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*Research article*

# Global existence of 3D rotating magnetohydrodynamic equations arising from Earth's fluid core

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**Abstract:** The paper is concerned with the three-dimensional magnetohydrodynamic equations in the rotational framework concerning with fluid flow of Earth's core and the variation of the Earth's magnetic field. By establishing new balances between the regularizing effects arising from viscosity dissipation and magnetic diffusion with the dispersive effects caused by the rotation of the Earth, we obtain the global existence and uniqueness of solutions of the Cauchy problem of the three-dimensional rotating magnetohydrodynamic equations in Besov spaces. Moreover, the spatial analyticity of solutions is verified by means of the Gevrey class approach.

**Keywords:** magnetohydrodynamic equations; rotating fluids; global solutions; Gevrey regularity

## 1. Introduction

Multiple pieces of evidence indicate that the Earth's magnetic field existed at least 3.45 billion years ago, and it is constantly changing, and the way in which it changes also changes. Especially, the geomagnetic reversals happen every several hundred centuries. Shortly after fulfilling the theory of general relativity, science master A. Einstein attributed the generation and maintenance of geomagnetic field to one of the major unsolved problems in the field of physics.

From a numerical, experimental, physical, or mathematical point of view, much work has been done trying to find the underlying mechanism and rules of operation of the geomagnetic field. Among numerous theories, the self-excitation dynamo theory is universally accepted, which is to say that the outer core of Earth is often thought of as a giant dynamo that generates the Earth's magnetic field due to the motion of the conductive fluid; see [1–3]. Specifically, the liquid metal circulation that constitutes the Earth's outer core moves under the weak magnetic field to generate electric current, and then the additional magnetic field generated by the electric current will strengthen the original weak magnetic field. Under the action of electromagnetic coupling effect, the magnetic field is continuously enhanced and amplified, and finally the Earth's magnetic field is formed.

On the other hand, Coriolis forces, generated by the rotation of the Earth, cannot be ignored. Physically, Coriolis forces deflect the upwelling fluid along corkscrewlike, or helical, paths, as though it were following the spiraling wire of a loose spring. Mathematically, Coriolis forces give rise to the so-called Poincaré waves, which are dispersive waves, see [4]. Based on a comprehensive consideration, one possible model is the following one describing the magnetohydrodynamic phenomena with a reasonable addition of the Coriolis forces:

$$\begin{cases} \partial_t u - \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla p = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \partial_t B - \Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \mathrm{div} u = 0, \quad \mathrm{div} B = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u|_{t=0} = u_0, \;\; B|_{t=0} = B_0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$, $B = (B_1, B_2, B_3)$, and $p$ are the fluid velocity field, the magnetic field, and the fluid pressure, respectively. $\Omega \in \mathbb{R}\backslash\{0\}$ denotes the Coriolis parameter, and $\Omega e_3 \times u$ represents the so-called Coriolis force with rotation axis $e_3 = (0, 0, 1)$. We refer readers to Section 10.2 of [4] for the derivation of system (1.1).

When $\Omega = 0$, Eq (1.1) becomes the 3D magnetohydrodynamic equations. Duvaut and Lions [5] established local well-posedness results of the 3D magnetohydrodynamic equations in $H^s(\mathbb{R}^n)$ with $s \geq n$ and global well-posedness results for the small initial data. Zhai et al. [6] established the global well-posedness of the 3D magnetohydrodynamic equations for initial data in critical Besov spaces and relaxed the smallness condition in the third components of the initial velocity field and initial magnetic field. For more relevant studies on the existence of solutions of the magnetohydrodynamic equations, we refer to [7–10]. However, the global well-posedness or global regularity for the 3D magnetohydrodynamic equations is still a challenging open problem.

When $\Omega \neq 0$ but $B \equiv 0$, Eq (1.1) becomes the 3D rotating Navier-Stokes equations. Chemin et al. [4, 11] showed that for any given $L^2(\mathbb{R}^2) + H^{\frac{1}{2}}(\mathbb{R}^3)$-initial data, there exists a positive constant $\Omega_0$ such that the 3D rotating Navier-Stokes equations are globally well-posed provided that $|\Omega| \geq \Omega_0$. Iwabuchi and Takada [12] proved that the 3D rotating Navier-Stokes equations is globally well-posed for $u_0 \in \dot{H}^s(\mathbb{R}^3)$ with $\frac{1}{2} < s < \frac{3}{4}$ satisfying

$$\|u_0\|_{\dot{H}^s} \leq C|\Omega|^{\frac{1}{2}(s-\frac{1}{2})}.$$

Later on, Koh et al. [13] and Sun et al. [14] relaxed the range of $s$ to $\frac{1}{2} < s < \frac{9}{10}$ and $\frac{1}{2} < s < 1$, respectively. We may refer to [15–18] for the global existence results on 3D rotating Navier-Stokes equations with uniformly small initial data. We also refer to [19–23] and the references therein for the global existence results on other models involving the Coriolis forces.

Recently, Ngo [24] studied Eq (1.1) with horizontal diffusion terms only and showed that the large Coriolis parameter implies global solvability for large initial data provided that $B_0$ is a perturbation of $e_3$. Ahn et al. [25] proved the existence and uniqueness of global solutions of Eq (1.1) for $u_0 \in H^s(\mathbb{R}^3)$ and $B_0 \in (L^2 \bigcap L^q)(\mathbb{R}^3)$ with $\frac{1}{2} < s < \frac{3}{4}$ and $3 < q < \min\{\frac{6}{3-2s}, \frac{27}{6+2s}\}$, when the Coriolis parameter is sufficiently large. Kim [26] proved the global existence and uniqueness of smooth solutions in $H^s(\mathbb{R}^3)$

with $\frac{1}{2} < s < \frac{3}{4}$ under large Coriolis parameter and further obtained the temporal decay estimates for the solutions.

We would like to point out that there is a great difference between the 3D rotational magnetohydrodynamic equations and the 3D magnetohydrodynamic equations or rotating Navier-Stokes equations. More specifically, the 3D magnetohydrodynamic equations (i.e., Eq (1.1) with $\Omega = 0$) are a purely dissipative system, and the 3D rotating Navier-Stokes equations (i.e., Eq (1.1) with $B \equiv 0$) are a system of dissipative-dispersive jointly type, whereas in Eq (1.1), the partition of flow field is of dissipative-dispersive jointly type and the partition of magnetic field is of dissipative type. The structural asymmetry of Eq (1.1) makes the problem much harder than that for the other ones.

The main aim of this paper is to investigate the global existence issue of the Cauchy problem of the 3D rotational magnetohydrodynamic equations and further verify the spatial analyticity for the obtained global solutions by adopting the famous Gevrey class approach (see [27–29]). Specifically, by establishing new balances between the regularizing effects arising from viscosity dissipation and magnetic diffusion with the dispersive effects caused by the rotation of the Earth, we shall show the existence and uniqueness of global mild solutions to Eq (1.1) for large initial data in Besov spaces under a large Coriolis parameter. Furthermore, let $\mathcal{X}$ be a Banach space and let $\Lambda_1$ be the pseudo-differential operator with symbol given by $|\xi|_1 := \sum_{i=1}^{3} |\xi_i|$; we will prove that the obtained solutions $(u, B) \in \mathcal{X}$ of problem (1.1) hold $\|e^{\sqrt{t}\Lambda_1} u\|_{\mathcal{X}} < \infty$, which implies the spatial analyticity of solutions. The main results of this paper are as follows.

**Theorem 1.1.** *Let $p \in (\frac{3}{2}, 2)$, $r \in [1, \infty]$, $\delta \in (2, \infty)$ and $\rho \in (2, 2\delta)$ satisfy*

$$0 < \frac{1}{\delta} < \min\Big\{\frac{2}{p} - 1, 2 - \frac{3}{p}\Big\} \quad \text{and} \quad \frac{1}{\rho} < 2 - \frac{3}{p}.$$

*Then there exist two positive constants, c and C, such that for $(u_0, B_0) \in \dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}(\mathbb{R}^3) \times \dot{B}_{p,r}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ satisfying $divu_0 = 0$, $divB_0 = 0$ and*

$$\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}} \le C|\Omega|^{\frac{1}{\delta}} \quad \text{and} \quad \|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}} \le c, \tag{1.2}$$

*Eq (1.1) has a unique global mild solution*

$$(u, B) \in \tilde{L}^{\delta}(0, \infty; e^{\theta\sqrt{t}\Lambda_1} \dot{B}_{p',r}^{-1+\frac{3}{p'}+\frac{2}{\delta}}(\mathbb{R}^3)) \times \tilde{L}^{\rho}(0, \infty; e^{\theta\sqrt{t}\Lambda_1} \dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\rho}}(\mathbb{R}^3)),$$

*with $\theta \in \{0, 1\}$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*

The next result involves the case of $p = 2$.

