Networks and Heterogeneous Media

http://www.aimspress.com/journal/nhm

NHM, 18(4): 1493–1527.
DOI: 10.3934/nhm.2023066
Received: 19 April 2023
Revised: 12 June 2023
Accepted: 16 June 2023
Published: 12 July 2023

Research article

Non-emergence of mono-cluster flocking and multi-cluster flocking of the thermodynamic Cucker–Smale model with a unit-speed constraint

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Abstract: This paper demonstrates several sufficient frameworks for the mono-cluster flocking, the non-emergence of mono-cluster flocking and the multi-cluster flocking of the thermodynamic Cucker–Smale model with a unit-speed constraint (say TCSUS). First, in a different way than [2], we present the admissible data for the mono-cluster flocking of TCSUS to occur. Second, we prove that when the coupling strength is less than some positive value, mono-cluster flocking does not occur in the TCSUS system with an integrable communication weight. Third, motivated from the study on coupling strengths where the mono-cluster flocking does not occur, we investigate appropriate sufficient frameworks to derive the multi-cluster flocking of the TCSUS system.

Keywords: Cucker–Smale; mono-cluster flocking; multi-agent system; multi-cluster flocking; thermodynamic; unit-speed

1. Introduction

Emergent dynamics in interacting multi-agent systems are frequently observed in nature. Examples include the aggregation of bacteria [39], flocking of birds and vehicular flocking [7, 14, 19, 34], schooling of fish [20, 38] and the synchronization of fireflies and pacemaker cells [1, 8, 21, 37, 43]. To more introduce related literature, we refer to [22, 35, 41, 42]. Herein, we are primarily concerned with "*flocking*" in which agents exhibit ordered movements and form appropriate groups. After the work of Vicsek et al. in [40], many studies on models representing flocking have been actively conducted for decades. Among them, the Cucker–Smale model [19] has received significant attention in math and physics communities due to its dissipative and simple velocity structure. Essentially, the Cucker–Smale model is a flocking dynamic system for position and velocity based on the Newtonian sense, which is governed by

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in \{1, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \psi(\|x_i - x_j\|) \left(v_j - v_i\right), \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where N denotes the number of particles, κ is a nonnegative coupling strength and ψ is a communication weight. To date, there have been many works examining this system and its variants due to its dissipative structure for velocity, such as the mean-field limit [5, 6, 25, 28, 30], kinetic models [9, 32], hydrodynamic descriptions [23, 24, 33], particle analysis [9, 10, 13–18], temperature field [26, 31] and relativistic setting [4–6, 27].

Since Eq (1.1), the authors of [12] noted that several Vicsek-type models with unit-speed constraints have been actively studied concerning heading angles in math community. To give a unit-speed constraint to Eq (1.1), the authors modified the velocity coupling term Eq $(1.1)_2$ so that the velocity of each agent has a unit-speed constraint as follows:

$$\psi(\|x_i - x_j\|) \left(v_j - v_i\right) \longrightarrow \psi(\|x_i - x_j\|) \left(v_j - \frac{\langle v_j, v_i \rangle v_i}{\|v_i\|^2}\right),$$

where the modified term is perpendicular to v_i. Thus, they proposed the following Cucker–Smale type model with constant speed and studied its flocking dynamics:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in \{1, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \psi(\|x_i - x_j\|) \left(v_j - \dfrac{\langle v_j, v_i \rangle v_i}{\|v_i\|^2}\right), \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{1.2}$$

Equation (1.2) has also been studied from several perspectives; for example, particle analysis [12], the emergence of the bi-cluster flocking in [17], multi-cluster flocking and critical coupling strength in [29], time-delay effect [11] and general digraph setting [36].

However, because the above literature [11, 12, 17, 29, 36] were only motivated by the original Cucker–Smale model (1.1) without considering internal energy, the author of [2] noted the extension of the above model to a temperature field to describe more realistic flocking dynamics. For this, as a backbone model, the author first adopted a thermodynamic Cucker–Smale model proposed by [26, 31] based on the theory of multi-temperature mixture of fluids under the space of homogeneity, which is given by the following second-order ODEs for *position-velocity-temperature* (x_i, v_i, T_i):

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N] := \{1, \cdots, N\}, \tag{1.3a}$$

$$\frac{dv_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^{N} \phi(\|x_i - x_j\|) \left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right), \tag{1.3b}$$

$$\frac{d}{dt}\left(T_i + \frac{1}{2}\|v_i\|^2\right) = \frac{\kappa_2}{N}\sum_{j=1}^{N}\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right), \tag{1.3c}$$

$$(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ - \{0\}, \tag{1.3d}$$

where $\sum_{i=1}^{N} T_i^0 =: NT^\infty$, N denotes the number of particles, κ_1, κ_2 are nonnegative coupling strengths and ψ, ζ are communication weights. Then, motivated from the derivation idea of Eq (1.2), by modifying the velocity coupling term Eq (1.3a) as

$$\phi(\|x_i - x_j\|)\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right) \longrightarrow \phi(\|x_i - x_j\|)\left(\frac{v_j}{T_j} - \frac{\langle v_j, v_i\rangle v_i}{T_j\|v_i\|^2}\right),$$

the author suggested the following TCSUS model in terms of *position-velocity-temperature* (x_i, v_i, T_i):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in \{1, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N}\displaystyle\sum_{j=1}^{N}\phi(\|x_i - x_j\|)\left(\frac{v_j}{T_j} - \frac{\langle v_j, v_i\rangle v_i}{T_j\|v_i\|^2}\right), \\ \dfrac{d}{dt}\left(T_i + \dfrac{1}{2}\|v_i\|^2\right) = \dfrac{\kappa_2}{N}\displaystyle\sum_{j=1}^{N}\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right), \\ (x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (\mathbb{R}_+ - \{0\}), \end{cases} \tag{1.4}$$

where $\sum_{i=1}^{N} T_i^0 =: NT^\infty$. Afterward, the author immediately verified that each agent in the system (1.4) has a unit-speed. Then, from the relations,

$$\|v_i\| = 1, \quad \frac{\langle v_j, v_i\rangle v_i}{T_j\|v_i\|^2} = \frac{\langle v_j, v_i\rangle v_i}{T_j} \quad \text{and} \quad \frac{d}{dt}\left(T_i + \frac{1}{2}\|v_i\|^2\right) = \frac{dT_i}{dt},$$

the author simply represented the system (1.4) as follows:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in \{1, \cdots, N\}, \tag{1.5a}$$

$$\frac{dv_i}{dt} = \frac{\kappa_1}{N}\sum_{j=1}^{N}\phi(\|x_i - x_j\|)\left(\frac{v_j - \langle v_j, v_i\rangle v_i}{T_j}\right), \tag{1.5b}$$

$$\frac{dT_i}{dt} = \frac{\kappa_2}{N}\sum_{j=1}^{N}\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right), \tag{1.5c}$$

$$(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (\mathbb{R}_+ - \{0\}), \tag{1.5d}$$

where $\sum_{i=1}^{N} T_i^0 =: NT^\infty$. Here, we set $\mathbb{R}_+ := [0, \infty)$ throughout the paper and we assume that two communication weights ϕ, $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ are nonnegative, locally Lipschitz continuous and monotonically decreasing and that $\mathbb{S}^{d-1}$ is the unit $(d-1)$-sphere isometrically embedded in $\mathbb{R}^d$; hence,

$$0 \le \phi(r) \le \phi(0) = 1, \quad (\phi(r_1) - \phi(r_2))(r_1 - r_2) \le 0, \ \forall r, r_1, r_2 \ge 0, \quad \phi(\cdot) \in C^{0,1}_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+),$$

$$0 \le \zeta(r) \le \zeta(0) = 1, \quad (\zeta(r_1) - \zeta(r_2))(r_1 - r_2) \le 0, \ \forall r, r_1, r_2 \ge 0, \quad \zeta(\cdot) \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}_+),$$

$$\mathbb{S}^{d-1} := \left\{ x := (x^1, \cdots, x^d) \middle| \sum_{i=1}^{d} |x^i|^2 = 1, \right\} \text{ where } x^i \text{ is the } i\text{-th component of } x \in \mathbb{R}^d.$$

The system (1.5) was studied in terms of mono-cluster flocking and bi-cluster flocking in [2] and collision avoidance [3], but the multi-cluster flocking of system (1.5) has not been studied yet. Indeed, the multi-cluster flocking phenomenon is ubiquitous in daily life. Examples include opinion disagreement, schools of fish invaded by predators and flight multi-formation. In addition, a phenomenon in which individuals with the same characteristics gather together can be an example of the multi-cluster flocking.

Therefore, this paper is mainly interested in the non-emergence of mono-cluster flocking in the system (1.5) under a sufficiently small coupling strength and extending the bi-cluster flocking of [2] to general multi-cluster flocking. For this, we first introduce several basic notions concerning mono- and multi-cluster flocking as follows:

Definition 1.1. *Let $Z = \{(x_i, v_i, T_i)\}_{i=1}^N$ be a solution to the system* (1.5).

(1) The configuration Z exhibits mono-cluster flocking if the following statements hold:

$$(i)\ (\textit{Group formation}) \iff \sup_{t\in\mathbb{R}_+} \max_{1\le i,j\le N} \|x_i(t) - x_j(t)\| < \infty,$$

$$(ii)\ (\textit{Velocity alignment}) \iff \lim_{t\to\infty} \max_{1\le i,j\le N} \|v_j(t) - v_i(t)\| = 0,$$

$$(iii)\ (\textit{Temperature equilibrium}) \iff \lim_{t\to\infty} \max_{1\le i,j\le N} |T_j(t) - T_i(t)| = 0.$$

(2) The configuration Z exhibits multi-cluster flocking if there exist n cluster groups $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ such that the following assertions hold for $1 \le n \le N$:

$$(i)\ |Z_\alpha| = N_\alpha \ge 1, \quad \sum_{\alpha=1}^{n} |Z_\alpha| = \sum_{\alpha=1}^{n} N_\alpha = N,$$

$$(ii)\ \sup_{t\in\mathbb{R}_+} \max_{1\le k,l\le N_\alpha} \|x_{\alpha k}(t) - x_{\alpha l}(t)\| < \infty, \quad \lim_{t\to\infty} \max_{1\le k,l\le N_\alpha} \|v_{\alpha k}(t) - v_{\alpha l}(t)\| = 0,$$

$$\lim_{t\to\infty} \max_{1\le k,l\le N_\alpha} |T_{\alpha k}(t) - T_{\alpha l}(t)| = 0, \quad n \ge 3, \quad 1 \le \alpha \le n,$$

$$(iii)\ \inf_{t\in\mathbb{R}_+} \min_{k,l} \|x_{\alpha k} - x_{\beta l}\| = \infty, \quad 1 \le k \le N_\alpha,\ 1 \le l \le N_\beta, \quad 1 \le \alpha \ne \beta \le n.$$

Then, we are primarily concerned with the following issue:

- (Main issue): How can we find sufficient conditions for the non-emergence of mono-cluster flocking in the system (1.5)? Additionally, under what sufficient conditions with respect to the initial data and system parameters can mono-cluster flocking emerge in system (1.5)?

The paper is organized as follows. Section 2 introduces several basic estimates for temperatures in system (1.5) and previous results studied in [2]. Section 3 gives a mono-cluster flocking estimate different from the previous paper [3] and proves the non-emergence of mono-cluster flocking under suitable sufficient conditions when ϕ is integrable in system (1.5). Next, we describe several sufficient frameworks for the mono-cluster flocking of system (1.5) when the communication weight ϕ is non-integrable. Section 4 reorganizes system (1.5) to the multi-cluster setting and derives some dissipative structures on each cluster group to demonstrate the multi-cluster flocking of system (1.5) under admissible data. Finally, Section 5 briefly summarizes the main results and discusses the remaining issues left for future work.

Notation. Throughout the paper, we denote the following notation for brevity:

$$\|\cdot\| = \text{standard } l_2\text{-norm}, \ \langle\cdot,\cdot\rangle = \text{standard inner product}, \ y^i = \text{i-th component of } y \in \mathbb{R}^d,$$
$$X := (x_1, \cdots, x_N), \quad V := (v_1, \cdots, v_N), \quad T := (T_1, \cdots, T_N), \quad \mathbb{R}_+ := [0, \infty),$$
$$D_Z(t) := \max_{1\le i,j\le N} \|z_i(t) - z_j(t)\| \quad \text{for} \quad Z = (z_1, \cdots, z_N) \in \{X, V, T\}.$$

2. Preliminaries

This section reviews several basic results for the subsystem (1.5c) to guarantee its global well-posedness; these estimates will be crucial throughout this paper. Afterward, we introduce the previous bi-cluster flocking results of system (1.5) studied in [2].

2.1. Basic estimates

This subsection deals with the entropy principle, the propagation of conserved quantity, and the uniform boundedness of temperature to the subsystem (1.5c). For this, we begin with defining the entropy of system (1.5).

