for any given positive small time $T < 1$, we have

$$\sup_{0 \leq t \leq T} \|h_n(t)\|_{H_\omega^1}^2 + \frac{\sigma}{2} \int_0^T \|\nabla_v h_n(\tau)\|_{H_\omega^1}^2 \, d\tau \leq C T^n, \tag{3.24}$$

which is system (3.10). This means that the approximate solution f_n is the Cauchy sequence in $C([0, T]; H_\omega^1(\Omega))$. Thus, it converges strongly to the limit function $f \in C([0, T]; H_\omega^1(\Omega))$ as $n \rightarrow \infty$.

With the help of estimates system (3.7) in Proposition 3.2 and (3.24) in Proposition 3.3, we conclude that there exists a constant $C > 0$ depending on d, σ , and weighed norms of initial datum, such that for any given positive small time $0 < T < 1$,

$$\sup_{0 \leq t \leq T} (\|h_n(t)\|_{L_\omega^1} + \|h_n(t)\|_{H_\omega^1}^2) \leq C T^n.$$

Therefore, the limit function f of Cauchy sequence $f_n \in C([0, T]; L_\omega^1(\Omega) \cap H_\omega^1(\Omega))$ is a local strong solution to the system (1.1). The uniqueness of the solution can be derived easily. Let f and $\bar{f}$ be the two strong solutions above corresponding to the same initial datum f_0 . Set

$$\mathcal{E}(t) := \|f(t) - \bar{f}(t)\|_{L_\omega^1(\Omega)} + \|f(t) - \bar{f}(t)\|_{H_\omega^1(\Omega)}^2.$$

Then, by the same argument as in Lemma 3.2-Proposition 3.3, $\mathcal{E}(t)$ satisfies Grönwall's inequality:

$$\mathcal{E}(t) + \int_0^t \|\nabla_v (f - \bar{f})(\tau)\|_{H_\omega^1}^2 \, d\tau \leq C \int_0^t \mathcal{E}(\tau) \, d\tau, \quad \mathcal{E}(0) = 0,$$

and the standard Grönwall's lemma implies that

$$\mathcal{E}(t) = 0, \quad i.e., \quad f \equiv \bar{f} \text{ in } C([0, T]; L_\omega^1(\Omega) \cap H_\omega^1(\Omega)),$$

which gives the uniqueness of the local solution.

3.3. Proof of Theorem 1.1.

When initial datum f_0 is smooth, the limit function f from Proposition 3.2–Proposition 3.3 is the unique local smooth solution to system (1.1). Combining with Lemma 2.1 and Proposition 2.1, one can extend the local smooth solution to be global-in-time. Hence, we obtain the global smooth solution.

When initial datum f_0 is not smooth, we first mollify the initial datum by convolution, i.e.,

$$f_0^\epsilon(x, v) = f_0(x, v) * j_\epsilon(x, v),$$

where $j_\epsilon(x, v)$ is the standard mollifier. Then, we consider the following modified system

$$\begin{cases} \partial_t f_n^\epsilon + v \cdot \nabla_x f_n^\epsilon - x \cdot \nabla_v f_n^\epsilon + \nabla_v \cdot (L[f_{n-1}^\epsilon] f_n^\epsilon) = \sigma \Delta_v f_n^\epsilon, & \text{in } [0, T] \times \Omega, \\ f_n^\epsilon(t, x, v)|_{t=0} = f_0^\epsilon(x, v), & \text{in } \{t = 0\} \times \Omega, \end{cases}$$

Following the basic idea about the proof of Theorem 3.1 and Theorem 3.2 in [24], we can also prove that there exists a sequence ϵ_k , with $\epsilon_k \rightarrow 0$, such that

$$f^{\epsilon_k} \rightarrow f \quad \text{in } C([0, \infty); L_\omega^1(\Omega) \cap H_\omega^1(\Omega))$$

and f satisfies system (1.1). Hence, the limit function $f(t, x, v)$ is the desired unique strong solution.

Author contributions

All authors contributed equally to the study and the writing of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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