Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 20(1): 213–231.
DOI: 10.3934/nhm.2025011
Received: 03 December 2024
Revised: 18 February 2025
Accepted: 19 February 2025
Published: 27 February 2025

Research article

Solutions of a class of higher order variable coefficient homogeneous differential equations

Peng E[1], Tingting Xu[1,*], Linhua Deng[1,*], Yulin Shan[2], Miao Wan[1] and Weihong Zhou[1,3,*]

[1] School of Mathematics and Computer Science, Yunnan Minzu University, Kunming 650504, Yunnan, China

[2] School of Mathematics and Computer Science, Chuxiong Normal University, Chuxiong 675000, Yunnan, China

[3] Key Laboratory for the Structure and Evolution of Celestial Objects, Chinese Academy of Sciences, Kunming, Yunnan, 650011, China

* **Correspondence:** Email: xutingting@ymu.edu.cn, linhua.deng@ymu.edu.cn, zwh@ymu.edu.cn.

Abstract: Recently, the variable coefficient homogeneous differential equations (VCHDE) have been widely applied to real-world problems, such as wave propagation and material science. However the exploration and research on higher-order VCHDE is relatively lagging. Given this, this work focuses on the solutions of fourth-order and nth-order VCHDE with polynomial coefficients. By means of the sufficient conditions for the existence of solutions to differential equations, a connection is established between the rank of the variable coefficient matrix and the existence of polynomial particular solutions. The main results show that: (1) the necessary and sufficient conditions for the existence of polynomial particular solutions of fourth-order VCHDE are derived; (2) the necessary and sufficient conditions for the existence of only one polynomial particular solution, or the existence of two, three, or four linearly independent polynomial particular solutions of fourth-order VCHDE are proved; (3) the necessary and sufficient conditions for the existence of only one polynomial particular solution, or the existence of two, three, or four linearly independent polynomial particular solutions of nth-order VCHDE are proved. These results not only extend the class of solvable differential equations, but also provide a new way of thinking about the existence of solutions to VCHDE.

Keywords: variable coefficient homogeneous differential equations; linearly independent solutions; rank of coefficient matrices; fourth-order and nth-order equations

1. Introduction

The variable coefficient homogeneous differential equations (VCHDE) provide a more vivid and comprehensive description of various phenomena and processes in the real world. Specifically, VCHDE are commonly applied in the study of fluctuation equations in physics to describe the propagation of fluctuations within a medium, where the variable coefficients are related to the density, elastic modulus, and other properties of the medium. For instance, they are used to derive design methods for high-resolution imaging optical systems [1]. Thus, exploring the existence of solutions to VCHDE is crucial for their broader applications.

In many early studies, certain progress had been made in researching the existence of solutions to lower-order VCHDE. Zhao and Ma applied the Leray–Schauder extension theorem to study a class of nonlinear fourth-order ordinary differential equation boundary value problems, obtaining existence results for the solutions [2]. Zhang et al. extended some results concerning the estimation of the growth of the full solution of a certain class of two-linear differential equations to nonlinear two-linear differential equations [3]. Yang and Chen et al. found the existence of solutions to impulsive mixed boundary value problems involving derivatives of Caputo fractions and Krasnoselskii's fixed point theorem, and the Arzela–Ascoli theorem gave some examples to illustrate the main results [4]. Alyusof and Jeelani et al. used the technical method of factorization to derive properties such as partial dichotomy, several properties of dichotomy of polynomials, partial differential expressions for Gould–Hopper–Frobenius–Genocchi polynomials, etc. [5]. Tian and Chen et al. obtained the general solution formula for the corresponding second-order second derivative equation from the prime decomposition of a quadratic matrix polynomial and the general solution formula for the corresponding second-order disjunctive equation [6]. Hasil and Vesel' et al. used the Riccati transformation, identified a new type of conditionally oscillatory linear differential equations together with the critical oscillation constant, and proved that they remained conditionally oscillatory with the same critical oscillation constant [7]. Bazighifan et al. studied the oscillation of solutions for a fourth-order neutral nonlinear differential equation, driven by a p-Laplace differential operator and obtained an oscillatory criterion for these equations [8]. Zheng et al. introduced the elastic transformation method into the process of solving ordinary differential equations. A class of first-order and a class of third-order ordinary differential equations with variable coefficients could be transformed into the Laguerre equation through elastic transformation. The general solution of these two types of ordinary differential equations was obtained [9]. Soto et al. studied a class of singular nonlinear partial differential equations and constructed unique solutions that were continuous in a certain interval [10]. According to the properties and initial conditions of the Chebyshev polynomials, Fan et al. could obtain the solutions of the initial value problems of the original first-order differential equations and the third-order differential equations by the elastic inverse transformation and then plot the curves [11]. Fan et al. studied the positive solutions of the periodic-parabolic logistic equation with indefinite weight function and a non homogeneous diffusion coefficient, and obtained the existence, uniqueness, and stability of positive periodic solutions [12]. Zhang et al. studied the initial value problem for higher-order Caputo type modified nonlinear differential equations with Erdelyi–Kober fractional derivatives. According to the transmutation method, the good solvability of the initial value problem for higher-order linear models was proved and explicit solutions were given. Some new Gronwall-type inequalities involving Erdelyi–Kober fractional integrals were then established. By applying these results and some fixed point theorems, the existence and uniqueness of positive solutions of nonlinear differential equations were proved [13]. The results of Albidah et al. showed that the solution was continuous in the domain of the problem in the limit of the given initial conditions, whereas the first-order derivatives were discontinuous at a point and lay in the domain of the delay equation [14]. Al-Mazmumy et al. improved the standard Adomian decomposition method by combining Taylor series with orthogonal polynomials [15]. Bendouma et al. investigated the existence of solutions to a system of fourth-order differential equations when a function lay on the right-hand side. The concept of a solution tube was proposed for these problems. According to this concept, the concepts of the upper and lower solutions of fourth-order differential equations were extended to systems of fourth-order differential equations [16]. Wang et al. studied a certain higher-order uncertain differential equation, focusing mainly on the second-order case. We introduced the notion of a path for uncertain differential equations and proved its properties. On this basis, we derived the inverse uncertainty distribution of the solution [17]. AlKandari et al. studied a certain oscillatory behaviour of the solutions of a class of multi-delayed third-order neutral differential equations and proposed a new oscillatory criterion that refined and simplified some previous results [18]. Turo et al. proved an existence and uniqueness theorem for first-order stochastic partial differential equations [19]. Zhang et al. numerically investigated the spatially variable coefficient fractional convection–diffusion wave equation with single and multiple delays when the exact solution satisfied certain regularities [20]. Borrego-Morell et al. studied linear second-order differential equations. Under some assumptions on a certain class of descending and ascending operators, it was shown that, for a certain sequence of polynomials to be orthogonal on the unit circle in order to satisfy the differential equations, the polynomials must be of a particular form involving extensions of Gaussian and confluent hypergeometric series [21]. Akhmetkaliyeva et al. considered the third-order differential equation with unconstrained coefficients, proved some new existence and uniqueness results, and gave exact estimates of the norm of the solution [22]. Padhi et al. obtained sufficient conditions for the existence of at least two non-negative periodic solutions for a system of first-order nonlinear functional differential equations, giving some applications to ecological [23]. Mohamed et al. studied the conditions under which Schrödinger-type operators with matrix potentials were separated and the Schrödinger equation had a unique solution in a weighted space [24]. Khaliq et al. presented a series of second-order methods for the numerical solution of fourth-order parabolic partial differential equations with variable coefficients in one spatial variable [25]. Mohammed et al. introduced a class of second-order ordinary differential equations with variable coefficients whose closed-form solutions could be obtained by the same methods used to solve differential equations with constant coefficients. General solutions in the homogeneous case were also discussed [26]. Lynch et al. presented a large class of second-order variable coefficient ordinary differential equations with known solutions. This class of equations formed the basis of a large number of examples that could be used to explore the theory of solutions to linear differential equations [27]. Camporesi et al. proposed an impulse response method based on the factorization of differential operators for solving linear ordinary differential equations with constant coefficients [28]. Jia et al. used the relationship between the higher-order variable coefficients, the conditions for constant coefficient of the fifth-order variable coefficient linear non-homogeneous equations were obtained by appropriate linear transformations [29]. To sum up, although these research achievements have made some contributions in the field of lower-order VCHDE, there is still much room for exploration in the research of higher-order VCHDE, especially in the aspect of extending the conditions for the existence of particular solutions to higher orders.

