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Research article

Dynamic analysis of the M/G/1 queueing system with multiple phases of operation

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Abstract: In this paper, we considered the M/G/1 queueing system with multiple phases of operation. First, we have proven the existence and uniqueness of the time-evolving solution for this queueing system. Second, by calculating the spectral distribution of the system operator, we proved that the solution converged at most strongly to its steady-state (static) solution. We also discussed the compactness of the system's corresponding semigroup. Additionally, we investigated the asymptotic behavior of dynamic indicators. Finally, to demonstrate the exponential convergence of the solution, we conducted some numerical analysis.

Keywords: queueing system with multiple phases of operation; C_0-semigroup; operator spectrum; asymptotic behavior; time-evolving queue length

1. Introduction

Queueing systems with multiple phases of operation are very useful in manufacturing systems, transportation systems, financial systems, etc.; see [1–4]. For example, in automobile manufacturing, raw materials first enter the stamping workshop for stamping and forming. Continuing with the welding assembly in the welding workshop is the second stage, then carry out surface treatment such as painting, is the third stage. Finally, the final assembly and quality inspection are carried out. In the airport boarding process, passengers first queue up at the check-in counter to complete check-in procedures, including checking in luggage. Then, queue up through the security checkpoint for security checks, is the second stage. Finally, they queue up at the boarding gate to wait for boarding. In hospital medical systems, patients first queue up at the registration counter to register. Then, the second stage is to go to the waiting area of the corresponding department to queue up for the doctor's diagnosis. If further examinations are needed, such as blood tests, X-rays, etc., one needs to queue outside the examination department to wait for the examination, which is the third stage. Finally, they take the examination results and return to the doctor for diagnosis or treatment, such as prescribing medication, intravenous infusion, etc.

In this paper, we consider the M/G/1 queueing system with multiple phases of operation, where M means that customer arrival follows a Poisson process, G represents the service rate of the server follows a general distribution, 1 indicates the number of servers in the system. Therefore, the M/G/1 queue is a single-server queue where customers arrive according to a Poisson process, and service times are independent and identically distributed general random variables. The process of establishing a mathematical model for the queuing system is as follows:

There are $n+1$ phases in this system, 0 is the idle period, and s ($s = 1, 2, \cdots, n$) are the operational phases. We assume that $N(t)$ denotes the number of customers in the system at time t, $J(t)$ denotes the phase in which the system operates at time t, $R_s(t)$ ($s = 1, 2, \cdots, n$) represents the elapsed service time of the customer currently receiving service during phase s, $F_s(x) = Prob\{R_s(t) \leq x\}$ ($s = 1, 2, \cdots, n$) denotes the probability distribution function corresponding to $R_s(t)$, and $\mu_s(x)dx$ is the service completion rate of the server in the interval $(x, x + dx]$ if the system is in phase s and satisfies $\mu_s(x) \geq 0$ and $\int_0^\infty \mu_s(x)dx = \infty$. The service time of the any two phases of service are mutually independent. Based on the properties of conditional probability and differentiation, we have

$$\mu_s(x)dx = Prob\{x < R_s(t) \leq x + dx \,\big|\, R_s(t) > x\} = -\frac{d(1 - F_s(x))}{1 - F_s(x)}.$$

Then, from this and $F_s(0) = 0$, we obtain the probability distribution function of the service time of the server in the phase s,

$$F_s(x) = 1 - e^{-\int_0^x \mu_s(\tau)d\tau}.$$

According to the definition of probability distribution function, we know that $F_s(x) \geq 0$ and $\lim_{x\to\infty} F_s(x) = 1$. Therefore,

$$\mu_s(x) \geq 0, \qquad \int_0^\infty \mu_s(x)dx = \infty.$$

We assume that, in phase s, the arrivals occur according to a Poisson process of rate $\lambda_s > 0$. In the idle period, the Poisson arrival rate is $\lambda_0 > 0$; upon arrival, the system moves to some operative phase s with probability q_s, and the customer service upon arrival begins immediately, where $q_s > 0$ and $\sum_{s=1}^n q_s = 1$. Then, according to the definition of Poisson process and exponential distribution, we have

$$Prob\{N(t) = k, J(t) = s\} = \frac{(\lambda_s t)^k}{k!}e^{-\lambda_s t}, \quad t \geq 0,\ k \geq 0,\ s = 0, 1, \cdots, n, \tag{1.1a}$$

$$\begin{aligned} &Prob\,\{\text{arriving one customer within the } \Delta t \text{ time in phase } s\} \\ &= \lambda_s \Delta t + o(\Delta t), \quad s = 1, 2, \cdots, n, \end{aligned} \tag{1.1b}$$

$$\begin{aligned} &Prob\,\{\text{arriving two or more customers within the } \Delta t \text{ time in phase } s\} \\ &= o(\Delta t), \quad s = 1, 2, \cdots, n, \end{aligned} \tag{1.1c}$$

$$\begin{aligned} &Prob\,\{\text{the server completing one service within the } \Delta t \text{ time in phase } s\} \\ &= \mu_s(x)\Delta t + o(\Delta t), \quad s = 1, 2, \cdots, n, \end{aligned} \tag{1.1d}$$

$$Prob\{\text{the server completing two or more services within the } \Delta t \text{ time in phase } s\} = o(\Delta t), \quad s = 1, 2, \cdots, n, \tag{1.1e}$$

where $o(\Delta t)$ denotes the infinitesimal quantity of Δt. Clearly, the process $\{(N(t), J(t), R_s(t)) : t \geq 0\}$ is a continuous-time Markov process with state space

$$\Gamma = \{(0, 0)\} \cup \{(k, s, x) \mid k = 1, 2, \cdots;\ s = 1, 2, \cdots, n,\ x \geq 0\}.$$

We define

$$p_{0,0}(t) = Prob\{N(t) = 0,\ J(t) = 0\}, \tag{1.2a}$$

$$p_{k,s}(x, t)dx = Prob\{N(t) = k,\ J(t) = s,\ x \leq R_s(t) < x + dx\}. \tag{1.2b}$$

Then, consider the changes in the system during Δt time. Based on the formula of total probability, the properties of Markov processes, and the above Eqs (1.1a)–(1.2b), we have (for convenience, assuming Δx is the same as Δt)

$$\begin{aligned}
p_{0,0}(t + \Delta t) &= Prob\left\{\begin{matrix}\text{within the } t + \Delta t\text{, no customers in the}\\ \text{system and the service desk being idle}\end{matrix}\right\}\\
&= Prob\left\{\begin{matrix}\text{at time } t\text{, no customers in the system, during } \Delta t\\ \text{no customers arriving and the system is idle}\end{matrix}\right\}\\
&\quad + Prob\left\{\begin{matrix}\text{at time } t \text{ there is one customer in the system,}\\ \text{during } \Delta t \text{ no customers arriving, server}\\ \text{completing one service within } \Delta t \text{ in phase 1}\end{matrix}\right\}\\
&\quad + Prob\left\{\begin{matrix}\text{at time } t \text{ there is one customer in the system,}\\ \text{during } \Delta t \text{ no customers arriving, server}\\ \text{completing one service within } \Delta t \text{ in phase 2}\end{matrix}\right\}\\
&\quad + \cdots\\
&\quad + Prob\left\{\begin{matrix}\text{at time } t \text{ there is one customer in the system,}\\ \text{during } \Delta t \text{ no customers arriving, server}\\ \text{completing one service within } \Delta t \text{ in phase n}\end{matrix}\right\}\\
&\quad + o(\Delta t)\\
&= p_{0,0}(t)(1 - \lambda_0 \Delta t) + \int_0^\infty p_{1,1}(x, t)\mu_1(x)dx\Delta t(1 - \lambda_1 \Delta t)\\
&\quad + \int_0^\infty p_{1,2}(x, t)\mu_2(x)dx\Delta t(1 - \lambda_2 \Delta t)\\
&\quad + \cdots\\
&\quad + \int_0^\infty p_{1,n}(x, t)\mu_n(x)dx\Delta t(1 - \lambda_n \Delta t) + o(\Delta t),
\end{aligned} \tag{1.3a}$$

$$
\begin{aligned}
&p_{1,s}(x+\Delta t, t+\Delta t)\\
&= Prob\left\{\begin{matrix}\text{at time } t+\Delta t\text{, there is one customer in the system,}\\ \text{the elapsed service time of the customer currently}\\ \text{receiving service during phase } s \text{ is } x+\Delta t\end{matrix}\right\}\\
&= Prob\left\{\begin{matrix}\text{at time } t\text{, there is one customer in the system and the service}\\ \text{time that has passed is } x\text{, no customers arrived within}\\ \Delta t \text{ and the server did not complete the service in phase } s\end{matrix}\right\}\\
&\quad +o(\Delta t)\\
&= p_{1,s}(x,t)(1-\lambda_s\Delta t)(1-\mu_s(x)\Delta t)+o(\Delta t), \quad ,1\le s\le n,
\end{aligned} \tag{1.3b}
$$

$$
\begin{aligned}
&p_{k,s}(x+\Delta t, t+\Delta t)\\
&= Prob\left\{\begin{matrix}\text{at time } t+\Delta t\text{, there are } k \text{ customers in the system,}\\ \text{the elapsed service time of the customer currently}\\ \text{receiving service during phase } s \text{ is } x+\Delta t\end{matrix}\right\}\\
&= Prob\left\{\begin{matrix}\text{at time } t\text{, there are } k \text{ customers in the system and the service}\\ \text{time that has passed is } x\text{, no customers arrived within}\\ \Delta t \text{ and the server did not complete the service in phase } s\end{matrix}\right\}\\
&\quad +Prob\left\{\begin{matrix}\text{at time } t\text{, there are } k-1 \text{ customers in the system and the}\\ \text{service time that has passed is } x\text{, one customer arrived in}\\ \Delta t \text{ and the server did not complete the service in phase } s\end{matrix}\right\}+o(\Delta t)\\
&= p_{k,s}(x,t)(1-\lambda_s\Delta t)(1-\mu_s(x)\Delta t)+p_{k-1,s}(x,t)\lambda_s\Delta t(1-\mu_s(x)\Delta t)\\
&\quad +o(\Delta t), \quad s=1,2,\cdots,n;\ k\ge 2.
\end{aligned} \tag{1.3c}
$$

We consider the boundary conditions as follows:

$$
\begin{aligned}
p_{1,s}(0,t+\Delta t)\Delta t &= Prob\left\{\begin{matrix}\text{at time } t+\Delta t\text{, there is one customer in the system}\\ \text{in phase } s \text{ and the service has not started yet}\end{matrix}\right\}\\
&= Prob\left\{\begin{matrix}\text{at time } t\text{, the system is in idle state, but}\\ \text{has just reached one customer and the system}\\ \text{has jumped from idle state to phase } s\end{matrix}\right\}\\
&\quad +Prob\left\{\begin{matrix}\text{at time } t\text{, there are two customers in}\\ \text{the system in phase } s \text{ and the server has}\\ \text{just completed one service within } \Delta t\end{matrix}\right\}\\
&\quad +o(\Delta t)\\
&= p_{0,0}(t)q_s\Delta t+\int_0^\infty p_{2,s}(x,t)\mu_s(x)dx\Delta t+o(\Delta t), \quad ,1\le s\le n,
\end{aligned} \tag{1.4a}
$$

$$
\begin{aligned}
p_{k,s}(0, t+\Delta t)\Delta t &= Prob\left\{\begin{array}{c}\text{at time } t+\Delta t\text{, there are } k \text{ customers in the system}\\ \text{in phase } s \text{ and the service has not yet started}\end{array}\right\}\\
&= Prob\left\{\begin{array}{c}\text{at time } t\text{, there are } k+1 \text{ customers}\\ \text{in the system in phase } s \text{ and the server}\\ \text{just completes one service within } \Delta t \text{ time}\end{array}\right\} + o(\Delta t) \qquad (1.4b)\\
&= \int_0^\infty p_{k+1,s}(x,t)\mu_s(x)dx\Delta t + o(\Delta t), \quad , 1 \le s \le n.
\end{aligned}
$$

Assume that the initial values

$$
p_{0,0}(0) = g_{0,0} \ge 0, \quad \text{and} \quad p_{k,s}(x,0) = g_{k,s}(x) \ge 0, \quad k \ge 1,\ s = 1, 2, \cdots, n, \qquad (1.5)
$$

satisfy

$$
g_{0,0} + \sum_{s=1}^{n}\sum_{k=1}^{\infty}\int_0^\infty g_{k,s}(x)dx = 1.
$$

Based on Eqs (1.2a)–(1.5) and the definition of partial differential derivatives, we obtain the following integro-partial differential equations [3]:

$$
\begin{cases}
\dfrac{dp_{0,0}(t)}{dt} = -\lambda_0 p_{0,0}(t) + \displaystyle\sum_{s=1}^{n}\int_0^\infty p_{1,s}(x,t)\mu_s(x)dx,\\
\partial_t p_{1,s}(x,t) + \partial_x p_{1,s}(x,t) = -[\lambda_s + \mu_s(x)]p_{1,s}(x,t),\\
\partial_t p_{k,s}(x,t) + \partial_x p_{k,s}(x,t) = -[\lambda_s + \mu_s(x)]p_{k,s}(x,t) + \lambda_s p_{k-1,s}(x,t),\ k \ge 2,\\
p_{1,s}(0,t) = q_s\lambda_0 p_{0,0}(t) + \displaystyle\int_0^\infty p_{2,s}(x,t)\mu_s(x)dx,\\
p_{k,s}(0,t) = \displaystyle\int_0^\infty p_{k+1,s}(x,t)\mu_s(x)dx, \quad k \ge 2,\\
p_{0,0}(0) = g_{0,0} \ge 0, \quad p_{k,s}(x,0) = g_{k,s}(x) \ge 0, \quad k \ge 1,\ 1 \le s \le n.
\end{cases} \qquad (1.6)
$$

In [3], several static indices for the system (1.6) such as the static queue length and static sojourn time distribution of an arbitrary customer were developed under the following static hypothesis:

- $\lim_{t\to\infty} p_{0,0}(t) = p_{0,0}$,
- $\lim_{t\to\infty} p_{k,s}(\cdot, t) = p_{k,s}(\cdot)$, $k \ge 1$, $1 \le s \le n$.

From the perspective of partial differential equations, the above hypothesis implies the following two hypotheses:

- (**H1**): The system (1.6) admits a time-evolving solution.
- (**H2**): The aforementioned time-evolving solution converges to its static.

