Networks and Heterogeneous Media

http://www.aimspress.com/journal/nhm

NHM, 19(3): 1336–1343.
DOI: 10.3934/nhm.2024057
Received: 23 August 2024
Revised: 25 November 2024
Accepted: 28 November 2024
Published: 04 December 2024

*Research article*

# On the critical points of solutions of PDE: The case of concentrating solutions on the sphere

**Francesca Gladiali***

Dipartimento SCFMN, Università di Sassari, Via Vienna 2, 07100 Sassari, Italy

* **Correspondence:** Email: fgladiali@uniss.it.

**Abstract:** In this paper, we are concerned with the number of critical points of solutions of nonlinear elliptic equations in a domain $D$ of the sphere and their index.

**Keywords:** critical points; multipeaks solutions; index; manifolds
**Mathematics Subject Classification:** 35B09, 35B40, 35Q

## 1. Introduction

In this paper, we continue the study of the critical points of positive solutions to nonlinear pde's, that we started in [1] and [2], extending it when the underlying domain is contained in a compact manifold $\mathcal{M}$. Our aim is to give an estimate on the possible number of maxima, minima, and saddle points of positive solutions to some nonlinear pde's and to their index of critical point. We examine first the case of the sphere in Theorem 1.1, and then we extend the result to a more general surface $\mathcal{M}$ in Theorem 1.4. We consider the mean field problem in a smooth domain $D \subset \mathcal{M}$

$$\begin{cases} -\Delta_g u = \rho \frac{h(x)e^u}{\int_D h(x)e^u dV_g} & \text{on } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.1}$$

where $\mathcal{M}$ is a compact Riemannian surface with metric $g$, $h(x)$ is a *smooth* function strictly positive on $D$, $\rho$ is a positive constant, $\Delta_g$ is the Laplace Beltrami operator on $\mathcal{M}$, and $dV_g$ is the volume form on $\mathcal{M}$.

The Mean Field equation appears in conformal geometry in the problem of understanding the possible Gauss curvatures $h(x)$ of metrics on $\mathcal{M}$ conformal to the standard metric. When $\mathcal{M} = S^2$, it is called the Nirenberg problem, and one can see as references the papers [3–6] and references therein. It arises also in some physical models as the mean-field limit of point vortices in the theory of Euler flows, as in [7–10]. And also in the abelian Chern–Simons–Higgs models, see [11–16].

To state our main result, we need to recall some properties of solutions to Eq (1.1). Let $D \subset \mathcal{M}$ be a smooth bounded domain. Assume $h$ is a smooth function such that $\inf_D h(x) > c$ for some $c > 0$. (By smooth, we mean locally analitic). By [17], problem (1.1) has a solution in $H_0^1(D)$ for any $\rho \in (8k\pi, 8(k+1)\pi)$ at least when $D$ is not simply connected, while it has a solution in $H_0^1(D)$ for any $\rho \in (0, 8\pi)$ for every $D$. The case of $\rho = 8\pi k$ is more complicated and depends on the domain $D$, on the value of $m$ and on the manifold $\mathcal{M}$. Here we assume to have a sequence of solutions $u_n$ to

$$\begin{cases} -\Delta_g u_n = \rho_n \frac{h(x)e^{u_n}}{\int_D h(x)e^{u_n} dV_g} & \text{on } D, \\ u_n = 0 & \text{on } \partial D, \end{cases} \tag{1.2}$$

such that $\rho_n \to 8\pi m$, for some $m \geq 1$, as $n \to \infty$. It is well known that, see [5, 17], there exist $m$ points $\{P_1, .., P_m\} \subset D$ and $m$ sequences of points $p_{i,n} \to P_i$ (as $n \to \infty$) for $i = 1, .., m$ such that $u_n(p_{i,n}) \to +\infty$ as $n \to \infty$; $u_n - \log \int_D h(x)e^{u_n} dV_g \to -\infty$ uniformly on compact sets of $\overline{D} \setminus \{P_1, .., P_m\}$ as $n \to \infty$. We say that $u_n$ is a sequence of $m$ *blowing-up solutions* to Eq (1.2). The value of $8\pi$ comes from the standard bubble solutions to

$$-\Delta v = e^v \ \text{ in } \mathbb{R}^2. \tag{1.3}$$

In fact, in a shrinking ball centered at $p_{i,n}$, it is possible to appropriately rescale the solution $u_n$ to Eq (1.2) and show that this rescaling converges in $C^2_{loc}(\mathbb{R}^2)$ to the solution $v$ of Eq (1.3). Thus, each concentration point $P_i$ contributes an amount of $8\pi$.