Generally speaking, previous research efforts have predominantly concentrated on lower-order VCHDE. The investigations into second-order and third-order VCHDE have, to a certain degree, established the groundwork and proffered ideas for higher-order scenarios. However, fourth-order VCHDE are far more intricate than their second-order and third-order counterparts, and frequently manifests in practical applications. For instance, Hu and Li et al. [30] furnished the sufficient conditions for the existence of polynomial particular solutions for a class of second-order linear VCHDE with polynomial coefficients through recursive formulas. Subsequently, Li and Jiang et al. [31] leveraged the concept of coefficient matrices to deduce the solutions of third-order linear VCHDE with polynomial coefficients, building upon the work of Hu et al. [30]. Notably, although fourth-order VCHDE are typically more complex than second or third order ones, they are inescapable in certain problems. This present work, for the first time, extends the necessary and sufficient conditions for the existence of particular solutions of VCHDE to the fourth-order and nth-order cases. By delving into the relationship between the coefficient matrix of the VCHDE and the existence of polynomial particular solutions, the necessary and sufficient conditions for the existence of solutions of fourth-order VCHDE and those for the nth-order VCHDE are derived. In this way, we are able to address the problem of the existence of solutions of VCHDE with polynomial coefficients for any order. It should be emphasized that although the ultimate conclusion pertains to the nth-order case, the exploration of the fourth-order VCHDE is indispensable. The fourth-order case serves as a crucial stepping-stone, bridging the gap between the lower-order studies and generalization to the nth-order, providing valuable insights and intermediate results that are essential for the overall theoretical framework.

The paper is organized as follows: Section 2 derives the existence of solutions to fourth-order VCHDE, Section 3 proves the existence of solutions to nth-order VCHDE, and Section 4 provides the discussion and conclusion.

2. Existence of special solutions to fourth-order VCHDE

In this section, we study the existence of particular solutions to fourth-order VCHDE subject to the following constraints.

Lemma 1. *Let A be an $n \times m$ matrix and let X be an m-dimensional column vector. There then exists a non-zero solution $X^* = 0$ to the system of homogeneous linear equations $AX = 0$, and the sufficient and necessary condition that its first component is not zero is:*

$$\operatorname{rank}\left(\widehat{A}\right) = \operatorname{rank}(A),$$

where $\widehat{A}$ is the submatrix of A minus the first column.

Now, the existence of solutions of fourth-order VCHDE is investigated

$$P(x)y^{''''} + Q(x)y^{'''} + R(x)y^{''} + S(x)y^{'} + T(x)y = 0, \tag{2.1}$$

based on the Lemma, where the coefficients of the differential equation are

$$\begin{pmatrix} P(x) \\ Q(x) \\ R(x) \\ S(x) \\ T(x) \end{pmatrix} = \begin{bmatrix} k_{1,n} & k_{1,n-1} & \cdots & k_{1,1} & k_{1,0} \\ k_{2,n} & k_{2,n-1} & \cdots & k_{2,1} & k_{2,0} \\ k_{3,n} & k_{3,n-1} & \cdots & k_{3,1} & k_{3,0} \\ k_{4,n} & k_{4,n-1} & \cdots & k_{4,1} & k_{4,0} \\ k_{5,n} & k_{5,n-1} & \cdots & k_{5,1} & k_{5,0} \end{bmatrix} \begin{pmatrix} x^n \\ x^{n-1} \\ \vdots \\ x \\ 1 \end{pmatrix}, \tag{2.2}$$

where $k_{i,j}(i = 1, 2, 3, 4, 5.\ j = 0, 1, \ldots, n)$ is a constant.

Theorem 1. *A necessary and sufficient condition for the existence of a particular solution of the polynomial type of the mth-order for the VCHDE* (2.1) *with fourth-order polynomial coefficients (where the coefficients are given by* (2.2)*) is*

$$k_{5,n} = 0, \quad \operatorname{rank}(\widehat{F_m}) = \operatorname{rank}(F_m),$$

where F_m is the $(m + n) \times (m + 1)$ matrix.

$$F_m = \begin{bmatrix} mk_{4,n} + k_{5,n-1} & k_{5,n} \quad 0 \quad \cdots \quad 0 \\ A_m^2 k_{3,n} + mk_{4,n-1} + k_{5,n-2} & \\ A_m^3 k_{2,n} + A_m^2 k_{3,n-1} + mk_{4,n-2} + k_{5,n-3} & \\ A_m^4 k_{1,n} + A_m^3 k_{2,n-1} + A_m^2 k_{3,n-2} + mk_{4,n-3} + k_{5,n-4} & \\ A_m^4 k_{1,n-1} + A_m^3 k_{2,n-2} + A_m^2 k_{3,n-3} + mk_{4,n-4} + k_{5,n-5} & \\ \vdots & F_{m-1} \\ A_m^4 k_{1,4} + A_m^3 k_{2,3} + A_m^2 k_{3,2} + mk_{4,1} + k_{5,0} & \\ A_m^4 k_{1,3} + A_m^3 k_{2,2} + A_m^2 k_{3,1} + mk_{4,0} & \\ A_m^4 k_{1,2} + A_m^3 k_{2,1} + A_m^2 k_{3,0} & \\ A_m^4 k_{1,1} + A_m^3 k_{2,0} & \\ A_m^4 k_{1,0} & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix},$$

$$F_1 = \begin{bmatrix} k_{4,n} + k_{5,n} & k_{5,n} \\ k_{4,n-1} + k_{5,n-1} & k_{5,n-1} \\ \vdots & \vdots \\ k_{4,1} + k_{5,1} & k_{5,1} \\ k_{4,0} & k_{5,0} \end{bmatrix}.$$