**Theorem 1.2.** *Let $q \in (2, 3)$, $r \in [1, \infty]$, $\delta \in (2, \infty)$ and $\rho \in (2, 2\delta)$ satisfy*

$$0 < \frac{1}{\delta} < \min\{\frac{1}{2} - \frac{1}{q}, \frac{3}{q} - 1\} \quad \text{and} \quad \frac{1}{\rho} < \frac{3}{q} - 1.$$

*Then there exist two positive constants, c and C, such that for $(u_0, B_0) \in \dot{B}_{2,r}^{\frac{1}{2}+\frac{2}{\delta}}(\mathbb{R}^3) \times \dot{B}_{q,r}^{-1+\frac{3}{q}}(\mathbb{R}^3)$ satisfying $divu_0 = 0$, $divB_0 = 0$, and*

$$\|u_0\|_{\dot{B}_{2,r}^{\frac{1}{2}+\frac{2}{\delta}}} \le C|\Omega|^{\frac{1}{\delta}} \quad \text{and} \quad \|B_0\|_{\dot{B}_{q,r}^{-1+\frac{3}{q}}} \le c, \tag{1.3}$$

*Eq* (1.1) *has a unique global mild solution*

$$(u, B) \in \tilde{L}^{\delta}(0, \infty; e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{q,r}^{-1+\frac{3}{q}+\frac{2}{\delta}}(\mathbb{R}^3)) \times \tilde{L}^{\rho}(0, \infty; e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{q,r}^{-1+\frac{3}{q}+\frac{2}{\rho}}(\mathbb{R}^3)),$$

*with* $\theta \in \{0, 1\}$.

**Remark 1.3.** *Theorems 1.1 and 1.2 with* $\theta = 0$ *indicate that for any given* $u_0 \in \dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}(\mathbb{R}^3)$ *and sufficiently small* $B_0 \in \dot{B}_{q,r}^{-1+\frac{3}{q}}(\mathbb{R}^3)$ *with specified p, r, δ, and q, Eq* (1.1) *admits a unique global mild solution if the Coriolis parameter* $|\Omega|$ *is large enough. Moreover, Theorems 1.1 and 1.2 with* $\theta = 1$ *imply these solutions possess Gevrey analyticity in the spatial variables. The main results in this paper are seen as a generalization of the global existence results on the 3D rotating Navier-Stokes equations to the 3D rotating magnetohydrodynamic equations.*

Throughout the paper, we denote by $c$ and $C$ the constants, which may differ in each line. $C = C(\cdot, \cdots, \cdot)$ denotes the constant, which depends only on the quantities appearing in parentheses.

## 2. Preliminaries

Let $\mathscr{S}(\mathbb{R}^3)$ be the set of Schwartz functions, $\mathscr{S}'(\mathbb{R}^3)$ be the set of tempered distributions, and $\{\psi_j\}_{j\in\mathbb{Z}}$ be a dyadic partition of unity satisfying

$$\text{supp}\,\widehat{\psi_0} \subset \{\xi \in \mathbb{R}^3 : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}, \quad 0 \le \widehat{\psi_0} \le 1, \quad \widehat{\psi_j}(\xi) := \widehat{\psi_0}(2^{-j}\xi),$$

and

$$\sum_{j\in\mathbb{Z}} \widehat{\psi_j}(\xi) = 1 \qquad \text{for } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Here $\hat{f}$ represents the Fourier transform of $f$. Define the Littlewood-Paley frequency localized operator $\Delta_j$ in $\mathscr{S}'(\mathbb{R}^3)$ by:

$$\Delta_j f := \psi_j * f \qquad \text{for } j \in \mathbb{Z} \text{ and } f \in \mathscr{S}'(\mathbb{R}^3).$$

Firstly, we give the definitions and product law of the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^3)$ and Chemin-Lerner spaces $\tilde{L}^{\delta}(0, \infty; \dot{B}^s_{p,r}(\mathbb{R}^3))$. The Chemin-Lerner spaces were first introduced by [30].

**Definition 2.1.** *(Besov Space, [31]) Let* $s \in \mathbb{R}$ *and* $1 \le p, r \le \infty$*, we define*

$$\|u\|_{\dot{B}^s_{p,r}} := \left\| \{2^{js}\|\Delta_j u\|_{L^p}\}_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}.$$

• *For* $s < \frac{3}{p}$ *(or* $s = \frac{3}{p}$*, if* $r = 1$*), we define* $\dot{B}^s_{p,r}(\mathbb{R}^3) := \{u \in \mathscr{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3] \mid \|u\|_{\dot{B}^s_{p,r}} < \infty\}$*, where* $\mathcal{P}[\mathbb{R}^3]$ *is the set of all polynomials on* $\mathbb{R}^3$*;*

• *If* $k \in \mathbb{N}$, $\frac{3}{p} + k \le s < \frac{3}{p} + k + 1$ *(or* $s = \frac{3}{p} + k + 1$*, if* $r = 1$*), then* $\dot{B}^s_{p,r}(\mathbb{R}^3)$ *is defined as the subset of* $\mathscr{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3]$ *such that* $\partial^{\delta}u \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ *with* $|\delta| = k$.

**Definition 2.2.** *(Chemin-Lerner Space, [30, 31]) For $s \in \mathbb{R}$ and $1 \le r, \delta \le \infty$, we define*

$$\|u\|_{\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r})} := \left\| \left\{ 2^{js} \|\Delta_j u\|_{L^\delta(0,\infty;L^p)} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}.$$

*We then define $\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r}(\mathbb{R}^3))$ as the set of temperate distributions $u$ on $(0,\infty)\times\mathbb{R}^3$ with $\lim_{j\to-\infty} S_j u = 0$ in $\mathscr{S}'((0,\infty)\times\mathbb{R}^3)$ and $\|u\|_{\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r})} < \infty$.*

**Lemma 2.3.** *(Product Law, [14, 32]) Let $\theta \in \{0,1\}$, $r \in [1,\infty]$ and $p_0 \in (1,\infty)$. Let $(p_1, p_2, \lambda_1, \lambda_2) \in [1,\infty]^4$ satisfy $\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \le \lambda_2$, $p_2 \le \lambda_1$, $\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{\lambda_1} \le 1$ and $\frac{1}{p_0} \le \frac{1}{p_2} + \frac{1}{\lambda_2} \le 1$. If $s_1 + s_2 + 3\inf\{0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\} > 0$, $s_1 + \frac{3}{\lambda_2} < \frac{3}{p_1}$ and $s_2 + \frac{3}{\lambda_1} < \frac{3}{p_2}$, then there is $C > 0$ such that*

$$\|uv\|_{e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{s_1+s_2-3(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0})}_{p_0,r}} \le C\|u\|_{e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{s_1}_{p_1,r}} \|v\|_{e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{s_2}_{p_2,\infty}}.$$

**Remark 2.4.** *We refer to [32] for Lemma 2.3 with the case of $\theta = 0$ and to [14] for Lemma 2.3 with the case of $\theta = 1$. Moreover, Lemma 2.3 can be generalized to $\tilde{L}^\delta(0,\infty;\dot{B}^s_{p,r}(\mathbb{R}^3))$ with $s, p, r$ behaving just as in Lemma 2.3 and index $\delta$ behaving as the rule of Hölder's inequality, see [31].*