Our main result is the following:

**Theorem 1.1.** *Let $\mathcal{M} = \mathcal{S}^2$ and $D \subset \mathcal{S}^2$ a smooth domain of Euler characteristic $\chi(D)$. Assume $h$ is a smooth function such that $\inf_D h(x) > c$ for some $c > 0$. Assume we have a sequence of solutions $u_n$ to Eq (1.2) such that $\rho_n \to 8\pi m$, for some $m \geq 1$, as $n \to \infty$. Then, $u_n$ is a sequence of $m$ blowing-up solutions, and for $n$ large enough*

$$\sharp\{\textit{critical point of } u_n \textit{ in } D\} \geq 2m - \chi(D). \tag{1.4}$$

*More precisely, we have that, for $n$ large, there exists exactly one critical point (a nondegenerate maximum) for $u_n$ in $B_\delta(P_i)$ $i = 1, .., m$ and $\delta$ small. Next, denoting by $D' = D \setminus \cup_{i=1}^m B_\delta(P_i)$ and $C_n$ the set of critical points of $u_n$ in $D'$ we have that $u_n$ admits at least $m - \chi(D)$ nondegenerate saddle points in $D'$ and*

$$\sum_{z_j \in C_n} \textit{index}_{z_j}(\nabla u_n) = \chi(D) - m. \tag{1.5}$$

We can also construct some examples when the estimate in Eq (1.4) is optimal or not.

**Corollary 1.2.** *There exists a domain $D \subset \mathcal{S}^2$ and a sequence of solutions $u_n$ to Eq (1.2) that blow-up at $m \geq 1$ points $\{P_1, \ldots, P_m\}$ such that, for $n$ large enough,*

$$\sharp\{\textit{critical point of } u_n \textit{ in } D\} = 2m - \chi(D). \tag{1.6}$$

*Moreover, all critical points of $u_n$ are nondegenerate; $m$ of them are local maxima and $m - \chi(D)$ saddle points.*

**Corollary 1.3.** *There exists a domain $D \subset \mathcal{S}^2$ and a sequence of solutions $u_n$ to Eq* (1.2) *that blow-up at $m \geq 1$ points $\{P_1, \ldots, P_m\}$ such that, for n large enough*

$$\sharp\{\textit{critical point of } u_n \textit{ in } D\} > 2m - \chi(D).$$

The results of Theorem 1.1 and its corollaries can be generalized to a more general smooth surfaces $\mathcal{M}$ when the underlying domain $D$ is all contained in a chart of local isotermal coordinates. And this is possible at least when $D$ is sufficiently small with respect to the coordinates. We will say that $D$ is *suitable* when there exists a system of local isotermal coordinates such that $D$ is contained in a unique chart. For a domain $D$ that is *suitable*, we can prove the following:

**Theorem 1.4.** *Let $\mathcal{M}$ be a smooth surface and $D \subset \mathcal{M}$ a suitable smooth domain of Euler characteristic $\chi(D)$. Assume the assumptions of Theorem 1.1 are satisfied. Then the results of Theorem 1, Corollary 1.2, 1.3 hold for the solutions $u_n$ to Eq* (1.2) *in $D \subset \mathcal{M}$.*