Proof. Let the coefficients be nth-degree polynomials, i.e.,

$$\begin{pmatrix} P(x) \\ Q(x) \\ R(x) \\ S(x) \\ T(x) \end{pmatrix} = \begin{bmatrix} k_{1,n} & k_{1,n-1} & \cdots & k_{1,1} & k_{1,0} \\ k_{2,n} & k_{2,n-1} & \cdots & k_{2,1} & k_{2,0} \\ k_{3,n} & k_{3,n-1} & \cdots & k_{3,1} & k_{3,0} \\ k_{4,n} & k_{4,n-1} & \cdots & k_{4,1} & k_{4,0} \\ k_{5,n} & k_{5,n-1} & \cdots & k_{5,1} & k_{5,0} \end{bmatrix} \begin{pmatrix} x^n \\ x^{n-1} \\ \vdots \\ x \\ 1 \end{pmatrix}. \tag{2.2}$$

When $m = 1$, the equation is assumed to have a first-order solution

$$y = K_1 x + K_0,$$

where $K_i (i = 0, 1)$ is a constant. Substituting this into Eq (2.1) yields

$$\begin{aligned}&P(x)y'''' + Q(x)y''' + R(x)y'' + S(x)y' + T(x)y\\ &= K_1 k_{5,n} x^{n+1} + \begin{pmatrix} x^n & x^{n-1} & \cdots & 1 \end{pmatrix} \begin{bmatrix} k_{4,n} + k_{5,n} & k_{5,n} \\ k_{4,n-1} + k_{5,n-1} & k_{5,n-1} \\ \vdots & \vdots \\ k_{4,1} + k_{5,1} & k_{5,1} \\ k_{4,0} & k_{5,0} \end{bmatrix} \begin{bmatrix} K_1 \\ K_0 \end{bmatrix}.\end{aligned}$$

Let

$$F_1 = \begin{bmatrix} k_{4,n} + k_{5,n} & k_{5,n} \\ k_{4,n-1} + k_{5,n-1} & k_{5,n-1} \\ \vdots & \vdots \\ k_{4,1} + k_{5,1} & k_{5,1} \\ k_{4,0} & k_{5,0} \end{bmatrix}.$$

The sufficient condition for the existence of a first-order solution to Eq (2.1) with coefficients (2.2) is given by the Lemma

$$k_{5,n} = 0, \quad \text{rank}(F_1) = \text{rank}(F_0).$$

When $m = 2$, the equation is assumed to have a second-order solution

$$y = K_2 x^2 + K_1 x + K_0,$$

where $K_i (i = 0, 1, 2.)$ is a constant. Substituting this into Eq (2.1) yields

$$\begin{aligned}&P(x)y'''' + Q(x)y''' + R(x)y'' + S(x)y' + T(x)y\\ &= K_2 k_{5,n} x^{n+2} + \begin{pmatrix} x^{n+1} & x^n & \cdots & x & 1 \end{pmatrix} \begin{bmatrix} 2k_{4,n} + k_{5,n-1} & k_{5,n} \quad 0 \\ 2k_{4,n-1} + k_{5,n-2} & \\ \vdots & F_1 \\ k_{4,1} + k_{5,1} & \\ k_{4,0} & \end{bmatrix} \begin{bmatrix} K_2 \\ K_1 \\ K_0 \end{bmatrix}.\end{aligned}$$

Let

$$F_2 = \begin{bmatrix} 2k_{4,n} + k_{5,n-1} & k_{5,n} \quad 0 \\ 2k_{4,n-1} + k_{5,n-2} & \\ \vdots & F_1 \\ k_{4,1} + k_{5,1} & \\ k_{4,0} & \end{bmatrix}.$$

The sufficient condition for the existence of an mth-order solution to Eq (2.1) with coefficients (2.2) is given by the lemma

$$k_{5,n} = 0, \quad \text{rank}(F_2) = \text{rank}(F_1).$$

By analogy, the equation is assumed to have mth-order solutions when $m \geq 4$

$$y = K_m x^m + K_{m-1}x^{m-1} + \cdots + K_1 x + K_0,$$

where $K_i (i = 0, 1, \ldots, m.)$ is a constant. Substituting this into Eq (2.1) yields

$$\begin{aligned}
&P(x)y'''' + Q(x)y''' + R(x)y'' + S(x)y' + T(x)y \\
&= K_m k_{5,n} x^{m+n} + \begin{pmatrix} x^{n+m-1} & x^{n+m-2} & \cdots & x^{m-3} & x^{m-4} \end{pmatrix} \times \\
&\left[\begin{array}{c|c} \begin{matrix} mk_{4,n} + k_{5,n-1} \\ A_m^2 k_{3,n} + mk_{4,n-1} + k_{5,n-2} \\ A_m^3 k_{2,n} + A_m^2 k_{3,n-1} + mk_{4,n-2} + k_{5,n-3} \\ A_m^4 k_{1,n} + A_m^3 k_{2,n-1} + A_m^2 k_{3,n-2} + mk_{4,n-3} + k_{5,n-4} \\ A_m^4 k_{1,n-1} + A_m^3 k_{2,n-2} + A_m^2 k_{3,n-3} + mk_{4,n-4} + k_{5,n-5} \\ \vdots \\ A_m^4 k_{1,4} + A_m^3 k_{2,3} + A_m^2 k_{3,2} + mk_{4,1} + k_{5,0} \\ A_m^4 k_{1,3} + A_m^3 k_{2,2} + A_m^2 k_{3,1} + mk_{4,0} \\ A_m^4 k_{1,2} + A_m^3 k_{2,1} + A_m^2 k_{3,0} \\ A_m^4 k_{1,1} + A_m^3 k_{2,0} \\ A_m^4 k_{1,0} \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} k_{5,n} \quad 0 \quad \cdots \quad 0 \\ \\ \\ \\ \\ F_{m-1} \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \end{array}\right] \begin{bmatrix} K_m \\ K_{m-1} \\ \vdots \\ K_1 \\ K_0 \end{bmatrix}.
\end{aligned}$$

Let

$$F_m = \left[\begin{array}{c|c} \begin{matrix} mk_{4,n} + k_{5,n-1} \\ A_m^2 k_{3,n} + mk_{4,n-1} + k_{5,n-2} \\ A_m^3 k_{2,n} + A_m^2 k_{3,n-1} + mk_{4,n-2} + k_{5,n-3} \\ A_m^4 k_{1,n} + A_m^3 k_{2,n-1} + A_m^2 k_{3,n-2} + mk_{4,n-3} + k_{5,n-4} \\ A_m^4 k_{1,n-1} + A_m^3 k_{2,n-2} + A_m^2 k_{3,n-3} + mk_{4,n-4} + k_{5,n-5} \\ \vdots \\ A_m^4 k_{1,4} + A_m^3 k_{2,3} + A_m^2 k_{3,2} + mk_{4,1} + k_{5,0} \\ A_m^4 k_{1,3} + A_m^3 k_{2,2} + A_m^2 k_{3,1} + mk_{4,0} \\ A_m^4 k_{1,2} + A_m^3 k_{2,1} + A_m^2 k_{3,0} \\ A_m^4 k_{1,1} + A_m^3 k_{2,0} \\ A_m^4 k_{1,0} \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} k_{5,n} \quad 0 \quad \cdots \quad 0 \\ \\ \\ \\ \\ F_{m-1} \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \end{array}\right].$$

In summary, it follows from the lemma that the sufficient condition for the existence of a particular solution to this VCHDE is

$$k_{5,n} = 0, \quad \mathrm{rank}(\widehat{F_m}) = \mathrm{rank}(F_m).$$

Proof completed (PC).