Definition 2.1. [26,31] *Let $\{(x_i, v_i, T_i)\}_{i=1}^N$ be a solution to the system* (1.5)*. Then, the entropy is defined as*

$$\mathcal{S}(t) := \sum_{i=1}^N \ln(T_i(t)) = \ln\left(\prod_{i=1}^N T_i(t)\right).$$

Then, we present the entropy principle and conserved temperature sum as below:

Proposition 2.1. [26, 31] *Assume that $\{(x_i, v_i, T_i)\}_{i=1}^N$ is a solution to the system* (1.5)*. Then, one has the following two assertions:*

1. *(Conserved temperature sum) The total sum $\sum_{i=1}^N T_i$ is conserved for $t \ge 0$.*

$$\sum_{i=1}^N T_i(t) = \sum_{i=1}^N T_i^0 = NT^\infty.$$

2. *(Entropy principle) Entropy $\mathcal{S}$ monotonically increases for $t \ge 0$:*

$$\frac{d\mathcal{S}}{dt} = \frac{\kappa_2}{2N}\sum_{i,j=1}^N \zeta(\|x_j - x_i\|)\left|\frac{1}{T_i} - \frac{1}{T_j}\right|^2 \ge 0.$$

Subsequently, we offer the following uniform boundedness consisting of strictly positive lower and upper bounds for temperatures to the system (1.5):

Proposition 2.2. [26] (Uniform boundedness for temperatures) *Let* $Z = \{(x_i, v_i, T_i)\}_{i=1}^N$ *be a solution to system* (1.5)*. Then,* $\min_{1\le i\le N} T_i(t)$ *monotonically increases and* $\max_{1\le i\le N} T_i(t)$ *monotonically decreases in time. In other words, for* $t \ge 0$*,*

$$0 < \min_{1\le i\le N} T_i^0 =: T_m^\infty \le T_i(t) \le \max_{1\le i\le N} T_i^0 =: T_M^\infty, \quad i = 1, \cdots, N.$$

Since Proposition 2.2 holds, ϕ, ζ are uniformly bounded, and the speed of each agent is unit. We directly obtain the well-posedness of system (1.5) from the standard Cauchy–Lipschitz theory.

2.2. Previous results

This subsection introduces the previous mono-cluster flocking and bi-cluster flocking estimated in [2]. First, we revisit the following mono-cluster flocking of the system (1.5) verified in [3]:

Proposition 2.3. [2] (Mono-cluster flocking) *Suppose that* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *is a global-in-time solution to the system* (1.5) *with the initial data* $\{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N$ *and assume that there exists a positive constant* $D_X^\infty > 0$ *that satisfies*

$$D_V^2(0) < \frac{T_m^\infty \phi(D_X^\infty)}{2T_M^\infty} \quad \textit{and} \quad D_X(0) + \frac{2T_M^\infty D_V(0)}{\kappa_1 \phi(D_X^\infty)} < D_X^\infty. \tag{2.1}$$

Then, we get that for $t \in \mathbb{R}_+$*,*

$$D_V^2(t) < 2D_V^2(0) \quad \textit{and} \quad D_X(t) < D_X^\infty,$$

which yields the following mono-cluster flocking estimate of system (1.5) *for* $t \in \mathbb{R}_+$*:*

$$D_V(t) \le D_V(0) \exp\left(-\frac{\kappa_1 \phi(D_X^\infty)}{2T_M^\infty}\right), \quad D_T(t) \le D_T(0) \exp\left(-\frac{\kappa_2 \zeta(D_X^\infty)}{(T_M^\infty)^2} t\right).$$

However, in Theorem 3.1, we can attain another mono-cluster flocking dynamics of system (1.5) by reducing the higher-order dissipative differential inequality in terms of velocity in Proposition 3.1 to a suitable lower-order inequality.

Subsequently, to describe the results of extending the mono-cluster flocking of Proposition 2.3 to bi-cluster flocking, we describe the admissible set $(\mathcal{H})$ proposed in [2]; for two cluster groups $Z_1 = \{(x_{1i}, v_{1i}, T_{1i})\}_{i=1}^{N_1}$ and $Z_2 = \{(x_{2j}, v_{2j}, T_{2j})\}_{j=1}^{N_2}$, we set the following three configuration vectors:

$$A_\alpha := (a_{\alpha 1}, \cdots a_{\alpha N_\alpha}) \quad \alpha = 1, 2, \text{ where } A \in \{X, V, T\}, \ a \in \{x, v, T\} \text{ and } A := (A_1, A_2).$$

Next, for $\alpha \in \{1, 2\}$, we denote L^∞ diameters regarding *position-velocity-temperature* for each cluster group

$$D_{X_\alpha} := \max_{1\le i,j\le N_\alpha} \|x_{\alpha i} - x_{\alpha j}\|, \quad D_{V_\alpha} := \max_{1\le i,j\le N_\alpha} \|v_{\alpha i} - v_{\alpha j}\|, \quad D_{T_\alpha} := \max_{1\le i,j\le N_\alpha} |T_{\alpha i} - T_{\alpha j}|$$

and we let

$$\mathcal{D}_X := D_{X_1} + D_{X_2}, \quad \mathcal{D}_V := D_{V_1} + D_{V_2}, \quad \mathcal{D}_T := D_{T_1} + D_{T_2}.$$

Then, the admissible set $(\mathcal{H})$ in terms of a system parameter and initial data is given by

$$(\mathcal{H}) =: \{(X(0), V(0), T(0)) \in \mathbb{R}^{2dN} \times (\mathbb{R}_+ - \{0\})^N \mid (\mathcal{H}_0), (\mathcal{H}_1), (\mathcal{H}_2) \text{ and } (\mathcal{H}_3) \text{ hold.}\}$$

• $(\mathcal{H}_0)$ (Basic notation): For simplicity, we set

$$\begin{aligned}
\Lambda_0 :=& \frac{2NT_M^\infty \mathcal{D}_V(0)}{\kappa_1 \min(N_1, N_2)\phi(\mathcal{D}_X^\infty)} + \frac{16N^2(T_M^\infty)^2\phi\left(\frac{r_0}{2}\right)}{\kappa_1(\min(N_1,N_2))^2(\phi(\mathcal{D}_X^\infty))^2 T_m^\infty} \\
&+ \frac{8NT_M^\infty \int_0^\infty \phi\left(s + \frac{r_0}{2}\right) ds}{\min(N_1,N_2)\phi(\mathcal{D}_X^\infty)T_m^\infty}, \\
r_0 :=& \min_{1\le i\le N_1, 1\le j\le N_2} (x_{1i}^k(0) - x_{2j}^k(0)), \quad \Lambda_1 := \frac{\kappa_1 \min(N_1,N_2)\phi(\mathcal{D}_X^\infty)}{2NT_M^\infty}, \\
\Lambda_2 :=& \frac{\kappa_1 N_1}{NT_m^\infty}\Lambda_1 + \frac{\kappa_1 N_2}{NT_m^\infty}\int_0^\infty \phi\left(s + \frac{r_0}{2}\right) ds, \\
\Lambda_3 :=& \frac{\kappa_1 N_2}{NT_m^\infty}\Lambda_1 + \frac{\kappa_1 N_1}{NT_m^\infty}\int_0^\infty \phi\left(s + \frac{r_0}{2}\right) ds, \\
\Lambda_4 :=& \frac{\min(N_1,N_2)\kappa_2\zeta(\mathcal{D}_X^\infty)}{N(T_M^\infty)^2}, \quad \Lambda_5 := 2\kappa_2\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right).
\end{aligned}$$

• $(\mathcal{H}_1)$ (Well prepared conditions): There exists a strictly positive number $\mathcal{D}_X^\infty > 0$ such that

$$\mathcal{D}_X^\infty > \mathcal{D}_X(0) + \Lambda_0 \quad \text{and} \quad \phi \text{ is integrable } \left(\Longleftrightarrow \int_0^\infty \phi(s)ds < \infty\right).$$

• $(\mathcal{H}_2)$ (Separated initial data): For $k \in [d]$ fixed in $\mathcal{H}_0$, the initial data and system parameters are chosen to be properly partitioned as follows:

$$r_0 > 0, \quad v_{1i}^k(0) - \Lambda_2 > \frac{1}{2}, \quad v_{2j}^k(0) + \Lambda_3 < -\frac{1}{2}.$$

• $(\mathcal{H}_3)$ (Small fluctuations and coupling strength): The perturbation of local velocity in each cluster group and the coupling strength are sufficiently small:

$$\frac{2\kappa_1}{T_m^\infty}\int_{\frac{r_0}{2}}^\infty \phi(s)ds < \mathcal{D}_V(0) \le \sqrt{\frac{T_m^\infty \min(N_1,N_2)\phi(\mathcal{D}_X^\infty)}{2\max(N_1,N_2)T_M^\infty}}.$$

When the admissible set $(\mathcal{H})$ is assumed, the author of [2] verified the following bi-cluster flocking of system (1.5):

Proposition 2.4. [2] (Bi-cluster flocking) *Suppose that* $Z_1 = \{(x_{1i}, v_{1i}, T_{1i})\}_{i=1}^{N_1}$ *and* $Z_2 = \{(x_{2j}, v_{2j}, T_{2j})\}_{j=1}^{N_2}$ *are a global-in-time solution to the bi-cluster dynamical system* (1.5). *Further, assume that the admissible set* $(\mathcal{H})$ *is valid. Then, we can get the following bi-cluster flocking result in time.*

1. $\min_{1\le i\le N_1, 1\le j\le N_2} \|x_{1i} - x_{2j}\| \ge t + \frac{r_0}{2}, \quad D_X(t) < D_X^\infty.$

2. $D_V(t) \le D_V(0)\exp(-\Lambda_1 t) + \frac{2\kappa_1}{T_m^\infty \Lambda_1}\exp\left(-\frac{\Lambda_1}{2}t\right)\phi\left(\frac{r_0}{2}\right) + \frac{2\kappa_1}{T_m^\infty \Lambda_1}\phi\left(\frac{t+r_0}{2}\right).$
3. $D_T(t) \le D_T(0)\exp(-\Lambda_4 t) + \Lambda_5 \exp\left(-\frac{\Lambda_4}{2}t\right)\zeta\left(\frac{r_0}{2}\right) + \Lambda_5\zeta\left(\frac{t+r_0}{2}\right).$

In Section 4, we extend the sufficient frameworks for the bi-cluster flocking of Proposition 2.4 to the multi-cluster flocking result.

3. Mono-cluster flocking

This section provides suitable sufficient frameworks for the mono-cluster flocking and gives sufficient conditions to guarantee the non-emergence of mono-cluster flocking to system (1.5) when ϕ is integrable. Finally, in the case of system (1.5) under non-integrable ϕ, we present a sufficient condition independent of coupling strength for mono-cluster flocking to arise.

3.1. Mono-cluster flocking

This subsection recalls a dissipative structure for *position-velocity-temperature* L^∞-diameters derived in [2] and gives a mono-cluster flocking result different from Proposition 2.3 which is the mono-cluster flocking of system (1.5) proven in [2]. For this, we begin with the following dissipative inequalities for system (1.5):

Proposition 3.1. [2] *Suppose that* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *is a solution to the system* (1.5). *Then, we have that for a.e.* $t \in \mathbb{R}_+ - \{0\}$,

$$\left|\frac{dD_X}{dt}\right| \le D_V, \quad \frac{dD_V}{dt} \le -\kappa_1\left(\frac{\phi(D_X)}{T_M^\infty} - \frac{D_V^2}{2T_m^\infty}\right)D_V, \quad \frac{dD_T}{dt} \le -\frac{\kappa_2\zeta(D_X)}{(T_M^\infty)^2}D_T.$$

Now, we are ready to study the new mono-cluster flocking result of system (1.5).

Theorem 3.1. (Mono-cluster flocking) *Assume that* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *is a solution to the system* (1.5). *Suppose that there exists a nonnegative number* $D_X^\infty \in \mathbb{R}_+$ *such that the following conditions hold:*

$$\begin{aligned} &D_V^2(0) < \frac{2\phi(D_X^\infty)T_m^\infty}{T_M^\infty}, \\ &D_X(0) - \frac{\sqrt{T_M^\infty T_m^\infty}}{\kappa_1\sqrt{2\phi(D_X^\infty)}}\log\left(\frac{2\phi(D_X^\infty)T_m^\infty - T_M^\infty D_V^2(0)}{\left(\sqrt{2\phi(D_X^\infty)T_m^\infty} + \sqrt{T_M^\infty}D_V(0)\right)^2}\right) \le D_X^\infty. \end{aligned} \tag{3.1}$$

Then, we attain the following assertions for $t \in \mathbb{R}_+$*:*

1. $D_X(t) \le D_X^\infty,$
2. $D_V(t) \le \left(\frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty} + \left(\frac{1}{D_V^2(0)} - \frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty}\right)\exp\left(\frac{2\kappa_1\phi(D_X^\infty)t}{T_M^\infty}\right)\right)^{-\frac{1}{2}},$
3. $D_T(t) \le D_T(0)\exp\left(-\frac{\kappa_2\zeta(D_X^\infty)}{(T_M^\infty)^2}t\right).$

Proof. (*i*) (The case of $D_V(t) > 0$ for $t \in \mathbb{R}_+$) First, we set $g(t)$ as

$$g(t) = \frac{1}{D_V^2(t)}.$$

It follows from the second assertion of Proposition 3.1 that

$$\frac{dg(t)}{dt} \geq \frac{2\kappa_1}{T_M^\infty}\phi(D_X(t))g(t) - \frac{\kappa_1}{T_m^\infty}, \quad a.e.\ t \in \mathbb{R}_+ - \{0\}. \tag{3.2}$$

Due to inequality (3.1) and the continuity of D_X, the following set:

$$S := \{s > 0 \mid (1) \text{ holds for } t \in (0, s)\}$$

is nonempty and we denote $t^* := \sup S > 0$. Next, we claim that

$$t^* = +\infty.$$

For the proof by contradiction, suppose that $t^* < \infty$. Then, we can obtain from inequality (3.2) and the definition of S that

$$\frac{dg(t)}{dt} \geq \frac{2\kappa_1}{T_M^\infty}\phi(D_X^\infty)g(t) - \frac{\kappa_1}{T_m^\infty}, \quad a.e.\ t \in (0, t^*).$$