Define Helmholtz projection $\mathbb{P} := (\delta_{ij} + \mathcal{R}_i\mathcal{R}_j)_{1\le i,j\le 3}$, where $\mathcal{R}_j$ is the $j$-th Riesz transform in $\mathbb{R}^3$. By the Duhamel principle, Eq (1.1) can be equivalently written as

$$\begin{cases} u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u(\tau)\otimes u(\tau))d\tau + \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot(B(\tau)\otimes B(\tau))d\tau, \\ B(t) = e^{t\Delta}B_0 - \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(u(\tau)\otimes B(\tau))d\tau + \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(B(\tau)\otimes u(\tau))d\tau, \end{cases} \tag{2.1}$$

where $\{T_\Omega(t)\}_{t\ge 0}$ is the so-called Stokes-Coriolis semigroup given specifically by

$$T_\Omega(t)f := \frac{1}{2}\mathcal{G}_+(\Omega t)[e^{t(\Delta}(I+\mathcal{R})f] + \frac{1}{2}\mathcal{G}_-(\Omega t)[e^{t\Delta}(I-\mathcal{R})f]. \tag{2.2}$$

Here $I$ denotes the identity operator, and $\mathcal{G}_\pm(t)$ represents the dispersive linear operator given explicitly by

$$\mathcal{G}_\pm(t)f(x) := e^{\pm it\frac{D_3}{|D|}}f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}}\hat{f}(\xi)d\xi, \quad x\in\mathbb{R}^3,\ t\in\mathbb{R}, \tag{2.3}$$

and $\mathcal{R}$ is the matrix of the Riesz transforms given explicitly by

$$\mathcal{R} := \begin{pmatrix} 0 & \mathcal{R}_3 & -\mathcal{R}_2 \\ -\mathcal{R}_3 & 0 & \mathcal{R}_1 \\ \mathcal{R}_2 & -\mathcal{R}_1 & 0 \end{pmatrix}.$$

We refer to [16] for the deduction of $\{T_\Omega(t)\}_{t\ge 0}$.

The following temporal decay estimates of $\{e^{t\Delta}\}_{t\ge 0}$ and $\{\mathcal{G}_\pm(\Omega t)\}_{t\in\mathbb{R}}$ are the keys to studying the global existence of solutions of Eq (1.1).

**Lemma 2.5.** *[33] For $-\infty < s_1 \le s_2 < +\infty$, $1 \le p_1 \le p_2 \le \infty$, and $1 \le r \le \infty$, there is $C = C(s_1, s_2, p) > 0$ such that*

$$\|\Delta_j e^{\Delta t} f\|_{L^{p_2}} \le C 2^{-(s_2-s_1)j} t^{-\frac{1}{2}(s_2-s_1)-\frac{3}{2}(\frac{1}{p_1}-\frac{1}{p_2})} \|\Delta_j f\|_{L^{p_1}},$$

*for all $t > 0$ and $j \in \mathbb{Z}$. Moreover, there holds*

$$\|e^{\Delta t} f\|_{\dot{B}^{s_2}_{p_2,r}} \le C t^{-\frac{1}{2}(s_2 - s_1) - \frac{3}{2}(\frac{1}{p_1} - \frac{1}{p_2})} \|f\|_{\dot{B}^{s_1}_{p_1,r}},$$

*for all $t > 0$.*

**Lemma 2.6.** *[13] For $s \in \mathbb{R}$, $1 \le p \le 2$, and $1 \le r \le \infty$, there is $C = C(p) > 0$ such that*

$$\|\Delta_j \mathcal{G}_\pm(t) f\|_{L^{p'}} \le C(1 + |t|)^{-(1-\frac{2}{p'})} 2^{j(\frac{3}{p} - \frac{3}{p'})} \|\Delta_j f\|_{L^p},$$

*for all $t \in \mathbb{R}$ and $j \in \mathbb{Z}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, there holds*

$$\|\mathcal{G}_\pm(t) f\|_{\dot{B}^s_{p',r}} \le C(1 + |t|)^{-(1-\frac{2}{p'})} \|f\|_{\dot{B}^{s+\frac{3}{p}-\frac{3}{p'}}_{p,r}},$$

*for all $t \in \mathbb{R}$ with $\frac{1}{p'} + \frac{1}{p} = 1$.*

The last two lemmas are the keys to studying the Gevrey analyticity of solutions.

**Lemma 2.7.** *[28] For $1 < p < \infty$ and $a \ge 0$, $E := e^{\frac{1}{2}a\Delta + \sqrt{a}\Lambda_1}$ is a multiplier that is bounded on $L^p$, and the norm of the operator is bounded uniformly in regard to $a$.*

**Lemma 2.8.** *[28] For $0 \le s \le t$, $E := e^{-[\sqrt{t-s} + \sqrt{s} - \sqrt{t}]\Lambda_1}$ is either the identity operator or an $L^1$ kernel whose norm is bounded independently to $s$ and $t$.*

## 3. Linear estimates

**Lemma 3.1.** *Let $r \in [1, \infty]$, $p \in (1, 2)$, and $\delta \in [1, \infty]$ satisfy*

$$0 < \frac{1}{\delta} < \frac{2}{p} - 1.$$

*Then there is $C = C(p, \delta) > 0$ such that*

$$\|T_\Omega(t) f\|_{\tilde{L}^\delta(0,\infty; e^{\theta\sqrt{t}\Lambda_1} \dot{B}^{-1+\frac{3}{p'}+\frac{2}{\delta}}_{p',r})} \le C|\Omega|^{-\frac{1}{\delta}} \|f\|_{\dot{B}^{-1+\frac{3}{p}+\frac{2}{\delta}}_{p,r}}, \tag{3.1}$$

*for $\theta \in \{0, 1\}$ and $\Omega \in \mathbb{R}\backslash\{0\}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

*Proof.* From Definition 2.2, we see

$$\|T_\Omega(t) f\|_{\tilde{L}^\delta(0,\infty; e^{\theta\sqrt{t}\Lambda_1} \dot{B}^{-1+\frac{3}{p'}+\frac{2}{\delta}}_{p',r})} = \left\| \left\{ 2^{(-1+\frac{3}{p'}+\frac{2}{\delta})j} \|\Delta_j e^{\theta\sqrt{t}\Lambda_1} T_\Omega(t) f\|_{L^\delta(0,\infty; L^{p'})} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}.$$

Since the matrix $\mathcal{R}$ is bounded on $L^q(\mathbb{R}^3)(1 < q < \infty)$, by the expression of $T_\Omega(t)$, we just need to verify that

$$\|\Delta_j e^{\theta\sqrt{t}\Lambda_1} \mathcal{G}_\pm(\Omega t) e^{t\Delta} f\|_{L^\delta(0,\infty; L^{p'})} \le C|\Omega|^{-\frac{1}{\delta}} 2^{j(\frac{3}{p} - \frac{3}{p'})} \|\Delta_j f\|_{L^p}.$$

In fact, applying Lemmas 2.5–2.7, we have

$$\begin{aligned}\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}\mathcal{G}_\pm(\Omega t)e^{t\Delta}f\|_{L^{p'}} &= \|e^{\theta\sqrt{t}\Lambda_1+\frac{t}{2}\Delta}\mathcal{G}_\pm(\Omega t)e^{\frac{t}{2}\Delta}\Delta_j f\|_{L^{p'}}\\ &\le C\|\mathcal{G}_\pm(\Omega t)e^{\frac{t}{2}\Delta}\Delta_j f\|_{L^{p'}}\\ &\le C2^{j(\frac{3}{p}-\frac{3}{p'})}(1+|\Omega|t)^{-(1-\frac{2}{p'})}\|\Delta_j f\|_{L^p}.\end{aligned}$$

Moreover, since $0<\frac{1}{\delta}<\frac{2}{p}-1$, it is obvious that

$$\Big(\int_0^\infty (1+|\Omega|t)^{-(1-\frac{2}{p'})\delta}dt\Big)^{\frac{1}{\delta}} \le C|\Omega|^{-\frac{1}{\delta}}.$$

This completes the proof.

By employing $TT^\star$ argument, we have the following estimate involving the case of $p=2$.