The existence of unique solutions to this fourth-order VCHDE is investigated next.

Theorem 2. *When $k_{4,n} \neq 0$, there is a unique number of polynomial particular solutions to VCHDE* (2.1) *with fourth-order polynomial coefficients of* (2.2) .

Proof. By Theorem 1, the differential Eq (2.1) has mth-order polynomial particular solutions if and only if

$$k_{5,n} = 0, \quad \mathrm{rank}(\widehat{F_m}) = \mathrm{rank}(F_m),$$

where

$$F_m = \begin{bmatrix} mk_{4,n} + k_{5,n-1} & k_{5,n} \quad 0 \quad \cdots \quad 0 \\ A_m^2 k_{3,n} + mk_{4,n-1} + k_{5,n-2} & \\ A_m^3 k_{2,n} + A_m^2 k_{3,n-1} + mk_{4,n-2} + k_{5,n-3} & \\ A_m^4 k_{1,n} + A_m^3 k_{2,n-1} + A_m^2 k_{3,n-2} + mk_{4,n-3} + k_{5,n-4} & \\ A_m^4 k_{1,n-1} + A_m^3 k_{2,n-2} + A_m^2 k_{3,n-3} + mk_{4,n-4} + k_{5,n-5} & \\ \vdots & F_{m-1} \\ A_m^4 k_{1,4} + A_m^3 k_{2,3} + A_m^2 k_{3,2} + mk_{4,1} + k_{5,0} & \\ A_m^4 k_{1,3} + A_m^3 k_{2,2} + A_m^2 k_{3,1} + mk_{4,0} & \\ A_m^4 k_{1,2} + A_m^3 k_{2,1} + A_m^2 k_{3,0} & \\ A_m^4 k_{1,1} + A_m^3 k_{2,0} & \\ A_m^4 k_{1,0} & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}.$$

It is clear that $f_{11} = mk_{4,n} + k_{5,n-1}$, $f_{1j} = 0$, $j = 2, 3, \ldots, m-1$ in the matrix $F_m = (f_{ij})$.
Then from $\mathrm{rank}(\widehat{F_m}) = \mathrm{rank}(F_m)$, we conclude

$$mk_{4,n} + k_{5,n-1} = 0.$$

In other words,

$$m = -\frac{k_{5,n-1}}{k_{4,n}} \in \mathbb{N}_+,$$

and the other elements on the diagonal of the matrix F_m satisfy

$$lk_{4,n} + k_{5,n-1} \neq 0, \quad l = 0, 1, \ldots, m-1.$$

Otherwise, it does not satisfy $\mathrm{rank}(\widehat{F_m}) = \mathrm{rank}(F_m)$,
where we let $\overline{F_m}$ be the matrix obtained by removing the first row of F_m.

Thus

$$\overline{F_m} = \begin{bmatrix} A_m^2 k_{3,n} + mk_{4,n-1} + k_{5,n-2} & \\ A_m^3 k_{2,n} + A_m^2 k_{3,n-1} + mk_{4,n-2} + k_{5,n-3} & \\ A_m^4 k_{1,n} + A_m^3 k_{2,n-1} + A_m^2 k_{3,n-2} + mk_{4,n-3} + k_{5,n-4} & \\ A_m^4 k_{1,n-1} + A_m^3 k_{2,n-2} + A_m^2 k_{3,n-3} + mk_{4,n-4} + k_{5,n-5} & \\ \vdots & \\ A_m^4 k_{1,4} + A_m^3 k_{2,3} + A_m^2 k_{3,2} + mk_{4,1} + k_{5,0} & F_{m-1} \\ A_m^4 k_{1,3} + A_m^3 k_{2,2} + A_m^2 k_{3,1} + mk_{4,0} & \\ A_m^4 k_{1,2} + A_m^3 k_{2,1} + A_m^2 k_{3,0} & \\ A_m^4 k_{1,1} + A_m^3 k_{2,0} & \\ A_m^4 k_{1,0} & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}.$$

It is clear that $rank(F_m) = rank(\overline{F_m}) = m$.
Subject to the above conditions, the VCHDE (2.1) has mth-order polynomial particular solutions.
Proof completed (PC).

The proof of Theorem 2 includes the proof of the following corollary, so by the same reasoning.

Corollary 1. *When $k_{4,n} \neq 0$, the sufficiently and necessary condition for the existence of mth-order polynomial particular solutions to the fourth-order VCHDE* (2.1) *with the coefficients* (2.2) *is*

$$m = -\frac{k_{5,n-1}}{k_{4,n}} \in \mathbb{N}_+, rank(\overline{F_m}) = m,$$

where we let $\overline{F_m}$ be the matrix obtained by removing the first row of F_m.

Theorem 3. *When $k_{3,n} \neq 0$, the sufficient and necessary condition for the coefficients* (2.2) *of fourth-order VCHDE* (2.1) *to have two linearly independent particular solutions is*

$$k_{4,n} = k_{5,n} = k_{5,n-1} = 0,$$

and

$$h(x) = A_x^2 k_{3,n} + xk_{4,n-1} + k_{5,n-2},$$

has two positive integer roots m_1, m_2, where $m_1 \neq m_2$, and

$$\text{rank}\left(F_{m_1}\right) = \text{rank}\left(F_{m_1-1}\right), \text{rank}\left(F_{m_2}\right) = \text{rank}\left(F_{m_2-1}\right).$$

Proof. With loss of generality, we can assume that $m_1 < m_2$. By Theorems 1 and 2, the equation with coefficients of (2.2) has two linearly independent particular solutions if and only if

$$k_{5,n} = 0, \quad k_{4,n} = 0, \quad \text{rank}\left(\widehat{F_m}\right) = \text{rank}\left(F_m\right), \quad m = m_1, m_2.$$

In other words,

$$mk_{4,n} + k_{5,n-1} = 0.$$

Thus

$$k_{5,n-1} = 0.$$

So under the above condition, the constraints at this point have

$$F_m = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ A_m^2 k_{3,n} + mk_{4,n-1} + k_{5,n-2} & & & \\ A_m^3 k_{2,n} + A_m^2 k_{3,n-1} + mk_{4,n-2} + k_{5,n-3} & & & \\ A_m^4 k_{1,n} + A_m^3 k_{2,n-1} + A_m^2 k_{3,n-2} + mk_{4,n-3} + k_{5,n-4} & & & \\ A_m^4 k_{1,n-1} + A_m^3 k_{2,n-2} + A_m^2 k_{3,n-3} + mk_{4,n-4} + k_{5,n-5} & & & \\ \vdots & & F_{m-1} & \\ A_m^4 k_{1,4} + A_m^3 k_{2,3} + A_m^2 k_{3,2} + mk_{4,1} + k_{5,0} & & & \\ A_m^4 k_{1,3} + A_m^3 k_{2,2} + A_m^2 k_{3,1} + mk_{4,0} & & & \\ A_m^4 k_{1,2} + A_m^3 k_{2,1} + A_m^2 k_{3,0} & & & \\ A_m^4 k_{1,1} + A_m^3 k_{2,0} & & & \\ A_m^4 k_{1,0} & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