Moreover, using Grönwall's lemma with the above inequality yields that

$$g(t) \geq \frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty} + \left(g(0) - \frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty}\right)\exp\left(\frac{2\kappa_1\phi(D_X^\infty)t}{T_M^\infty}\right), \quad t \in [0, t^*].$$

This induces that for $t \in [0, t^*]$,

$$D_V(t) \leq \left(\frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty} + \left(\frac{1}{D_V^2(0)} - \frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty}\right)\exp\left(\frac{2\kappa_1\phi(D_X^\infty)t}{T_M^\infty}\right)\right)^{-\frac{1}{2}}. \tag{3.3}$$

Accordingly, we combine inequality (3.3) with the first assertion of Proposition 3.1 to estimate that for $t \in [0, t^*]$,

$$\begin{aligned}
&D_X(t)\\
&\leq D_X(0) + \int_0^t D_V(s)\,ds\\
&\leq D_X(0) + \int_0^t \left(\frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty} + \left(\frac{1}{D_V^2(0)} - \frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty}\right)\exp\left(\frac{2\kappa_1\phi(D_X^\infty)s}{T_M^\infty}\right)\right)^{-\frac{1}{2}} ds\\
&< D_X(0) + \int_0^\infty \left(\frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty} + \left(\frac{1}{D_V^2(0)} - \frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty}\right)\exp\left(\frac{2\kappa_1\phi(D_X^\infty)s}{T_M^\infty}\right)\right)^{-\frac{1}{2}} ds\\
&= D_X(0) - \frac{\sqrt{T_M^\infty T_m^\infty}}{\kappa_1\sqrt{2\phi(D_X^\infty)}}\log\left(\frac{2\phi(D_X^\infty)T_m^\infty - T_M^\infty D_V^2(0)}{\left(\sqrt{2\phi(D_X^\infty)T_m^\infty} + \sqrt{T_M^\infty}D_V(0)\right)^2}\right) \leq D_X^\infty,
\end{aligned}$$

which contradicts to $t^* < \infty$. Therefore, $t^* = \infty$ and for $t \in \mathbb{R}_+$,

$$D_X(t) \le D_X^\infty. \tag{3.4}$$

Hence, one has for $t \in \mathbb{R}_+$,

$$D_V(t) \le \left(\frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty} + \left(\frac{1}{D_V^2(0)} - \frac{T_M^\infty}{2\phi(D_X^\infty)T_m^\infty} \right) \exp\left(\frac{2\kappa_1 \phi(D_X^\infty)t}{T_M^\infty} \right) \right)^{-\frac{1}{2}}.$$

In addition, because the third assertion of Proposition 3.1 and inequality (3.4) hold, we derive that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\frac{dD_T}{dt} \le -\frac{\kappa_2 \zeta(D_X)}{(T_M^\infty)^2} D_T \le -\frac{\kappa_2 \zeta(D_X^\infty)}{(T_M^\infty)^2} D_T,$$

which implies that for $t \in \mathbb{R}_+$,

$$D_T(t) \le D_T(0) \exp\left(-\frac{\kappa_2 \zeta(D_X^\infty)}{(T_M^\infty)^2} \right).$$

(ii) (The case of $D_V(t) = 0$ for some $t \in \mathbb{R}_+$) We define s_* by

$$s_* := \inf\{t \in \mathbb{R}_+ \mid D_V(t) = 0\}.$$

Then, $s_* \in \mathbb{R}_+$ and applying the Cauchy–Lipschitz theory implies that

$$D_V(t) = 0, \quad t \ge s_*.$$

Finally, if we follow the arguments employed in the first case, we immediately reach the desired mono-cluster flocking estimate.

Before we end this subsection, we provide the following remark:

Remark 3.1. *Although* $\frac{T_m^\infty \phi(D_X^\infty)}{2T_M^\infty}$ *of Eq* (2.1) *and* $\frac{2\phi(D_X^\infty)T_m^\infty}{T_M^\infty}$ *of Eq* (3.1) *satisfy the following inequality for* $D_X^\infty \ge 0$*:*

$$\frac{T_m^\infty \phi(D_X^\infty)}{2T_M^\infty} \le \frac{2\phi(D_X^\infty)T_m^\infty}{T_M^\infty},$$

but the following term diverges to $-\infty$ *when* $2\phi(D_X^\infty)T_m^\infty$ *and* $T_M^\infty D_V^2(0)$ *are close to each other in Eq* (3.1)*:*

$$\log\left(\frac{2\phi(D_X^\infty)T_m^\infty - T_M^\infty D_V^2(0)}{\left(\sqrt{2\phi(D_X^\infty)T_m^\infty} + \sqrt{T_M^\infty} D_V(0) \right)^2} \right).$$

Thus, it is unknown which of Proposition 2.3 and Theorem 3.1 yields better mono-cluster flocking result.

3.2. Non-emergence of mono-cluster flocking

This subsection guarantees the non-emergence of mono-cluster flocking of the system (1.5) with integrable ϕ and sufficient small κ_1. For this, we employ the main strategies implemented in [29] for the targeted system (1.5).

3.2.1. Basic frameworks

This subsubsection offers basic notations and preliminary estimates to show the non-emergence of the mono-cluster flocking of system (1.5) when ϕ is integrable. First, we consider the following subdivided $n \geq 2$ configurations $\{Z_\alpha^0\}_{\alpha=1}^n$ of $Z^0 = \{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N$ satisfying

$$(x_{\alpha i}^0, v_{\alpha i}^0, T_{\alpha i}^0), (x_{\alpha j}^0, v_{\alpha j}^0, T_{\alpha j}^0) \in Z_\alpha^0 \iff v_{\alpha i}^0 = v_{\alpha j}^0,$$

where

$$|Z_\alpha^0| =: N_\alpha \geq 1, \quad Z^0 = \dot\cup_{\alpha=1}^n Z_\alpha^0.$$

In other words, we primarily deal with the initial configuration Z^0 that is not in a mono-cluster flocking state. Subsequently, we reorganize the system (1.5) to distinguish the n-dynamics initiated from n-subdivided initial configurations Z_α^0 as follows:

$$\begin{cases} \dfrac{dx_{\alpha i}}{dt} = v_{\alpha i}, \quad t > 0, \quad i = 1, \cdots, N_\alpha, \quad \alpha = 1, \cdots, n, \quad n \geq 2, \\ \dfrac{dv_{\alpha i}}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N_\alpha} \phi(\|x_{\alpha i} - x_{\alpha j}\|) \left(\dfrac{v_{\alpha j} - \langle v_{\alpha j}, v_{\alpha i} \rangle v_{\alpha i}}{T_{\alpha j}} \right) \\ \qquad\qquad + \dfrac{\kappa_1}{N} \displaystyle\sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \phi(\|x_{\alpha i} - x_{\beta j}\|) \left(\dfrac{v_{\beta j} - \langle v_{\beta j}, v_{\alpha i} \rangle v_{\alpha i}}{T_{\beta j}} \right), \\ \dfrac{dT_{\alpha i}}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N_\alpha} \zeta(\|x_{\alpha i} - x_{\alpha j}\|) \left(\dfrac{1}{T_{\alpha i}} - \dfrac{1}{T_{\alpha j}} \right) \\ \qquad\qquad + \dfrac{\kappa_2}{N} \displaystyle\sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \zeta(\|x_{\alpha i} - x_{\beta j}\|) \left(\dfrac{1}{T_{\alpha i}} - \dfrac{1}{T_{\beta j}} \right), \\ (x_{\alpha i}(0), v_{\alpha i}(0), T_{\alpha i}(0)) = (x_{\alpha i}^0, v_{\alpha i}^0, T_{\alpha i}^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (\mathbb{R}_+ - \{0\}). \end{cases} \tag{3.5}$$

In the following, we denote local averages and local deviations for $\alpha = 1, \cdots, n$

$$x_\alpha^{cen} = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} x_{\alpha i}, \quad v_\alpha^{cen} = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} v_{\alpha i}, \quad \hat{x}_{\alpha i} := x_{\alpha i} - x_\alpha^{cen}, \quad \hat{v}_{\alpha i} := v_{\alpha i} - v_\alpha^{cen},$$

and we set the following notation to estimate the degree of separation between n-subdivided initial configuration sets $\{Z_\alpha^0\}_{\alpha=1}^n$.

$$\begin{aligned} &D(x^0) := \max_{\alpha \neq \beta, i, j} \|x_{\alpha i}^0 - x_{\beta j}^0\|, \quad \theta_0 := \min_{\alpha \neq \beta} \arccos \langle v_\alpha^{cen}(0), v_\beta^{cen}(0) \rangle, \\ &\lambda_0 := \min \Big(\cos((\delta + \epsilon)\theta_0) - \cos((1 - 4\delta - \epsilon)\theta_0), \\ &\qquad\qquad \cos(\delta\theta_0) - \cos((1 - \delta)\theta_0) - (D(x^0) + 2T_0) \cdot \frac{(N-1)\kappa_1}{N T_m^\infty} \Big), \end{aligned}$$

where two auxiliary parameters $\epsilon, \delta \in (0, 1)$ will be specified later such that $\lambda_0 > 0$ in Section 3.2.2 and we define T_0 as

$$T_0 := \max_{\alpha \neq \beta, i, j} \left\{ 0, -\frac{\langle x_{\alpha i}^0 - x_{\beta j}^0, v_\alpha^{cen}(0) \rangle}{\lambda_0} \right\}.$$

We observe that $D(x^0)$, θ_0 and λ_0 are dependent on given initial data non-mono-cluster flocking state. As we will see later, T_0 is indeed the time when two agents belonging to different cluster groups begin to move away from each other linearly and λ_0 is needed to estimate T_0. For the detailed descriptions, see Section 3.2.2.

Next, we set the coupling strength $\tilde{\kappa}_0$ dependent on given initial data $Z^0 = \{(x_i^0, v_i^0)\}_{i=1}^N$ of the system (1.5) as follows:

(i) (The case of $\min_{\alpha\neq\beta,i,j}\langle(x_{\alpha i}^0 - x_{\beta j}^0), v_\alpha^{cen}\rangle < 0$): We define $\tilde{\kappa}_0$ as

$$\tilde{\kappa}_0 = \min\Bigg(\frac{NT_m^\infty(1-\cos(\delta\theta_0))}{2(N-1)T_0}, \frac{NT_m^\infty(\cos(\delta\theta_0)-\cos((1-\delta)\theta_0)-\lambda_0)}{(N-1)(D(x^0)+2T_0)}, \frac{\lambda_0(\cos(\delta\theta_0)-\cos((\delta+\epsilon)\theta_0))}{(1-\gamma_N)\int_0^\infty \phi(s)ds}\Bigg), \quad \text{where} \quad \gamma_N := \frac{\min_\alpha N_\alpha}{N}.$$

(ii) (The case of $\min_{\alpha\neq\beta,i,j}\langle(x_{\alpha i}^0 - x_{\beta j}^0), v_\alpha^{cen}\rangle \geq 0$): We define $\tilde{\kappa}_0$ as

$$\tilde{\kappa}_0 = \frac{\tilde{\lambda}_0(1-\cos(\tilde{\delta}\theta_0))}{(1-\gamma_N)\int_0^\infty \phi(s)ds}, \quad \text{where} \quad \tilde{\lambda}_0 := \cos(\tilde{\delta}\theta_0) - \cos((1-\tilde{\delta})\theta_0).$$

Herein, an auxiliary parameter $\tilde{\delta} \in (0,1)$ will be determined such that $\tilde{\lambda}_0 > 0$ later in Section 3.2.2.

Finally, we present the definitions of $\triangle_{\alpha i,\beta j}(t)$ and v_α^{min}, which will be crucially used to verify the non-emergence of mono-cluster flocking in the system (1.5). We let

$$\triangle_{\alpha i,\beta j}(t) := \langle x_{\alpha i}(t) - x_{\beta j}(t), v_\alpha^{cen}(t)\rangle, \quad v_\alpha^{min} := \min_{1\leq i\leq N_\alpha} \langle v_{\alpha i}(t), e_\alpha(T_0)\rangle,$$

where $e_\alpha(t) := \frac{v_\alpha^{cen}(t)}{\|v_\alpha^{cen}(t)\|}$. Note that $\triangle_{\alpha i,\beta j}(t)$ shows how well $Z_\alpha(t)$ and $Z_\beta(t)$ are separated from each other at time t. Therefore, rigorous estimates concerning $\triangle_{\alpha i,\beta j}(t)$ are important to obtain the non-emergence of mono-cluster flocking in the system (1.5).

3.2.2. Non-emergence of mono-cluster flocking

In what follows, we demonstrate the non-emergence of the mono-cluster flocking of the TCSUS system (1.5). For this, we assume that $T_0 > 0$ throughout the subsubsection. If otherwise, it is a trivial case when $T_0 = 0$ (see Theorem 3.2). Now, we begin with the following preparatory lemmas:

Lemma 3.1. *Suppose that Z_α is a solution to the system* (3.5) *with given initial data Z_α^0 that is a non-mono-cluster flocking state for each $\alpha \in \{1, \cdots, n\}$. Assume that there exists a positive number $\delta \in \left(0, \frac{1}{3}\right)$ such that*

$$0 < \kappa_1 < \frac{NT_m^\infty(1-\cos(\delta\theta_0))}{2(N-1)T_0}.$$

Then, one has for $t \in [0, T_0]$ and $\alpha \neq \beta$,

1. $\langle v_{\alpha i}, v_\alpha^{cen}\rangle > \cos(\delta\theta_0)$, $\quad \langle v_{\beta j}, v_\alpha^{cen}\rangle < \cos((1-\delta)\theta_0)$,
2. $\langle v_{\alpha i}, v_{\beta j}\rangle < \cos((1-\delta)\theta_0)$, $\quad \langle e_\alpha, e_\beta\rangle < \cos((1-3\delta)\theta_0)$.