**Lemma 3.2.** *Let $r\in[1,\infty]$, $q\in(2,\infty)$, and $\delta\in[2,\infty]$ satisfy*

$$0<\frac{1}{\delta}<\frac{1}{2}-\frac{1}{q}.$$

*Then there is $C=C(q,\delta)>0$ such that*

$$\|T_\Omega(t)f\|_{\tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{q,r}^{-1+\frac{3}{q}+\frac{2}{\delta}})} \le C|\Omega|^{-\frac{1}{\delta}}\|f\|_{\dot{B}_{2,r}^{\frac{1}{2}+\frac{2}{\delta}}},$$

*for $\theta\in\{0,1\}$ and $\Omega\in\mathbb{R}\backslash\{0\}$.*

*Proof.* From Definition 2.2, we have

$$\|T_\Omega(t)f\|_{\tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{q,r}^{-1+\frac{3}{q}+\frac{2}{\delta}})} = \Big\|\Big\{2^{(-1+\frac{3}{q}+\frac{2}{\delta})j}\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}T_\Omega(t)f\|_{L^\delta(0,\infty;L^q)}\Big\}_{j\in\mathbb{Z}}\Big\|_{\ell^r(\mathbb{Z})}.$$

Since the matrix $\mathcal{R}$ is bounded on $L^q(\mathbb{R}^3)$ $(1<q<\infty)$, by the expression of $T_\Omega(t)$, we just need to verify that

$$\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}\mathcal{G}_\pm(\Omega t)e^{t\Delta}f\|_{L^\delta(0,\infty;L^q)} \le C|\Omega|^{-\frac{1}{\delta}}2^{j(\frac{3}{2}-\frac{3}{q})}\|\Delta_j f\|_{L^2}, \quad \text{for each } \ j\in\mathbb{Z}.$$

In fact, from Lemma 2.7, we see

$$\begin{aligned}\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}\mathcal{G}_\pm(\Omega t)e^{t\Delta}f\|_{L^q} &= \|e^{\theta\sqrt{t}\Lambda_1+\frac{t}{2}\Delta}\mathcal{G}_\pm(\Omega t)e^{\frac{t}{2}\Delta}\Delta_j f\|_{L^q}\\ &\le C\|\mathcal{G}_\pm(\Omega t)e^{\frac{t}{2}\Delta}\Delta_j f\|_{L^q}.\end{aligned}$$

We claim for $q\in(2,\infty)$ and $0<\frac{1}{\delta}<\frac{1}{2}-\frac{1}{q}$ that there holds

$$\|\mathcal{G}_\pm(\Omega t)e^{\frac{t}{2}\Delta}\Delta_j f\|_{L^\delta(0,\infty;L^q)} \le C|\Omega|^{-\frac{1}{\delta}}2^{(\frac{3}{2}-\frac{3}{q})j}\|\Delta_j f\|_{L^2}. \tag{3.2}$$

Furthermore, applying the $TT^\star$ argument (see [34, 35]), we just need to prove that

$$\Big|\int_0^\infty\int_{\mathbb{R}^3}\mathcal{G}_\pm(\Omega t)e^{\frac{t}{2}\Delta}\Delta_j f(x)\overline{\phi(t,x)}dxdt\Big| \le C|\Omega|^{-\frac{1}{\delta}}2^{j(\frac{3}{2}-\frac{3}{q})}\|\Delta_j f\|_{L^2}\|\phi\|_{L^{\delta'}(0,\infty;L^{q'})},$$

for $\phi \in C_0^\infty((0,\infty)\times\mathbb{R}^3)$ with $\frac{1}{q'}+\frac{1}{q}=1$ and $\frac{1}{\delta}+\frac{1}{\delta'}=1$.

In fact, we define a new operator $\widetilde{\Delta}_j$ given specifically by

$$\widetilde{\Delta}_j f := (\psi_{j-1}+\psi_j+\psi_{j+1}) * f \quad \text{for each } j\in\mathbb{Z}.$$

It is obvious that $\widetilde{\Delta}_j\Delta_j=\Delta_j$ for all $j\in\mathbb{Z}$.

It follows from the Hölder inequality that

$$\begin{aligned}\Big|\int_0^\infty\int_{\mathbb{R}^3}\mathcal{G}_\pm(\Omega t)e^{\frac{t}{2}\Delta}\Delta_j f(x)\overline{\phi(t,x)}dxdt\Big| &= \Big|\int_0^\infty\int_{\mathbb{R}^3}\Delta_j f(x)\overline{\mathcal{G}_\mp(\Omega t)e^{\frac{t}{2}\Delta}\widetilde{\Delta}_j\phi(t,x)}dxdt\Big| \\ &\le \|\Delta_j f\|_{L^2}\Big\|\int_0^\infty\mathcal{G}_\mp(\Omega t)e^{\frac{t}{2}\Delta}\widetilde{\Delta}_j\phi(t)dt\Big\|_{L^2},\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\Big\|\int_0^\infty\mathcal{G}_\mp(\Omega t)e^{\frac{t}{2}\Delta}\widetilde{\Delta}_j\phi(t)dt\Big\|_{L^2}^2 \\ &=\int_{\mathbb{R}^3}\int_0^\infty\int_0^\infty\mathcal{G}_\mp(\Omega t)e^{\frac{t}{2}\Delta}\widetilde{\Delta}_j\phi(t,x)\overline{\mathcal{G}_\mp(\Omega\tau)e^{\frac{\tau}{2}\Delta}\widetilde{\Delta}_j\phi(\tau,x)}dtd\tau dx \\ &\le\int_0^\infty\int_0^\infty\|\phi(t)\|_{L^{q'}}\Big\|\mathcal{G}_\pm(\Omega(t-\tau))e^{\frac{t+\tau}{2}\Delta}\widetilde{\Delta}_j\phi(\tau)\Big\|_{L^q}d\tau dt.\end{aligned}\tag{3.4}$$

Moreover, applying Lemmas 2.5 and 2.6, we have

$$\Big\|\mathcal{G}_\pm(\Omega(t-\tau))e^{\frac{t+\tau}{2}\Delta}\widetilde{\Delta}_j\phi(\tau)\Big\|_{L^q}\le C(1+|\Omega||t-\tau|)^{-(1-\frac{2}{q})}2^{3(1-\frac{2}{q})j}\big\|\widetilde{\Delta}_j\phi(\tau)\big\|_{L^{q'}}.\tag{3.5}$$

Substituting Eq (3.5) into Eq (3.4), applying the Hölder inequality and Young inequality yields that

$$\begin{aligned}&\Big\|\int_0^\infty\mathcal{G}_\mp(\Omega t)e^{\frac{t}{2}\Delta}\widetilde{\Delta}_j\phi(t)dt\Big\|_{L^2}^2 \\ &\le C2^{3(1-\frac{2}{q})j}\|\phi\|_{L^{\delta'}(0,\infty;L^{q'})}\Big[\int_0^\infty\Big(\int_0^\infty(1+|\Omega||t-\tau|)^{-(1-\frac{2}{q})}\|\phi(\tau)\|_{L^{q'}}d\tau\Big)^\delta dt\Big]^{\frac{1}{\delta}} \\ &\le C2^{3(1-\frac{2}{q})j}\|\phi\|_{L^{\delta'}(0,\infty;L^{q'})}^2\Big(\int_0^\infty(1+|\Omega|t)^{-\frac{\delta}{2}(1-\frac{2}{q})}dt\Big)^{\frac{2}{\delta}}.\end{aligned}$$

Moreover, because of $0<\frac{1}{\delta}<\frac{1}{2}-\frac{1}{q}$, it is obvious that

$$\Big(\int_0^\infty(1+|\Omega|t)^{-\frac{\delta}{2}(1-\frac{2}{q})}dt\Big)^{\frac{2}{\delta}}\le C|\Omega|^{-\frac{2}{\delta}}.$$

Therefore, we immediately obtain

$$\Big\|\int_0^\infty\mathcal{G}_\mp(\Omega t)e^{\frac{t}{2}\Delta}\widetilde{\Delta}_j\phi(t)dt\Big\|_{L^2}^2\le C|\Omega|^{-\frac{2}{\delta}}2^{3j(1-\frac{2}{q})}\|\phi\|_{L^{\delta'}(0,\infty;L^{q'})}^2.\tag{3.6}$$

Substituting Eq (3.6) into Eq (3.3) completes the proof.