It is clear that for $F_m = (f_{ij})$, we have $f_{ii} = 0$, $f_{i,i+1} = 0$, for $i = 1, 2, \ldots, m$, and

$$\operatorname{rank}(F_{m_1-1}) = \operatorname{rank}\left(\widehat{F_{m_1}}\right), \quad \operatorname{rank}(F_{m_2-1}) = \operatorname{rank}\left(\widehat{F_{m_2}}\right),$$

from $\operatorname{rank}(F_{m_1}) = \operatorname{rank}\left(\widehat{F_{m_1}}\right)$ and $\operatorname{rank}(F_{m_2}) = \operatorname{rank}\left(\widehat{F_{m_2}}\right)$, we obtain

$$\operatorname{rank}(F_{m_1}) = \operatorname{rank}(F_{m_1-1}), \quad \operatorname{rank}(F_{m_2}) = \operatorname{rank}(F_{m_2-1}),$$

$$A_{m_1}^2 k_{3,n} + m_1 k_{4,n-1} + k_{5,n-2} = 0,$$

$$A_{m_2}^2 k_{3,n} + m_2 k_{4,n-1} + k_{5,n-2} = 0,$$

In other words,

$$h(m_1) = 0, \quad h(m_2) = 0.$$

Because $h(x)$ is a quadratic polynomial, m_1, m_2 are exactly two different positive integer roots of $h(x)$.

Proof completed (PC).

Using the same ideas and methods as in Theorem 3, we continue to explore the rank of the coefficient matrix F_m, proving the following corollary, so by the same reasoning.

Corollary 2. *When $k_{2,n} \neq 0$, the sufficiently and necessary condition for a fourth-order VCHDE* (2.1) *with the coefficients* (2.2) *to have three linearly independent particular solutions is*

$$k_{4,n} = k_{5,n} = k_{5,n-1} = 0,$$
$$k_{3,n} = k_{4,n-1} = k_{5,n-2} = 0,$$
$$h_1(x) = A_x^3 k_{2,n} + A_x^2 k_{3,n-1} + x k_{4,n-2} + k_{5,n-3},$$

where there exist three positive integer roots m_1, m_2, m_3, where m_1, m_2, m_3 are pairwise distinct, and

$$\begin{aligned} \operatorname{rank}(F_{m_1}) &= \operatorname{rank}(F_{m_1-2}), \\ \operatorname{rank}(F_{m_2}) &= \operatorname{rank}(F_{m_2-2}), \\ \operatorname{rank}(F_{m_3}) &= \operatorname{rank}(F_{m_3-2}). \end{aligned}$$

Corollary 3. *When $k_{1,n} \neq 0$, the sufficiently and necessary condition for a fourth-order VCHDE* (2.1) *with the coefficients* (2.2) *to have four linearly independent particular solutions is*

$$k_{4,n} = k_{5,n} = k_{5,n-1} = 0,$$
$$k_{3,n} = k_{4,n-1} = k_{5,n-2} = 0,$$
$$k_{2,n} = k_{3,n-1} = k_{4,n-2} = k_{5,n-3} = 0,$$
$$h_2(x) = A_x^4 k_{1,n} + A_x^3 k_{2,n-1} + A_x^2 k_{3,n-2} + x k_{4,n-3} + k_{5,n-4},$$

where there exist four positive integer roots m_1, m_2, m_3, m_4, where m_1, m_2, m_3, m_4 are pairwise distinct, and

$$\begin{aligned} \operatorname{rank}(F_{m_1}) &= \operatorname{rank}(F_{m_1-3}), \\ \operatorname{rank}(F_{m_2}) &= \operatorname{rank}(F_{m_2-3}), \\ \operatorname{rank}(F_{m_3}) &= \operatorname{rank}(F_{m_3-3}), \\ \operatorname{rank}(F_{m_4}) &= \operatorname{rank}(F_{m_4-3}). \end{aligned}$$

3. Existence of special solutions to nth-order VCHDE

In this section, we study the existence of solutions to VCHDE of the nth-order.

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = 0, \tag{3.1}$$

where the coefficients of the VCHDE are

$$\begin{pmatrix} P_n(x) \\ P_{n-1}(x) \\ \vdots \\ P_1(x) \\ P_0(x) \end{pmatrix} = \begin{bmatrix} k_{1,n} & k_{1,n-1} & \dots & k_{1,1} & k_{1,0} \\ k_{2,n} & k_{2,n-1} & \dots & k_{2,1} & k_{2,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n,n} & k_{n,n-1} & \dots & k_{n,1} & k_{n,0} \\ k_{n+1,n} & k_{n+1,n-1} & \dots & k_{n+1,1} & k_{n+1,0} \end{bmatrix} \begin{pmatrix} x^n \\ x^{n-1} \\ \vdots \\ x \\ 1 \end{pmatrix}. \tag{3.2}$$

We extend the conclusions for all particular solutions of VCHDE of the fourth-order to VCHDE of the nth-order.

Theorem 4. *A necessary and sufficient condition for the existence of a particular solution of the polynomial type of the mth-order for the VCHDE* (3.1) *with nth-order polynomial coefficients (where the coefficients are given by* (3.2)*) is*

$$k_{n+1,n} = 0, \quad \operatorname{rank}\left(\widehat{F_m}\right) = \operatorname{rank}\left(F_m\right),$$

where F_m is the $(m+n) \times (m+1)$ matrix.

$F_m =$

$$\begin{bmatrix} mk_{n,n} + k_{n+1,n-1} & k_{n+1,n} & 0 & \cdots & 0 \\ A_m^2 k_{n-1,n} + mk_{n,n-1} + k_{n+1,n-2} & & & & \\ A_m^3 k_{n-2,n} + A_m^2 k_{n-1,n-1} + mk_{n,n-2} + k_{n+1,n-3} & & & & \\ \vdots & & & & \\ A_m^n k_{1,n} + A_m^{n-1} k_{2,n-1} + \cdots + A_m^3 k_{n-2,3} + A_m^2 k_{n-1,2} + mk_{n,1} + k_{n+1,0} & F_{m-1} & & & \\ A_m^n k_{1,n-1} + A_m^{n-1} k_{2,n-2} + \cdots + A_m^3 k_{n-2,2} + A_m^2 k_{n-1,1} + mk_{n,0} & & & & \\ A_m^n k_{1,n-2} + A_m^{n-1} k_{2,n-3} + \cdots + A_m^3 k_{n-2,1} + A_m^2 k_{n-1,0} & & & & \\ \vdots & & & & \\ A_m^n k_{1,0} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix},$$

$$F_1 = \begin{bmatrix} k_{n,n} + k_{n+1,n} & k_{n+1,n} \\ k_{n,n-1} + k_{n+1,n-1} & k_{n+1,n-1} \\ \vdots & \vdots \\ k_{n,1} + k_{n+1,1} & k_{n+1,1} \\ k_{n,0} & k_{n+1,0} \end{bmatrix}.$$

Proof. If there exist mth-order polynomial particular solutions to the nth-order VCHDE (3.1) with the coefficients (3.2)

$$y = K_m x^m + K_{m-1} x^{m-1} + \ldots + K_1 x + K_0,$$

where $K_i (i = 0, 1, \ldots, m)$ is a constant.