Next, we consider the analog of problem (1.1) in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain with $k \geq 0$ holes. Assume $V(x)$ is a smooth function such that $\inf_\Omega V(x) > c$ for some $c > 0$. Assume we have a sequence of solutions $\tilde{u}_n$ to

$$\begin{cases} -\Delta \tilde{u}_n = \lambda_n V(x) e^{\tilde{u}_n} & \text{in } \Omega, \\ \tilde{u}_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

such that $\lambda_n \int_\Omega V(x) e^{\tilde{u}_n} dx \to 8\pi m$ as $n \to \infty$, for some integer $m \geq 1$. Then, there exist $m$ points $\{P_1, .., P_m\} \subset \Omega$ and $m$ sequences of points $x_{i,n} \to P_i$ (as $n \to \infty$) for $i = 1, .., m$, such that $\tilde{u}_n(x_{i,n}) \to +\infty$ as $n \to \infty$; $\tilde{u}_n \to K(x) := 8\pi \sum_{i=1}^m G(x, P_i)$ uniformly on compact sets of $\overline{\Omega} \setminus \{P_1, .., P_m\}$ as $n \to \infty$, where $G(x, y)$ is the Green function of the domain $\Omega$ with Dirichlet boundary conditions and pole in $y \in \Omega$. We say that $\tilde{u}_n$ is a sequence of *m blowing-up solutions* to Eq (1.7).
In this case, we can prove the following result:

**Theorem 1.5.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain with $k \geq 0$ holes. Assume we have a sequence of solutions $\tilde{u}_n$ to Eq* (1.7) *such that $\lambda_n \int_\Omega V(x) e^{\tilde{u}_n} dx \to 8\pi m$ as $n \to \infty$, for some integer $m \geq 1$. Then, when n is large enough,*

$$\sharp\{\textit{critical point of } \tilde{u}_n \textit{ in } \Omega\} \geq 2m + k - 1. \tag{1.8}$$

*More precisely, we have that, for n large, there exists exactly one critical point (a nondegenerate maximum) for $\tilde{u}_n$ in $B_\rho(P_i)$ $i = 1, .., m$ and $\rho$ small. Next, denoting by $\Omega' = \Omega \setminus \cup_{i=1}^m B_\rho(P_i)$ and $C_n$ the set of critical points of $\tilde{u}_n$ in $\Omega'$ we have that $\tilde{u}_n$ admit at least $m + k - 1$ nondegenerate saddle points in $\Omega'$ and*

$$\sum_{z_j \in C_n} index_{z_j}(\nabla \tilde{u}_n) = 1 - k - m. \tag{1.9}$$

## 2. Proofs

In this section, we collect the proofs of the previous results.

*Proof of Theorem 1.1.* We take a point $N \in \mathcal{S}^2$, such that $N \notin \bar{D}$, to be the north pole of the sphere. Then we introduce the standard coordinates on the sphere $\mathcal{S}^2$. We use the stereographic projection

$\psi : \mathcal{S}^2 \setminus \{N\} \to \mathbb{R}^2$. We let $\Omega := \psi(D) \subset \mathbb{R}^2$ and $v_n(x) := u_n(\psi^{-1}(x))$ for $x \in \Omega$. In these coordinates the functions $v_n$ satisfy

$$\begin{cases} -\Delta v_n = \rho_n \frac{h(x)e^{v_n}}{\int_\Omega h(x)e^{v_n}e^{\psi(x)}dx} e^{\psi(x)} & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where

$$e^{\psi(x)} := \frac{4}{(1+|x|^2)^2}$$

is the conformal factor. Next we let

$$\lambda_n = \frac{\rho_n}{\int_\Omega h(x)e^{v_n}e^{\psi(x)}dx}$$

and $V(x) := h(x)e^{\psi(x)}$. Then $v_n$ solves

$$\begin{cases} -\Delta v_n = \lambda_n V(x)e^{v_n} & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

for some $V(x)$ (which is locally analytic in $\Omega$) and for some $\lambda_n \in (0, \infty)$. Moreover, since $\rho_n \to 8\pi m$ as $n \to \infty$, then

$$\lim_{n\to\infty} \lambda_n \int_\Omega V(x)e^{v_n}dx = 8\pi m.$$

Finally, $\Omega = \psi(D)$ and $\chi(\Omega) = \chi(D)$. Since $\Omega \subset \mathbb{R}^2$ and it is smooth, then $\chi(D) = 1 - k$ where $k$ is the number of the holes of $\Omega$. Then, the claim follows from Theorem 1.5.