In this article, we investigate the aforementioned static hypothesis, that is, (**H1**) and (**H2**), and consider the asymptotic behavior of the dynamic indices of the system (1.6). It is worth noting that when

$n = 1$, system (1.6) becomes the classical M/G/1 queuing system [5], and a detailed dynamic analysis of this classical queuing system was conducted in [6–10]. Gupur et al. [6] studied the well-posedness of the queueing system [5] and obtained the strong convergence of the solution of the system when the service rate is constant. When the service rate is a bounded function, similar results to [6] are obtained by [7]. In [8–10], the exponential convergence of the solution of the queueing system [5] was studied. Therefore, our results include the above findings.

In the study of partial differential equations, many solutions of partial differential equations can function as semigroups in Banach space. By studying the properties of semigroups, important properties such as the existence, uniqueness, and stability of solutions to partial differential equations can be obtained; see [11–15]. In this article, first, we convert the system (1.6) into an abstract Cauchy problem in a natural state Banach space. Then, using the semigroup theory on Banach spaces, we show that the system operator generates a positive C_0−semigroup of contractions on the state space. Consequently, we verify that the system (1.6) admits a unique positive time-evolving solution, which shows that (**H1**) holds under certain conditions.

Second, we investigate the asymptotic behavior of the solution. To this aim, we need to know the spectrum of the system operator; see [16–18]. For example, in order to study the asymptotic stability of multilayer thermal wave systems, Avalos et al. [16] conducted spectral analysis on the system and found that the system operator had no spectral points on the imaginary axis. Drogoul and Veltz [17] proved that 0 is the unique spectral point of the spike neural network operator on the imaginary axis, thus obtaining the exponential stability of the system. In [18], it was proved that 0 is the point spectrum of the system operator of the queuing system and is the unique spectral point, thus obtaining the solution corresponding to this queuing system that strongly converges to its static solution.

Moreover, it is a challenge to find spectrum on the imaginary axis. Here, we apply Greiner's [19] boundary perturbation method to fully describe the spectral distribution of the system operator on the imaginary axis. If the service rates are constants, then we obtain that the system operator of the system (1.6) has uncountable eigenvalues on the left-half complex plane. Consequently, these spectral results imply that the time-evolving solution of system (1.6) at most strongly converges to its static solution. In other words, (**H2**) holds only in the context of strong convergence.

Finally, we discuss the asymptotic behavior of the dynamic indices of the system (1.6). By using cone theory and positive operator theory, we prove that the time-evolving queue length of the system (1.6) converges to its static queue length under some conditions. This asymptotic result includes the result of [3].

The remaining part of this article is organized as follows. In the next section, we rewrite the system (1.6) as an abstract Cauchy problem in a Banach space and provide the well-posedness. In Section 3, we provide a complete asymptotic behavior of the solution of system (1.6). In Section 4, we discuss the dynamic indices of the system (1.6). To demonstrate the exponential convergence of the solution, we conduct some numerical analysis in Section 5. We conclude this article in the last section.

2. Well-posedness

We choose the state Banach space of system (1.6) as follows:

$$X=\left\{(p_1,p_2,\cdots,p_n)\left|\begin{array}{l}p_1=(p_{0,0},p_{1,1},p_{2,1},\cdots),p_s=(p_{1,s},p_{2,s},\cdots),2\le s\le n,\\ p_{0,0}\in\mathbb{R},p_{k,s}\in L^1[0,\infty),\|(p_1,p_2,\cdots,p_n)\|\\ =|p_{0,0}|+\sum_{s=1}^{n}\sum_{k=1}^{\infty}\|p_{k,s}\|_{L^1[0,\infty)}<\infty\end{array}\right.\right\}.$$

Define the maximal operator of the system (1.6) by

$$\begin{aligned}&A_m(p_1,p_2,\cdots,p_n)\\ &=\left(\begin{pmatrix}-\lambda_0p_{0,0}+\sum_{s=1}^{n}\varphi_sp_{1,s}\\ B_1p_{1,1}\\ \lambda_1p_{1,1}+B_1p_{2,1}\\ \lambda_1p_{2,1}+B_1p_{3,1}\\ \vdots\end{pmatrix},\begin{pmatrix}B_2p_{1,2}\\ \lambda_2p_{1,2}+B_2p_{2,2}\\ \lambda_2p_{2,2}+B_2p_{3,2}\\ \lambda_2p_{3,2}+B_2p_{4,2}\\ \vdots\end{pmatrix},\cdots,\begin{pmatrix}B_np_{1,n}\\ \lambda_np_{1,n}+B_np_{2,n}\\ \lambda_np_{2,n}+B_np_{3,n}\\ \lambda_np_{3,n}+B_np_{4,n}\\ \vdots\end{pmatrix}\right),\end{aligned}$$

with domain

$$D(A_m)=\left\{(p_1,p_2,\cdots,p_n)\in X\mid\frac{dp_{k,s}}{dx}\in L^1[0,\infty),\ \sum_{s=1}^{n}\sum_{k=1}^{\infty}\left\|\frac{dp_{k,s}}{dx}\right\|_{L^1[0,\infty)}<\infty\right\},$$

where $p_{k,s}$ are absolutely continuous functions, $k\ge 1,\ 1\le s\le n$, and

$$\begin{cases}B_sv&=-\dfrac{dv(x)}{dx}-[\lambda_s+\mu_s(x)]v(x),\quad v\in W^{1,1}[0,\infty),\\ \varphi_sf&=\displaystyle\int_0^\infty f(x)\mu_s(x)dx,\quad f\in L^1[0,\infty).\end{cases}$$

We choose the boundary space as $\partial X=\underbrace{l^1\times l^1\times\cdots\times l^1}_{n}$ and define boundary operators $\Psi,\Phi:D(A_m)\to\partial X$ of the system (1.1a) by

$$\Psi(p_1,p_2,\cdots,p_n)=\left(\begin{pmatrix}p_{1,1}(0)\\ p_{2,1}(0)\\ \vdots\end{pmatrix},\begin{pmatrix}p_{1,2}(0)\\ p_{2,2}(0)\\ \vdots\end{pmatrix},\cdots,\begin{pmatrix}p_{1,n}(0)\\ p_{2,n}(0)\\ \vdots\end{pmatrix}\right),$$

$$\begin{aligned}&\Phi(p_1,p_2,\cdots,p_n)\\ &=\left(\begin{pmatrix}q_1\lambda_0p_{0,0}+\varphi_1p_{2,1}\\ \varphi_1p_{3,1}\\ \varphi_1p_{4,1}\\ \vdots\end{pmatrix},\begin{pmatrix}q_2\lambda_0p_{0,0}+\varphi_2p_{2,2}\\ \varphi_2p_{3,2}\\ \varphi_2p_{4,2}\\ \vdots\end{pmatrix},\cdots,\begin{pmatrix}q_n\lambda_0p_{0,0}+\varphi_np_{2,n}\\ \varphi_np_{3,n}\\ \varphi_np_{4,n}\\ \vdots\end{pmatrix}\right).\end{aligned}$$

Now, we introduce the system operator $(A_\Phi,D(A_\Phi))$ of the system (1.6) by

$$\begin{cases}A_\Phi(p_1,p_2,\cdots,p_n)=A_m(p_1,p_2,\cdots,p_n),\\ D(A_\Phi)=\{(p_1,p_2,\cdots,p_n)\in D(A_m)|\Psi(p_1,p_2,\cdots,p_n)=\Phi(p_1,p_2,\cdots,p_n)\}.\end{cases}\tag{2.1}$$

Then, the system (1.6) can be written as an abstract Cauchy problem in the Banach space X:

$$\begin{cases} \dfrac{d(p_1, p_2, \cdots, p_n)(\cdot, t)}{dt} = A_\Phi(p_1, p_2, \cdots, p_n)(\cdot, t), \quad t \in (0, \infty), \\ (p_1, p_2, \cdots, p_n)(\cdot, 0) = (g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot)), \end{cases} \tag{2.2}$$

where $g_1(\cdot) = (g_{0,0}, g_{1,1}(\cdot), g_{2,1}(\cdot), \cdots), g_s(\cdot) = (g_{1,s}(\cdot), g_{2,s}(\cdot), \cdots),\ s = 2, 3, \cdots, n.$

Theorem 2.1. *Let A_Φ be defined by Eq (2.1). If $\mu_s(x) \le \overline{\mu}_s = \sup_{x\in[0,\infty)} \mu_s(x) < \infty$ $(1 \le s \le n)$, then A_Φ generates a positive C_0-semigroup $e^{A_\Phi t}$ of contractions on X.*

Proof. For self contained and conciseness, we only sketch the proof of Theorem 2.1. To start, we divided operator A_Φ into three parts $A_\Phi = A + U + E$, where

$$A(p_1, \cdots, p_n) = \left(\begin{pmatrix} -\lambda_0 p_{0,0} \\ -\frac{dp_{1,1}}{dx} \\ -\frac{dp_{2,1}}{dx} \\ \vdots \end{pmatrix}, \begin{pmatrix} -\frac{dp_{1,2}(x)}{dx} \\ -\frac{dp_{2,2}(x)}{dx} \\ -\frac{dp_{3,2}(x)}{dx} \\ \vdots \end{pmatrix}, \cdots\cdots, \begin{pmatrix} -\frac{dp_{1,n}(x)}{dx} \\ -\frac{dp_{2,n}(x)}{dx} \\ -\frac{dp_{3,n}(x)}{dx} \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \left\{ (p_1, \cdots, p_n) \in X \,\middle|\, \begin{array}{l} \frac{dp_{k,s}(x)}{dx} \in L^1[0, \infty),\ p_{k,s}(x) \text{ are absolutely} \\ \text{continuous functions and } p_s(0) = \Gamma_{0,s} p_1 \\ + \int_0^\infty \Gamma_{1,s} p_s dx,\ k \ge 1;\ 1 \le s \le n \end{array} \right\},$$

where

$$\Gamma_{1,1} = \begin{pmatrix} e^{-x} & 0 & 0 & 0 & \cdots \\ q_1\lambda_0 e^{-x} & 0 & \mu_1 & 0 & \cdots \\ 0 & 0 & 0 & \mu_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Gamma_{0,s} = \begin{pmatrix} q_s\lambda_0 & 0 & \cdots & \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Gamma_{1,s} = \begin{pmatrix} 0 & \mu_s & 0 & \cdots \\ 0 & 0 & \mu_s & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad 2 \le s \le n.$$

$$U(p_1, \cdots, p_n) = \left(\begin{pmatrix} 0 \\ -(\lambda_1 + \mu_1)p_{1,1}(x) \\ -(\lambda_1 + \mu_1)p_{2,1}(x) + \lambda_1 p_{1,1}(x) \\ -(\lambda_1 + \mu_1)p_{3,1}(x) + \lambda_1 p_{2,1}(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -(\lambda_2 + \mu_2)p_{1,2}(x) \\ -(\lambda_2 + \mu_2)p_{2,2}(x) + \lambda_2 p_{1,2}(x) \\ -(\lambda_2 + \mu_2)p_{3,2}(x) + \lambda_2 p_{2,2}(x) \\ \vdots \end{pmatrix}, \cdots\cdots, \begin{pmatrix} -(\lambda_n + \mu_n)p_{1,n}(x) \\ -(\lambda_n + \mu_n)p_{2,n}(x) + \lambda_n p_{1,n}(x) \\ -(\lambda_n + \mu_n)p_{3,n}(x) + \lambda_n p_{2,n}(x) \\ \vdots \end{pmatrix} \right),$$

$$E(p_1, \cdots, p_n) = \begin{pmatrix} \sum_{s=1}^n \mu_s \int_0^\infty p_{1,s}(x)dx \\ 0 \\ \vdots \end{pmatrix},$$

$$D(U) = X, \quad D(E) = X.$$

We can verify that $\|(\gamma I - A)^{-1}\| < \frac{1}{\gamma - \mathbb{M}_0}$ for all $\gamma > \mathbb{M}_0 = \max\{\overline{\mu}_1, \overline{\mu}_2, \cdots, \overline{\mu}_n\}$, where I represents the identity operator. Moreover, it is easy to prove that $D(A)$ is dense in X. Then, using the Helle-Yosida theorem (see [20, Theorem 4.20]), we obtain that A generates a C_0-semigroup. Due to the operators U and E being linear bounded, with perturbation theory of the semigroup, we know that A_Φ generates a C_0-semigroup $e^{A_\Phi t}$. Thus, we complete the proof of this theorem.