**Lemma 3.3.** *Let $r \in [1,\infty]$, $p \in (1,2)$, and $\delta \in (2,\infty)$. Then there is $C = C(p,\delta) > 0$ such that*

$$\Big\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{\tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{-1+\frac{3}{p'}+\frac{2}{\delta}}_{p',r})} \le C\|f\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{-2+\frac{3}{p}+\frac{4}{\delta}}_{p,r})},$$

*for $\theta \in \{0,1\}$ and $\Omega \in \mathbb{R}\backslash\{0\}$, where $\frac{1}{p'} + \frac{1}{p} = 1$.*

*Proof.* From Definition 2.2, we have

$$\begin{aligned}
&\Big\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{\tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{-1+\frac{3}{p'}+\frac{2}{\delta}}_{p',r})} \\
&= \Big\| \Big\{ 2^{(-1+\frac{3}{p'}+\frac{2}{\delta})j} \Big\| \Delta_j e^{\theta\sqrt{t}\Lambda_1} \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{L^\delta(0,\infty;L^{p'})} \Big\}_{j\in\mathbb{Z}} \Big\|_{\ell^r(\mathbb{Z})}.
\end{aligned}$$

Since operator $\mathbb{P}$ and matrix $\mathcal{R}$ are bounded on $L^q(\mathbb{R}^3)(1<q<\infty)$, by the expression of $T_\Omega(t)$, we just need to verify that

$$\Big\| \int_0^t \Big\| e^{\theta\sqrt{t}\Lambda_1}\mathcal{G}_\pm(\Omega(t-\tau))e^{(t-\tau)\Delta}\nabla\Delta_j f(\tau)\Big\|_{L^{p'}} d\tau \Big\|_{L^\delta(0,\infty)} \le C2^{j(2-\frac{6}{p'}+\frac{2}{\delta})}\Big\|\Delta_j f\Big\|_{L^{\frac{\delta}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}L^p)}.$$

First, applying Lemmas 2.7 and 2.8, we deduce that

$$\begin{aligned}
&\Big\| \int_0^t \Big\| e^{\theta\sqrt{t}\Lambda_1}\mathcal{G}_\pm(\Omega(t-\tau))e^{(t-\tau)\Delta}\nabla\Delta_j f(\tau)\Big\|_{L^{p'}} d\tau \Big\|_{L^\delta(0,\infty)} \\
&= \Big\| \int_0^t \Big\| e^{\theta(\sqrt{t}-\sqrt{\tau}-\sqrt{t-\tau})\Lambda_1} e^{\theta\sqrt{t-\tau}\Lambda_1+\frac{t-\tau}{2}\Delta}\mathcal{G}_\pm(\Omega(t-\tau))e^{\frac{t-\tau}{2}\Delta}\nabla e^{\theta\sqrt{\tau}\Lambda_1}\Delta_j f(\tau)\Big\|_{L^{p'}} d\tau \Big\|_{L^\delta(0,\infty)} \\
&\le C\Big\| \int_0^t \Big\| \mathcal{G}_\pm(\Omega(t-\tau))e^{\frac{t-\tau}{2}\Delta}\nabla e^{\theta\sqrt{\tau}\Lambda_1}\Delta_j f(\tau)\Big\|_{L^{p'}} d\tau \Big\|_{L^\delta(0,\infty)}.
\end{aligned} \tag{3.7}$$

Second, it follows from Lemmas 2.5 and 2.6 and the Bernstein inequality that

$$\begin{aligned}
&\Big\| \int_0^t \Big\| \mathcal{G}_\pm(\Omega(t-\tau))e^{\frac{t-\tau}{2}\Delta}\nabla e^{\theta\sqrt{\tau}\Lambda_1}\Delta_j f(\tau)\Big\|_{L^{p'}} d\tau \Big\|_{L^\delta(0,\infty)} \\
&\le C2^{j(2-\frac{6}{p'}+\frac{2}{\delta})}\Big\| \int_0^t (1+|\Omega|(t-\tau))^{-(1-\frac{2}{p'})}(t-\tau)^{-(1-\frac{1}{\delta})}\Big\|\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\Big\|_{L^p}d\tau\Big\|_{L^\delta(0,\infty)} \\
&\le C2^{j(2-\frac{6}{p'}+\frac{2}{\delta})}\Big\| \int_0^t (t-\tau)^{-(1-\frac{1}{\delta})}\Big\|\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\Big\|_{L^p}d\tau\Big\|_{L^\delta(0,\infty)}.
\end{aligned}$$

Furthermore, due to $1 < \frac{\delta}{2} < \delta < \infty$, $0 < 1-\frac{1}{\delta} < 1$ and $\frac{1}{\delta} = \frac{2}{\delta} - [1-(1-\frac{1}{\delta})]$, by the Hardy-Littlewood-Sobolev inequality, we see that there is $C = C(\delta) > 0$ such that

$$\Big\| \int_0^t (t-\tau)^{-(1-\frac{1}{\delta})}\Big\|\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\Big\|_{L^p}d\tau\Big\|_{L^\delta(0,\infty)} \le C\Big\|\Delta_j f\Big\|_{L^{\frac{\delta}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}L^p)}.$$

This completes the proof.

**Lemma 3.4.** *Let $r \in [1,\infty]$, $p \in (1,2)$, $\delta \in (2,\infty)$, and $\rho \in (2,2\delta)$. Then there is $C = C(p,\delta) > 0$ such that*

$$\Big\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{\tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{-1+\frac{3}{p'}+\frac{2}{\delta}}_{p',r})} \le C\|f\|_{\tilde{L}^{\frac{\rho}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{-2+\frac{3}{p}+\frac{4}{\rho}}_{p,r})},$$

*for $\theta \in \{0,1\}$ and $\Omega \in \mathbb{R}\backslash\{0\}$, where $\frac{1}{p}+\frac{1}{p'}=1$.*

*Proof.* From Definition 2.2, we have

$$\begin{aligned}&\Big\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{\tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{-1+\frac{3}{p'}+\frac{2}{\delta}}_{p',r})}\\ &= \Big\| \Big\{ 2^{j(-1+\frac{3}{p'}+\frac{2}{\delta})} \Big\| \Delta_j e^{\theta\sqrt{t}\Lambda_1} \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{L^\delta(0,\infty;L^{p'})} \Big\}_{j\in\mathbb{Z}} \Big\|_{l^r(\mathbb{Z})}.\end{aligned}$$

By the expression of $T_\Omega(t)$, we just need to verify that

$$\begin{aligned}&\Big\| \int_0^t \big\| e^{\theta\sqrt{t}\Lambda_1}\mathcal{G}_\pm(\Omega(t-\tau))e^{(t-\tau)\Delta}\Delta_j\nabla f(\tau)\big\|_{L^{p'}}d\tau \Big\|_{L^\delta(0,\infty)}\\ &\le C2^{j(-1+\frac{3}{p}-\frac{3}{p'}-\frac{2}{\delta}+\frac{4}{\rho})}\big\|\Delta_j f\big\|_{L^{\frac{\rho}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}L^p)}.\end{aligned}$$

In fact, through the similar process of Eq (3.7), we have

$$\begin{aligned}&\Big\| \int_0^t \big\| e^{\theta\sqrt{t}\Lambda_1}\mathcal{G}_\pm(\Omega(t-\tau))e^{(t-\tau)\Delta}\nabla\Delta_j f(\tau)\big\|_{L^{p'}}d\tau \Big\|_{L^\delta(0,\infty)}\\ &\le C\Big\| \int_0^t \big\| \mathcal{G}_\pm(\Omega(t-\tau))e^{\frac{t-\tau}{2}\Delta}\nabla\Delta_j e^{\theta\sqrt{\tau}\Lambda_1} f(\tau)\big\|_{L^{p'}}d\tau \Big\|_{L^\delta(0,\infty)}.\end{aligned}$$