*Proof of Corollaries 1.2 and 1.3.* In [2], the authors construct a suitable domain $\Omega_1 \subset \mathbb{R}^2$ in which $v_n$ (the solution to Eq (2.1), with $V(x) = 1$) has exactly $2m + k - 1$ nondegenerate critical points ($m$ maxima and $m + k - 1$ saddle points). The very same construction can be done for the case of solutions to Eq (2.1) for a positive, smooth $V(x)$. Pulling back the domain $\Omega$ on the sphere $\mathcal{S}^2$ gives the desired example on $\mathcal{S}$.

In Theorem 1.4 in [2], an example of a domain $\Omega_2 \subset \mathbb{R}^2$ in which $v_n$ has at least $2m + k + 1 = 2m - \chi(\Omega_2) + 2$ nondegenerate critical points is given. This provides the example in Corollary 1.3.

*Proof of Theorem 1.4.* We consider the local isotermal coordinates such that $D$ is contained in a unique chart. In these coordinates, Eq (1.2) becomes Eq (2.1), where $e^{\psi(x)}$ is the conformal factor which is locally analitic since we are assuming $\mathcal{M}$ is smooth. The proof follows as in the case of Theorem 1.1 and its corollaries.

*Proof of Theorem 1.5.* The proof is similar to the proof of Theorem 1.1 in [2]. First, we prove the following statement:

There exists $\rho > 0$ such that $\tilde{u}_n$ has a unique nondegenerate critical point (the maximum) in $B_\rho(P_i)$ for $i = 1, \dots, m$ when $n$ is large enough.

By contradiction, we assume that there exists $\xi_n \in B_{\rho_n}(x_{i,n}) \setminus \{x_{i,n}\}$ such that $\rho_n \to 0$ and $\nabla u_n(\xi_n) = 0$. We have to distinguish two cases: i) $\xi_n \in B_{R\delta_{i,n}}(x_{i,n})$ for some $i \in \{1, .., m\}$, for some $R > 0$; ii) $\xi_n \notin B_{R\delta_{i,n}}(x_{i,n})$ for any $R > 0$, where $\delta_{i,n}$ satisfies

$$\delta_{i,n}^2 \lambda_n V(x_{i,n}) e^{\tilde{u}_n(x_{i,n})} = 1.$$

For $i = 1, \ldots, m$, we let

$$\widehat{u_{i,n}}(x) := \tilde{u}_n(\delta_{i,n}x + x_{i,n}) - \tilde{u}_n(x_{i,n}).$$

The function $\widehat{u_{i,n}}(x)$ satisfies

$$-\Delta\widehat{u_{i,n}} = \frac{V(\delta_{i,n}x + x_{i,n})}{V(x_{i,n})}\, e^{\widehat{u_{i,n}}} \quad \text{in } B_{\frac{R}{\delta_{i,n}}}(0).$$

It is standard that

$$\widehat{u_{i,n}} \to U(x) := \log \frac{1}{\left(1 + \frac{|x|^2}{8}\right)^2} \quad \text{in } C^2_{loc}(\mathbb{R}^2),$$

see, [18–22]. Denote by $\widehat{\xi_n} := \frac{\xi_n - x_{i,n}}{\delta_{i,n}}$. Then $\nabla\widehat{u_{i,n}}(\widehat{\xi_n}) = \nabla\tilde{u}_n(\xi_n) = 0$ and $|\widehat{\xi_n}| \le R$. Up to a subsequence, $\widehat{\xi_n} \to \widehat{\xi}$ and by the previous convergence, $\nabla U(\widehat{\xi}) = 0$. Then the definition of $U(x)$ implies $\widehat{\xi} = 0$. This is not possible, since $x = 0$ is a nondegenerate maximum point for the function $U(x)$ and the functions $\widehat{u_{i,n}}$ have a maximum in $x = 0$ for every $n$. This also shows that the point $x_{i,n}$ is a nondegenerate maximum for $\tilde{u}_n(x)$ and that $index_{x_{i,n}}(\nabla\tilde{u}_n) = 1$.
Case ii). In this case, we have that $\xi_n \to P_i$. Denoting by $r_n := |\xi_n - x_{i,n}|$, we have that $\frac{\delta_{i,n}}{r_n} \to 0$ as $n \to 0$. We define the function $\bar{u}_{i,n}(x) := \tilde{u}_n(r_n x + x_{i,n}) + 4\log r_n$. Green's representation formula gives