Next, we investigate the isometry of $e^{A_\Phi t}$. It is easy to obtain that X^*, the dual space of X, is as follows:

$$X^* = \left\{ (p_1^*, \cdots, p_n^*) \;\middle|\; \begin{array}{l} p_1^* = (p_{0,0}^*, p_{1,1}^*, p_{2,1}^*, \cdots), p_s^* = (p_{1,s}^*, p_{2,s}^*, \cdots), \ 2 \le s \le n, \\ p_{0,0}^* \in \mathbb{R}, \ p_{k,s}^* \in L^\infty[0,\infty), \|(p_1^*, \cdots, p_n^*)\| \\ = \sup\left\{\sup |p_{0,0}^*|, \ \sup_{\substack{k\ge 1 \\ 1\le s\le n}} \|p_{k,s}^*\|_{L^\infty[0,\infty)}\right\} < \infty \end{array} \right\}.$$

If we take a set $X_+ = \{(p_1, p_2, \cdots, p_n) \in X \mid p_{0,0} \ge 0, \ p_{k,s}(\cdot) \ge 0, \ k \ge 1, \ 1 \le s \le n\}$ in X, then, Theorem 2.1 ensures $e^{A_\Phi t} X_+ \subset X_+$. Now, for any $(p_1, p_2, \cdots, p_n) \in D(A_\Phi) \cap X_+$, we choose

$$(p_1^*, p_2^*, \cdots, p_n^*) = \|(p_1, p_2, \cdots, p_n)\| \left(\begin{pmatrix} 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ \vdots \end{pmatrix} \right).$$

It is not difficult to calculate that for $(p_1^*, p_2^*, \cdots, p_n^*) \in X^*$ and $(p_1, p_2, \cdots, p_n) \in X_+$, we have

$$\begin{aligned} &\langle (p_1, p_2, \cdots, p_n), (p_1^*, p_2^*, \cdots, p_n^*) \rangle \\ &= \|(p_1, p_2, \cdots, p_n)\| \left[p_{0,0} + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x) dx \right] = \|(p_1, p_2, \cdots, p_n)\|^2. \end{aligned}$$

This shows that $(p_1^*, p_2^*, \cdots, p_n^*) \in Q(p_1, p_2, \cdots, p_n)$, where

$$Q(p_1, p_2, \cdots, p_n) = \left\{ (p_1^*, p_2^*, \cdots, p_n^*) \in X^* \;\middle|\; \begin{array}{l} \langle (p_1, p_2, \cdots, p_n), (p_1^*, p_2^*, \cdots, p_n^*) \rangle \\ = \|(p_1, p_2, \cdots, p_n)\|^2 \\ = \|(p_1^*, p_2^*, \cdots, p_n^*)\|^2 \end{array} \right\}.$$

In addition, we obtain for $(p_1, p_2, \cdots, p_n) \in D(A_\Phi)$ and $(p_1^*, p_2^*, \cdots, p_n^*) \in Q(p_1, p_2, \cdots, p_n)$ that

$$\begin{aligned} &\langle A_\Phi (p_1, p_2, \cdots, p_n), (p_1^*, p_2^*, \cdots, p_n^*) \rangle = \|(p_1, p_2, \cdots, p_n)\| \\ &\times \left\{ -\lambda_0 p_{0,0} + \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x) \mu_s(x) dx + \sum_{s=1}^{n} \sum_{k=2}^{\infty} \int_0^\infty \lambda_s p_{k-1,s}(x) dx \right. \\ &\left. + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty \left[-\frac{d p_{k,s}(x)}{dx} - (\lambda_s + \mu_s(x)) p_{k,s}(x) \right] dx \right\} = 0. \end{aligned} \tag{2.3}$$

Then, Eq (2.3) implies that A_Φ is conservative with respect to $Q(\cdot)$. Theorem 3.6.1 of [21] stated that: Assume that A_Φ is densely defined, conservative with respect to $Q(\cdot) : D(A_\Phi) \to X^*$ and $(\gamma I - A_\Phi) D(A_\Phi) = X$ for some $\gamma > 0$. Then, for some $g \in D((A_\Phi)^2)$, the corresponding semigroup to Cauchy problem (2.2) is isometric. Hence, we obtain the following result.

Theorem 2.2. *Let $\mu_s(\cdot)$ satisfy $\mu_s(x) \leq \sup_{x\in[0,\infty)} \mu_s(x) < \infty$, $1 \leq s \leq n$. If the initial value $(p_1, p_2, \cdots, p_n)(\cdot, 0) = (g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot))$ of the system (2.2) belongs to $D(A_\Phi^2)$, then semigroup $e^{A_\Phi t}$ is isometric for $(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot))$. That is,*

$$\|e^{A_\Phi t}(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot))\| = \|(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot))\|, \quad t \in [0, \infty). \tag{2.4}$$

By combining Theorems 2.1 and 2.2, we obtain the main result in this section.

Theorem 2.3. *Let $\mu_s(\cdot)$ satisfy $\mu_s(x) \leq \sup_{x\in[0,\infty)} \mu_s(x) < \infty, 1 \leq s \leq n$. If the initial value $(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot))$ of system (2.2) belongs to $D(A_\Phi^2)$, then system (2.2) admits a unique positive time-evolving solution $(p_1(\cdot, t), p_2(\cdot, t), \cdots, p_n(\cdot, t))$ which satisfies*

$$\|(p_1(\cdot, t), p_2(\cdot, t), \cdots, p_n(\cdot, t))\| = 1, \quad \forall t \in [0, \infty). \tag{2.5}$$

Proof. Due to $(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot)) \in D(A_\Phi^2)$ and $(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot)) \in X_+$, it is easy to see that $(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot)) \in D(A_\Phi^2) \cap X_+$. By Theorem 1.81 of [12], we see that the system (1.6) has a unique positive time-evolving solution $(p_1(\cdot, t), p_2(\cdot, t), \cdots, p_n(\cdot, t))$, which can be expressed as

$$(p_1(\cdot, t), p_2(\cdot, t), \cdots, p_n(\cdot, t)) = e^{A_\Phi t}(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot)), \quad \forall t \in [0, \infty). \tag{2.6}$$

From this together with Eq (2.5), we obtain

$$\|(p_1(\cdot, t), p_2(\cdot, t), \cdots, p_n(\cdot, t))\| = \|e^{A_\Phi t}(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot))\| = 1, \quad \forall t \in [0, \infty). \tag{2.7}$$

This illustrates the physical meaning of $(p_1(\cdot, \cdot), p_2(\cdot, \cdot), \cdots, p_n(\cdot, \cdot))$.

3. Asymptotic behavior of the time-evolving solution

In this section, our main objective is to address the issue of asymptotic behavior of the time-evolving solution that we stated in Eq (2.6). In this regard, we prove that the time-evolving solution of the system (1.6) strongly converges but not exponentially converges to its static solution. In other words, hypothesis (**H2**) holds only in the context of strong convergence.

3.1. Strong convergence of the time-evolving solution

The main result of this subsection is given by the following Theorem 3.1.

Theorem 3.1. *Let $\mu_s(x) : [0, \infty) \to [0, \infty)$ be a measurable function that satisfies*

$$0 < \inf_{x\in[0,\infty)} \mu_s(x) \leq \mu_s(x) \leq \sup_{x\in[0,\infty)} \mu_s(x) < \infty, \quad 1 \leq s \leq n.$$

Then, the time-evolving solution of the system (2.2) strongly converges to its static solution. In other words,

$$\lim_{t\to\infty} \|(p_1, p_2, \cdots, p_n)(\cdot, t) - \langle (p_1^*, p_2^*, \cdots, p_n^*), (g_1, g_2, \cdots, g_n)\rangle (p_1, p_2, \cdots, p_n)(\cdot)\| = 0,$$

where $(p_1^*, p_2^*, \cdots, p_n^*)$ and $(p_1, p_2, \cdots, p_n)$ are the eigenvectors associated to zero, respectively.

To prove the above Theorem 3.1, we need to find the spectra of A_Φ along the imaginary axis. For this, we first provide the following seven lemmas.

Lemma 3.1. *Let A_Φ be defined by Eq (2.1). If $\mu_s(x) : [0,\infty) \to [0,\infty)$ is measurable function that satisfies*

$$0 < \underline{\mu}_s = \inf_{x\in[0,\infty)} \mu_s(x) \le \mu_s(x) \le \overline{\mu}_s = \sup_{x\in[0,\infty)} \mu_s(x) < \infty, \quad 1 \le s \le n,$$

then, zero is an eigenvalue of A_Φ with geometric multiplicity one.

Proof. We need to solve $A_\Phi(p_1, p_2, \cdots, p_n) = 0$ for unknown $(p_1, p_2, \cdots, p_n)$. This equation is equivalent to

$$\lambda_0 p_{0,0} = \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x)\mu_s(x)dx, \tag{3.1a}$$

$$\frac{dp_{1,s}(x)}{dx} = -[\lambda_s + \mu_s(x)]p_{1,s}(x), \tag{3.1b}$$

$$\frac{dp_{k,s}(x)}{dx} = -[\lambda_s + \mu_s(x)]p_{k,s}(x) + \lambda_s p_{k-1,s}(x), k \ge 2, \tag{3.1c}$$

$$p_{1,s}(0) = q_s\lambda_0 p_{0,0} + \int_0^\infty p_{2,s}(x)\mu_s(x)dx, \tag{3.1d}$$

$$p_{k,s}(0) = \int_0^\infty p_{k+1,s}(x)\mu_s(x)dx, \quad k \ge 2. \tag{3.1e}$$

Solve Eqs (3.1b) and (3.1c) to obtain

$$p_{k,s}(x) = e^{-\lambda_s x - \int_0^x \mu_s(\tau)d\tau} \sum_{j=1}^{k} \frac{(\lambda_s x)^{j-1}}{(j-1)!} p_{k-j+1,s}(0), \quad k \ge 1. \tag{3.2}$$

If we take $p_{k,s}(0) = 2^{-(k+1)}p_{1,s}(0),\ p_{1,s}(0) = q_s\lambda_0 p_{0,0} > 0$ and define

$$c_{k,s} := \int_0^\infty \frac{(\lambda_s x)^k}{k!} e^{-\lambda_s x - \int_0^x \mu_s(\tau)d\tau}dx, \quad d_{k,s} := \int_0^\infty \mu_s(x)\frac{(\lambda_s x)^k}{k!} e^{-\lambda_s x - \int_0^x \mu_s(\tau)d\tau}dx,\ k \ge 1,$$

then $p_{k,s}(0) = 2^{-(k+1)}p_{1,s}(0)$ satisfies the boundary conditions (3.1d) and (3.1e). Therefore, since the Cauchy product of series, the formula $\int_0^\infty \mu_s(x)e^{-\int_0^x \mu_s(\xi)d\xi}dx = 1$, and

$$\sum_{s=1}^{n}\sum_{k=1}^{\infty} p_{k,s}(0) = \lambda_0 q_{0,0}, \quad \sum_{k=1}^{\infty} c_{k,s} = \int_0^\infty e^{-\int_0^x \mu_s(\tau)d\tau}dx, \quad \sum_{k=1}^{\infty} d_{k,s} = 1,$$

we have

$$\begin{aligned} \sum_{s=1}^{n}\sum_{k=1}^{\infty}\int_0^\infty |p_{k,s}(x)|dx &= \sum_{s=1}^{n}\sum_{k=1}^{\infty}\sum_{j=1}^{k} c_{j,s}p_{k-j+1,s}(0) \\ &= \lambda_0 p_{0,0}\int_0^\infty e^{-\int_0^x \mu_s(\tau)d\tau}dx \le \frac{\lambda_0}{\underline{\mu}_s}p_{0,0} < \infty. \end{aligned} \tag{3.3}$$

Eq (3.3) means that zero is an eigenvalue of A_Φ. Moreover, by Eqs (3.1a) and (3.1d)–(3.2), we see that the geometric multiplicity of zero is one.

Now, we use the idea of [19] to describe the other spectrum of A_Φ along the imaginary axis. For this objective, we define the operator $(A_0, D(A_0))$ by

$$\begin{cases} A_0(p_1, p_2, \cdots, p_n) = A_m(p_1, p_2, \cdots, p_n), \\ D(A_0) = \{(p_1, p_2, \cdots, p_n) \in D(A_m) | \Psi(p_1, p_2, \cdots, p_n) = 0\}. \end{cases}$$

and discuss the inverse of A_0.

For given $(y_1, y_2, \cdots, y_n) \in X$, consider $(\gamma I - A_0)(p_1, p_2, \cdots, p_n) = (y_1, y_2, \cdots, y_n)$ of unknown $(p_1, p_2, \cdots, p_n) \in D(A_0)$. This equation can be equivalently written as the following system of equations

$$(\gamma + \lambda_0) p_{0,0} = y_{0,0} + \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x) \mu_s(x) dx, \tag{3.4a}$$

$$\frac{dp_{1,s}(x)}{dx} = -[\gamma + \lambda_s + \mu_s(x)] p_{1,s}(x) + y_{1,s}(x), \tag{3.4b}$$

$$\frac{dp_{k,s}(x)}{dx} = -[\gamma + \lambda_s + \mu_s(x)] p_{k,s}(x) + \lambda_s p_{k-1,s}(x) + y_{k,s}(x), k \geq 2, \tag{3.4c}$$

$$p_{k,s}(0) = 0, \ k \geq 1; \ 1 \leq s \leq n. \tag{3.4d}$$

By solving Eqs (3.4a)–(3.4c) and using Eq (3.4d), we obtain

$$p_{1,s}(x) = e^{-(\gamma+\lambda_s)x - \int_0^x \mu_s(\tau) d\tau} \int_0^x y_{1,s}(\tau) e^{(\gamma+\lambda_s)\tau + \int_0^\tau \mu_s(\xi) d\xi} d\tau, \tag{3.5a}$$

$$\begin{aligned} p_{k,s}(x) &= e^{-(\gamma+\lambda_s)x - \int_0^x \mu_s(\tau) d\tau} \int_0^x y_{k,s}(\tau) e^{(\gamma+\lambda_s)\tau + \int_0^\tau \mu_s(\xi) d\xi} d\tau \\ &\quad + \lambda_s e^{-(\gamma+\lambda_s)x - \int_0^x \mu_s(\tau) d\tau} \int_0^x p_{k-1,s}(\tau) e^{(\gamma+\lambda_s)\tau + \int_0^\tau \mu_s(\xi) d\xi} d\tau, k \geq 2, \end{aligned} \tag{3.5b}$$

$$\begin{aligned} p_{0,0} &= \frac{1}{\gamma + \lambda_0} y_{0,0} + \frac{1}{\gamma + \lambda_0} \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x) \mu_s(x) dx \\ &= \frac{1}{\gamma + \lambda_0} y_{0,0} + \frac{1}{\gamma + \lambda_0} \sum_{s=1}^{n} \int_0^\infty \mu_s(x) \\ &\quad \times \left[e^{-(\gamma+\lambda_s)x - \int_0^x \mu_s(\tau) d\tau} \int_0^x y_{1,s}(\tau) e^{(\gamma+\lambda_s)\tau + \int_0^\tau \mu_s(\xi) d\xi} d\tau \right] dx. \end{aligned} \tag{3.5c}$$

Denoting by

$$E_s f(x) = e^{-(\gamma+\lambda_s)x - \int_0^x \mu_s(\tau) d\tau} \int_0^x f(\tau) e^{(\gamma+\lambda_s)\tau + \int_0^\tau \mu_s(\xi) d\xi} d\tau, \ f \in L^1[0, \infty), \tag{3.6}$$

then the Eqs (3.5a)–(3.5c) and $\varphi_s f(x) = \int_0^\infty f(x)\mu_s(x)dx, f \in L^1[0,\infty)$ give, if the resolvent of A_0 exists,

$$\begin{aligned}
&(\gamma I - A_0)^{-1}(y_1, y_2, \cdots, y_n) \\
&= \left(\begin{pmatrix} \frac{1}{\gamma+\lambda_0} \sum_{s=1}^n \varphi_s E_s y_{1,s}(x) \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{1}{\gamma+\lambda_0} & 0 & 0 & 0 & \cdots \\ 0 & E_1 & 0 & 0 & \cdots \\ 0 & \lambda_1 E_1^2 & E_1 & 0 & \cdots \\ 0 & \lambda_1^2 E_1^3 & \lambda_1 E_1^2 & E_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} y_{0,0} \\ y_{1,1}(x) \\ y_{2,1}(x) \\ y_{3,1}(x) \\ \vdots \end{pmatrix}, \right. \\
&\left. \begin{pmatrix} E_2 & 0 & 0 & \cdots \\ \lambda_2 E_2^2 & E_2 & 0 & \cdots \\ \lambda_2^2 E_2^3 & \lambda_2 E_2^2 & E_2 & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} y_{1,2}(x) \\ y_{2,2}(x) \\ y_{3,2}(x) \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} E_n & 0 & 0 & \cdots \\ \lambda_n E_n^2 & E_n & 0 & \cdots \\ \lambda_n^2 E_n^3 & \lambda_n E_n^2 & E_n & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} y_{1,n}(x) \\ y_{2,n}(x) \\ y_{3,n}(x) \\ \vdots \end{pmatrix} \right).
\end{aligned} \tag{3.7}$$

The following Lemma 3.2 indicates the resolvent set $\rho(A_0)$ of A_0.