Furthermore, from the Bernstein inequality and Lemmas 2.5 and 2.6, we have

$$\begin{aligned}&\Big\| \int_0^t \big\| \mathcal{G}_\pm(\Omega(t-\tau))e^{\frac{t-\tau}{2}\Delta}\nabla\Delta_j e^{\theta\sqrt{\tau}\Lambda_1} f(\tau)\big\|_{L^{p'}}d\tau \Big\|_{L^\delta(0,\infty)}\\ &\le C2^{j(-1+\frac{3}{p}-\frac{3}{p'}-\frac{2}{\delta}+\frac{4}{\rho})}\Big\| \int_0^t (1+|\Omega|(t-\tau))^{-(1-\frac{2}{p'})}(t-\tau)^{-(1+\frac{1}{\delta}-\frac{2}{\rho})}\big\|\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\big\|_{L^p}d\tau\Big\|_{L^\delta(0,\infty)}\\ &\le C2^{j(-1+\frac{3}{p}-\frac{3}{p'}-\frac{2}{\delta}+\frac{4}{\rho})}\Big\| \int_0^t (t-\tau)^{-(1+\frac{1}{\delta}-\frac{2}{\rho})}\big\|\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\big\|_{L^p}d\tau\Big\|_{L^\delta(0,\infty)}.\end{aligned}$$

Moreover, due to $1 < \frac{\rho}{2} < \delta < \infty$, $0 < 1+\frac{1}{\delta}-\frac{2}{\rho} < 1$ and $\frac{1}{\delta} = \frac{2}{\rho} - [1-(1+\frac{1}{\delta}-\frac{2}{\rho})]$, by the Hardy-Littlewood-Sobolev inequality, we see that there is $C = C(\delta) > 0$ such that

$$\Big\| \int_0^t (t-\tau)^{-(1+\frac{1}{\delta}-\frac{2}{\rho})}\big\|\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\big\|_{L^p}d\tau\Big\|_{L^\delta(0,\infty)} \le C\big\|\Delta_j f\big\|_{L^{\frac{\rho}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}L^p)}.$$

This completes the proof.

**Lemma 3.5.** *Let $r \in [1,\infty]$, $p \in (1,\infty)$, and $\rho \in [1,\infty]$. Then there is $C = C(p,\rho) > 0$ such that*

$$\|e^{t\Delta}f\|_{\tilde{L}^\rho(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}^{-1+\frac{3}{p}+\frac{2}{\rho}}_{p,r})} \le C\|f\|_{\dot{B}^{-1+\frac{3}{p}}_{p,r}},$$

*for $\theta \in \{0,1\}$.*

*Proof.* From Definition 2.2, we have

$$\left\|e^{t\Delta}f\right\|_{\tilde{L}^\rho(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\rho}})} = \left\|\left\{2^{(-1+\frac{3}{p}+\frac{2}{\rho})j}\left\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}e^{t\Delta}f\right\|_{L^\rho(0,\infty;L^p)}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}. \tag{3.8}$$

Applying Lemma 2.7 implies that

$$\begin{aligned}\left\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}e^{t\Delta}f\right\|_{L^\rho(0,\infty;L^p)} &= \left\|\Delta_j e^{\theta\sqrt{t}\Lambda_1+\frac{t}{2}\Delta}e^{\frac{t}{2}\Delta}f\right\|_{L^\rho(0,\infty;L^p)}\\ &\le C\left\|\Delta_j e^{\frac{t}{2}\Delta}f\right\|_{L^\rho(0,\infty;L^p)}.\end{aligned} \tag{3.9}$$

Moreover, it follows from Lemma 2.4 of [31] that there are $c > 0$ and $C > 0$ such that

$$\begin{aligned}\left\|\Delta_j e^{\frac{t}{2}\Delta}f\right\|_{L^\rho(0,\infty;L^p)} &\le C\|e^{-ct2^{2j}}\|_{L^\rho(0,\infty)}\left\|\Delta_j f\right\|_{L^p}\\ &\le C2^{-\frac{2}{\rho}j}\left\|\Delta_j f\right\|_{L^p}.\end{aligned} \tag{3.10}$$

Substituting Eqs (3.10) and (3.9) into Eq (3.8) completes the proof.

**Lemma 3.6.** *Let $r \in [1,\infty]$, $p \in (1,\infty)$, $\rho \in [1,\infty]$, and $\gamma \in [1,\rho]$. Then there is $C = C(p,\rho,\gamma) > 0$ such that*

$$\left\|\int_0^t e^{(t-\tau)\Delta}\nabla f(\tau)d\tau\right\|_{\tilde{L}^\rho(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\rho}})} \le C\|f\|_{\tilde{L}^\gamma(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-2+\frac{3}{p}+\frac{2}{\gamma}})},$$

*for $\theta \in \{0,1\}$.*

*Proof.* From Definition 2.2, we see

$$\begin{aligned}&\left\|\int_0^t e^{(t-\tau)\Delta}\nabla f(\tau)d\tau\right\|_{\tilde{L}^\rho(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\rho}})}\\ &= \left\|\left\{2^{j(-1+\frac{3}{p}+\frac{2}{\rho})}\left\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}\int_0^t e^{(t-\tau)\Delta}\nabla f(\tau)d\tau\right\|_{L^\rho(0,\infty;L^p)}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}.\end{aligned} \tag{3.11}$$

Through the similar process of Eq (3.7), we have

$$\begin{aligned}\left\|\Delta_j e^{\theta\sqrt{t}\Lambda_1}\int_0^t e^{(t-\tau)\Delta}\nabla f(\tau)d\tau\right\|_{L^\rho(0,\infty;L^p)} &\le \left\|\int_0^t \left\|e^{\theta\sqrt{t}\Lambda_1}e^{(t-\tau)\Delta}\nabla\Delta_j f(\tau)\right\|_{L^p}d\tau\right\|_{L^\rho(0,\infty)}\\ &\le C\left\|\int_0^t \left\|e^{\frac{t-\tau}{2}\Delta}\nabla\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\right\|_{L^p}d\tau\right\|_{L^\rho(0,\infty)}.\end{aligned} \tag{3.12}$$

Moreover, it follows from Lemma 2.4 of [31], Bernstein's inequality, and Young's inequality that

$$\begin{aligned}\left\|\int_0^t \left\|e^{\frac{t-\tau}{2}\Delta}\nabla\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\right\|_{L^p}d\tau\right\|_{L^\rho(0,\infty)} &\le C2^j\left\|\int_0^t e^{-C(t-\tau)2^{2j}}\left\|\Delta_j e^{\theta\sqrt{\tau}\Lambda_1}f(\tau)\right\|_{L^p}d\tau\right\|_{L^\rho(0,\infty)}\\ &\le C2^j\|e^{-Ct2^{2j}}\|_{L^m(0,\infty)}\left\|\Delta_j f\right\|_{L^\gamma(0,\infty;e^{\theta\sqrt{t}\Lambda_1}L^p)}\\ &\le C2^j 2^{-2j(1+\frac{1}{\rho}-\frac{1}{\gamma})}\left\|\Delta_j f\right\|_{L^\gamma(0,\infty;e^{\theta\sqrt{t}\Lambda_1}L^p)},\end{aligned} \tag{3.13}$$

where $\frac{1}{m} = 1 + \frac{1}{\rho} - \frac{1}{\gamma}$. Substituting Eqs (3.12) and (3.13) into Eq (3.11) completes the proof.