$$\begin{aligned}
\bar{u}_{i,n}(x) &= \lambda_n \int_\Omega G(r_n x + x_{i,n}, y)V(y)e^{\tilde{u}_n(y)}dy + 4\log r_n \\
&= \underbrace{\lambda_n \int_{\Omega\setminus\cup_i B_R(x_{i,n})} G(r_n x + x_{i,n}, y)V(y)e^{\tilde{u}_n(y)}dy}_{:=I_1} \\
&\quad + \underbrace{\sum_{j\neq i} \lambda_n \int_{B_R(x_{j,n})} G(r_n x + x_{i,n}, y)V(y)e^{\tilde{u}_n(y)}dy}_{:=I_2} \\
&\quad + \underbrace{\lambda_n \int_{B_R(x_{i,n})} G(r_n x + x_{i,n}, y)V(y)e^{\tilde{u}_n(y)}dy + 4\log r_n}_{:=I_3}.
\end{aligned}$$

First we observe that $I_1 = o(1)$ as $n \to \infty$ since $\tilde{u}_n(y)$ is bounded in $\Omega \setminus \cup_i B_R(x_{i,n})$ and $\lambda_n \to 0$. The second term can be estimated as:

$$\begin{aligned}
I_2 &= \sum_{j\neq i} \lambda_n \int_{B_R(x_{j,n})} G(r_n x + x_{i,n}, y)V(y)e^{\tilde{u}_n(y)}dy \\
&= \sum_{j\neq i} \int_{B_{\frac{R}{\delta_{j,n}}}(0)} G(r_n x + x_{i,n}, \delta_{j,n}y + x_{j,n})\frac{V(\delta_{j,n}y + x_{j,n})}{V(x_{j,n})}e^{\widehat{u_{j,n}}(y)}dy \\
&= \sum_{j\neq i} 8\pi G(P_i, P_j) + o(1).
\end{aligned}$$

The last term is given by:

$$
\begin{aligned}
I_3 =&\lambda_n \int_{B_R(x_{i,n})} G(r_n x + x_{i,n}, y)V(y)e^{\tilde{u}_n(y)}dy + 4\log r_n \\
=& \int_{B_{\frac{R}{\delta_{i,n}}}(0)} G(r_n x + x_{i,n}, \delta_{i,n}y + x_{i,n})\frac{V(\delta_{i,n}y + x_{i,n})}{V(x_{i,n})}e^{\widehat{u}_{i,n}(y)}dy + 4\log r_n \\
=& 8\pi H(P_i, P_i) - \frac{1}{2\pi}\int_{B_{\frac{R}{\delta_{i,n}}}(0)} \log|r_n x + \delta_{i,n}y|\frac{V(\delta_{i,n}y + x_{i,n})}{V(x_{i,n})}e^{\widehat{u}_{i,n}(y)}dy \\
&+ 4\log r_n + o(1) \\
=& 8\pi H(P_i, P_i) - \frac{1}{2\pi}\int_{B_{\frac{R}{\delta_{i,n}}}(0)} \log|x + \frac{\delta_{i,n}}{r_n}y|\frac{V(\delta_{i,n}y + x_{i,n})}{V(x_{i,n})}e^{\widehat{u}_{i,n}(y)}dy \\
&- \frac{1}{2\pi}\log r_n \int_{B_{\frac{R}{\delta_{i,n}}}(0)} \frac{V(\delta_{i,n}y + x_{i,n})}{V(x_{i,n})}e^{\widehat{u}_{i,n}(y)}dy + 4\log r_n + o(1) \\
=& 8\pi H(P_i, P_i) + 4\log\frac{1}{|x|} + \log r_n\left(4 - \frac{1}{2\pi}\int_{B_{\frac{R}{\delta_{i,n}}}(0)} \frac{V(\delta_{i,n}y + x_{i,n})}{V(x_{i,n})}e^{\widehat{u}_{i,n}(y)}dy\right) + o(1) \\
=& 8\pi H(P_i, P_i) + 4\log\frac{1}{|x|} + o(1).
\end{aligned}
$$