Lemma 3.2. *Let $\mu_s(x) : [0,\infty) \to [0,\infty)$ be a measurable function that satisfies*

$$0 < \underline{\mu}_s = \inf_{x\in[0,\infty)} \mu_s(x) \le \mu_s(x) \le \overline{\mu}_s = \sup_{x\in[0,\infty)} \mu_s(x) < \infty, \quad 1 \le s \le n.$$

Then, $\{\gamma \in \mathbb{C} \mid \mathrm{Re}(\gamma) + \lambda_0 > 0, \mathrm{Re}(\gamma) + \underline{\mu}_s > 0\} \subset \rho(A_0)$.

Proof. For all $f \in L^1[0,\infty)$, by performing integration by parts to Eq (3.6), it is easy to obtain that E_s satisfies the following inequality:

$$\|E_s\| \le \frac{1}{\mathrm{Re}(\gamma) + \lambda_s + \underline{\mu}_s}.$$

Then, using the inequality $\|\varphi_s\| \le \sup_{x\in[0,\infty)} \mu_s(x)$, we calculate for any $(y_1, y_2, \cdots, y_n) \in X$ that

$$\begin{aligned}
&\|(\gamma I - A_0)^{-1}(y_1, y_2, \cdots, y_n)\| \\
&\le \frac{1}{\mathrm{Re}(\gamma)+\lambda_0}|y_{0,0}| + \frac{1}{\mathrm{Re}(\gamma)+\lambda_0} \sum_{s=1}^n \|\varphi_s\|\|E_s\|\|y_{1,s}\|_{L^1[0,\infty)} \\
&\quad + \sum_{k=1}^\infty \lambda_1^{k-1}\|E_1\|^k \sum_{j=1}^\infty \|y_{j,1}\|_{L^1[0,\infty)} + \sum_{k=1}^\infty \lambda_2^{k-1}\|E_2\|^k \sum_{j=1}^\infty \|y_{j,2}\|_{L^1[0,\infty)} + \cdots \\
&\quad + \sum_{k=1}^\infty \lambda_n^{k-1}\|E_n\|^k \sum_{j=1}^\infty \|y_{j,n}\|_{L^1[0,\infty)} \\
&\le \frac{1}{\mathrm{Re}(\gamma)+\lambda_0}|y_{0,0}| + \frac{1}{\mathrm{Re}(\gamma)+\lambda_0} \sum_{s=1}^n \frac{\overline{\mu}_s}{\mathrm{Re}(\gamma)+\lambda_s+\underline{\mu}_s}\|y_{1,m}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\mathrm{Re}(\gamma)+\lambda_1+\underline{\mu}_1} \sum_{k=1}^\infty \left(\frac{\lambda_1}{\mathrm{Re}(\gamma)+\lambda_1+\underline{\mu}_1} \right)^{k-1} \sum_{j=1}^\infty \|y_{j,1}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\mathrm{Re}(\gamma)+\lambda_2+\underline{\mu}_2} \sum_{k=1}^\infty \left(\frac{\lambda_2}{\mathrm{Re}(\gamma)+\lambda_2+\underline{\mu}_2} \right)^{k-1} \sum_{j=1}^\infty \|y_{j,2}\|_{L^1[0,\infty)} + \cdots
\end{aligned}$$

$$+\frac{1}{\mathrm{Re}(\gamma)+\lambda_n+\underline{\mu}_n}\sum_{k=1}^{\infty}\left(\frac{\lambda_n}{\mathrm{Re}(\gamma)+\lambda_n+\underline{\mu}_n}\right)^{k-1}\sum_{j=1}^{\infty}\|y_{j,n}\|_{L^1[0,\infty)}$$

$$=\frac{1}{\mathrm{Re}(\gamma)+\lambda_0}|y_{0,0}|+\frac{1}{\mathrm{Re}(\gamma)+\lambda_0}\sum_{s=1}^{n}\frac{\overline{\mu}_s}{\mathrm{Re}(\gamma)+\lambda_s+\underline{\mu}_s}\|y_{1,s}\|_{L^1[0,\infty)}$$

$$+\sum_{s=1}^{n}\frac{1}{\mathrm{Re}(\gamma)+\underline{\mu}_s}\sum_{j=1}^{\infty}\|y_{j,s}\|_{L^1[0,\infty)}$$

$$\leq \sup\left\{\frac{1}{\mathrm{Re}(\gamma)+\lambda_0}+\frac{1}{\mathrm{Re}(\gamma)+\lambda_0}\frac{\overline{\mu}_1}{\mathrm{Re}(\gamma)+\lambda_1+\underline{\mu}_1}+\frac{1}{\mathrm{Re}(\gamma)+\underline{\mu}_1},\right.$$

$$\frac{1}{\mathrm{Re}(\gamma)+\lambda_0}\frac{\overline{\mu}_2}{\mathrm{Re}(\gamma)+\lambda_2+\underline{\mu}_2}+\frac{1}{\mathrm{Re}(\gamma)+\underline{\mu}_2},\cdots,$$

$$\left.\frac{1}{\mathrm{Re}(\gamma)+\lambda_0}\frac{\overline{\mu}_n}{\mathrm{Re}(\gamma)+\lambda_n+\underline{\mu}_n}+\frac{1}{\mathrm{Re}(\gamma)+\underline{\mu}_n}\right\}\|(y_1,y_2,\cdots,y_n)\|. \tag{3.8}$$

That is, inequality (3.8) means that the result of this lemma is correct.

Next, we use the following Lemma 3.3 to provide a specific expression for the Dirichlet operator.

Lemma 3.3. *Let* $\gamma\in\{\gamma\in\mathbb{C}\mid \mathrm{Re}(\gamma)+\lambda_0>0,\mathrm{Re}(\gamma)+\underline{\mu}_s>0\}$. *Then, we have* $(p_1,p_2,\cdots,p_n)\in\ker(\gamma I-A_m)$ *if, and only if,*

$$p_{0,0}=\frac{1}{\gamma+\lambda_0}\sum_{s=1}^{n}p_{1,s}(0)\int_0^{\infty}\mu_s(x)e^{-(\gamma+\lambda_s)x-\int_0^x\mu_s(\tau)d\tau}dx, \tag{3.9a}$$

$$p_{k,s}(x)=e^{-(\gamma+\lambda_s)x-\int_0^{\infty}\mu_s(\tau)d\tau}\sum_{j=1}^{k}\frac{(\lambda_s x)^{j-1}}{(j-1)!}p_{k-j+1,s}(0),\ k\geq 1, \tag{3.9b}$$

$$p_s(0)=(p_{1,s}(0),p_{2,s}(0),p_{3,s}(0),\cdots)\in l^1,\quad 1\leq s\leq n. \tag{3.9c}$$

Proof. If $(p_1,p_2,\cdots,p_n)\in\ker(\gamma I-A_m)$, then $(\gamma I-A_m)(p_1,p_2,\cdots,p_n)=0$, that is,

$$(\gamma+\lambda_0)p_{0,0}=\sum_{s=1}^{n}\int_0^{\infty}p_{1,s}(x)\mu_s(x)dx, \tag{3.10a}$$

$$\frac{dp_{1,s}(x)}{dx}=-[\gamma+\lambda_s+\mu_s(x)]p_{1,s}(x), \tag{3.10b}$$

$$\frac{dp_{k,s}(x)}{dx}=-[\gamma+\lambda_s+\mu_s(x)]p_{k,s}(x)+\lambda_s p_{k-1,s}(x). \tag{3.10c}$$

By solving Eqs (3.10a)–(3.10c), we obtain

$$p_{k,s}(x)=e^{-(\gamma+\lambda_s)x-\int_0^{x}\mu_s(\tau)d\tau}\sum_{j=1}^{k}\frac{(\lambda_s x)^{j-1}}{(j-1)!}p_{k-j+1,s}(0),\ k\geq 1, \tag{3.11a}$$

$$p_{0,0} = \frac{1}{\gamma + \lambda_0} \sum_{s=1}^{n} p_{1,s}(0) \int_0^\infty \mu_s(x) e^{-(\gamma+\lambda_s)x - \int_0^x \mu_s(\tau)d\tau} dx. \tag{3.11b}$$

Since $(p_1, p_2, \cdots, p_n) \in \ker(\gamma I - A_m)$, according to the Sobolev embedding theorem [22], we can easily obtain

$$\begin{aligned} \sum_{k=1}^{\infty} |p_{k,s}(0)| &\le \sum_{k=1}^{\infty} \|p_{k,s}\|_{L^\infty[0,\infty)} \\ &\le \sum_{k=1}^{\infty} \left(\|p_{k,s}\|_{L^1[0,\infty)} + \left\| \frac{dp_{k,s}}{dx} \right\|_{L^1[0,\infty)} \right) < \infty. \end{aligned} \tag{3.12}$$

Hence, Eqs (3.11a)–(3.12) show that Eqs (3.9a)–(3.9c) are true.

On the other hand, if Eqs (3.9a)–(3.9c) hold, due to the formula

$$\int_0^\infty e^{-cx} x^k dx = c^{-(k+1)} k!$$

it holds true for any $c > 0$ and positive integer $k \ge 1$, and performing integration by parts, we deduce

$$\begin{aligned} \|p_{k,s}\|_{L^1[0,\infty)} &\le \int_0^\infty e^{-(\mathrm{Re}(\gamma)+\lambda_s)x - \int_0^x \mu_s(\tau)d\tau} \sum_{j=1}^{k} \frac{(\lambda_s x)^{j-1}}{(j-1)!} |p_{k-j+1,s}(0)| dx \\ &\le \sum_{j=1}^{k} \frac{\lambda_s^{j-1}}{(j-1)!} |p_{k-j+1,s}(0)| \int_0^\infty x^{j-1} e^{-[\mathrm{Re}(\gamma)+\lambda_s + \inf_{x\in[0,\infty)} \mu_s(x)]x} dx \\ &= \sum_{j=1}^{k} \frac{\lambda_s^{j-1}}{[\mathrm{Re}(\gamma) + \lambda_s + \underline{\mu}_s]^j} |p_{k-j+1,s}(0)|. \end{aligned}$$

Then, by the Cauchy product of series, we calculate that

$$\begin{aligned} \sum_{k=1}^{\infty} \|p_{k,s}\|_{L^1[0,\infty)} &\le \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{\lambda_s^{j-1}}{[\mathrm{Re}(\gamma) + \lambda_s + \underline{\mu}_s]^j} |p_{k-j+1,s}(0)| \\ &= \frac{1}{\mathrm{Re}(\gamma) + \lambda_s + \underline{\mu}_s} \sum_{j=1}^{\infty} \left(\frac{\lambda_s}{\mathrm{Re}(\gamma) + \lambda_s + \underline{\mu}_s} \right)^{j-1} \sum_{k=1}^{\infty} |p_{k,s}(0)| \\ &= \frac{1}{\mathrm{Re}(\gamma) + \underline{\mu}_s} \sum_{k=1}^{\infty} |p_{k,s}(0)| < \infty, \end{aligned} \tag{3.13}$$

for any $\gamma > -\underline{\mu}_s$. Inequalities (3.12) and (3.13) show that $(p_1, p_2, \cdots, p_n) \in X$. In addition, by Eq (3.9b), we have

$$\begin{aligned} \frac{dp_{1,s}(x)}{dx} &= -[\gamma + \lambda_s + \mu_s(x)] p_{1,s}(0) e^{-(\gamma+\lambda_s)x - \int_0^x \mu_s(\tau)d\tau} \\ &= -[\gamma + \lambda_s + \mu_s(x)] p_{1,s}(x), \end{aligned} \tag{3.14a}$$

$$\begin{aligned}\frac{dp_{k,s}(x)}{dx} &= -[\gamma+\lambda_s+\mu_s(x)]e^{-(\gamma+\lambda_s)x-\int_0^x\mu_s(\tau)d\tau}\sum_{j=1}^{k}\frac{(\lambda_s x)^{j-1}}{(j-1)!}p_{k-j+1,s}(0)\\ &\quad +\lambda_s e^{-(\gamma+\lambda_s)x-\int_0^x\mu_s(\tau)d\tau}\sum_{j=1}^{k-1}\frac{(\lambda_s x)^{j-1}}{(j-1)!}p_{k-j,s}(0)\\ &= -[\gamma+\lambda_s+\mu_s(x)]p_{k,s}(x)+\lambda_s p_{k-1,s}(x),\quad k\geq 2.\end{aligned}\tag{3.14b}$$

Combining the above Eqs (3.14a) and (3.14b) with inequality (3.13), we obtain

$$\sum_{k=1}^{\infty}\left\|\frac{dp_{k,s}}{dx}\right\|_{L^1[0,\infty)}\leq\left(|\gamma|+2\lambda_s+\sup_{x\in[0,\infty)}\mu_s(x)\right)\sum_{k=1}^{\infty}\|p_{k,s}\|_{L^1[0,\infty)}<\infty.$$

This inequality implies that

$$\sum_{s=1}^{n}\sum_{k=1}^{\infty}\left\|\frac{dp_{k,s}}{dx}\right\|_{L^1[0,\infty)}<\infty.\tag{3.15}$$