Proof. To estimate the first assertion of (1), we first see that

$$\begin{aligned}\frac{dv_{\alpha i}}{dt} &= \frac{\kappa_1}{N}\sum_{j=1}^{N}\phi(\|x_{\alpha i}-x_j\|)\left(\frac{v_j-\langle v_j, v_{\alpha i}\rangle v_{\alpha i}}{T_j}\right)\\ &= \frac{\kappa_1}{N}\sum_{j\neq\alpha i}^{N}\phi(\|x_{\alpha i}-x_j\|)\left(\frac{v_j-\langle v_j, v_{\alpha i}\rangle v_{\alpha i}}{T_j}\right).\end{aligned}$$

Then, the triangle inequality and $\phi \le 1$ yield that

$$\left\|\frac{dv_{\alpha i}}{dt}\right\| \le \frac{(N-1)\kappa_1}{NT_m^\infty},$$

where we used Proposition 2.2 and $\|v_j - \langle v_j, v_{\alpha i}\rangle v_{\alpha i}\| \le 1$. Thus, it follows that

$$\left|\frac{d}{dt}\langle v_{\alpha i}, v_\alpha^{cen}\rangle\right| \le \frac{2(N-1)\kappa_1}{NT_m^\infty},$$

which implies by the condition for κ_1 and construction of Z_α^0 that for $t \in [0, T_0]$,

$$\begin{aligned}\langle v_{\alpha i}(t), v_\alpha^{cen}(t)\rangle &= \langle v_{\alpha i}(0), v_\alpha^{cen}(0)\rangle + \int_0^t \frac{d}{ds}\langle v_{\alpha i}(s), v_\alpha^{cen}(s)\rangle ds\\ &\ge \langle v_{\alpha i}(0), v_\alpha^{cen}(0)\rangle - \frac{2(N-1)\kappa_1 T_0}{NT_m^\infty}\\ &= 1 - \frac{2(N-1)\kappa_1 T_0}{NT_m^\infty} > \cos(\delta\theta_0).\end{aligned}$$

To prove the second assertion of (1), we employ the same method as in the proof of the first assertion of (1) as follows:

$$\left\|\frac{dv_{\beta j}}{dt}\right\| \le \frac{(N-1)\kappa_1}{NT_m^\infty} \quad \text{and then,} \quad \left|\frac{d}{dt}\langle v_{\beta j}, v_\alpha^{cen}\rangle\right| \le \frac{2(N-1)\kappa_1}{NT_m^\infty}.$$

From the definitions of Z_α^0 and θ_0, we get that for $t \in [0, T_0]$,

$$\begin{aligned}\langle v_{\beta j}, v_\alpha^{cen}\rangle &\le \langle v_{\beta j}(0), v_\alpha^{cen}(0)\rangle + \frac{2(N-1)\kappa_1 T_0}{NT_m^\infty} = \langle v_\beta^{cen}(0), v_\alpha^{cen}(0)\rangle + \frac{2(N-1)\kappa_1 T_0}{NT_m^\infty}\\ &\le \cos(\theta_0) + \frac{2(N-1)\kappa_1 T_0}{NT_m^\infty} \le \cos(\theta_0) + 1 - \cos(\delta\theta_0) < \cos((1-\delta)\theta_0),\end{aligned}$$

where we used the assumption for κ_1. Next, following the proof of (1), we can also attain the first assertion of (2) for $t \in [0, T_0]$:

$$\langle v_{\alpha i}, v_{\beta j}\rangle < \cos((1-\delta)\theta_0).$$

Finally, to verify the second assertion of (2), we combine (1) and the first assertion of (2) to attain that for $t \in [0, T_0]$,

$$\begin{aligned}\arccos(\langle e_\alpha, e_\beta\rangle) &\ge -\arccos(\langle e_\alpha, v_{\alpha i}\rangle) + \arccos(\langle v_{\alpha i}, v_{\beta j}\rangle) - \arccos(\langle v_{\beta j}, e_\beta\rangle)\\ &> (1-\delta)\theta_0 - 2\delta\theta_0 = (1-3\delta)\theta_0.\end{aligned}$$

Therefore, $\langle e_\alpha, e_\beta\rangle < \cos((1-3\delta)\theta_0)$ for $t \in [0, T_0]$ and we conclude this lemma.

The following lemma plays a key role in deriving the desired result:

Lemma 3.2. *Let Z_α be a solution to the system* (3.5) *with given initial data Z_α^0 that is a non-mono-cluster flocking state for each $\alpha = 1, \cdots, n$. Suppose that there exists a positive number $\delta \in \left(0, \frac{1}{3}\right)$ such that*

$$0 < \kappa_1 < \min\left(\frac{NT_m^\infty(1-\cos(\delta\theta_0))}{2(N-1)T_0}, \frac{NT_m^\infty(\cos(\delta\theta_0) - \cos((1-\delta)\theta_0) - \lambda_0)}{(N-1)(D(x^0)+2T_0)}\right), \quad \lambda_0 > 0.$$

Then, we obtain that

$$\min_{\alpha\neq\beta,i,j} \triangle_{\alpha i,\beta j}(T_0) > 0.$$

Proof. First, we note that

$$\|x_{\alpha i}(t) - x_{\beta j}(t)\| = \left\| x_{\alpha i}(0) - x_{\beta j}(0) + \int_0^t (v_{\alpha i}(s) - v_{\beta j}(s))ds \right\| \leq D(x^0) + 2T_0.$$

Hence, we have from the arguments studied in Lemma 3.1 and the definition of λ_0 that

$$\begin{aligned}\frac{d}{dt}\triangle_{\alpha i,\beta j} &= \langle v_{\alpha i}, v_\alpha^{cen}\rangle - \langle v_{\beta j}, v_\alpha^{cen}\rangle + \langle x_{\alpha i} - x_{\beta j}, \dot{v}_\alpha^{cen}\rangle \\ &> \cos(\delta\theta_0) - \cos((1-\delta)\theta_0) - (D(x^0) + 2T_0)\frac{(N-1)\kappa_1}{NT_m^\infty} \geq \lambda_0 > 0,\end{aligned}$$

which leads to the following result using the definition of T_0:

$$\triangle_{\alpha i,\beta j}(t) > \triangle_{\alpha i,\beta j}(0) + \lambda_0 t \quad \text{and thus,} \quad \triangle_{\alpha i,\beta j}(T_0) > \triangle_{\alpha i,\beta j}(0) + \lambda_0 T_0 > 0.$$

From the above relation, we take $\min_{\alpha\neq\beta,i,j}$ to derive that

$$\min_{\alpha\neq\beta,i,j} \triangle_{\alpha i,\beta j}(T_0) > 0.$$

We reach the desired lemma.

Subsequently, to prove the main result using the bootstrapping argument, we denote $\bar{T}_0$

$$\bar{T}_0 := \sup\left\{ t \in (T_0, \infty) \;\middle|\; \min_{\alpha,i}\langle v_{\alpha i}(s), e_\alpha(T_0)\rangle > \cos((\delta+\epsilon)\theta_0), \quad s \in [T_0, t) \right\},$$

where an auxiliary parameter $\epsilon \in (0, 1)$ will be determined in Lemma 3.3. Here, we observe from Lemma 3.2 that $e_\alpha(T_0)$ is well-defined. In addition, $\bar{T}_0$ is well-defined due to Lemma 3.1. Indeed,

$$\langle v_{\alpha i}(T_0), e_\alpha(T_0)\rangle > \cos(\delta\theta_0) > \cos((\delta+\epsilon)\theta_0).$$

From now on, we claim that

$$\bar{T}_0 = \infty.$$

Lemma 3.3. *Assume that Z_α is a solution to the system* (3.5) *given initial data Z_α^0 that is a non-mono-cluster flocking state for each $\alpha = 1, \cdots, n$. Suppose that there exist positive numbers ϵ and δ that satisfy*

$$0 < \delta < \frac{1-2\epsilon}{5}, \quad \epsilon \in \left(0, \frac{1}{2}\right), \quad 0 < \kappa_1 < \frac{NT_m^\infty(1-\cos(\delta\theta_0))}{2(N-1)T_0}, \quad \lambda_0 > 0.$$

Then, for $t \in [T_0, \bar{T}_0]$,

$$\max_{\alpha,\beta,j}\langle v_{\beta j}, e_\alpha(T_0)\rangle < \cos((1-4\delta-\epsilon)\theta_0), \quad \min_{\alpha\neq\beta,i,j}\langle v_{\alpha i} - v_{\beta j}, e_\alpha(T_0)\rangle > \lambda_0.$$

Proof. To get the first assertion, from the definition of $\bar{T}_0$ and Lemma 3.1, we estimate that

$$\begin{aligned}\arccos(\langle v_{\beta j}, e_\alpha(T_0)\rangle) &\geq \arccos(\langle e_\beta(T_0), e_\alpha(T_0)\rangle) - \arccos(\langle v_{\beta j}, e_\beta(T_0)\rangle)\\ &> (1-3\delta)\theta_0 - (\delta+\epsilon)\theta_0 = (1-4\delta-\epsilon)\theta_0.\end{aligned}$$

This leads us to deduce that

$$\max_{\alpha,\beta,j}\langle v_{\beta j}, e_\alpha(T_0)\rangle < \cos((1-4\delta-\epsilon)\theta_0).$$

Additionally, the definition of $\bar{T}_0$ and the first assertion yield that

$$\min_{\alpha\neq\beta,i,j}\langle (v_{\alpha i} - v_{\beta j}), e_\alpha(T_0)\rangle > \cos((\delta+\epsilon)\theta_0) - \cos((1-4\delta-\epsilon)\theta_0) \geq \lambda_0.$$

We need the following lemma to verify that $\bar{T}_0 = \infty$:

Lemma 3.4. *Let Z_α be a solution to the system* (3.5) *given initial data Z_α^0 that is a non-mono-cluster flocking state for each $\alpha = 1, \cdots, n$. Assume that there exist positive numbers ϵ and δ that satisfy $0 < \delta < \frac{1-2\epsilon}{5}$, $\epsilon \in \left(0, \frac{1}{2}\right)$, and*

$$0 < \kappa_1 < \min\left(\frac{NT_m^\infty(1-\cos(\delta\theta_0))}{2(N-1)T_0}, \frac{NT_m^\infty(\cos(\delta\theta_0) - \cos((1-\delta)\theta_0) - \lambda_0)}{(N-1)(D(x^0)+2T_0)}\right), \quad \lambda_0 > 0.$$

Then, we reach that

$$\phi_M(t) := \max_{\alpha\neq\beta,i,j}\phi(\|x_{\beta j} - x_{\alpha i}\|) \leq \phi(\lambda_0(t-T_0)), \quad t \in [T_0, \bar{T}_0).$$

Proof. By applying Lemma 3.2 and Lemma 3.3, we induce that for $t \in [T_0, \bar{T}_0)$,

$$\begin{aligned}\|x_{\alpha i} - x_{\beta j}\| &\geq \langle (x_{\alpha i} - x_{\beta j}), e_\alpha(T_0)\rangle\\ &= \langle (x_{\alpha i}(T_0) - x_{\beta j}(T_0)), e_\alpha(T_0)\rangle + \int_{T_0}^t \langle (v_{\alpha i}(s) - v_{\beta j}(s)), e_\alpha(T_0)\rangle ds\\ &> \int_{T_0}^t \langle (v_{\alpha i}(s) - v_{\beta j}(s)), e_\alpha(T_0)\rangle ds > \lambda_0(t-T_0).\end{aligned}$$

Then, this leads to the following result for $t \in [T_0, \bar{T}_0)$ due to the monotonicity of ϕ:

$$\phi_M(t) := \max_{\alpha\neq\beta,i,j}\phi(\|x_{\beta j} - x_{\alpha i}\|) \leq \phi(\lambda_0(t-T_0)).$$

Hence, we conclude the desired lemma.

Subsequently, we estimate the time derivative of v_α^{min} to demonstrate the main result.