In the last line we use that, by [5],

$$
\lambda_n \int_{B_R(x_{i,n})} V(y)e^{\tilde{u}_n(y)}dy = 8\pi + o(\lambda_n).
$$

Putting together the previous estimates, we have that

$$
\bar{u}_{i,n}(x) \to V(x) := 4\log\frac{1}{|x|} + 8\pi H(P_i, P_i) + \sum_{j\neq i} G(P_i, P_j) \text{ in } C^1_{loc}(\mathbb{R}^2).
$$

We let $\bar{\xi}_n := \frac{\xi_n - x_{i,n}}{r_n}$. Then $\nabla\bar{u}_{i,n}(\bar{\xi}_n) = \nabla\tilde{u}_n(\xi_n) = 0$ and $|\bar{\xi}_n| = 1$. Up to a subsequence, $\bar{\xi}_n \to \bar{\xi}$. The previous convergence gives $\nabla V(\bar{\xi}) = 0$. This is a contradiction. Now we let $\Omega' = \Omega \setminus \cup_{i=1}^m B_\rho(P_i)$ and we give an estimate on the critical points of $\tilde{u}_n$ in $\Omega'$. To this end, we observe that a solution $\tilde{u}_n(x)$ to Eq (1.7) is real analytic in a neighborhood of $x_0$, for every $x_0 \in \Omega$, (one can see [23]). Moreover it is known that $\tilde{u}_n(x) \to K(x) := \sum_{i=1}^m 8\pi G(x, P_i)$ in $\Omega'$. The function $K(x)$ is harmonic and non-trivial in $\Omega'$. Then it has only a finite number of critical points $\{z_1, \ldots, z_l\}$, which are saddle points of finite multiplicity $m_j \geq 1$ and $index_{z_j}(\nabla K) \leq -1$. Whenever $index_{z_j}(\nabla K) = -1$, then $z_j$ is a nondegenerate saddle point; see Proposition 5.1 in [2]. Moreover, we can adapt the proof of Proposition 5.2 in [2] getting that $\tilde{u}_n$, for $n$ large enough, has only a finite number of isolated critical points that we denote by $\{z_{1,n}, \ldots, z_{l_n,n}\}$. These points converge to the critical points $\{z_1, \ldots, z_l\}$ of $K(x)$. Moreover $index_{z_{j,n}}(\nabla u_n) \in \{-1, 0, 1\}$ and, whenever the index is 1, then $z_{j,n}$ is a nondegenerate maximum, while whenever the index is $-1$, $z_{j,n}$ is a nondegenerate saddle point.

Finally, as in Proposition 5.3 and 5.4 in [2], we use the Poincarè Hopf formula with $v = \nabla\tilde{u}_n$ in $\Omega$ (observe that by Hopf Lemma $\nabla\tilde{u}_n \cdot \nu < 0$) to have $\sum index_{z_{j,n}}(\nabla u_n) = \chi(\Omega)$, and by the first assertion,

$m + \sum_{C_n} index_{z_{j,n}}(\nabla u_n) = \chi(\Omega) = 1 - k$. The previous result on the critical points of $\tilde{u}_n$ then implies that $\tilde{u}_n$ has at least $m - k - 1$ nondegenerate saddle points (of index $-1$) in $C_n$ and concludes the proof.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This research was supported by the e.INS—Ecosystem of Innovation for Next Generation Sardinia (cod. ECS 00000038) and Studio di modelli nelle scienze della vita DM 737/2021 risorse 2022–2023, Uniss (CUP J55F21004240001).

## Conflict of interest

The authors declare there is no conflict of interest.