For ease of computation, without loss of generality, let us assume $m \geq n > 0$. Then bringing y into the Eq (3.1) yields

$$\begin{aligned} & P_n(x) y^{(n)} + P_{n-1}(x) y^{(n-1)} + \cdots + P_1(x) y' + P_0(x) y \\ &= K_m [A_m^n P_n(x) x^{m-n} + A_m^{n-1} P_{n-1}(x) x^{m-n+1} + \cdots + m P_1(x) x^{m-1} + P_0(x) x^m] + E_{m-1} \\ &= K_m k_{n+1,n} x^{m+n} + K_m [A_m^n (x^m \quad x^{m-1} \quad \cdots \quad x^{m-n+1} \quad x^{m-n}) \begin{pmatrix} k_{1,n} \\ k_{1,n-1} \\ \vdots \\ k_{1,0} \\ k_{1,0} \end{pmatrix} \end{aligned}$$

$$+A_m^{n-1}\begin{pmatrix} x^{m+1} & x^m & \cdots & x^{m-n+2} & x^{m-n+1}\end{pmatrix}\begin{pmatrix} k_{2,n} \\ k_{2,n-1} \\ \vdots \\ k_{2,1} \\ k_{2,0}\end{pmatrix}+\cdots$$

$$+m\begin{pmatrix} x^{n+m-1} & x^{n+m-2} & ... & x^m & x^{m-1}\end{pmatrix}\begin{pmatrix} k_{n,n} \\ k_{n,n-1} \\ \vdots \\ k_{n,1} \\ k_{4,0}\end{pmatrix}$$

$$+\begin{pmatrix} x^{n+m-1} & x^{n+m-2} & \cdots & x^{m+1} & x^m\end{pmatrix}\begin{pmatrix} k_{n+1,n-1} \\ k_{n+1,n-2} \\ \vdots \\ k_{n+1,1} \\ k_{n+1,0}\end{pmatrix}+E_{m-1}$$

$$=K_m k_{n+1,n}x^{m+n}+\begin{pmatrix} x^{n+m-1} & x^{n+m-2} & \cdots & x^{m-n+1} & x^{m-n}\end{pmatrix}F_m\begin{pmatrix} K_m \\ K_{m-1} \\ \vdots \\ K_1 \\ K_0\end{pmatrix},$$

where F_m is the $(m+n)\times(m+1)$ matrix.

$F_m =$

$$\begin{bmatrix} mk_{n,n}+k_{n+1,n-1} & k_{n+1,n} & 0 & \cdots & 0 \\ A_m^2 k_{n-1,n}+mk_{n,n-1}+k_{n+1,n-2} & & & & \\ A_m^3 k_{n-2,n}+A_m^2 k_{n-1,n-1}+mk_{n,n-2}+k_{n+1,n-3} & & & & \\ \vdots & & & & \\ A_m^n k_{1,n}+A_m^{n-1}k_{2,n-1}+\cdots+A_m^3 k_{n-2,3}+A_m^2 k_{n-1,2}+mk_{n,1}+k_{n+1,0} & F_{m-1} & & & \\ A_m^n k_{1,n-1}+A_m^{n-1}k_{2,n-2}+\cdots+A_m^3 k_{n-2,2}+A_m^2 k_{n-1,1}+mk_{n,0} & & & & \\ A_m^n k_{1,n-2}+A_m^{n-1}k_{2,n-3}+\cdots+A_m^3 k_{n-2,1}+A_m^2 k_{n-1,0} & & & & \\ \vdots & & & & \\ A_m^n k_{1,0} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix},$$

$$F_1 = \begin{bmatrix} k_{n,n} + k_{n+1,n} & k_{n+1,n} \\ k_{n,n-1} + k_{n+1,n-1} & k_{n+1,n-1} \\ \vdots & \vdots \\ k_{n,1} + k_{n+1,1} & k_{n+1,1} \\ k_{n,0} & k_{n+1,0} \end{bmatrix}.$$

By Lemma 1, it follows that if and only if it satisfies

$$k_{n+1,n} = 0, rank\left(\widehat{F_m}\right) = rank(F_m),$$

There exist mth polynomial particular solutions to the nth-order VCHDE (3.1) with the coefficients (3.2).

Proof completed (PC).

We can now extend the conclusions about the specific solutions of the fourth-order VCHDE (2.1) to the conclusions about the specific solutions of the nth-order VCHDE (3.1), and therefore the following conclusions can be proved as well.

Theorem 5. *When $k_{n,n} \neq 0$, there is a unique number of polynomial particular solutions to VCHDE* (3.1) *with nth-order polynomial coefficients of* (3.2).

Corollary 4. *When $k_{n,n} \neq 0$, the sufficiently and necessary condition for the existence of mth-order polynomial particular solutions to the nth-order VCHDE* (3.1) *with the coefficients* (3.2) *is*

$$m = -\frac{k_{n+1,n-1}}{k_{n,n}} \in N_{+} rank(\widehat{F_m}) = m,$$

where $\widehat{F_m}$ is the matrix obtained by removing the first row of F_m.

Theorem 6. *When $k_{n-1,n} \neq 0$, the sufficient and necessary condition for the coefficients* (3.2) *of nth-order VCHDE* (3.1) *to have two linearly independent particular solutions is*

$$k_{n,n} = k_{n+1,n} = k_{n+1,n-1} = 0,$$

$$h(x) = A_m^2 k_{n-1,n} + m k_{n,n-1} + k_{n+1,n-2},$$

has two positive integer roots m_1, m_2, where $m_1 \neq m_2$, and

$$\text{rank}\left(F_{m_1}\right) = \text{rank}\left(F_{m_1-1}\right), \text{rank}\left(F_{m_2}\right) = \text{rank}\left(F_{m_2-1}\right).$$