## 4. Proofs of main results

*Proof of Theorem 1.1.* Because of $0 < \frac{1}{\delta} < \frac{2}{p} - 1$, by Lemmas 3.1 and 3.5, we see that there has $C_0 > 0$ and $C_1 > 0$ such that

$$\|T_\Omega(t)u_0\|_{\tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p',r}^{-1+\frac{3}{p'}+\frac{2}{\delta}})} \le C_0|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}}, \tag{4.1}$$

and

$$\|e^{t\Delta}B_0\|_{\tilde{L}^\rho(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\rho}})} \le C_1\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}}. \tag{4.2}$$

Let

$$N_1(w,v) := \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot[w(\tau)\otimes v(\tau)]d\tau,$$

and

$$N_2(w,v) := \int_0^t e^{(t-\tau)\Delta}\nabla\cdot[w(\tau)\otimes v(\tau)]d\tau.$$

Now, we define the mapping $\mathscr{B}$ by

$$\mathscr{B}(u,B)(t) := (\mathscr{B}_1(u,B)(t), \mathscr{B}_2(u,B)(t)),$$

where

$$\mathscr{B}_1(u,B)(t) := T_\Omega(t)u_0 - N_1(u,u)(t) + N_1(B,B)(t),$$

and

$$\mathscr{B}_2(u,B)(t) := e^{t\Delta}B_0 - N_2(u,B)(t) + N_2(B,u)(t).$$

And we define the solution space $Z$ by

$$Z := \Big\{(u,B) \in X\times Y := \tilde{L}^\delta(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p',r}^{-1+\frac{3}{p'}+\frac{2}{\delta}}(\mathbb{R}^3)) \times \tilde{L}^\rho(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\rho}}(\mathbb{R}^3)) :$$

$$\|u\|_X \le 2C_0|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}},\ \ \|B\|_Y \le 2C_1\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}}\Big\},$$

with $\|(u,B)\|_Z := \|u\|_X + \|B\|_Y$,

Since $\delta \in (2,\infty)$ and $\rho \in (\delta, 2\delta)$, employing Lemmas 3.3, 3.4, and 3.6, we see that there are $C_i > 0$ $(i = 2,3,4)$ such that

$$\|N_1(u,u)\|_X \le C_2\|u\otimes u\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-2+\frac{3}{p}+\frac{4}{\delta}})}, \tag{4.3}$$

$$\|N_1(B,B)\|_X \le C_3\|B\otimes B\|_{\tilde{L}^{\frac{\rho}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-2+\frac{3}{p}+\frac{4}{\rho}})}, \tag{4.4}$$

and

$$\|N_2(u,B)\|_Y + \|N_2(B,u)\|_Y \le C_4\|u\otimes B\|_{\tilde{L}^\gamma(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-2+\frac{3}{p}+\frac{2}{\delta}+\frac{2}{\rho}})}, \tag{4.5}$$

with $\frac{1}{\gamma} = \frac{1}{\delta} + \frac{1}{\rho}$.

Moreover, because of $\frac{3}{2} < p < 2$ and $\frac{1}{\delta} < 2 - \frac{3}{p}$, by taking $s_1 = s_2 = -1 + \frac{3}{p'} + \frac{2}{\delta}$, $p_0 = p$, $p_1 = p_2 = p'$ and $\lambda_1 = \lambda_2 = \frac{p'}{p'-2}$, from Lemma 2.3 and Remark 2.4, we see that there is $C_5 > 0$ such that

$$\|u \otimes u\|_{\tilde{L}^{\frac{\delta}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-2+\frac{3}{p}+\frac{4}{\delta}})} \leq C_5\|u\|_X^2. \tag{4.6}$$

Because of $\rho > 2$ and $p < 2$, by taking $s_1 = s_2 = -1 + \frac{3}{p} + \frac{2}{\rho}$, $p_0 = p_2 = p_1 = p$, $\lambda_1 = \lambda_2 = \infty$, from Lemma 2.3 and Remark 2.4, we see that there is $C_6 > 0$ such that

$$\|B \otimes B\|_{\tilde{L}^{\frac{\rho}{2}}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-2+\frac{3}{p}+\frac{4}{\rho}})} \leq C_6\|B\|_Y^2. \tag{4.7}$$

Due to $p < 2$, $\delta > 2$, $\frac{1}{\rho} < 2 - \frac{3}{p}$ and $\frac{1}{\gamma} := \frac{1}{\rho} + \frac{1}{\delta}$, by taking $s_1 = -1 + \frac{3}{p'} + \frac{2}{\delta}$, $s_2 = -1 + \frac{3}{p} + \frac{2}{\rho}$, $p_0 = p_2 = p$, $p_1 = p'$, $\lambda_1 = \frac{p'}{p'-2}$ and $\lambda_2 = \infty$, from Lemma 2.3 and Remark 2.4, we see that there has $C_7 > 0$ such that

$$\|u \otimes B\|_{\tilde{L}^{\gamma}(0,\infty;e^{\theta\sqrt{t}\Lambda_1}\dot{B}_{p,r}^{-2+\frac{3}{p}+\frac{2}{\delta}+\frac{2}{\rho}})} \leq C_7\|u\|_X\|B\|_Y. \tag{4.8}$$

Therefore, combining Eqs (4.1)–(4.8) implies

$$\begin{aligned}\|\mathscr{B}_1(u,B)\|_X &\leq C_0|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}} + C_2C_5\|u\|_X^2 + C_3C_6\|B\|_Y^2 \\ &\leq C_0|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}}\Big\{1 + 4C_0C_2C_5|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}} \\ &\quad + 4C_0^{-1}C_1^2C_3C_6|\Omega|^{\frac{1}{\delta}}\|u_0\|^{-1}_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}}\|B_0\|^2_{\dot{B}_{p,r}^{-1+\frac{3}{p}}}\Big\},\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\|\mathscr{B}_2(u,B)\|_Y &\leq C_1\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}} + C_4C_7\|u\|_X\|B\|_Y \\ &\leq C_1\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}}\Big\{1 + 4C_0C_4C_7|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}}\Big\},\end{aligned} \tag{4.10}$$

for every $(u, B) \in Z$.

Moreover, it follows from the similar argument that

$$\begin{aligned}&\|\mathscr{B}(u_1,B_1) - \mathscr{B}(u_2,B_2)\|_Z \\ &\leq \Big\|\int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot[u_1(\tau)\otimes(u_1(\tau)-u_2(\tau)) + (u_1(\tau)-u_2(\tau))\otimes u_2(\tau)]d\tau\Big\|_X \\ &\quad + \Big\|\int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot[B_1(\tau)\otimes(B_1(\tau)-B_2(\tau)) + (B_1(\tau)-B_2(\tau))\otimes B_2(\tau)]d\tau\Big\|_X \\ &\quad + \Big\|\int_0^t e^{(t-\tau)\Delta}\nabla\cdot[u_1(\tau)\otimes(B_1(\tau)-B_2(\tau)) + (u_1(\tau)-u_2(\tau))\otimes B_2(\tau)]d\tau\Big\|_Y \\ &\quad + \Big\|\int_0^t e^{(t-\tau)\Delta}\nabla\cdot[B_1(\tau)\otimes(u_1(\tau)-u_2(\tau)) + (B_1(\tau)-B_2(\tau))\otimes u_2(\tau)]d\tau\Big\|_Y \\ &\leq \Big\{4C_0C_2C_5|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}} + 4C_1C_4C_7\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}}\Big\}\|u_1-u_2\|_X \\ &\quad + \Big\{4C_1C_3C_6\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}} + 4C_0C_4C_7|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}}\Big\}\|B_1-B_2\|_Y,\end{aligned} \tag{4.11}$$

for every $(u_1, B_1)$ and $(u_2, B_2)$ in $Z$.

Hence, if $(u_0, B_0) \in \dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}(\mathbb{R}^3) \times \dot{B}_{p,r}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ satisfies

$$\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}} \le \min\left\{\frac{1}{16C_0C_2C_5}|\Omega|^{\frac{1}{\delta}}, \frac{1}{16C_0C_4C_7}|\Omega|^{\frac{1}{\delta}}\right\},$$

and

$$\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}} \le \min\left\{\frac{C_0^{\frac{1}{2}}}{16C_1C_3^{\frac{1}{2}}C_6^{\frac{1}{2}}}|\Omega|^{-\frac{1}{2\delta}}\|u_0\|^{\frac{1}{2}}_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}}, \frac{1}{16C_1C_3C_6}, \frac{1}{16C_1C_4C_7}\right\},$$

Eqs (4.9)–(4.11) imply that

$$\|\mathscr{B}_1(u, B)\|_X \le 2C_0|\Omega|^{-\frac{1}{\delta}}\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}+\frac{2}{\delta}}}, \quad \|\mathscr{B}_2(u, B)\|_Y \le 2C_1\|B_0\|_{\dot{B}_{p,r}^{-1+\frac{3}{p}}},$$

and

$$\|\mathscr{B}(u_1, B_1) - \mathscr{B}(u_2, B_2)\|_Z < \frac{1}{2}\|(u_1, B_1) - (u_2, B_2)\|_Z,$$

for all $(u_1, B_1)$ and $(u_2, B_2)$ in $Z$. Then, applying the contraction mapping principle implies that there is a unique global mild solution $(u, B) \in Z$ to problem (1.1).