## References

1. F. Gladiali, M. Grossi, On the number of critical points of solutions of semilinear equations in $\mathbb{R}^2$, *Am. J. Math.*, **144** (2022), 1221–1240. https://doi.org/10.1353/ajm.2022.0028
2. F. Gladiali, M. Grossi, On the critical points of solutions of pde in non-convex settings: The case of concentrating solutions, *J. Funct. Anal.*, **287** (2024), 110620. https://doi.org/10.1016/j.jfa.2024.110620
3. S. Y. A. Chang, M. J. Gursky, P. C. Yang, The scalar curvature equation on 2- and 3-spheres, *Calc. Var. Partial Differ. Equ.*, **1** (1993), 205–229. https://doi.org/10.1007/BF01191617
4. S. Y. A. Chang, P. C. Yang, Prescribing Gaussian curvature on $S^2$, *Acta Math.*, **159** (1987), 215–259. https://doi.org/10.1007/BF02392560
5. C. C. Chen, C. S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, *Commun. Pure Appl. Math.*, **55** (2002), 728–771. https://doi.org/10.1002/cpa.3014
6. J. L. Kazdan, F. W. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures, *Ann. Math.*, **101** (1975), 317–331. https://doi.org/10.2307/1970993
7. E. Caglioti, P. L. Lions, C. Marchioro, M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description, *Commun. Math. Phys.*, **143** (1992), 501–525. https://doi.org/10.1007/BF02099262
8. E. Caglioti, P. L. Lions, C. Marchioro, M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description. Part II, *Commun. Math. Phys.*, **174** (1995), 229–260. https://doi.org/10.1007/BF02099602
9. S. Chanillo, M. Kiessling, Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry, *Commun. Math. Phys.*, **160** (1994), 217–238. https://doi.org/10.1007/BF02103274

10. M. K. H. Kiessling, Statistical mechanics of classical particles with logarithmic interactions, *Commun. Pure Appl. Math.*, **46** (1993), 27–56. https://doi.org/10.1002/cpa.3160460103

11. L. A. Caffarelli, Y. S. Yang, Vortex condensation in the Chern-Simons Higgs model: An existence theorem, *Commun. Math. Phys.*, **168** (1995), 321–336. https://doi.org/10.1007/BF02101552

12. W. Ding, J. Jost, J. Li, G. Wang, Existence results for mean field equations, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **16** (1999), 653–666. https://doi.org/10.1016/S0294-1449(99)80031-6

13. J. Hong, Y. Kim, P. Y. Pac, Multivortex solutions of the abelian Chern-Simons-Higgs theory, *Phys. Rev. Lett.*, **64** (1990), 2230–2233. https://doi.org/10.1103/PhysRevLett.64.2230

14. M. Nolasco, G. Tarantello, Double vortex condensates in the Chern-Simons-Higgs theory, *Calc. Var. Partial Differ. Equ.*, **9** (1999), 31–94. https://doi.org/10.1007/s005260050132

15. M. Nolasco, G. Tarantello, Vortex condensates for the SU(3) Chern-Simons theory, *Commun. Math. Phys.*, **213** (2000), 599–639. https://doi.org/10.1007/s002200000252

16. J. Spruck, Y. S. Yang, Topological solutions in the self-dual Chern-Simons theory: Existence and approximation, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **12** (1995), 75–97. https://doi.org/10.1016/S0294-1449(16)30168-8

17. J. Li, L. Sun, Y. Yang, The boundary value problem for the mean field equation on a compact Riemann surface, *Sci. China Math.*, **66** (2023), 115–142. https://doi.org/10.1007/s11425-021-1962-5

18. M. Grossi, H. Ohtsuka, T. Suzuki, Asymptotic non-degeneracy of the multiple blow-up solutions to the Gel'fand problem in two space dimensions, *Adv. Differ. Equ.*, **16** (2011), 145–164. https://doi.org/10.57262/ade/1355854333

19. Y. Y. Li, Harnack type inequality: The method of moving planes, *Commun. Math. Phys.*, **200** (1999), 421–444. https://doi.org/10.1007/s002200050536

20. K. Nagasaki, T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities, *Asymptot. Anal.*, **3** (1990), 173–188. https://doi.org/10.3233/ASY-1990-3205

21. H. Ohtsuka, T. Sato, Refinement of asymptotic behavior of the eigenvalues for the linearized Liouville-Gel'fand problem, *Nonlinear Anal.*, **240** (2024), 113464. https://doi.org/10.1016/j.na.2023.113464

22. T. Suzuki, Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **9** (1992), 367–397. https://doi.org/10.1016/S0294-1449(16)30232-3

23. Y. Hashimoto, A remark on the analyticity of the solutions for non-linear elliptic partial differential equations, *Tokio J. Math.*, **29** (2006), 271–281. https://doi.org/10.3836/tjm/1170348166

AIMS Press