Corollary 5. *When $k_{n-2,n} \neq 0$, the sufficiently necessary condition for an nth-order VCHDE* (3.1) *with the coefficients* (3.2) *to have three linearly independent particular solutions is*

$$k_{n,n} = k_{n+1,n} = k_{n+1,n-1} = 0,$$

$$k_{n-1,n} = k_{n,n-1} = k_{n+1,n-2} = 0,$$

$$h_1(x) = A_m^3 k_{n-2,n} + A_m^2 k_{n-1,n-1} + m k_{n,n-2} + k_{n+1,n-3},$$

where there exist three positive integer roots m_1, m_2, m_3, where m_1, m_2, m_3 are pairwise distinct, and

$$\begin{aligned} \operatorname{rank}(F_{m_1}) &= \operatorname{rank}(F_{m_1-2}), \\ \operatorname{rank}(F_{m_2}) &= \operatorname{rank}(F_{m_2-2}), \\ \operatorname{rank}(F_{m_3}) &= \operatorname{rank}(F_{m_3-2}). \end{aligned}$$

Corollary 6. *When $k_{n-3,n} \neq 0$, the sufficiently necessary condition for a fourth-order VCHDE* (3.1) *with the coefficients* (3.2) *to have four linearly independent particular solutions is*

$$k_{n,n} = k_{n+1,n} = k_{n+1,n-1} = 0,$$

$$k_{n-1,n} = k_{n,n-1} = k_{n+1,n-2} = 0,$$

$$k_{n-2,n} = k_{n-1,n-1} = k_{n,n-2} = k_{n+1,n-3} = 0,$$

$$h_2(x) = A_m^4 k_{n-3,n} + A_m^3 k_{n-2,n-1} + A_m^2 k_{n-1,n-2} + m k_{n,n-3} + k_{n+1,n-4},$$

where there exist four positive integer roots m_1, m_2, m_3, m_4, where m_1, m_2, m_3, m_4 are pairwise distinct, and

$$\begin{aligned} \operatorname{rank}(F_{m_1}) &= \operatorname{rank}(F_{m_1-3}), \\ \operatorname{rank}(F_{m_2}) &= \operatorname{rank}(F_{m_2-3}), \\ \operatorname{rank}(F_{m_3}) &= \operatorname{rank}(F_{m_3-3}), \\ \operatorname{rank}(F_{m_4}) &= \operatorname{rank}(F_{m_4-3}). \end{aligned}$$

4. Discussion and conclusions

In this work, we used the lemma about the necessary and sufficient conditions for the existence of the solutions of homogeneous differential equations, and the relationship between the coefficient matrix of VCHDE and the existence of polynomial particular solutions was studied. Especially, compared with prior works [30, 31], this work extends, for the first time, the necessary and sufficient conditions for the existence of particular solutions of VCHDE to the fourth-order and nth-order cases. The main results are as follows:

(1) The necessary and sufficient conditions for the existence of solutions of fourth-order variable coefficient VCHDE and the necessary and sufficient conditions for the existence of solutions of nth-order VCHDE were obtained.

(2) The necessary and sufficient conditions for the existence of only one polynomial particular solution, or the existence of two, three, or four linearly independent polynomial particular solutions of fourth-order VCHDE are proved.

(3) The necessary and sufficient conditions for the existence of only one polynomial particular solution, or the existence of two, three, or four linearly independent polynomial particular solutions of nth-order VCHDE are proved.

These results not only expand the class of solvable differential equations but also provide a new way of thinking about the existence of solutions to polynomial coefficient VCHDE and a systematic and unified analytical framework for studying VCHDE.

However, the results of this work have not yet been validated numerically, and we plan to validate the results in future work. Many early studies have demonstrated that numerical validation methods are indeed feasible. Hu designed an algorithm to automatically determine and solve the polynomial particular or general solutions of the equation [30]. The algorithm judges and calculates on the basis of the equation's coefficient conditions, including evaluating the polynomial coefficient values, calculating the roots of relevant functions, and constructing matrices to determine the rank and determinant. Examples are verified in Maple software. Li designed an algorithm for solving the third-order VCHDE with polynomial coefficients [31]. The process entails inputting the polynomial coefficients, making multi-step judgments according to the coefficients' conditions, such as verifying that the coefficient elements are zero, calculating the roots of relevant functions, and constructing matrices to judge the rank, ultimately outputting the expressions of particular or general solutions. MATLAB software is used to solve and verify specific examples. Specifically, by substituting specific coefficient values into the equation and employing numerical computation methods to solve it, followed by comparing the outcomes with the theoretical results, the existence of particular solutions under various conditions can be vividly illustrated. Thus, this method can be applied in our future validation work.

Additionally, the current work has limitations in terms of the polynomial coefficients and the number of linearly independent solutions. Therefore, we plan to address these limitations in future work. On one hand, we will consider various types of VCHDE, including those with non-polynomial coefficients, such as exponential and trigonometric functions, and explore the feasibility of establishing similar conditions for the existence of solutions. On the other hand, the influence of boundary value conditions on the equations is an important aspect. If VCHDEs 2.1 and 3.1 are subject to boundary value conditions, they may alter the existence and uniqueness of solutions, requiring reanalysis and the derivation of relevant conclusions. Zhao et al. confirmed that the interaction between boundary value conditions and the rank of the coefficient matrix has a significant effect on VCHDE [2].

The theoretical results presented in this work have potential applications to real-world problems. In materials science, VCHDE can describe processes like heat conduction and mass diffusion in materials, with coefficients varying according to the material's physical properties. Wang's optical system design research shows precise analysis of differential equation solutions' properties is crucial in material physical process modeling. This study's results help determine the heat conduction equation solutions' existence, the aiding understanding of heat transfer in alloys [1]. In communication, signal propagation models in complex media often involve VCHDE. For instance, in time-varying channels, signal attenuation and velocity may change with time or space. Zhang et al.'s research on nonlinear differential equation solutions shows that this study's findings may help analyze propagation model solutions' characteristics, providing theoretical support for signal processing and communication system design [20]. However, further in-depth studies and real-data verification are needed for specific applications.

Authors contribution

The first author was mainly responsible for the formulas' derivation and programming. The second author guided the crucial steps from theoretical hypothesis to derivation of the conclusion. The third author participated in the writing and review of the paper. The fourth author provided technical support for the research. The fifth author was in charge of the literature research and data analysis. The sixth author took part in the formulation of the research plan.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

We thank the anonymous referee for valuable and helpful comments and suggestions. This work is supported by the National Nature Science Foundation of China (12463009), the Yunnan Fundamental Research Projects (grant Nos. 202301AV070007, 202401AU070026, 202401AT070955, and 202401AU070296), the "Yunnan Revitalization Talent Support Program" Innovation Team Project (grant No. 202405AS350012), the Scientific Research Foundation Project of Yunnan Education Department (2023J0624, 2024Y469, 2025Y0720, 2025Y0721, and 2025J0502), and the GHfund A (202407016295).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. P. Y. Wang, F. Feng, Design of high-resolution imaging optical system based on differential equation method, *Laser J.*, **41** (2020), 184–187. https://doi.org/10.14016/j.cnki.jgzz.2020.06.184

2. J. Zhao, R. Y. Ma, Existence and uniqueness of solutions for a class of fourth-order differential equation boundary value problems, *Pure Math. Appl. Math.*, **37** (2021), 81–90. https://doi.org/10.3969/j.issn.1008-5513.2021.01.008