Lemma 3.5. *Let Z_α be a solution to the system* (3.5) *given initial data Z_α^0 that is a non-mono-cluster flocking state for each $\alpha = 1, \cdots, n$. Then, for $\alpha = 1, \cdots, n$, it follows that for $t \in [T_0, \bar{T}_0)$,*

$$\dot{v}_\alpha^{min} \geq -\frac{\kappa_1(1-\gamma_N)\phi_M}{T_m^\infty}.$$

Proof. First, we fix $\alpha \in \{1, \cdots, n\}$; then, we select index $i_\alpha := i_\alpha(t) \in \{1, \cdots, N_\alpha\}$ at time t such that

$$v_\alpha^{min} = \langle v_{\alpha i_\alpha}, e_\alpha(T_0)\rangle.$$

Then, if we use system (3.5), Proposition 2.2, and the definitions of i_α and $\bar{T}_0$, we obtain that

$$\begin{aligned}
\dot{v}_\alpha^{min} &= \langle \dot{v}_{\alpha i_\alpha}, e_\alpha(T_0)\rangle \\
&= \frac{\kappa_1}{N}\sum_{j=1}^{N_\alpha}\left\langle \phi(\|x_{\alpha i_\alpha} - x_{\alpha j}\|)\left(\frac{v_{\alpha j} - \langle v_{\alpha j}, v_{\alpha i_\alpha}\rangle v_{\alpha i_\alpha}}{T_{\alpha j}}\right), e_\alpha(T_0)\right\rangle \\
&\quad + \frac{\kappa_1}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\left\langle \phi(\|x_{\alpha i_\alpha} - x_{\beta j}\|)\left(\frac{v_{\beta j} - \langle v_{\beta j}, v_{\alpha i_\alpha}\rangle v_{\alpha i_\alpha}}{T_{\beta j}}\right), e_\alpha(T_0)\right\rangle \\
&\geq \frac{\kappa_1}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\left\langle \phi(\|x_{\alpha i_\alpha} - x_{\beta j}\|)\left(\frac{v_{\beta j} - \langle v_{\beta j}, v_{\alpha i_\alpha}\rangle v_{\alpha i_\alpha}}{T_{\beta j}}\right), e_\alpha(T_0)\right\rangle \\
&\geq -\frac{\kappa_1\phi_M}{T_m^\infty}\cdot\frac{(N-N_\alpha)}{N} \geq -\frac{\kappa_1(1-\gamma_N)\phi_M}{T_m^\infty},
\end{aligned}$$

where we employed

$$\|v_{\beta j} - \langle v_{\beta j}, v_{\alpha i_\alpha}\rangle v_{\alpha i_\alpha}\| \leq 1.$$

Thus, we get the desired lemma.

Finally, we are ready to study the non-emergence of the mono-cluster flocking of system (3.5) under the integrable communication weight ϕ, i.e.,

$$\|\phi\|_{L_1} = \int_0^\infty \phi(s)ds < \infty.$$

Theorem 3.2. (Non-emergence of mono-cluster flocking) *Assume that Z_α is a solution to the system* (3.5) *with given initial data Z_α^0 that is a non-mono-cluster flocking state for each $\alpha = 1, \cdots, n$. Suppose that $T_0 > 0$ and there exist positive numbers ϵ and δ that satisfy $0 < \delta < \frac{1-2\epsilon}{5}$ and $\epsilon \in \left(0, \frac{1}{2}\right)$ such that*

$$0 < \kappa_1 < \tilde{\kappa}_0, \quad \lambda_0 > 0.$$

Then, we attain that

$$\min_{\alpha\neq\beta, i, j}\sup_{t\in\mathbb{R}_+}\|x_{\alpha i} - x_{\beta j}\| = \infty, \quad \min_{\alpha\neq\beta, i, j}\liminf_{t\to\infty}\|v_{\alpha i} - v_{\beta j}\| > 0.$$

Meanwhile, when $T_0 = 0$, we let $\tilde{\lambda} > 0$ and $\tilde{\delta} \in \left(0, \frac{1}{2}\right)$. Then, we can reach the same results as above.

Proof. To demonstrate the desired results, we divide them by the following dichotomy:

$$T_0 > 0 \quad \text{or} \quad T_0 = 0.$$

(i) (The case of $T_0 > 0$) For the proof by contradiction, suppose that $\bar{T}_0 < \infty$. Then, there exist $\alpha \in \{1, \cdots, n\}$ and $i_\alpha \in \{1, \cdots, N_\alpha\}$ such that

$$\langle v_{\alpha i_\alpha}(\bar{T}_0), e_\alpha(T_0)\rangle = \cos((\delta + \epsilon)\theta_0).$$

Then, we use Lemmas 3.1, 3.4 and 3.5 to obtain that for $t \in [T_0, \bar{T}_0]$,

$$\begin{aligned}
\langle v_{\alpha i_\alpha}, e_\alpha(T_0)\rangle &\geq v_\alpha^{min} \geq v_\alpha^{min}(T_0) - \frac{\kappa_1(N - N_\alpha)}{NT_m^\infty}\int_{T_0}^t \phi_M(s)ds \\
&\geq \cos(\delta\theta_0) - \frac{\kappa_1(N - N_\alpha)}{NT_m^\infty\lambda_0}\|\phi\|_{L_1} \geq \cos(\delta\theta_0) - \frac{\kappa_1(1 - \gamma_N)}{T_m^\infty\lambda_0}\|\phi\|_{L_1} \\
&> \cos((\delta + \epsilon)\theta_0),
\end{aligned}$$

which gives a contradiction; therefore, $\bar{T}_0 = \infty$. Then, the second assertion of Lemmas 3.3 and 3.4 with $\bar{T}_0 = \infty$ yield the desired result.
(ii) (The case of $T_0 = 0$) This case is trivial, but we provide the proof rigorously to compare with the proof regarding the first assertion. Let

$$T_0^* := \sup\left\{t \in \mathbb{R}_+ - \{0\} \;\middle|\; \min_{\alpha,i}\langle v_{\alpha i}, e_\alpha(0)\rangle > \cos(\tilde{\delta}\theta_0), \quad t \in [0, t)\right\}, \text{ where } \tilde{\delta} \in \left(0, \frac{1}{2}\right).$$

It follows from the definition of Z_α that $T_0^* > 0$ exists. For the proof by contradiction, suppose that $T_0^* < \infty$. Next, we employ the same method as utilized in proof of the first assertion of Lemma 3.1 to estimate that

$$\langle v_{\beta j}(t), e_\alpha(0)\rangle < \cos(1 - \tilde{\delta})\theta_0, \quad t \in [0, T_0^*).$$

Hence, we have

$$\min_{\alpha\neq\beta,i,j}\langle v_{\alpha i}(t) - v_{\beta j}(t), e_\alpha(0)\rangle > \cos(\tilde{\delta}\theta_0) - \cos((1 - \tilde{\delta})\theta_0) =: \tilde{\lambda}_0 > 0.$$

Then, similarly to the proof of Lemma 3.4, one can show that

$$\phi_M(t) \leq \phi(\tilde{\lambda}_0 t), \quad t \in [0, T_0^*]$$

and thus, for $t \in [0, T_0^*)$, we can get the following estimates using the same methodologies as in the proof of Lemma 3.5:

$$\begin{aligned}
\langle v_{\alpha i_\alpha}, e_\alpha(0)\rangle &\geq v_\alpha^{min} \geq v_\alpha^{min}(0) - \frac{\kappa_1(N - N_\alpha)}{NT_m^\infty}\int_0^t \phi_M(s)ds \\
&= 1 - \frac{\kappa_1(N - N_\alpha)}{NT_m^\infty}\int_0^t \phi_M(s)ds \geq 1 - \frac{\kappa_1(N - N_\alpha)}{NT_m^\infty\lambda_0}\|\phi\|_{L_1} > \cos(\tilde{\delta}\theta_0),
\end{aligned}$$

which leads to a contradiction. Therefore, $T_0^* = \infty$. Finally, if the arguments of Lemmas 3.3 and 3.4 are applied to the case of $T_0 = 0$, we conclude the desired result.

3.3. Mono-cluster flocking under non-integrable ϕ

This subsection demonstrates a different sufficient framework than Section 3.1 for mono-cluster flocking to emerge in the system (1.5) when ϕ is non-integrable by using the previous results of [3].

Proposition 3.2. [3] *Let $\{(x_i, v_i, T_i)\}_{i=1}^N$ be a solution to the system* (1.5) *such that*

$$\mathcal{A}(v)(0) := \min_{1\le i,j\le N} \langle v_i^0, v_j^0 \rangle > 0, \quad D_V(0) < \frac{\kappa_1 \mathcal{A}(v)(0)}{T_M^\infty} \int_{D_X(0)}^\infty \phi(s) ds.$$

Then, there exists a nonnegative number $D_X^\infty \in \mathbb{R}_+$ satisfying for $t \in \mathbb{R}_+$,

1. *(Group formation) $D_X(t) \le D_X^\infty$,*
2. *(Velocity alignment) $D_V(t) \le D_V(0) \exp\left(-\frac{\kappa_1 \mathcal{A}(v)(0)\phi(D_X^\infty)}{T_M^\infty} t\right)$,*
3. *(Temperature equilibrium) $D_T(t) \le D_T(0) \exp\left(-\frac{\kappa_2 \zeta(D_X^\infty)}{(T_M^\infty)^2} t\right)$.*

Proof. We employ the same methodologies as the proofs of Lemma 3.1 and Theorem 3.2 in [3] to obtain the desired result. Although the previous paper [3] dealt with the singular communication weight ϕ to system (1.5), the proofs of Lemma 3.1 and Theorem 3.2 in [3] can be applied, even assuming the regular communication weight case covered in this paper.

Due to Proposition 3.2, we note the following remark.

Remark 3.2. *It is easy to check that we can remove the condition,*

$$D_V(0) < \frac{\kappa_1 \mathcal{A}(v)(0)}{T_M^\infty} \int_{D_X(0)}^\infty \phi(s) ds,$$

when ϕ is non-integrable. In other words, when ϕ is non-integrable, the mono-cluster flocking of the system (1.5) *emerges under the only assumption $\mathcal{A}(v)(0) > 0$.*

Finally, we present the following mono-cluster flocking of system (1.5) under non-integrable ϕ:

Theorem 3.3. (Mono-cluster flocking under non-integrable ϕ) *Assume that $\{(x_i, v_i, T_i)\}_{i=1}^N$ is a solution to the system* (1.5) *under non-integrable ϕ and suppose that*

$$\mathcal{A}(v)(0) := \min_{1\le i,j\le N} \langle v_i^0, v_j^0 \rangle > 0.$$

Then, there exists a nonnegative number $D_X^\infty \in \mathbb{R}_+$ such that for $t \in \mathbb{R}_+$,

1. *(Group formation) $D_X(t) \le D_X^\infty$,*
2. *(Velocity alignment) $D_V(t) \le D_V(0) \exp\left(-\frac{\kappa_1 \mathcal{A}(v)(0)\phi(D_X^\infty)}{T_M^\infty} t\right)$,*
3. *(Temperature equilibrium) $D_T(t) \le D_T(0) \exp\left(-\frac{\kappa_2 \zeta(D_X^\infty)}{(T_M^\infty)^2} t\right)$.*

4. Multi-cluster flocking

This section provides several sufficient frameworks for the multi-cluster flocking of the system (1.5). In Section 3, we studied that mono-cluster flocking does not occur when the coupling strength κ_1 is less than a certain positive value in system (1.5) with integrable ϕ. In Section 3.2.1, we employed suitable subdivided configurations, $\{Z_\alpha^0\}_{\alpha=1}^n$, so that all initial velocities are equal to each other in each group and deduced some sufficient conditions guaranteeing the non-emergence of the mono-cluster flocking of the system. Accordingly, we may wonder what the sufficient conditions are for multi-cluster flocking to occur, so it is necessary to check how little coupling strength is required for multi-cluster flocking to occur in system (1.5). To achieve this, we reorganize the system (1.5) under integrable ϕ to a multi-cluster setting and then derive suitable dissipative differential inequalities with respect *position–velocity–temperature*. Finally, using bootstrapping arguments for these inequalities, we deduce appropriate sufficient conditions in terms of the initial data and system parameters to guarantee the mono-cluster flocking of system (1.5). As a direct consequence, we also prove that the velocity and temperature of all agents in each cluster group converge to the same values.

4.1. Reorganization of system (1.5) *and basic materials*

This subsection converts the TCSUS model (1.5) into some multi-cluster setting. Afterward, we present basic estimates for the averages of *position-velocity-temperature*. For this, we begin by reorganizing the system (1.5) to the following multi-cluster setting:

$$\frac{dx_{\alpha i}}{dt} = v_{\alpha i}, \quad t > 0, \quad i \in \{1, \cdots, N_\alpha\}, \quad \alpha \in \{1, \cdots, n\}, \quad n \geq 3, \tag{4.1a}$$

$$\dot{v}_{\alpha i} = \frac{\kappa_1}{N} \sum_{j=1}^{N_\alpha} \phi(\|x_{\alpha i} - x_{\alpha j}\|) \frac{(v_{\alpha j} - \langle v_{\alpha i}, v_{\alpha j}\rangle v_{\alpha i})}{T_{\alpha j}} \tag{4.1b}$$

$$+\frac{\kappa_1}{N} \sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \phi(\|x_{\alpha i} - x_{\beta j}\|) \frac{(v_{\beta j} - \langle v_{\alpha i}, v_{\beta j}\rangle v_{\alpha i})}{T_{\beta j}}, \tag{4.1c}$$

$$\dot{T}_{\alpha i} = \frac{\kappa_2}{N} \sum_{j=1}^{N_\alpha} \zeta(\|x_{\alpha i} - x_{\alpha j}\|) \left(\frac{1}{T_{\alpha i}} - \frac{1}{T_{\alpha j}}\right) + \frac{\kappa_2}{N} \sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \zeta(\|x_{\alpha i} - x_{\beta j}\|) \left(\frac{1}{T_{\alpha i}} - \frac{1}{T_{\beta j}}\right), \tag{4.1d}$$

$$(x_{\alpha i}(0), v_{\alpha i}(0), T_{\alpha i}(0)) \in Z_\alpha^0 \times T_{\alpha i}^0 \subset \mathbb{R}^d \times \mathbb{S}^{d-1} \times (\mathbb{R}_+ - \{0\}). \tag{4.1e}$$

For each cluster group $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$, we denote the following three configuration vectors:

$$A_\alpha := (a_{\alpha 1}, \cdots a_{\alpha N_\alpha}),\ 1 \leq \alpha \leq n,\ \text{where } A \in \{X, V, T\},\ \ a \in \{x, v, T\},\ A := (A_1, \cdots, A_\alpha).$$

Next, we define *position-velocity-temperature* L^∞-diameters to each cluster group as follows:

(i) (The *position-velocity-temperature* diameters to the α-th cluster group)

$$D_{X_\alpha} := \max_{1 \leq i,j \leq N_\alpha} \|x_{\alpha i} - x_{\alpha j}\|, \quad D_{V_\alpha} := \max_{1 \leq i,j \leq N_\alpha} \|v_{\alpha i} - v_{\alpha j}\|, \quad D_{T_\alpha} := \max_{1 \leq i,j \leq N_\alpha} |T_{\alpha i} - T_{\alpha j}|.$$

(ii) (The local averages of velocity and temperature in each cluster group)

$$v_\alpha^{cen} := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha} v_{\alpha i}, \quad T_\alpha^{cen} := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha} T_{\alpha i}.$$

Before we end this subsection, we offer the following lemma regarding the local averages of velocity and temperature for each cluster group. This lemma will be crucially used to prove that the velocity and temperature of all agents in each cluster group converges to some unified values.