Hence, Eqs (3.13)–(3.15) indicate that $(p_1,p_2,\cdots,p_n)\in D(A_m)$ and

$$(\gamma I-A_m)(p_1,p_2,\cdots,p_n)=0.$$

Clearly, by the definition it is not difficult to show that the boundary operator Ψ is surjective. In addition, for all $\gamma\in\rho(A_0)$, the operator

$$\Psi\big|_{\ker(\gamma I-A_m)}:\ker(\gamma I-A_m)\to\partial X,$$

is invertible. Now, for any $\gamma\in\rho(A_0)$, we introduce the Dirichlet operator by

$$D_\gamma:=\left(\Psi\big|_{\ker(\gamma I-A_m)}\right)^{-1}:\partial X\to\ker(\gamma I-A_m).$$

Then, using Lemma 3.3, for any $\gamma\in\rho(A_0)$, we can obtain the following specific expression for D_γ:

$$\begin{aligned}&D_\gamma(p_1(0),p_2(0),\cdots,p_n(0))\\ &=\left(\begin{pmatrix}\frac{1}{\gamma+\lambda_0}\sum_{s=1}^n p_{1,s}(0)\varphi_s\varepsilon_{1,s}\\ 0\\ 0\\ 0\\ \vdots\end{pmatrix}+\begin{pmatrix}0&0&0&0&\cdots\\ \varepsilon_{1,1}&0&0&0&\cdots\\ \varepsilon_{2,1}&\varepsilon_{1,1}&0&0&\cdots\\ \varepsilon_{3,1}&\varepsilon_{2,1}&\varepsilon_{1,1}&0&\cdots\\ \vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}p_{1,1}(0)\\ p_{2,1}(0)\\ p_{3,1}(0)\\ p_{4,1}(0)\\ \vdots\end{pmatrix},\right.\\ &\left.\begin{pmatrix}\varepsilon_{1,2}&0&0&\cdots\\ \varepsilon_{2,2}&\varepsilon_{1,2}&0&\cdots\\ \varepsilon_{3,2}&\varepsilon_{2,2}&\varepsilon_{1,2}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}p_{1,2}(0)\\ p_{2,2}(0)\\ p_{3,2}(0)\\ \vdots\end{pmatrix},\cdots,\begin{pmatrix}\varepsilon_{1,n}&0&0&\cdots\\ \varepsilon_{2,n}&\varepsilon_{1,n}&0&\cdots\\ \varepsilon_{3,n}&\varepsilon_{2,n}&\varepsilon_{1,n}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}p_{1,n}(0)\\ p_{2,n}(0)\\ p_{3,n}(0)\\ \vdots\end{pmatrix}\right),\end{aligned}\tag{3.16}$$

where

$$\varepsilon_{j,s}=\frac{(\lambda_s x)^{j-1}}{(j-1)!}e^{-(\gamma+\lambda_s)x-\int_0^x\mu_s(\tau)d\tau},\quad j\geq 1,\ 1\leq s\leq n.$$

Finally, using the expression of Dirichlet operator (3.16) and the boundary operator Φ, we can calculate the specific expression of ΦD_γ as follows:

$$\Phi D_\gamma(p_1(0), p_2(0), \cdots, p_n(0)) = (J_1, J_2, \cdots, J_n)(p_1(0), p_2(0), \cdots, p_n(0)), \tag{3.17}$$

where

$$J_k = \begin{pmatrix} \frac{q_k\lambda_0}{\gamma+\lambda_0}\sum_{s=1}^n p_{1,s}(0)\varphi_s\varepsilon_{1,s} \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \varphi_k\varepsilon_{2,k} & \varphi_k\varepsilon_{1,k} & 0 & \cdots \\ \varphi_k\varepsilon_{3,k} & \varphi_k\varepsilon_{2,k} & \varphi_k\varepsilon_{1,k} & \cdots \\ \varphi_k\varepsilon_{4,k} & \varphi_k\varepsilon_{3,k} & \varphi_k\varepsilon_{2,k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} p_{1,k}(0) \\ p_{2,k}(0) \\ p_{3,k}(0) \\ \vdots \end{pmatrix}, \ 1 \le k \le n.$$

The following Lemma 3.4 was found in [23], and we use this lemma along with the above results in this subsection to provide spectrum $\sigma(A_\Phi)$ of A_Φ on the imaginary axis.

Lemma 3.4. *Assume $\gamma \in \rho(A_0)$. If there exists γ_0 that satisfies $1 \notin \sigma(\Phi D_{\gamma_0})$, then $\gamma \in \sigma(A_\Phi)$ if, and only if, $1 \in \sigma(\Phi D_\gamma)$.*

Lemma 3.5. *Let A_Φ be defined by Eq (2.1). If $\mu_s(x) : [0,\infty) \to [0,\infty)$ is a measurable function that satisfies*

$$0 < \inf_{x\in[0,\infty)} \mu_s(x) \le \mu_s(x) \le \sup_{x\in[0,\infty)} \mu_s(x) < \infty, \quad 1 \le s \le n,$$

then, we have $i\mathbb{R} \cap \sigma(A_\Phi) = \{0\}$, $i^2 = -1$.

Proof. If we take $\gamma = ib$, $i^2 = -1$, $b \in \mathbb{R} \setminus \{0\}$, then applying the Riemann-Lebesgue lemma, we obtain that there exists $\mathbb{M} > 0$ that satisfies

$$\left|\int_0^\infty \mu_s(x)\frac{(\lambda_s x)^{k-1}}{(k-1)!}e^{-(ib+\lambda_s)x-\int_0^x \mu_s(\tau)d\tau}dx\right| < \int_0^\infty \mu_s(x)\frac{(\lambda_s x)^{k-1}}{(k-1)!}e^{-\lambda_s x-\int_0^x \mu_s(\tau)d\tau}dx, \tag{3.18}$$

for all $|b| > \mathbb{M}$. Hence, using inequality (3.18) and the formulas $\sum_{s=1}^n q_s = 1$ and

$$\int_0^\infty \mu_s(x)e^{-\int_0^x \mu_s(\tau)d\tau}dx = 1,$$

we calculate for $p_s(0) = (p_{1,s}(0), p_{2,s}(0), p_{3,s}(0), \cdots) \in l^1 \setminus \{0\}$ that

$$\begin{aligned} ||\Phi D_{ib}(p_1(0), p_2(0), \cdots, p_n(0))|| &\le \frac{\lambda_0}{\sqrt{b^2+\lambda_0^2}}\sum_{s=1}^n |p_{1,s}(0)||\varphi_s\varepsilon_{1,s}| \\ &+ \sum_{s=1}^n\sum_{k=2}^\infty |\varphi_s\varepsilon_{k,s}||p_{1,s}(0)| + \sum_{s=1}^n\sum_{k=1}^\infty |\varphi_s\varepsilon_{k,s}|\sum_{j=2}^\infty |p_{j,s}(0)| \\ &< \sum_{s=1}^n\sum_{k=1}^\infty |\varphi_s\varepsilon_{k,s}|\sum_{j=1}^\infty |p_{j,s}(0)| \\ &= \sum_{s=1}^n\sum_{k=1}^\infty \left|\int_0^\infty \mu_s(x)\frac{(\lambda_s x)^{k-1}}{(k-1)!}e^{-(ib+\lambda_s)x-\int_0^x \mu_s(\tau)d\tau}dx\right|\sum_{j=1}^\infty |p_{j,s}(0)| \end{aligned}$$

$$< \sum_{s=1}^{n}\sum_{k=1}^{\infty}\int_0^{\infty}\mu_s(x)\frac{(\lambda_s x)^{k-1}}{(k-1)!}e^{-\lambda_s x-\int_0^x\mu_s(\tau)d\tau}dx\sum_{j=1}^{\infty}|p_{j,s}(0)|$$

$$= \sum_{s=1}^{n}\int_0^{\infty}\mu_s(x)\sum_{k=1}^{\infty}\frac{(\lambda_s x)^{k-1}}{(k-1)!}e^{-\lambda_s x-\int_0^x\mu_s(\tau)d\tau}dx\sum_{j=1}^{\infty}|p_{j,s}(0)|$$

$$= \sum_{s=1}^{n}\int_0^{\infty}\mu_s(x)e^{-\int_0^x\mu_s(\tau)d\tau}dx\sum_{j=1}^{\infty}|p_{j,s}(0)|$$

$$= \sum_{s=1}^{n}\sum_{j=1}^{\infty}|p_{j,s}(0)| = \|(p_1(0),p_2(0),\cdots,p_n(0))\|. \tag{3.19}$$

That is, $\|\Phi D_{ib}\| < 1$ for all $|b| > \mathbb{M}$. Since $\lambda_0 > 0$ and $\underline{\mu}_s > 0$, there exists $\gamma_1 = \min\{\lambda_0, \underline{\mu}_s\} > 0$ such that $\{\gamma \in \mathbb{C} \mid \mathrm{Re}(\gamma) > -\gamma_1\} \subset \rho(A_0)$. This means that $\gamma = ib \in \rho(A_0)$. Then, by the above inequality (3.19), we know that the spectral radius $r(\Phi D_{ib})$ of operator ΦD_{ib} satisfies $r(\Phi D_{ib}) \le \|\Phi D_{ib}\| < 1$ if $|b| > \mathbb{M}$. In other words, $1 \notin \sigma(\Phi D_{ib})$ for $|b| > \mathbb{M}$. This indicates that there must be $\gamma_0 = 2|b|$ satisfying $1 \notin \sigma(\Phi D_{\gamma_0})$. Consequently, using Lemma 3.4, we obtain $\gamma = ib \notin \sigma(A_\Phi)$ for $|b| > \mathbb{M}$, i.e.,

$$\{ib\big||b| > \mathbb{M}\} \subset \rho(A_\Phi) \ \text{ and } \{ib\big||b| \le \mathbb{M}\} \subset \sigma(A_\Phi) \cap i\mathbb{R}.$$

On the other hand, since $e^{A_\Phi t}$ is a positive uniformly bounded semigroup (Theorem 2.1), using Corollary 2.3 of [24], we obtain that $\sigma(A_\Phi) \cap i\mathbb{R}$ is imaginary additively cyclic, which states that $ib \in \sigma(A_\Phi) \cap i\mathbb{R}$, and we deduce that $ibk \in \sigma(A_\Phi) \cap i\mathbb{R}$ for every integer k. Therefore, combining the above discussion with the inclusion relationship $\{ib\big||b| \le \mathbb{M}\} \subset \sigma(A_\Phi) \cap i\mathbb{R}$ and Lemma 3.1, we have $\sigma(A_\Phi) \cap i\mathbb{R} = \{0\}$.

Lemma 3.6. *The specific expression of the adjoint operator $A_\Phi{}^*$ of A_Φ is as follows:*

$$A_\Phi^*(p_1^*,p_2^*,\cdots,p_n^*) = \left(\begin{pmatrix}\lambda_0\sum_{s=1}^n q_s p_{1,s}^*(0)\\ 0\\ \mu_1(x)p_{1,1}^*(0)\\ \mu_1(x)p_{2,1}^*(0)\\ \ddots\end{pmatrix} + \begin{pmatrix}-\lambda_0 & 0 & 0 & 0 & \cdots\\ \mu_1(x) & \phi_1 & \lambda_1 & 0 & \cdots\\ 0 & 0 & \phi_1 & \lambda_1 & \cdots\\ 0 & 0 & 0 & \phi_1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}p_{0,0}^*\\ p_{1,1}^*(x)\\ p_{2,1}^*(x)\\ p_{3,1}^*(x)\\ \vdots\end{pmatrix},\right.$$

$$\begin{pmatrix}\mu_2(x)p_{0,0}^*\\ \mu_2(x)p_{1,2}^*(0)\\ \mu_2(x)p_{2,2}^*(0)\\ \ddots\end{pmatrix} + \begin{pmatrix}\phi_2 & \lambda_2 & 0 & 0 & \cdots\\ 0 & \phi_2 & \lambda_2 & 0 & \cdots\\ 0 & 0 & \phi_2 & \lambda_2 & \cdots\\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}\begin{pmatrix}p_{1,2}^*(x)\\ p_{2,2}^*(x)\\ p_{3,2}^*(x)\\ \vdots\end{pmatrix},$$

$$\cdots\cdots,$$

$$\left.\begin{pmatrix}\mu_n(x)p_{0,0}^*\\ \mu_n(x)p_{1,n}^*(0)\\ \mu_n(x)p_{2,n}^*(0)\\ \ddots\end{pmatrix} + \begin{pmatrix}\phi_n & \lambda_n & 0 & 0 & \cdots\\ 0 & \phi_n & \lambda_n & 0 & \cdots\\ 0 & 0 & \phi_n & \lambda_n & \cdots\\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}\begin{pmatrix}p_{1,n}^*(x)\\ p_{2,n}^*(x)\\ p_{3,n}^*(x)\\ \vdots\end{pmatrix}\right),$$

with domain $D(A_\Phi^*) = \{(p_1^*, p_2^*, \cdots, p_n^*) \in X^* \mid \frac{dp_{k,s}^*(x)}{dx}$ existing and $p_{k,s}^*(\infty) = h\}$, where $\phi_s = \frac{d}{dx} - [\lambda_s + \mu_s(x)]$ and h is a positive constant that is independent of k and s.