3. F. R. Zhang, L. L. Wu, J. Yang, W. R. Lv, Onentire solutions of certain type of nonlinear differential equations, *AIMS Math.*, **5** (2020), 6124–6134. https://doi.org/10.3934/math.2020393

4. J. Yang, G. P. Chen, Existence of solutions for impulsive hybrid boundary value problems to fractional differential systems, *AIMS Math.*, **6** (2021), 8895–8911 https://doi.org/10.3934/math.2021516

5. R. Alyusof, M. B. Jeelani, Some families of differential equations associated with the Gould-Hopper-Frobenius-Genocchi polynomials, *AIMS Math.*, **7** (2021), 4851–4860. https://doi.org/10.3934/math.2022270

6. Y. B. Tian, S. Chen, Prime decomposition of quadratic matrix polynomials, *AIMS Math.*, **6** (2021), 9911–9918. https://doi.org/10.3934/math.2021576

7. P. Hasil, M. Vesel´, Conditionally oscillatory linear differential equations with coeffcients containing powers of natural logarithm, *AIMS Math.*, **7** (2022), 10681–10699. https://doi.org/10.3934/math.2022596

8. O. Bazighifan, Nonlinear differential equations of fourth-order: Qualitative properties of the solutions, *AIMS Math.*, **5** (2020), 6436–6447. https://doi.org/10.3934/math.2020414

9. P. S. Zheng, J. Luo, S. C. Li, X. X. Dong, Elastic transformation method for solving ordinary differential equations with variable coeffcients, *AIMS Math.*, **7** (2021), 1307–1320. https://doi.org/10.3934/math.2022077

10. J. P. O. Soto, J. E. C. Lope, M. P. F. Ona, Uniformly analytic solutions to a class of singular partial differential equations, *AIMS Math.*, **7** (2022), 10400–10421. https://doi.org/10.3934/math.2022580

11. L. Fan , S. C. Li, D. F. Shao, X. Q. Fu, P. Liu, Q. M. Gui, Elastic transformation method for solving the initial value problem of variable coeffcient nonlinear ordinary differential equations, *AIMS Math.*, **7** (2022), 11972–11991. https://doi.org/ 10.3934/math.2022667

12. M. M. Fan, J. W. Sun, Positive solutions for the periodic-parabolic problem with large diffusion, *AIMS Math.*, **19** (2024), 1116–1132. https://doi.org/ 10.3934/nhm.2024049

13. K. Q. Zhang, Existence and uniqueness of positive solution of a nonlinear differential equation with higher order Erdelyi-Kober operators, *AIMS Math.*, **9** (2024), 1358–1372. https://doi.org/ 10.3934/math.2024067

14. A. B. Albidah, I. M. Alsulami, E. R. El-Zahar, A. Ebaid, Advances in mathematical analysis for solving inhomogeneous scalar differential equation, *AIMS Math.*, **9** (2024), 23331–23343. https://doi.org/ 10.3934/math.20241134

15. M. Al-Mazmumy, M. A. Alyami, M. Alsulami, A. S. Alsulami, S. S. Redhwan, An adomian decomposition method with some orthogonal polynomials to solve nonhomogeneous fractional differential equations (FDEs), *AIMS Math.*, **9** (2024), 30548–30571. https://doi.org/ 10.3934/math.20241475

16. B. Bendouma, F. Z. Ladrani, K. Bouhali, A. Hammoudi, L. Alkhalifa, Solution-tube and existence results for fourth-order differential equations system, *AIMS Math.*, **9** (2024), 32831–32848. https://doi.org/ 10.3934/math.20241571

17. Z. M. Wang, Z. Liu, Z. K. Han, X. Y. Guo, Q. B. Wang, The inverse uncertainty distribution of the solutions to a class of higher-order uncertain differential equations, *AIMS Math.*, **9** (2024), 33023–33061. https://doi.org/ 10.3934/math.20241579

18. M. AlKandari, Nonlinear differential equations with neutral term: Asymptotic behavior of solutions, *AIMS Math.*, **19** (2024), 1116–1132. https://doi.org/ 10.3934/nhm.2024049

19. J. Turo, Study of first order stochastic partial differential equations using integral contractors, *Appl. Anal.*, **70** (2000), 281–291. https://doi.org/10.1080/00036819808840691

20. Q. F. Zhang, L. L. Liu, C. J. Zhang, Compact scheme for fractional diffusion-wave equation with spatial variable coefficient and delays, *Appl. Anal.*, **101** (2020), 1911–1932. https://doi.org/10.1080/00036811.2020.1789600

21. J. Borrego-Morell, A. S. Ranga, Orthogonal polynomials on the unit circle satisfying a second-order differential equation with varying polynomial coefficients, *Integr. Transforms Spec. Funct.*, **28** (2016), 39–55. https://doi.org/10.1080/10652469.2016.1249866

22. R. D. Akhmetkaliyeva, L. E. Persson, K. N. Ospanov, P. Wall, Some new results concerning a class of third-order differential equations, *Appl. Anal.*, **94** (2014), 419–434. https://doi.org/10.1080/00036811.2014.898375

23. S. Padhi, S. Pati, Multiple periodic solutions for system of first-order differential equation, *Appl. Anal.*, **88** (2009), 1005–1014. https://doi.org/10.1080/00036810903114775

24. A. S. Mohamed, Existence and uniqueness of the solution, separation for certain second order elliptic differential equation, *Appl. Anal.*, **76** (2000), 179–184. https://doi.org/10.1080/00036810008840875

25. A. Q. M. Khaliq, E. H. Twizell, A family of second order methods for variable coefficient fourth order parabolic partial differential equations, *Int. J. Comput. Math.*, **23** (1987), 63–76. https://doi.org/10.1080/00207168708803608

26. A. Mohammed, A. Zeleke, Extending the constant coefficient solution technique to variable coefficient ordinary differential equations, *PRIMUS*, **25** (2015), 485–494. https://doi.org/10.1080/10511970.2015.1025160

27. M. A. M. Lynch, H. Y. Gao, Setting up second-order variable coefficient differential equation problems with known general solution, *Int. J. Math. Educ. Sci. Technol.*, **31** (2000), 727–732. https://doi.org/10.1080/002073900434396

28. R. Camporesi, Linear ordinary differential equations with constant coefficients. Revisiting the impulsive response method using factorization, *Int. J. Math. Educ. Sci. Technol.*, **42** (2011), 497–514. https://doi.org/10.1080/0020739X.2010.543162

29. Q. J. Jia, A solution of higher-order variable coefficient non-homogeneous linear differential equation, *Pure Math. Appl. Math.*, **30** (2014), 234–239. https://doi.org/10.3969/j.issn.1005-8036.2012.02.007

30. Y. Z. Hu, S. M. Li, Y. Luo, A study on the solution method of second-order polynomial coefficient linear differential equations, *Univ. Math.*, **31** (2015), 27–33. https://doi.org/10.3969/j.issn.1672-1454.2015.03.006

31. S. M. Li, Y. Jiang, Solution method for third-order polynomial coefficient homogeneous linear differential equations, *Univ. Math.*, **39** (2023), 69–75. https://doi.org/10.3969/j.issn.1672-1454.2023.01.003