Lemma 4.1. *Assume that* $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ *is a solution to the system* (4.1)*. Then, each local average* $(x_\alpha^{cen}, v_\alpha^{cen}, T_\alpha^{cen})$ *satisfies the following relations:*

$$\begin{cases} \dfrac{dx_\alpha^{cen}}{dt} = v_\alpha^{cen}, \quad t>0, \quad \alpha \in \{1,\cdots,n\}, \quad n\geq 3,\\ N_\alpha \dot{v}_\alpha^{cen} = \dfrac{\kappa_1}{N}\displaystyle\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\alpha}\phi(\|x_{\alpha i}-x_{\alpha j}\|)\frac{v_{\alpha i}\|v_{\alpha j}-v_{\alpha i}\|^2}{2T_{\alpha j}}\\ \qquad\qquad +\dfrac{\kappa_1}{N}\displaystyle\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i}-x_{\beta j}\|)\left(v_{\beta j}-v_{\alpha i}+\frac{v_{\alpha i}\|v_{\beta j}-v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}},\\ N_\alpha \dot{T}_\alpha^{cen} = \dfrac{\kappa_2}{N}\displaystyle\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha i}-x_{\beta j}\|)\left(\frac{1}{T_{\alpha i}}-\frac{1}{T_{\beta j}}\right). \end{cases}$$

Proof. The first assertion is trivial. For the second assertion, we take $\sum_{i=1}^{N_\alpha}$ to $\dot{v}_{\alpha i}$ and use the standard trick of interchanging i and j and dividing 2 and

$$1 - \langle v_{\alpha i}, v_{\alpha j}\rangle = \frac{\|v_{\alpha i}-v_{\alpha j}\|^2}{2}.$$

For the third assertion, we take $\sum_{i=1}^{N_\alpha}$ to $\dot{T}_{\alpha i}$ and again use the standard trick as above.

4.2. Dissipative inequalities

In the following, we derive several dissipative differential inequalities with respect to *position–velocity–temperature* to obtain suitable sufficient frameworks in terms of system parameters and initial data for the multi-cluster flocking of system (4.1). For this, we define

$$D_X := \sum_{\alpha=1}^{n} D_{X_\alpha}, \quad D_V := \sum_{\alpha=1}^{n} D_{V_\alpha}, \quad D_T := \sum_{\alpha=1}^{n} D_{T_\alpha}.$$

Note that the above diameter functionals D_X, D_V and D_T measure the total deviations of position, velocity and temperature to each cluster group Z_α, respectively.

To reduce the TCSUS system (4.1) to its appropriate dissipative structure, we employ the following functionals: For $\alpha = 1,\cdots,n$,

$$\Phi_{\alpha ij}(t) := \frac{\phi(\|x_{\alpha i}-x_{\alpha j}\|)}{N_\alpha} + \left(1 - \frac{\sum_{j=1}^{N_\alpha}\phi(\|x_{\alpha i}-x_{\alpha j}\|)}{N_\alpha}\right)\delta_{ij},$$

where δ_{ij} denotes the Kronecker delta. Next, for simplicity, we set

$$\phi_{\alpha ij} := \phi(\|x_{\alpha i} - x_{\alpha j}\|).$$

Then, we can easily check that $\Phi_{\alpha ij}$ satisfies the following properties:

1. $\Phi_{\alpha ij} \geq \dfrac{\phi_{\alpha ij}}{N_\alpha}, \quad \displaystyle\sum_{j=1}^{N_\alpha} \Phi_{\alpha ij} = 1, \quad \Phi_{\alpha ij} = \Phi_{\alpha ji},$
2. $\displaystyle\sum_{j=1}^{N_\alpha} \Phi_{\alpha ij} \frac{(v_{\alpha j} - \langle v_{\alpha j}, v_{\alpha i}\rangle v_{\alpha i})}{T_{\alpha j}} = \sum_{j=1}^{N_\alpha} \frac{\phi_{\alpha ij}}{N_\alpha} \frac{(v_{\alpha j} - \langle v_{\alpha j}, v_{\alpha i}\rangle v_{\alpha i})}{T_{\alpha j}}.$

Similarly, we can observe that the functional $\Psi_{\alpha ij}$ defined by

$$\Psi_{\alpha ij}(t) := \frac{\zeta(\|x_{\alpha i} - x_{\alpha j}\|)}{N_\alpha} + \left(1 - \frac{\sum_{j=1}^{N_\alpha} \zeta(\|x_{\alpha i} - x_{\alpha j}\|)}{N_\alpha}\right)\delta_{ij}, \quad \zeta_{\alpha ij} := \zeta(\|x_{\alpha i} - x_{\alpha j}\|)$$

satisfies the following relations:

1. $\Psi_{\alpha ij} \geq \dfrac{\zeta_{\alpha ij}}{N_\alpha}, \quad \displaystyle\sum_{j=1}^{N_\alpha} \Psi_{\alpha ij} = 1, \quad \Psi_{\alpha ij} = \Psi_{\alpha ji},$
2. $\displaystyle\sum_{j=1}^{N_\alpha} \Psi_{\alpha ij} \left(\frac{1}{T_{\alpha i}} - \frac{1}{T_{\alpha j}}\right) = \sum_{j=1}^{N_\alpha} \frac{\zeta_{\alpha ij}}{N_\alpha} \left(\frac{1}{T_{\alpha i}} - \frac{1}{T_{\alpha j}}\right).$

We note that the above functionals of this type have already been used several times in previous literature [2, 6, 25, 27, 28]. Unlike the aforementioned previous papers, the above functionals can be applied to a multi-cluster setting.

In what follows, we induce dissipative differential inequalities in terms of D_X, D_V and D_T, respectively, to deduce several sufficient frameworks for the multi-cluster flocking estimate of system (4.1).

Lemma 4.2. (Dissipative structure) *Suppose that* $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ *is a solution to the system* (4.1). *If we set* ϕ_M *and* ζ_M *as*

$$\phi_M(t) := \max_{\alpha\neq\beta,i,j} \phi(\|x_{\beta j} - x_{\alpha i}\|), \quad \zeta_M(t) := \max_{\alpha\neq\beta,i,j} \zeta(\|x_{\beta j} - x_{\alpha i}\|).$$

Then, we have the following three differential inequalities for a.e. $t \in \mathbb{R}_+ - \{0\}$*:*

1. $\left|\dfrac{dD_X}{dt}\right| \leq D_V,$
2. $\dfrac{dD_V}{dt} \leq -\dfrac{\kappa_1 \min(N_1, \cdots, N_\alpha)\phi(D_X)}{NT_M^\infty} D_V + \dfrac{\kappa_1 \max(N_1, \cdots, N_\alpha) D_V^3}{2NT_m^\infty} + \dfrac{2\kappa_1(n-1)\phi_M}{T_m^\infty},$
3. $\dfrac{dD_T}{dt} \leq -\dfrac{\kappa_2 \min(N_1, \cdots, N_\alpha)\zeta(D_X)}{N(T_M^\infty)^2} D_T + 2\kappa_2(n-1)\zeta_M \left(\dfrac{1}{T_m^\infty} - \dfrac{1}{T_M^\infty}\right).$

Proof. Cauchy–Schwarz's inequality immediately yields the first assertion. Next, to prove the third assertion, we choose two indices, M_t and m_t, depending on t, such that

$$D_{T_\alpha}(t) = T_{\alpha M_t}(t) - T_{\alpha m_t}(t), \quad 1 \leq m_t, M_t \leq N_\alpha.$$

Now, we recall the subsystem (4.1c) as follows:

$$\dot{T}_{\alpha i}=\frac{\kappa_2}{N}\sum_{j=1}^{N_\alpha}\zeta(\|x_{\alpha i}-x_{\alpha j}\|)\left(\frac{1}{T_{\alpha i}}-\frac{1}{T_{\alpha j}}\right)+\frac{\kappa_2}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha i}-x_{\beta j}\|)\left(\frac{1}{T_{\alpha i}}-\frac{1}{T_{\beta j}}\right).$$

Then, for a.e. $t\in\mathbb{R}_+-\{0\}$, one can show that by using the definitions of M_t and m_t

$$\begin{aligned}\frac{dD_{T_\alpha}}{dt}=&\frac{\kappa_2}{N}\sum_{j=1}^{N_\alpha}\zeta(\|x_{\alpha M_t}-x_{\alpha j}\|)\left(\frac{1}{T_{\alpha M_t}}-\frac{1}{T_{\alpha j}}\right)\\&-\frac{\kappa_2}{N}\sum_{j=1}^{N_\alpha}\zeta(\|x_{\alpha m_t}-x_{\alpha j}\|)\left(\frac{1}{T_{\alpha m_t}}-\frac{1}{T_{\alpha j}}\right)\\&+\frac{\kappa_2}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha M_t}-x_{\beta j}\|)\left(\frac{1}{T_{\alpha M_t}}-\frac{1}{T_{\beta j}}\right)\\&-\frac{\kappa_2}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha m_t}-x_{\beta j}\|)\left(\frac{1}{T_{\alpha m_t}}-\frac{1}{T_{\beta j}}\right)\\=:&\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.\end{aligned}$$

(i) (Estimate of $\mathcal{I}_1+\mathcal{I}_2$) Similar to the proof of Lemma 3.2 in [2], for a.e. $t\in\mathbb{R}_+-\{0\}$,

$$\mathcal{I}_1+\mathcal{I}_2\leq-\frac{\kappa_2N_\alpha\zeta(D_{X_\alpha})}{N(T_M^\infty)^2}D_{T_\alpha}.$$

(ii) (Estimate of $\mathcal{I}_3+\mathcal{I}_4$) From Proposition 2.2 and the definitions of ϕ_M and ϕ_m, for a.e. $t\in\mathbb{R}_+-\{0\}$,

$$\begin{aligned}\mathcal{I}_3+\mathcal{I}_4\leq&\frac{\kappa_2}{N}\left|\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha M_t}-x_{\beta j}\|)\left(\frac{1}{T_{\alpha M_t}}-\frac{1}{T_{\beta j}}\right)\right|\\&+\frac{\kappa_2}{N}\left|\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha m_t}-x_{\beta j}\|)\left(\frac{1}{T_{\alpha m_t}}-\frac{1}{T_{\beta j}}\right)\right|\\\leq&\frac{2\kappa_2(N-N_\alpha)\zeta_M}{N}\left(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}\right).\end{aligned}$$

Thus, combining $\mathcal{I}_1+\mathcal{I}_2$ and $\mathcal{I}_3+\mathcal{I}_4$ yields that for a.e. $t\in\mathbb{R}_+-\{0\}$,

$$\frac{dD_{T_\alpha}}{dt}\leq-\frac{\kappa_2N_\alpha\zeta(D_{X_\alpha})}{N(T_M^\infty)^2}D_{T_\alpha}+\frac{2\kappa_2(N-N_\alpha)\zeta_M}{N}\left(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}\right).$$

Therefore, we take the summation from α=1 to n to the above inequality to get that for a.e. $t\in\mathbb{R}_+-\{0\}$,

$$\frac{dD_T}{dt}\leq-\frac{\kappa_2\min(N_1,\cdots,N_\alpha)\zeta(D_X)}{N(T_M^\infty)^2}D_T+2\kappa_2(n-1)\zeta_M\left(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}\right).$$

To verify the second assertion, we select two indices M_t and m_t depending on t satisfying

$$D_{V_\alpha}(t)=\|v_{\alpha M_t}(t)-v_{\alpha m_t}(t)\|,\quad 1\leq m_t,M_t\leq N_\alpha.$$

We recall the following velocity coupling Eq (4.1b):

$$\begin{aligned}\dot{v}_{\alpha i} =& \frac{\kappa_1}{N}\sum_{j=1}^{N_\alpha}\phi(\|x_{\alpha i}-x_{\alpha j}\|)\frac{(v_{\alpha j}-\langle v_{\alpha i}, v_{\alpha j}\rangle v_{\alpha i})}{T_{\alpha j}} \\ &+ \frac{\kappa_1}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i}-x_{\beta j}\|)\frac{(v_{\beta j}-\langle v_{\alpha i}, v_{\beta j}\rangle v_{\alpha i})}{T_{\beta j}}.\end{aligned}$$