Proof. For $(p_1, p_2, \cdots, p_n) \in D(A_\Phi)$ and $(p_1^*, p_2^*, \cdots, p_n^*) \in D(A_\Phi^*)$, using integration by parts, we calculate that

$$\begin{aligned}
&\langle A_\Phi(p_1, p_2, \cdots, p_n), (p_1^*, p_2^*, \cdots, p_n^*)\rangle = \left[-\lambda_0 p_{0,0} + \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x)\mu_s(x)dx\right] p_{0,0}^* \\
&\quad + \sum_{s=1}^{n} \left\{ \sum_{k=1}^{\infty} \int_0^\infty \left[-\frac{dp_{k,s}(x)}{dx} - (\lambda_s + \mu_s(x))p_{k,s}(x)\right] p_{k,s}^*(x)dx \right. \\
&\quad \left. + \sum_{k=2}^{\infty} \int_0^\infty \lambda_s p_{k-1,s}(x) p_{k,s}^*(x)dx \right\} \\
&= -\lambda_0 p_{0,0} p_{0,0}^* + p_{0,0}^* \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x)\mu_s(x)dx + \sum_{s=1}^{n}\sum_{k=1}^{\infty} p_{k,s}(0)p_{k,s}^*(0) \\
&\quad + \sum_{s=1}^{n}\sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x)\left[\frac{dp_{k,s}^*(x)}{dx} - (\lambda_s + \mu_s(x))p_{k,s}^*(x)\right]dx \\
&\quad + \lambda_s \sum_{s=1}^{n}\sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x)p_{k+1,s}^*(x)dx \\
&= -\lambda_0 p_{0,0} p_{0,0}^* + p_{0,0}^* \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x)\mu_s(x)dx \\
&\quad + \sum_{s=1}^{n}\left(q_s\lambda_0 p_{0,0}^* + \int_0^\infty p_{2,s}^*(x)\mu_s(x)dx\right)p_{1,s}^*(0) + \sum_{s=1}^{n}\sum_{k=2}^{\infty}\int_0^\infty p_{k+1,s}(x)\mu_s(x)dx p_{k,s}^*(0) \\
&\quad + \sum_{s=1}^{n}\sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x)\left[\frac{dp_{k,s}^*(x)}{dx} - (\lambda_s + \mu_s(x))p_{k,s}^*(x)\right]dx \\
&\quad + \sum_{s=1}^{n}\lambda_s\sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x)p_{k+1,s}^*(x)dx \\
&= -\lambda_0 p_{0,0} p_{0,0}^* + p_{0,0}^* \sum_{s=1}^{n} \int_0^\infty p_{1,s}(x)\mu_s(x)dx \\
&\quad + \lambda_0 p_{0,0}\sum_{s=1}^{n} q_s p_{1,s}^*(0) + \sum_{s=1}^{n}\sum_{k=1}^{\infty}\int_0^\infty p_{k+1,s}(x)\mu_s(x)dx p_{k,s}^*(0) \\
&\quad + \sum_{s=1}^{n}\sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x)\left[\frac{dp_{k,s}^*(x)}{dx} - (\lambda_s + \mu_s(x))p_{k,s}^*(x)\right]dx \\
&\quad + \sum_{s=1}^{n}\lambda_s\sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x)p_{k+1,s}^*(x)dx \\
&= \langle (p_0, p_1, \cdots, p_n), A_\Phi^*(p_0^*, p_1^*, \cdots, p_n^*)\rangle.
\end{aligned} \tag{3.20}$$

Then, from the last equation of the above Eq (3.20), we can obtain ${A_\Phi}^*$.

Lemma 3.7. *The zero is an eigenvalue of A_Φ^* with geometric multiplicity one.*

Proof. We consider $A_\Phi^*(p_1^*, p_2^*, \cdots, p_n^*) = 0$. This equation is equivalent to

$$-\lambda_0 p_{0,0}^* + \lambda_0 \sum_{s=1}^{n} q_s p_{1,s}^*(0) = 0, \tag{3.21a}$$

$$\frac{dp_{1,s}^*(x)}{dx} - [\lambda_s + \mu_s(x)]p_{1,s}^*(x) + \lambda_s p_{2,s}^*(x) + \mu_s(x)p_{0,0}^* = 0, \tag{3.21b}$$

$$\frac{dp_{k,s}^*(x)}{dx} - [\lambda_s + \mu_s(x)]p_{k,s}^*(x) + \lambda_s p_{k+1,s}^*(x) + \mu_s(x)p_{k-1,s}^*(0) = 0, k \geq 2, \tag{3.21c}$$

$$p_{k,s}^*(\infty) = h, \quad k \geq 1,\ 1 \leq s \leq n. \tag{3.21d}$$

By the above equations, it is easy to investigate that

$$(p_1^*, p_2^*, \cdots, p_n^*)_{(h)} := \left(\begin{pmatrix} h \\ \vdots \end{pmatrix}, \begin{pmatrix} h \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} h \\ \vdots \end{pmatrix} \right) \in D(A_\Phi^*),$$

is a positive solution of Eqs (3.21a)–(3.21d). In addition, Eqs (3.21a)–(3.21d) are equivalent to

$$p_{0,0}^* = \sum_{s=1}^{n} q_s p_{1,s}^*(0), \tag{3.22a}$$

$$p_{2,s}^*(x) = \frac{1}{\lambda_s}\left[-\frac{dp_{1,s}^*(x)}{dx} + (\lambda_s + \mu_s(x))p_{1,s}^*(x) - \mu_s(x)p_{0,0}^* \right], \tag{3.22b}$$

$$p_{k+1,s}^*(x) = \frac{1}{\lambda_s}\left[-\frac{dp_{k,s}^*(x)}{dx} + (\lambda_s + \mu_s(x))p_{k,s}^*(x) - \mu_s(x)p_{k-1,s}^*(0) \right], k \geq 2. \tag{3.22c}$$

Clearly, Eqs (3.22a)–(3.22c) imply that the geometric multiplicity of zero is one.

Proof of Theorem 3.1: Theorem 2.1 shows that semigroup $e^{A_\Phi t}$ is a uniformly bounded C_0−semigroup on Banach space X. In addition, using Lemmas 3.1, 3.5, and 3.7, we obtain that $\sigma_p(A_\Phi) \cap i\mathbb{R} = \sigma_p(A_\Phi^*) \cap i\mathbb{R} = \{0\}$ and $\{\gamma \in \mathbb{C} \mid \gamma = ib, b \neq 0, b \in \mathbb{R}\} \subset \rho(A_\Phi)$, and zero is an eigenvalue of A_Φ^* with algebraic multiplicity one. Hence, due to Theorem 1.96 of [12], we obtain that the time-evolving solution of system (2.2) converges strongly to its static solution. In other words,

$$\lim_{t\to\infty} \|(p_1, p_2, \cdots, p_n)(\cdot, t) - \langle (p_1^*, p_2^*, \cdots, p_n^*), (g_1, g_2, \cdots, g_n)\rangle (p_1, p_2, \cdots, p_n)(\cdot)\| = 0,$$

where $(p_1^*, p_2^*, \cdots, p_n^*)$ and $(p_1, p_2, \cdots, p_n)$ are the eigenvectors associated to zero in Lemmas 3.7 and 3.1, respectively.

3.2. Exponential convergence of the time-evolving solution

To prove the exponential convergence of the time-evolving solution, we need to find the spectral distribution of A_Φ on the left-half complex plane. For this objective, we first provide the following Lemma 3.8.

Lemma 3.8. *If $\lambda_s < \mu_s,\ 1 \le s \le n$, then each point in*

$$\Lambda := \left\{\gamma \in \mathbb{C} \,\middle|\, \left|\gamma + \lambda_s + \mu_s \pm \sqrt{(\gamma + \lambda_s + \mu_s)^2 - 4\lambda_s\mu_s}\right| < 2\mu_s, \mathrm{Re}(\gamma) + \mu_s > 0 \right\} \cup \{0\},$$

is an eigenvalue of A_Φ with geometric multiplicity one, in particular

$$\left(-\min_{1\le s\le n}\{\mu_s\}, \min_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s\}\right)\bigcup\left[\max_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s\}, 0\right] \subset \sigma(A_\Phi).$$

Proof. For each $\gamma \in \Lambda$, we consider the equation $A_\Phi(p_1, p_2, \cdots, p_n) = \gamma(p_1, p_2, \cdots, p_n)$ of unknown $(p_1, p_2, \cdots, p_n) \in D(A_\Phi)$. This is equivalent to the following system:

$$(\gamma + \lambda_0)p_{0,0} = \sum_{s=1}^{n} \mu_s \int_0^\infty p_{1,s}(x)dx, \tag{3.23a}$$

$$\frac{dp_{1,s}(x)}{dx} = -(\gamma + \lambda_s + \mu_s)p_{1,s}(x), \tag{3.23b}$$

$$\frac{dp_{k,s}(x)}{dx} = -(\gamma + \lambda_s + \mu_s)p_{k,s}(x) + \lambda_s p_{k-1,s}(x), \quad k \ge 2, \tag{3.23c}$$

$$p_{1,s}(0) = q_s\lambda_0 p_{0,0} + \mu_s \int_0^\infty p_{2,s}(x)dx, \tag{3.23d}$$

$$p_{k,s}(0) = \mu_s \int_0^\infty p_{k+1,s}(x)dx, \quad k \ge 2. \tag{3.23e}$$

Solving Eqs (3.23b) and (3.23c), we have

$$p_{k,s}(x) = e^{-(\gamma+\lambda_s+\mu_s)x} \sum_{j=1}^{k} \frac{(\lambda_s x)^{k-j}}{(k-j)!} p_{j,s}(0), \quad k \ge 1;\ 1 \le s \le n. \tag{3.24}$$

From this together with the formula

$$\int_0^\infty x^{k-j} e^{-(\gamma+\lambda_s+\mu_s)x} dx = \frac{(k-j)!}{(\gamma + \lambda_s + \mu_s)^{k+1-j}},\ \mathrm{Re}(\gamma) + \lambda_s + \mu_s > 0,$$

we obtain

$$\int_0^\infty p_{k,s}(x)dx = \sum_{j=1}^{k} \frac{\lambda_s^{k-j}}{(\gamma + \lambda_s + \mu_s)^{k+1-j}} p_{j,s}(0), \quad k \ge 1. \tag{3.25}$$

We can thus combine Eqs (3.23e) and (3.25) to obtain

$$p_{k,s}(0) = \mu_s \sum_{j=1}^{k+1} \frac{\lambda_s^{k+1-j}}{(\gamma + \lambda_s + \mu_s)^{k+2-j}} p_{j,s}(0), \quad k \ge 2. \tag{3.26}$$

This yields

$$p_{k+1,s}(0) - \frac{\lambda_s}{\gamma + \lambda_s + \mu_s} p_{k,s}(0) = \frac{\mu_s}{\gamma + \lambda_s + \mu_s} p_{k+2,s}(0), \quad k \geq 2.$$

Clearly, the above equation is equivalent to

$$p_{k+2,s}(0) = \frac{\gamma + \lambda_s + \mu_s}{\mu_s} p_{k+1,s}(0) - \frac{\lambda_s}{\mu_s} p_{k,s}(0), \quad k \geq 2. \tag{3.27}$$

For any complex number ξ_s and η_s, $1 \leq s \leq n$, if we set

$$p_{k+2,s}(0) - \xi_s p_{k+1,s}(0) = \eta_s [p_{k+1,s}(0) - \xi_s p_{k,s}(0)], \; k \geq 2, \tag{3.28}$$

then it is easy to see that ξ_s and η_s satisfy the following two equations:

$$\xi_s + \eta_s = \frac{\gamma + \lambda_s + \mu_s}{\mu_s}, \quad \xi_s \eta_s = \frac{\lambda_s}{\mu_s}. \tag{3.29}$$

From Eq (3.29), it is easy to determine that

$$\xi_s = \frac{\gamma + \lambda_s + \mu_s + \sqrt{(\gamma + \lambda_s + \mu_s)^2 - 4\lambda_s \mu_s}}{2\mu_s}, \tag{3.30a}$$

$$\eta_s = \frac{\gamma + \lambda_s + \mu_s - \sqrt{(\gamma + \lambda_s + \mu_s)^2 - 4\lambda_s \mu_s}}{2\mu_s}. \tag{3.30b}$$

Note that from Eq (3.28), we observe that

$$p_{k+2,s}(0) - \xi_s p_{k+1,s}(0) = \eta_s^{k-1} [p_{3,s}(0) - \xi_s p_{2,s}(0)], \quad k \geq 2. \tag{3.31}$$

Then, by reusing the above equations and organizing it, we obtain

$$\begin{aligned} p_{k+2,s}(0) &= \xi_s p_{k+1,s}(0) - \xi_s [p_{k+1,s}(0) - \xi_s p_{k,s}(0)] - \xi_s^2 [p_{k,s}(0) - \xi_s p_{k-1,s}(0)] \\ &\quad - \xi_s^3 [p_{k-1,s}(0) - \xi_s p_{k-2,s}(0)] - \xi_s^{k-2} [p_{4,s}(0) - \xi_s p_{3,s}(0)] \\ &= [\eta_s^{k-1} + \xi_s \eta_s^{k-2} + \cdots + \xi_s^{k-2} \eta_s + \xi_s^{k-1}] p_{3,s}(0) \\ &\quad - [\eta_s^{k-2} + \xi_s \eta_s^{k-3} + \cdots + \xi_s^{k-3} \eta_s + \xi_s^{k-2}] \xi_s \eta_s p_{2,s}(0), \quad k \geq 2. \end{aligned} \tag{3.32}$$

If $\xi_s = \eta_s$, then Eq (3.32) is simplified as

$$p_{k+2,s}(0) = k \xi_s^{k-1} p_{3,s}(0) - (k-1) \xi_s^k p_{2,s}(0), \quad k \geq 2.$$

The inequality $|p_{k+2,s}(0)| \leq k|\xi_s|^{k-1}|p_{3,s}(0)| + (k-1)|\xi_s|^k|p_{2,s}(0)|$ can be obtained by taking the absolute value of the above equation. Then, taking the sum of $k = 2$ to ∞ for this inequality, we obtain

$$\sum_{k=2}^{\infty} |p_{k+2,s}(0)| \leq |p_{3,s}(0)| \sum_{k=2}^{\infty} k|\xi_s|^{k-1} + |p_{2,s}(0)| \sum_{k=2}^{\infty} (k-1)|\xi_s|^k. \tag{3.33}$$

If $\xi_s \neq \eta_s$, then Eq (3.32) can be written as

$$p_{k+2,s}(0) = \frac{p_{3,s}(0) - \eta_s p_{2,s}(0)}{\xi_s - \eta_s}\xi_s^k - \frac{p_{3,s}(0) - \xi_s p_{2,s}(0)}{\xi_s - \eta_s}\eta_s^k.$$

This means that

$$\sum_{k=2}^{\infty} |p_{k+2,s}(0)| \leq \left|\frac{p_{3,s}(0) - \eta_s p_{2,s}(0)}{\xi_s - \eta_s}\right| \sum_{k=2}^{\infty} |\xi_s|^k + \left|\frac{p_{3,s}(0) - \xi_s p_{2,s}(0)}{\xi_s - \eta_s}\right| \sum_{k=2}^{\infty} |\eta_s|^k. \tag{3.34}$$

Moreover, take the $L^1[0, \infty)-$norm on both sides of Eq (3.24), and using the formula

$$\int_0^{\infty} x^{k-j} e^{-(\mathrm{Re}(\gamma)+\lambda_s+\mu_s)x} dx = \frac{(k-j)!}{(\mathrm{Re}(\gamma) + \lambda_s + \mu_s)^{k+1-j}},$$

for all $\mathrm{Re}(\gamma) + \mu_s > 0$, we have

$$\begin{aligned} \|p_{k,s}\|_{L^1[0,\infty)} &\leq \sum_{j=1}^{k} \frac{\lambda_s^{k-j}}{(k-j)!} |p_{j,s}(0)| \int_0^{\infty} x^{k-j} e^{-(\mathrm{Re}(\gamma)+\lambda_s+\mu_s)x} dx \\ &= \frac{1}{\mathrm{Re}(\gamma) + \lambda_s + \mu_s} \sum_{j=1}^{k} \left(\frac{\lambda_s}{\mathrm{Re}(\gamma) + \lambda_s + \mu_s}\right)^{k-j} |p_{j,s}(0)|. \end{aligned} \tag{3.35}$$