*Proof of Theorem 1.2.* The proof of Theorem 1.2 is identical to that of Theorem 1.1. We omit the proof.

## Author contributions

J. Sun handled the review and supervision. W. Wang was responsible for writing the original draft. D. Zhao worked on validating.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

## References


1. W. M. Elsässer, Induction effects in terrestrial magnetism part I. Theory, *Phys. Rev.*, **69** (1946), 106–116. https://doi.org/10.1103/PhysRev.69.106
2. W. M. Elsässer, Induction effects in terrestrial magnetism part II. The secular variation, *Phys. Rev.*, **70** (1946), 202–212. https://doi.org/10.1103/PhysRev.70.202
3. R. T. Merrill, M. W. McElhinny, P. L. McFadden, *The Magnetic Field of the Earth: Paleomagnetism, the Core, and the Deep Mantle*, Academic Press, 1998.
4. J. Y. Chemin, B. Desjardins, I. Gallagher, E. Grenier, *Mathematical Geophysics: An Introduction to Rotating Fluids and the Navier-Stokes Equations*, The Clarendon Press Oxford University Press, Oxford, 2006.
5. G. Duvaut, J. L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Ration. Mech. Anal.*, **46** (1972), 241–279. https://doi.org/10.1007/BF00250512
6. X. Zhai, Y. Li, W. Yan, Global well-posedness for the 3-D incompressible MHD equations in the critical Besov spaces, *Commun. Pure Appl. Anal.*, **14** (2015), 1865–1884. https://doi.org/10.3934/cpaa.2015.14.1865
7. J. Chemin, D. McCormick, J. Robinson, J. Rodrigo, Local existence for the nonresistive MHD equations in Besov spaces, *Adv. Math.*, **286** (2016), 1–31. https://doi.org/10.1016/j.aim.2015.09.004
8. C. Fefferman, D. McCormick, J. Robinson, J. Rodrigo, Higher order commutator estimates and local existence for the non-resistive MHD equations and related models, *J. Funct. Anal.*, **267** (2014), 1035–1056. https://doi.org/10.1016/j.jfa.2014.03.021
9. Q. Liu, J. Zhao, Global well-posedness for the generalized magneto-hydrodynamic equations in the critical Fourier-Herz spaces, *J. Math. Anal. Appl.*, **420** (2014), 1301–1315. https://doi.org/10.1016/j.jmaa.2014.06.031
10. X. Zhai, Stability for the 2D incompressible MHD equations with only magnetic diffusion, *J. Differ. Equations*, **374** (2023), 267–278. https://doi.org/https://doi.org/10.1016/j.jde.2023.07.033
11. J. Y. Chemin, B. Desjardin, I. Gallagher, E. Grenier, Anisotropy and dispersion in rotating fluids, *Stud. Math. Appl.*, **31** (2002), 171–192. https://doi.org/10.1016/S0168-2024(02)80010-8
12. T. Iwabuchi, R. Takada, Global solutions for the Navier-Stokes equations in the rotational framework, *Math. Ann.*, **357** (2013), 727–741. https://doi.org/10.1007/s00208-013-0923-4
13. Y. Koh, S. Lee, R. Takada, Dispersive estimates for the Navier-Stokes equations in the rotational framework, *Adv. Differ. Equations*, **19** (2014), 857–878. https://doi.org/10.57262/ade/1404230126
14. J. Sun, M. Yang, S. Cui, Existence and analyticity of mild solutions for the 3D rotating Navier-Stokes equations, *Ann. Mat. Pura Appl.*, **196** (2017), 1203–1229. https://doi.org/10.1007/s10231-016-0613-4
15. Y. Giga, K. Inui, A. Mahalov, J. Saal, Uniform global solvability of the rotating Navier-Stokes equations for nondecaying initial data, *Indiana Univ. Math. J.*, **57** (2008), 2775–2791. https://doi.org/10.1512/iumj.2008.57.3795

16. M. Hieber, Y. Shibata, The Fujita-Kato approach to the Navier-Stokes equations in the rotational framework, *Math. Z.*, **265** (2010), 481–491. https://doi.org/10.1007/s00209-009-0525-8

17. T. Iwabuchi, R. Takada, Global well-posedness and ill-posedness for the Navier-Stokes equations with the Coriolis force in function spaces of Besov type, *J. Funct. Anal.*, **267** (2014), 1321–1337. https://doi.org/10.1016/j.jfa.2014.05.022

18. P. Konieczny, T. Yoneda, On dispersive effect of the Coriolis force for the stationary Navier-Stokes equations, *J. Differ. Equations*, **250** (2011), 3859–3873. https://doi.org/10.1016/j.jde.2011.01.003

19. F. Charve, Global well-posedness and asymptotics for a geophysical fluid system, *Commun. Partial Differ. Equations*, **29** (2004), 1919–1940. https://doi.org/10.1081/PDE-200043510

20. J. Sun, S. Cui, Sharp well-posedness and ill-posedness of the three-dimensional primitive equations of geophysics in Fourier-Besov spaces, *Nonlinear Anal. Real World Appl.*, **48** (2019), 445–465. https://doi.org/10.1016/j.nonrwa.2019.02.003

21. J. Sun, C. Liu, M. Yang, Global solutions to 3D rotating Boussinesq equations in Besov spaces, *J. Dyn. Differ. Equations*, **32** (2020), 589–603. https://doi.org/10.1007/s10884-019-09747-0

22. J. Sun, C. Liu, M. Yang, Global existence for three-dimensional time-fractional Boussinesq-Coriolis equations, *Fract. Calc. Appl. Anal.*, **27** (2024), 1759–1778. https://doi.org/10.1007/s13540-024-00272-6

23. J. Sun, M. Yang, Global well-posedness for the viscous primitive equations of geophysics, *Boundary Value Probl.*, **21** (2016), 16. https://doi.org/10.1186/s13661-016-0526-6

24. V. S. Ngo, A global existence result for the anisotropic rotating magnetohydrodynamical systems, *Acta Appl. Math.*, **150** (2017), 1–42. https://doi.org/10.1007/s10440-016-0092-z

25. J. Ahn, J. Kim, J. Lee, Global solutions to 3D incompressible rotational MHD system, *J. Evol. Equations*, **21** (2021), 235–246. https://doi.org/10.1007/s00028-020-00576-z

26. J. Kim, Rotational effect on the asymptotic stability of the MHD system, *J. Differ. Equations*, **319** (2022), 288–311. https://doi.org/10.1016/j.jde.2022.02.033

27. C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier-Stokes equations, *J. Funct. Anal.,* **87** (1989), 359–369. https://doi.org/10.1016/0022-1236(89)90015-3

28. H. Bae, A. Biswas, E. Tadmor, Analyticity and decay estimates of the Navier-Stokes equations in critical Besov spaces, *Arch. Ration. Mech. Anal.*, **205** (2012), 963–991. https://doi.org/10.1007/s00205-012-0532-5

29. M. Oliver, E. S. Titi, Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in $\mathbb{R}^n$, *J. Funct. Anal.*, **172** (2000), 1–18. https://doi.org/10.1006/jfan.1999.3550

30. J. Y. Chemin, N. Lerner, Flow of non-Lipschitz vector fields and Navier-Stokes equations(French), *J. Differ. Equations*, **121** (1995), 314–328. https://doi.org/10.1006/jdeq.1995.1131

31. H. Bahouri, J. Y. Chemin, R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer-Verlag, Berlin, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-16830-7

32. H. Abidi, M. Paicu, Existence globale pour un fluide inhomogéne (French), *Ann. Inst. Fourier*, **57** (2007), 883–917. https://doi.org/10.5802/aif.2280

33. H. Kozono, T. Ogawa, Y. Taniuchi, Navier-Stokes equations in the Besov space near $L^1$ and BMO, *Kyushu J. Math.*, **57** (2003), 303–324. https://doi.org/10.2206/kyushujm.57.303

34. R. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions to the wave equation, *Duke Math. J.*, **44** (1977), 705–714. https://doi.org/10.1215/S0012-7094-77-04430-1

35. P. A Tomas, A restriction theorem for the Fourier transform, *Bull. Am. Math. Soc.*, **81** (1975), 477–478. https://doi.org/10.1090/S0002-9904-1975-13790-6