Hence, we attain that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\begin{aligned}\frac{1}{2}\frac{dD_{V_\alpha}^2}{dt} &= \langle v_{\alpha M_t}-v_{\alpha m_t}, \dot{v}_{\alpha M_t}-\dot{v}_{\alpha m_t}\rangle \\ &= \left\langle v_{\alpha M_t}-v_{\alpha m_t}, \frac{\kappa_1}{N}\sum_{j=1}^{N_\alpha}\phi_{\alpha M_t j}\left(\frac{v_{\alpha j}-\langle v_{\alpha M_t}, v_{\alpha j}\rangle v_{\alpha M_t}}{T_{\alpha j}}\right)\right. \\ &\qquad\qquad \left. - \frac{\kappa_1}{N}\sum_{j=1}^{N_\alpha}\phi_{\alpha m_t j}\left(\frac{v_{\alpha j}-\langle v_{\alpha m_t}, v_{\alpha j}\rangle v_{\alpha m_t}}{T_{\alpha j}}\right)\right\rangle \\ &\quad + \left\langle v_{\alpha M_t}-v_{\alpha m_t}, \frac{\kappa_1}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha M_t}-x_{\beta j}\|)\left(\frac{v_{\beta j}-\langle v_{\alpha M_t}, v_{\beta j}\rangle v_{\alpha M_t}}{T_{\beta j}}\right)\right. \\ &\qquad\qquad \left. - \frac{\kappa_1}{N}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha m_t}-x_{\beta j}\|)\left(\frac{v_{\beta j}-\langle v_{\alpha m_t}, v_{\beta j}\rangle v_{\alpha m_t}}{T_{\beta j}}\right)\right\rangle \\ &=: \mathcal{J}_1 + \mathcal{J}_2.\end{aligned}$$

(*iii*) (Estimate of $\mathcal{J}_1$) In the same way as the proof of Lemma 3.2 of [2], for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\mathcal{J}_1 \le -\frac{\kappa_1 N_\alpha}{N}\left(\frac{\phi(D_{X_\alpha})}{T_M^\infty} - \frac{D_{V_\alpha}^2}{2T_m^\infty}\right)D_{V_\alpha}^2.$$

(*iiii*) (Estimate of $\mathcal{J}_2$) We employ the following identities:

$$\|v_{\beta j}-\langle v_{\alpha M_t}, v_{\beta j}\rangle v_{\alpha M_t}\| \le 1, \quad \|v_{\beta j}-\langle v_{\alpha m_t}, v_{\beta j}\rangle v_{\alpha m_t}\| \le 1$$

with Cauchy–Schwarz's inequality and Proposition 2.2 to estimate that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\begin{aligned}\mathcal{J}_2 &\le \frac{\kappa_1 D_{V_\alpha}}{N}\left\|\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha M_t}-x_{\beta j}\|)\left(\frac{v_{\beta j}-\langle v_{\alpha M_t}, v_{\beta j}\rangle v_{\alpha M_t}}{T_{\beta j}}\right)\right\| \\ &\quad + \frac{\kappa_1 D_{V_\alpha}}{N}\left\|\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha m_t}-x_{\beta j}\|)\left(\frac{v_{\beta j}-\langle v_{\alpha m_t}, v_{\beta j}\rangle v_{\alpha m_t}}{T_{\beta j}}\right)\right\| \\ &\le \frac{2\kappa_1(N-N_\alpha)\phi_M D_{V_\alpha}}{NT_m^\infty}.\end{aligned}$$

Then, we combine $\mathcal{J}_1$ and $\mathcal{J}_2$ to derive that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\frac{dD_{V_\alpha}}{dt} \le -\frac{\kappa_1 N_\alpha}{N}\left(\frac{\phi(D_{X_\alpha})}{T_M^\infty} - \frac{D_{V_\alpha}^2}{2T_m^\infty}\right)D_{V_\alpha} + \frac{2\kappa_1(N-N_\alpha)\phi_M}{NT_m^\infty}.$$

We take the summation from α=1 to n to the above inequality to obtain that

$$\frac{dD_V}{dt} \le -\frac{\kappa_1 \min(N_1,\cdots,N_\alpha)\phi(D_X)}{NT_M^\infty}D_V + \frac{\kappa_1 \max(N_1,\cdots,N_\alpha)D_V^3}{2NT_m^\infty} + \frac{2\kappa_1(n-1)\phi_M}{T_m^\infty},$$

because the monotonicity of ϕ implies that

$$D_V^3 \ge \sum_\alpha D_{V_\alpha}^3, \quad \min(\phi(D_{X_1}),\cdots,\phi(D_{X_\alpha})) \ge \phi(D_X).$$

Finally, we demonstrate the second assertion.

Remark 4.1. *In Lemma* 4.2*, the two terms below*

$$\begin{aligned}&-\frac{\kappa_1 \min(N_1,\cdots,N_\alpha)\phi(D_X)}{NT_M^\infty}D_V + \frac{\kappa_1 \max(N_1,\cdots,N_\alpha)D_V^3}{2NT_m^\infty},\\ &-\frac{\kappa_2 \min(N_1,\cdots,N_\alpha)\zeta(D_X)}{N(T_M^\infty)^2}D_T\end{aligned}$$

are related to the velocity alignment and temperature equilibrium for each cluster group of system (4.1)*, respectively. Meanwhile, the following terms in Lemma* 4.2

$$\frac{2(n-1)\phi_M\kappa_1}{T_m^\infty}, \qquad 2(n-1)\zeta_M\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\kappa_2$$

show the tendency of the velocities and temperatures of system (4.1) *to separated into n multi-cluster groups in system* (4.1).

4.3. Multi-cluster flocking

This subsection describes suitable sufficient frameworks $(\mathcal{H})$ for the multi-cluster flocking estimate and then, under $(\mathcal{H})$, we demonstrate the multi-cluster flocking of the proposed system (4.1). For this, we first display the admissible data $(\mathcal{H})$ as follows:

$$(\mathcal{H}) := \{(X(0), V(0), T(0)) \in \mathbb{R}^{2dN} \times (\mathbb{R}_+ - \{0\})^N \mid (\mathcal{H}_0), (\mathcal{H}_1), (\mathcal{H}_2) \text{ and } (\mathcal{H}_3) \text{ hold.}\}$$

(i) $(\mathcal{H}_0)$ (Notation): For brevity, we denote the following notation:

$$\begin{aligned}&\Lambda := \frac{D_V(0)}{\Lambda_0} + \frac{4(n-1)\kappa_1}{T_m^\infty \Lambda_0^2}\phi\left(\frac{r_0}{2}\right) + \frac{4(n-1)\kappa_1}{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\,T_m^\infty \Lambda_0}\int_{\frac{r_0}{2}}^\infty \phi(s)ds,\\ &\delta \in (0,1), \quad \bar{\Lambda}_0 := \frac{\kappa_2 \min(N_1,\cdots,N_\alpha)\zeta(D_X^\infty)}{N(T_M^\infty)^2},\end{aligned}$$

$$\Lambda_0 := \frac{\delta\kappa_1 \min(N_1,\cdots,N_\alpha)\phi(D_X^\infty)}{NT_M^\infty},$$
$$\Lambda_\alpha := \frac{\kappa_1 N_\alpha}{NT_m^\infty}\Lambda_0 + \frac{\kappa_1(N-N_\alpha)}{NT_m^\infty\,(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))}\int_{\frac{r_0}{2}}^\infty \phi(s)ds,$$
$$r_0 := \min_{\alpha<\beta,i,j}(x_{\beta j}^k(0)-x_{\alpha i}^k(0)) \text{ for some fixed } 1\le k\le d.$$

(*ii*) ($\mathcal{H}_1$) (Well prepared conditions): There exists a strictly positive number $D_X^\infty > 0$ such that

$$D_X^\infty \ge D_X(0)+\Lambda, \quad \text{and} \quad \phi \text{ is integrable i.e., } \|\phi\|_{L^1}<\infty.$$

(*iii*) ($\mathcal{H}_2$) (Separated initial data): For fixed $1\le k\le d$ in ($\mathcal{H}_0$), there exist real sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ such that the initial data and system parameters are selected to be split suitably as follows:

$$r_0>0,\quad a_1<b_1<a_2<b_2\cdots<a_n<b_n,\quad I_\alpha:=[a_\alpha,b_\alpha]\subset[-1,1],\quad I_\alpha\cap I_\beta=\emptyset\ (\beta\ne\alpha),$$
$$[v_{\alpha i}^k(0)-\Lambda_\alpha, v_{\alpha i}^k(0)+\Lambda_\alpha]\subset I_\alpha:=[a_\alpha,b_\alpha]\subset[-1,1],\quad \alpha,\beta=1,\cdots,n,\quad i=1,\cdots,N_\alpha.$$

(*iii*) ($\mathcal{H}_3$) (Small fluctuations and coupling strength): The local velocity perturbation for each cluster group and coupling strength are sufficiently small as follows:

$$\frac{2\kappa_1(\sqrt{1+\delta}+1)(n-1)\int_{\frac{r_0}{2}}^\infty \phi(s)ds}{\delta\,(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))\,T_m^\infty} < D_V(0) \le \sqrt{\frac{2(1-\delta)\phi(D_X^\infty)\min(N_1,\cdots,N_\alpha)T_m^\infty}{(1+\delta)\max(N_1,\cdots,N_\alpha)T_M^\infty}}.$$

Next, we give a brief comment regarding ($\mathcal{H}$). The assumption ($\mathcal{H}_1$) is that the sufficient condition guarantees a group formation to each cluster group. Note that ($\mathcal{H}_2$) implies that position initial data for each cluster group should be sufficiently separate from each other to verify the multi-cluster flocking result. Indeed, if $v_{\alpha i}^k(0)$ is covered by $I_\alpha := [a_\alpha, b_\alpha]$, then we take sufficiently small κ_1 so that $[v_{\alpha i}^k(0)-\Lambda_\alpha, v_{\alpha i}^k(0)+\Lambda_\alpha]\subset I_\alpha$ because Λ_α is linearly proportional to κ_1. ($\mathcal{H}_3$) describes that the velocity perturbation between each pair of cluster groups is sufficiently small to deduce the velocity alignment for each cluster group. Here, we can find the admissible data satisfying the assumption ($H3$) when κ_1 is sufficiently small. Moreover, under sufficiently large r_0 and suitable temperature initial data and small coupling strength regime, we can check that the sufficient framework ($\mathcal{H}$) is admissible data.

To prove the multi-cluster flocking result, we define the following set:

$$S := \left\{ s>0 \,\middle|\, \min_{\alpha\ne\beta,i,j}\|x_{\alpha i}(t)-x_{\beta j}(t)\| \ge \left(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\right)t+\frac{r_0}{2},\quad t\in[0,s)\right\},$$

where $d(I_\alpha, I_{\alpha+1})$ is a distance between adjacent intervals I_α and $I_{\alpha+1}$. Herein, we observe that S is nonempty due to the assumption ($\mathcal{H}_2$) and the continuity of $\|x_{\alpha i}(t)-x_{\beta j}(t)\|$, and we set

$$\sup S =: T^*.$$

Lemma 4.3. *Assume that $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ is a solution to the system* (4.1)*. Suppose that* ($\mathcal{H}_0$)*,* ($\mathcal{H}_1$)*, and* ($\mathcal{H}_3$) *hold. Then, it follows that*

$$D_V(t) < \sqrt{(1+\delta)}D_V(0),\quad D_X(t)\le D_X^\infty,\quad t\in[0,T^*]. \tag{4.2}$$

Proof. First, we consider

$$S' := \left\{ s > 0 \,\middle|\, \text{the desired estimates Eq (4.2) hold}, \quad t \in [0, s], \quad s \le T^* \right\}.$$

Let $\sup S' =: T^{**}$ and suppose that $T^{**} < T^*$ for the proof by contradiction. Then, one has for $t \in [0, T^{**}]$,

$$D_V^2(t) \le (1+\delta) D_V^2(0) \le \frac{2(1-\delta)\phi(D_X^\infty)\min(N_1, \cdots, N_\alpha) T_m^\infty}{\max(N_1, \cdots, N_\alpha) T_M^\infty}$$

and

$$-\frac{\kappa_1 \min(N_1, \cdots, N_\alpha)\phi(D_X)}{N T_M^\infty} \le -\frac{\kappa_1 \min(N_1, \cdots, N_\alpha)\phi(D_X^\infty)}{N T_M^\infty}.$$

Then, for a.e. $t \in (0, T^{**})$, the second assertion of Lemma 4.2 and the above estimates lead to the following inequalities:

$$\begin{aligned}
\frac{dD_V}{dt} &\le -\frac{\kappa_1 \min(N_1, \cdots, N_\alpha)\phi(D_X)}{N T_M^\infty} D_V + \frac{\kappa_1 \max(N_1, \cdots, N_\alpha) D_V^3}{2N T_m^\infty} + \frac{2\kappa_1 (n-1)\phi_M}{T_m^\infty} \\
&\le -\frac{\delta \kappa_1 \min(N_1, \cdots, N_\alpha)\phi(D_X^\infty)}{N T_M^\infty} D_V + \frac{2\kappa_1 (n-1)\phi_M}{T_m^\infty} \\
&= -\Lambda_0 D_V + \frac{2\kappa_1 (n-1)\phi_M}{T_m^\infty}.
\end{aligned}$$