Therefore, for all $\mathrm{Re}(\gamma) + \mu_s > 0$, by the above inequalities and Cauchy product of series, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \|p_{k,s}\|_{L^1[0,\infty)} &\leq \frac{1}{\mathrm{Re}(\gamma) + \lambda_s + \mu_s} \sum_{k=1}^{\infty} \sum_{j=1}^{k} \left(\frac{\lambda_s}{\mathrm{Re}(\gamma) + \lambda_s + \mu_s}\right)^{k-j} |p_{j,s}(0)| \\ &= \frac{1}{\mathrm{Re}(\gamma) + \lambda_s + \mu_s} \sum_{k=1}^{\infty} |p_{k,s}(0)| \sum_{j=1}^{\infty} \left(\frac{\lambda_s}{\mathrm{Re}(\gamma) + \lambda_s + \mu_s}\right)^{j-1} \\ &= \frac{1}{\mathrm{Re}(\gamma) + \mu_s} \sum_{k=1}^{\infty} |p_{k,s}(0)|. \end{aligned} \tag{3.36}$$

In addition, from Eqs (3.23a), (3.25), (3.23d), and (3.26), it is easy to calculate that

$$p_{0,0} = \frac{1}{\gamma + \lambda_0} \sum_{s=1}^{n} \frac{\mu_s}{\gamma + \lambda_s + \mu_s} p_{1,s}(0), \tag{3.37a}$$

$$p_{1,s}(0) = q_s\lambda_0 p_{0,0} + \mu_s \left[\frac{\lambda_s}{(\gamma + \lambda_s + \mu_s)^2} p_{1,s}(0) + \frac{1}{\gamma + \lambda_s + \mu_s} p_{2,s}(0)\right], \tag{3.37b}$$

$$p_{2,s}(0) = \frac{(\gamma + \lambda_s + \mu_s)^2 - \lambda_s\mu_s}{\mu_s(\gamma + \lambda_s + \mu_s)} p_{1,s}(0) \quad - \frac{(\gamma + \lambda_s + \mu_s) q_s \lambda_0}{\mu_s} p_{0,0}, \tag{3.37c}$$

$$p_{3,s}(0) = \frac{(\gamma + \lambda_s + \mu_s)^2 - \lambda_s\mu_s}{\mu_s(\gamma + \lambda_s + \mu_s)} p_{2,s}(0) - \left(\frac{\lambda_s}{\gamma + \lambda_s + \mu_s}\right)^2 p_{1,s}(0). \tag{3.37d}$$

Finally, by Eqs (3.30a) and (3.30b), it is easy to see that $\gamma \in \Lambda$ if, and only if, $\mathrm{Re}(\gamma) + \mu_s > 0$ and $|\xi_s| < 1,\ |\eta_s| < 1,\ 1 \le s \le n.$ Therefore, if $\gamma \in \Lambda$, then from Eqs (3.36), (3.33), (3.34), (3.37a), and (3.37d), we obtain

$$\|(p_1, p_2, \cdots, p_n)\| = |p_{0,0}| + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \|p_{k,s}\|_{L^1[0,\infty)} < \infty. \tag{3.38}$$

The Eq (3.38) shows that for any γ in Λ is an eigenvalue of A_Φ. Moreover, Eqs (3.24), (3.26), (3.32), (3.37a), and (3.37d) mean that the geometric multiplicity of every $\gamma \in \Lambda$ is one.

Next, we observe the case $\gamma \in \mathbb{R}$. Since Theorem 2.3 implies that $(0, \infty) \subset \rho(A_\Phi)$, the real spectrum of A_Φ in the interval $(-\infty, 0]$ exists. We discuss the real spectrum of A_Φ in the following three cases.

Case 1: $(\gamma + \lambda_s + \mu_s)^2 > 4\lambda_s\mu_s$ if, and only if, $|\gamma + \lambda_s + \mu_s| > 2\sqrt{\lambda_s\mu_s}$. Since $\gamma + \mu_s > 0$ and $\gamma + \lambda_s + \mu_s > 2\sqrt{\lambda_s\mu_s}$, we have $\gamma > 2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s$. From this together with $\lambda_s < \mu_s$ and $\gamma + \mu_s > 0$, it is easy to calculate that

$$\begin{aligned}
&\gamma < 0 \Rightarrow 4\mu_s(\gamma + \lambda_s) - 4\lambda_s\mu_s < 0\\
&\Rightarrow (\gamma + \lambda_s)^2 + 2\mu_s(\gamma + \lambda_s) + \mu_s^2 - 4\lambda_s\mu_s < (\gamma + \lambda_s)^2 - 2\mu_s(\gamma + \lambda_s) + \mu_s^2\\
&\Rightarrow \sqrt{(\gamma + \lambda_s + \mu_s)^2 - 4\lambda_s\mu_s} < -(\gamma + \lambda_s - \mu_s)\\
&\Rightarrow \gamma + \lambda_s + \mu_s + \sqrt{(\gamma + \lambda_s + \mu_s)^2 - 4\lambda_s\mu_s} < 2\mu_s\\
&\Rightarrow 0 < \xi_s = \frac{\gamma + \lambda_s + \mu_s + \sqrt{(\gamma + \lambda_s + \mu_s)^2 - 4\lambda_s\mu_s}}{2\mu_s} < 1,\\
&0 < \eta_s = \frac{\gamma + \lambda_s + \mu_s - \sqrt{(\gamma + \lambda_s + \mu_s)^2 - 4\lambda_s\mu_s}}{2\mu_s} < \xi_s < 1.
\end{aligned} \tag{3.39}$$

This implies $\left(\max_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s\}, 0\right) \subset \sigma(A_\Phi)$. Then, from this together with Lemma 3.1, we obtain

$$\left(\max_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s\}, 0\right] \subset \sigma(A_\Phi).$$

Case 2: $(\gamma + \lambda_s + \mu_s)^2 = 4\lambda_s\mu_s$ if, and only if, $|\gamma + \lambda_s + \mu_s| = 2\sqrt{\lambda_s\mu_s}$. Since $\gamma + \mu_s > 0$ and $\gamma + \lambda_s + \mu_s = 2\sqrt{\lambda_s\mu_s}$, we deduce $\gamma = 2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s$. Then, using $\lambda_s < \mu_s$ and $\gamma + \mu_s > 0$, we have

$$0 < \xi_s = \eta_s = \frac{\gamma + \lambda_s + \mu_s}{2\mu_s} = \frac{2\sqrt{\lambda_s\mu_s}}{2\mu_s} = \sqrt{\frac{\lambda_s}{\mu_s}} < 1.$$

This shows that $\max\{2\sqrt{\lambda_1\mu_1} - \lambda_1 - \mu_1, \cdots, 2\sqrt{\lambda_n\mu_n} - \lambda_n - \mu_n\}$ is an eigenvalue of A_Φ.

Case 3: $(\gamma + \lambda_s + \mu_s)^2 < 4\lambda_s\mu_s$ if, and only if, $-2\sqrt{\lambda_s\mu_s} < \gamma + \lambda_s + \mu_s < 2\sqrt{\lambda_s\mu_s}$. Since $\gamma + \mu_s > 0$ and $0 < \gamma + \lambda_s + \mu_s < 2\sqrt{\lambda_s\mu_s}$, we obtain $\gamma < 2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s$. Then, from this together with $\lambda_s < \mu_s$, $\gamma + \mu_s > 0$, and $i^2 = -1$, we have

$$\xi_s, \eta_s = \frac{\gamma + \lambda_s + \mu_s \pm i\sqrt{4\lambda_s\mu_s - (\gamma + \lambda_s + \mu_s)^2}}{2\mu_s}.$$

Therefore,

$$|\xi_s| = |\eta_s| = \frac{\sqrt{(\gamma + \lambda_s + \mu_s)^2 + 4\lambda_s\mu_s - (\gamma + \lambda_s + \mu_s)^2}}{2\mu_s} = \sqrt{\frac{\lambda_s}{\mu_s}} < 1. \tag{3.40}$$

Hence, this implies that

$$\Big(-\min_{1\le s\le n}\{\mu_s\},\min_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s}-\lambda_s-\mu_s\}\Big)\subset\sigma(A_\Phi).$$

Consequently, by summing up the above three cases, we obtain

$$\Big(-\min_{1\le s\le n}\{\mu_s\},\min_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s}-\lambda_s-\mu_s\}\Big)\bigcup\Big[\max_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s}-\lambda_s-\mu_s\},0\Big]\subset\sigma(A_\Phi).$$

Let $\omega_0(A_\Phi)$, $\omega_{ess}(A_\Phi)$, $s(A_\Phi)$ represent the growth bound, the essential growth bound, and spectral bound of A_Φ, respectively. The spectral mapping theorem [11] means that

$$\sigma_p(e^{A_\Phi t})=e^{t\sigma_p(A_\Phi)}\cup\{0\},$$

Hence, from this property and Lemma 3.8, we obtain that $e^{A_\Phi t}$ has uncountable eigenvalues. Therefore, $e^{A_\Phi t}$ is not compact and it is not eventually compact by Corollary V.3.2 of [11].

Additionally, due to $e^{A_\Phi t}$ being a C_0-semigroup on X with generator A_Φ, using Corollary IV.2.11 of [11], we know that $\omega_0=\max\{\omega_{ess},s(A_\Phi)\}$ and $\sigma(A_\Phi)\cap\{\gamma\in\mathbb{C}\mid\mathrm{Re}(\gamma)\ge w\}$ is finite for every $w>\omega_{ess}$. Using Lemma 3.8, we can obtain that the spectrum determined condition $\omega_0=s(A_\Phi)$ holds and $\omega_0=s(A_\Phi)=0$ (we suggest that readers refer to the proof of Theorem 4.1 in [25] for similar proofs in this part). Hence, using the aforementioned discussions, we have $\omega_{ess}=0$. Then, by Proposition 3.5 of [11], we derive that $e^{A_\Phi t}$ is not quasi-compact.

The main result of this subsection is given by the following Theorem 3.2.

Theorem 3.2. *Let $\mu_s(\cdot):=\mu_s$ be a constant and $\lambda_s<\mu_s$, $1\le s\le n$. Then, the time-evolving solution of the system (2.2) cannot exponentially converge to its static solution. That is to say, there are no constants $\mathcal{M}>0$ and $\varepsilon>0$ such that*

$$\begin{aligned}&\left\|e^{A_\Phi t}\left((p_1,p_2,\cdots,p_n)_{(0)}+A_\Phi(p_1,p_2,\cdots,p_n)\right)-(p_1,p_2,\cdots,p_n)_{(0)}\right\|\\&\le\mathcal{M}e^{-\varepsilon t}\|(p_1,p_2,\cdots,p_n)\|,\end{aligned}$$

for any $t\ge 0$ and $(p_1,p_2,\cdots,p_n)\in D(A_\Phi)$, where $(p_1,p_2,\cdots,p_n)_{(0)}$ is the eigenvector associated to zero.

Proof. Assume that $(p_1,p_2,\cdots,p_n)_{(0)}$ and $(p_1,p_2,\cdots,p_n)_{(r)}$ are the eigenvectors of 0 and

$$r\max_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s}-\lambda_s-\mu_s\}:=r\beta_s,$$

in Lemma 3.8, respectively, for any $r\in(0,1)$. Hence, using $A_\Phi(p_1,p_2,\cdots,p_n)_{(0)}=0$ and $A_\Phi(p_1,p_2,\cdots,p_n)_{(r)}=r\beta_s(p_1,p_2,\cdots,p_n)_{(r)}$, we have

$$\begin{aligned}&e^{A_\Phi t}\left[(p_1,p_2,\cdots,p_n)_{(0)}+A_\Phi(p_1,p_2,\cdots,p_n)_{(r)}\right]\\&=e^{A_\Phi t}(p_1,p_2,\cdots,p_n)_{(0)}+e^{A_\Phi t}A_\Phi(p_1,p_2,\cdots,p_n)_{(r)}\\&=(p_1,p_2,\cdots,p_n)_{(0)}+e^{A_\Phi t}r\beta_s(p_1,p_2,\cdots,p_n)_{(\delta)}\\&=(p_1,p_2,\cdots,p_n)_{(0)}+r\beta_s e^{r\beta_s t}(p_1,p_2,\cdots,p_n)_{(r)}.\end{aligned}\tag{3.41}$$

Therefore,

$$\begin{aligned}&\left\|e^{A_\Phi t}\left((p_1,p_2,\cdots,p_n)_{(0)}+A_\Phi(p_1,p_2,\cdots,p_n)_{(r)}\right)-(p_1,p_2,\cdots,p_n)_{(0)}\right\|\\&=r|\beta_s|e^{r\beta_s t}\left\|(p_1,p_2,\cdots,p_n)_{(r)}\right\|,\quad \forall t\geq 0,\ \forall r\in(0,1).\end{aligned}\tag{3.42}$$

That is, there are no constants $\mathcal{M}>0$ and $\varepsilon>0$ such that

$$\begin{aligned}&\left\|e^{A_\Phi t}\left((p_1,p_2,\cdots,p_n)_{(0)}+A_\Phi(p_1,p_2,\cdots,p_n)\right)-(p_1,p_2,\cdots,p_n)_{(0)}\right\|\\&\leq \mathcal{M}e^{-\varepsilon t}\|(p_1,p_2,\cdots,p_n)\|.\end{aligned}$$

for any $t\geq 0$ and $(p_1,p_2,\cdots,p_n)\in D(A_\Phi)$.

In neural network [17] and reliability model [12], it has been proven that the semigroup corresponding to these systems is a quasi-compact strongly continuous semigroup, thus they obtain the dynamic solution of the corresponding system that strongly converges to its steady-state solution. Therefore, the result of Theorem 3.2 is significantly different from those in [12, 17].

4. Asymptotic behavior of the time-evolving queue length

Define the time-evolving queue length of system (1.6) by

$$L(t)=p_{0,0}(t)+\sum_{s=1}^{n}\sum_{k=1}^{\infty}\int_0^\infty p_{k,s}(x,t)dx.$$

Then, by combining Theorems 2.1, 2.3 and 3.1 and Lemma 3.1, we can obtain the asymptotic behavior of $L(t)$.

Theorem 4.1. *Let $\mu_s(x):[0,\infty)\to[0,\infty)$ be a measurable function that satisfies*

$$0<\inf_{x\in[0,\infty)}\mu_s(x)\leq\mu_s(x)\leq\sup_{x\in[0,\infty)}\mu_s(x)<\infty,\ 1\leq s\leq n.$$

If the initial value $(g_1(\cdot),g_2(\cdot),\cdots,g_n(\cdot))$ of the system (1.6) and the eigenvector $(\tilde{p}_1(\cdot),\tilde{p}_2(\cdot),\cdots,\tilde{p}_n(\cdot))$ corresponding to zero satisfying $\tilde{p}_s(\cdot)\geq u_s(\cdot)$, then time-evolving queue length $L(\cdot)$ of system (1.6) converges to its static queue length. That is to say,

$$\lim_{t\to\infty}L(t)=\tilde{p}_{0,0}+\sum_{s=1}^{n}\sum_{k=1}^{\infty}\int_0^\infty\tilde{p}_{k,s}(x)dx.$$

Proof. For any $(p_1,p_2,\cdots,p_n)$ and $(y_1,y_2,\cdots,y_n)$ in X, we introduce an order relation "$\geq$" by

$$\begin{aligned}&(p_1,p_2,\cdots,p_n)\geq(y_1,y_2,\cdots,y_n)\\&\Longleftrightarrow p_s\geq y_s,\ 1\leq s\leq n\\&\Longleftrightarrow p_{0,0}\geq y_{0,0}\text{ and }p_{k,s}(x)\geq y_{k,s}(x),\ x\in[0,\infty),\quad k\geq 1,\ 1\leq s\leq n.\end{aligned}$$

Then, it is not difficult to show that "$\geq$" is a partial order relation in X. Therefore, $(X,\geq)$ is a poset. Let $(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)$ be the positive eigenvector associated to zero of A_Φ (Lemma 3.1). Let

$(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n)$ and the initial value $(g_1, g_2, \cdots, g_n)$ of the system (2.2) satisfy the aforementioned partial order relation

$$\tilde{p}_{0,0} \geq g_{0,0}, \; \tilde{p}_{k,s}(x) \geq g_{k,s}(x), \; x \in [0, \infty), \quad k \geq 1, \; 1 \leq s \leq n.$$

That is, $(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \geq (g_1, g_2, \cdots, g_n)$. Due to $e^{A_\Phi t}$ being a positive linear operator (Theorem 2.1), it is a monotone increasing operator. In addition, by Theorem 2.3 and Lemma 3.1, we know that

$$\begin{cases} (p_1(\cdot, t), p_2(\cdot, t), \cdots, p_n(\cdot, t)) = e^{A_\Phi t}(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot)), \; t \geq 0, \\ e^{A_\Phi t}(\tilde{p}_1(\cdot), \tilde{p}_2(\cdot), \cdots, \tilde{p}_n(\cdot)) = (\tilde{p}_1(\cdot), \tilde{p}_2(\cdot), \cdots, \tilde{p}_n(\cdot)), \; t \geq 0. \end{cases}$$

Therefore, from this together with the partial order relation $(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \geq (g_1, g_2, \cdots, g_n)$, we have

$$\begin{aligned} & e^{A_\Phi t}(\tilde{p}_1(\cdot), \tilde{p}_2(\cdot), \cdots, \tilde{p}_n(\cdot)) \geq e^{A_\Phi t}(g_1(\cdot), g_2(\cdot), \cdots, g_n(\cdot)) \\ & \Longrightarrow (\tilde{p}_1(\cdot), \tilde{p}_2(\cdot), \cdots, \tilde{p}_n(\cdot)) \geq (p_1(\cdot, t), p_2(\cdot, t), \cdots, p_n(\cdot, t)) \\ & \Longrightarrow \tilde{p}_{0,0} \geq p_{0,0}(t), \quad \tilde{p}_{k,s}(\cdot) \geq p_{k,s}(\cdot, t), \quad k \geq 1, \\ & \Longrightarrow \infty > \tilde{p}_{0,0} + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty \tilde{p}_{k,s}(x) dx \geq p_{0,0}(t) + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty p_{k,s}(x, t) dx, \; t \geq 0. \end{aligned}$$

Theorem 3.1 includes the following result:

$$\lim_{t \to \infty} \left[|p_{0,0}(t) - \tilde{p}_{0,0}| + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty |p_{k,s}(x, t) - \tilde{p}_{k,s}(x)| dx \right] = 0.$$

Hence, by Lemma 3.1 and the Lebesgue theorem, we obtain

$$\begin{aligned} & \lim_{t \to \infty} \left| L(t) - \left[\tilde{p}_{0,0} + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty \tilde{p}_{k,s}(x) dx \right] \right| \\ & \leq \lim_{t \to \infty} \left[|p_{0,0}(t) - \tilde{p}_{0,0}| + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty |p_{k,s}(x, t) - \tilde{p}_{k,s}(x)| dx \right] = 0. \end{aligned}$$

This inequality shows that

$$\lim_{t \to \infty} L(t) = \tilde{p}_{0,0} + \sum_{s=1}^{n} \sum_{k=1}^{\infty} \int_0^\infty \tilde{p}_{k,s}(x) dx.$$

Remark 4.1. In Theorem 4.1, the static queue length is obtained by using the eigenvector that related to zero (Lemma 3.1). This is the same as the result obtained by introducing probability generating functions in [3].

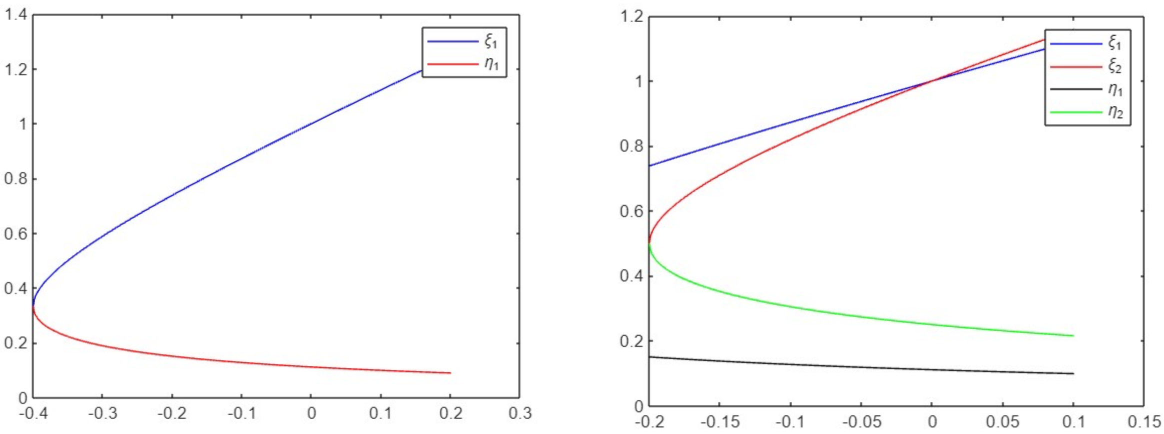
Similarly, we can obtain the other time-evolving indicators of system (1.6) such as the convergence of the time-evolving average number of customers to its static average number of customers.

5. Numerical analysis

To prove the correctness of the exponential convergence results in this article, we perform numerical analysis on the spectral results in Lemma 3.8. The numerical analysis results are shown in Figures 1 and 2. We obtain these numerical results using Matlab.

In Figure 1(a), we consider that system (1.6) only has the idle period and one operational phase, that is, $n = 1$ in system (1.6). In other words, we consider the classical queuing model [5], and the point spectrum results of the system operator A_Φ of this queueing model [5] are obtained in detail in [8–10]. If we take $\lambda_1 = 0.1,\ \mu_1 = 0.9$, then it is easy to see that $\frac{\lambda_1}{\mu_1} = \frac{0.1}{0.9} < 1$ and $2\sqrt{\lambda_1\mu_1} - \lambda_1 - \mu_1 = -0.4$. In addition, Figure 1(a) means that $0 < \xi_1 < 1$ and $0 < \eta_1 < 1$ for all $\gamma \in (-0.4, 0)$. Hence, by Eqs (3.34) and (3.36), we see that every $\gamma \in (-0.4, 0)$ is the point spectrum of A_Φ.

In Figure 1(b), we consider that system (1.6) only has the idle period and two operational phases, that is, $n = 2$ in system (1.6). In this case, we take $\lambda_1 = 0.1,\ \mu_1 = 0.9$ and $\lambda_2 = 0.2,\ \mu_2 = 0.8$. Then, these values satisfy $\frac{\lambda_1}{\mu_1} = \frac{1}{9},\ \frac{\lambda_2}{\mu_2} = \frac{1}{4}$, and $\max_{s=1,2}\{2\sqrt{\lambda_s\mu_s} - \lambda_s - \mu_s\} = -0.2$. Moreover, when $\gamma \in (-0.2, 0)$, from Figure 1(b) we see that ξ_1, ξ_2, η_1, and η_2 satisfy $0 < \xi_n < 1$ and $0 < \eta_n < 1$. Therefore, by Eqs (3.34) and (3.36), we know that all $\gamma \in (-0.2, 0)$ are the point spectrum of A_Φ. Figure 1 means that the point spectrum results in Lemma 3.8 are correct if $\lambda_s < \mu_s$, $s = 1, 2$. Therefore, the exponential convergence result of Theorem 3.2 is valid.



(a) For $n = 1, \lambda_1 = 0.1, \mu_1 = 0.9$.

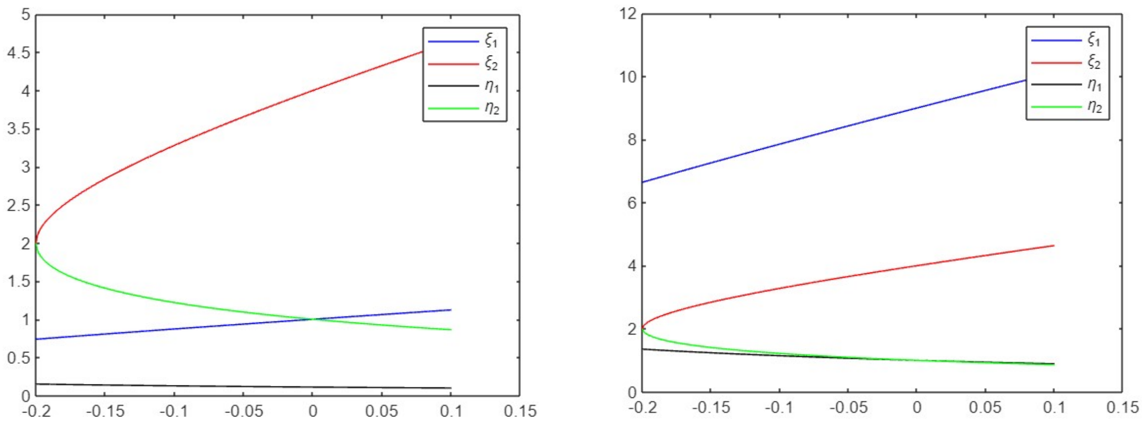
(b) For $n = 2, \lambda_1 = 0.1, \mu_1 = 0.9, \lambda_2 = 0.2, \mu_2 = 0.8$.

Figure 1. For Lemma 3.8.

In the following, we check whether the condition $\lambda_s < \mu_s,\ 1 \le s \le n$ in Lemma 3.8 is necessary. In Figure 2, we consider that system (1.6) only has the idle period and two operational phases, that is, $n = 2$ in system (1.6). If we take $\lambda_1 = 0.1,\ \mu_1 = 0.9$ and $\lambda_2 = 0.8,\ \mu_2 = 0.2$, then we have $\lambda_1 < \mu_1$ and $\lambda_2 > \mu_2$. In this case, Figure 2(a) means that $0 < \xi_1 < 1,\ 0 < \eta_1 < 1$, and $\xi_2 > 1,\ \eta_2 > 1$ for all $\gamma \in (-0.2, 0)$.

If we take $\lambda_1 = 0.9,\ \mu_1 = 0.1$ and $\lambda_2 = 0.8,\ \mu_2 = 0.2$, then we have $\lambda_n > \mu_n$. In this case, Figure 2(b) implies that $\xi_n > 1$ and $\eta_n > 1$ for all $\gamma \in (-0.2, 0)$. Of course, we can choose some different λ_n and μ_n, at least one of which satisfies $\lambda_{s_0} > \mu_{s_0}$ for some $s_0 = 1, 2, \cdots, n$, to obtain the same conclusion. As a result, Figure 2 means that we cannot obtain whether it is $\gamma \in \sigma_p(A_\Phi)$, or even whether it is $\gamma \in \sigma(A_\Phi)$

under the above circumstances, where $\gamma \in (\max_{1\le s\le n}\{2\sqrt{\lambda_s\mu_s}-\lambda_s-\mu_s\},0)$. Therefore, in Lemma 3.8 (or in Theorem 3.2), we must consider the condition $\lambda_s<\mu_s$. This condition is also the stability condition obtained for system (1.6) in reference [3].



(a) For $n=2, \lambda_1=0.1, \mu_1=0.9, \lambda_2=0.8, \mu_2=0.2$.

(b) For $n=2, \lambda_1=0.9, \mu_1=0.1, \lambda_2=0.8, \mu_2=0.2$.

Figure 2. For Lemma 3.8.

6. Conclusions

In this article, we conduct a dynamic analysis of the M/G/1 queueing system with multiple phases of operation. Using operator semigroup theory, we prove that there exists a unique time-evolving solution for this system. We obtain the spectral distribution of the system operator on the imaginary axis and prove that the system operator has an infinite number of eigenvalues on the left-half of the complex plane. As a result, the above solution converges at most strongly to its static solution. We also discuss the compactness of the system's corresponding semigroup by using these spectral results. However, we have not obtained the complete spectrum of the system operator on the left-half of the complex plane. This is the work we will continue to do in the future. Additionally, we obtain that the dynamic queue length of the model strongly converges to its static queue length.

The method described in this article can only be applied to queuing systems established using the supplementary variable method and described by partial differential equations. For example, we cannot use the method proposed in this paper for the queuing systems in [1, 26, 27].

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of Interest

The authors declare there is no conflict of interest.

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