This gives from Grönwall's lemma that for $t \in [0, T^{**}]$,

$$\begin{aligned}
D_V(t) \le & D_V(0)\exp(-\Lambda_0 t) + \frac{2\kappa_1(n-1)}{T_m^\infty \Lambda_0}\exp\left(-\frac{\Lambda_0}{2}t\right)\phi\left(\frac{r_0}{2}\right) \\
&+ \frac{2\kappa_1(n-1)}{T_m^\infty \Lambda_0}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, t + r_0}{2}\right),
\end{aligned} \tag{4.3}$$

where we used the definition of S and the fact that

$$\phi_M \le \phi\left(\left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})\right) t + \frac{r_0}{2}\right).$$

Moreover, we again employ Grönwall's lemma to reach that for $t \in [0, T^{**}]$,

$$D_V(t) \le D_V(0)\exp(-\Lambda_0 t) + \frac{2\kappa_1(n-1)}{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, T_m^\infty} \int_{\frac{r_0}{2}}^\infty \phi(s)ds. \tag{4.4}$$

Next, using the definition of T^{**} yields that

$$D_V^2(T^{**}) = (1+\delta) D_V^2(0) \quad \text{or} \quad D_X(T^{**}) = D_X^\infty.$$

In the former case, it is contradictory to $(\mathcal{H}_3)$ because inequality (4.4) implies that

$$D_V(T^{**}) = \sqrt{1+\delta} D_V(0) \le D_V(0) + \frac{2\kappa_1(n-1)}{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, T_m^\infty} \int_{\frac{r_0}{2}}^\infty \phi(s)ds.$$

In the latter case, we estimate from inequality (4.3) that for $t \in [0, T^{**}]$,

$$\begin{aligned} D_X(t) &\le D_X(0) + \int_0^t D_V(s)ds \\ &\le D_X(0) + \int_0^t \Big[D_V(0)\exp(-\Lambda_0 s) + \frac{2\kappa_1(n-1)}{T_m^\infty \Lambda_0} \exp\left(-\frac{\Lambda_0}{2}s\right)\phi\left(\frac{r_0}{2}\right) \\ &\qquad\qquad + \frac{2\kappa_1(n-1)}{T_m^\infty \Lambda_0}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right)\Big] ds \\ &< D_X(0) + \Lambda \le D_X^\infty. \end{aligned} \tag{4.5}$$

Accordingly, $D_X(T^{**}) < D_X^\infty$, which is contradictory. Finally, $\sup S' = T^{**} = T^*$. We have reached the desired lemma.

Subsequently, we claim that $T^* = \infty$, which is crucial to derive the multi-cluster flocking estimate of the system (4.1).

Theorem 4.1. *Following Lemma 4.3, we further assume that* $(\mathcal{H}_2)$ *holds. Then, we get that*

$$T^* = \infty.$$

This is equivalent to

$$\min_{\alpha\neq\beta,i,j} \|x_{\alpha i}(t) - x_{\beta j}(t)\| \ge \left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})\right)t + \frac{r_0}{2}, \quad t \in \mathbb{R}_+.$$

Proof. For the proof by contradiction, suppose that $T^* < \infty$. From the definition of S, we select four indices that satisfy

$$1 \le \alpha^* < \beta^* \le n, \quad i^* \in \{1, \cdots, N_{\alpha^*}\} \quad \text{and} \quad j^* \in \{1, \cdots, N_{\beta^*}\}$$

such that

$$\|x_{\alpha^* i^*}(T^*) - x_{\beta^* j^*}(T^*)\| = \left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})\right)T^* + \frac{r_0}{2}.$$

Then, we show that for the $k \in \{1, \cdots, d\}$ chosen in $(\mathcal{H}_0)$,

$$\begin{aligned} \|x_{\alpha^* i^*}(T^*) - x_{\beta^* j^*}(T^*)\| &\ge x^k_{\beta^* j^*}(T^*) - x^k_{\alpha i^*}(T^*) \\ &= x^k_{\beta^* j^*}(0) - x^k_{\alpha^* i^*}(0) + \int_0^{T^*} \left(v^k_{\beta^* j^*}(t) - v^k_{\alpha^* i^*}(t)\right)dt \\ &\ge r_0 + \int_0^{T^*} \left(v^k_{\beta^* j^*}(t) - v^k_{\alpha^* i^*}(t)\right)dt. \end{aligned}$$

Next, we integrate system (4.1b) and employ the following relation:

$$\|v_{\alpha j} - \langle v_{\alpha i}, v_{\alpha j}\rangle v_{\alpha i}\|^2 = 1 - \langle v_{\alpha i}, v_{\alpha j}\rangle^2 = (1 - \langle v_{\alpha i}, v_{\alpha j}\rangle)(1 + \langle v_{\alpha i}, v_{\alpha j}\rangle) \le D^2_{V_\alpha}$$

to attain that for $t \in [0, T^*]$,

$$|v^k_{\alpha i}(t) - v^k_{\alpha i}(0)| \le \|v_{\alpha i}(t) - v_{\alpha i}(0)\| \le \int_0^t \|\dot{v}_{\alpha i}\| ds$$

$$
\begin{aligned}
&\leq \frac{\kappa_1 N_\alpha}{NT_m^\infty}\int_0^t D_{V_\alpha}(s)ds + \frac{\kappa_1(N-N_\alpha)}{NT_m^\infty}\int_0^t \phi_M(s)ds \\
&\leq \frac{\kappa_1 N_\alpha}{NT_m^\infty}\int_0^\infty D_{V_\alpha}(s)ds + \frac{\kappa_1(N-N_\alpha)}{NT_m^\infty}\int_0^\infty \phi\left(\min_{1\leq\alpha\leq n-1} d(I_\alpha, I_{\alpha+1})s + \frac{r_0}{2}\right)ds \\
&\leq \frac{\kappa_1 N_\alpha}{NT_m^\infty}\int_0^\infty D_{V}(s)ds + \frac{\kappa_1(N-N_\alpha)}{NT_m^\infty}\int_0^\infty \phi\left(\min_{1\leq\alpha\leq n-1} d(I_\alpha, I_{\alpha+1})s + \frac{r_0}{2}\right)ds \\
&\leq \frac{\kappa_1 N_\alpha}{NT_m^\infty}\Lambda + \frac{\kappa_1(N-N_\alpha)}{NT_m^\infty}\int_0^\infty \phi\left(\min_{1\leq\alpha\leq n-1} d(I_\alpha, I_{\alpha+1})s + \frac{r_0}{2}\right)ds \\
&= \frac{\kappa_1 N_\alpha}{NT_m^\infty}\Lambda + \frac{\kappa_1(N-N_\alpha)}{NT_m^\infty\left(\min_{1\leq\alpha\leq n-1} d(I_\alpha, I_{\alpha+1})\right)}\int_{\frac{r_0}{2}}^\infty \phi(s)ds =: \Lambda_\alpha,
\end{aligned}
$$

where we used $\phi \leq 1$, $\|v_{\beta j} - \langle v_{\alpha i}, v_{\beta j}\rangle v_{\alpha i}\| \leq 1$, and Λ was estimated in inequality (4.5). Therefore, it follows by $(\mathcal{H}_2)$ that for $\alpha = 1, \cdots, n$,

$$
\begin{aligned}
v_{\alpha i}^k(0) + \Lambda_\alpha &\geq v_{\alpha i}^k(0) + |v_{\alpha i}^k(t) - v_{\alpha i}^k(0)| \geq v_{\alpha i}^k(t) = v_{\alpha i}^k(0) + v_{\alpha i}^k(t) - v_{\alpha i}^k(0) \\
&\geq v_{\alpha i}^k(0) - |v_{\alpha i}^k(t) - v_{\alpha i}^k(0)| \geq v_{\alpha i}^k(0) - \Lambda_\alpha \Longrightarrow v_{\alpha i}^k(t) \in I_\alpha.
\end{aligned}
$$

Then, we derive that using the assumption $(\mathcal{H}_2)$,

$$
\begin{aligned}
\|x_{\alpha^* i^*}(T^*) - x_{\beta^* j^*}(T^*)\| &\geq r_0 + \int_0^{T^*}\left(v_{\beta^* j^*}^k(t) - v_{\alpha^* i^*}^k(t)\right)dt \\
&> \frac{r_0}{2} + \min_{1\leq\alpha\leq n-1} d(I_\alpha, I_{\alpha+1})T_*,
\end{aligned}
$$

which gives a contradiction to $T^* < \infty$. Consequently, we conclude that $T^* = \infty$.

Now, we are ready to prove the multi-cluster flocking dynamics under sufficient framework $(\mathcal{H})$ by applying Lemma 4.3 and Theorem 4.1. In addition, we verify that there exist common velocity and temperature convergence values depending on the decay rates of the integrable communication weights ϕ and ζ, respectively, in each cluster group.

Theorem 4.2. *Let $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ be a solution to the system* (4.1) *and suppose that the frameworks $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$ hold. Then, we obtain the following assertions for $t \in \mathbb{R}_+$:*

1. *(Velocity alignment for each cluster group)*

$$
\begin{aligned}
D_V(t) &\leq D_V(0)\exp(-\Lambda_0 t) + \frac{2\kappa_1(n-1)}{T_m^\infty \Lambda_0}\exp\left(-\frac{\Lambda_0}{2}t\right)\phi\left(\frac{r_0}{2}\right) \\
&\quad + \frac{2\kappa_1(n-1)}{T_m^\infty \Lambda_0}\phi\left(\frac{(\min_{1\leq\alpha\leq n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right).
\end{aligned}
$$

2. *(Temperature equilibrium for each cluster group)*

$$
\begin{aligned}
D_T(t) &\leq D_T(0)\exp(-\bar{\Lambda}_0 t) + 2\kappa_2(n-1)\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\exp\left(-\frac{\bar{\Lambda}_0}{2}t\right)\zeta\left(\frac{r_0}{2}\right) \\
&\quad + 2\kappa_2(n-1)\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\zeta\left(\frac{(\min_{1\leq\alpha\leq n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right).
\end{aligned}
$$

Proof. We apply the second assertion of Lemma 4.2, the definition of the set S, and Theorem 4.1 to have that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\begin{aligned}\frac{dD_V}{dt} &\le -\Lambda_0 D_V + \frac{2\kappa_1(n-1)}{T_m^\infty}\phi_M \\ &\le -\Lambda_0 D_V + \frac{2\kappa_1(n-1)}{T_m^\infty}\phi\left(\frac{r_0}{2} + \min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})t\right).\end{aligned}$$

From inequality (4.3), we recall that for $t \in \mathbb{R}_+$,

$$\begin{aligned}D_V(t) \le D_V(0)\exp(-\Lambda_0 t) &+ \frac{2\kappa_1(n-1)}{T_m^\infty \Lambda_0}\exp\left(-\frac{\Lambda_0}{2}t\right)\phi\left(\frac{r_0}{2}\right) \\ &+ \frac{2\kappa_1(n-1)}{T_m^\infty\Lambda_0}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, t + r_0}{2}\right).\end{aligned}$$

Hence, we reach the desired first assertion. To prove the second assertion, we employ the third assertion of Lemma 4.2, Theorem 4.1, and the second assertion of Lemma 4.3 to get that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\begin{aligned}\frac{dD_T}{dt} &\le -\frac{\kappa_2 \min(N_1, \cdots, N_\alpha)\zeta(D_X)}{N(T_M^\infty)^2} D_T + 2\kappa_2(n-1)\zeta_M\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right) \\ &\le -\bar{\Lambda}_0 D_T + 2\kappa_2(n-1)\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\zeta\left(\left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})\right)t + \frac{r_0}{2}\right).\end{aligned}$$

We use Grönwall's lemma to yield that for $t \in \mathbb{R}_+$,

$$\begin{aligned}D_T(t) \le D_T(0)\exp(-\bar{\Lambda}_0 t) &+ 2\kappa_2(n-1)\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\exp\left(-\frac{\bar{\Lambda}_0}{2}t\right)\zeta\left(\frac{r_0}{2}\right) \\ &+ 2\kappa_2(n-1)\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, t + r_0}{2}\right).\end{aligned}$$

We conclude the desired second assertion.

As a direct consequence, we present the following result that the velocity and temperature of each agent in each cluster group converge to some same nonnegative value, respectively:

Corollary 4.1. *Assume that $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ is a solution to system* (4.1). *Then, under the sufficient frameworks ($\mathcal{H}_0$), ($\mathcal{H}_1$), ($\mathcal{H}_2$), and ($\mathcal{H}_3$), there exist some convergence values v_α^∞ and T_α^∞ for $\alpha = 1, \cdots, n$ that satisfy that for $t \in \mathbb{R}_+$,*

1. *(Velocity convergence value for each cluster group)*

$$\begin{aligned}\|v_{\alpha i}(t) - v_\alpha^\infty\| = \mathcal{O}\Bigg(\exp\left(-\frac{\Lambda_0}{2}t\right) &+ \phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) \\ &+ \int_t^\infty \phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) ds\Bigg).\end{aligned}$$

2. *(Temperature convergence value for each cluster group)*

$$|T_{\alpha i}(t) - T_\alpha^\infty| = O\Bigg(\exp\left(-\frac{\bar{\Lambda}_0}{2}t\right) + \zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) + \int_t^\infty \zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) ds\Bigg).$$

Proof. Remember from Lemma 4.1 that

$$N_\alpha \dot{v}_\alpha^{cen} = \frac{\kappa_1}{N}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\alpha}\phi(\|x_{\alpha i} - x_{\alpha j}\|)\frac{v_{\alpha i}\|v_{\alpha j} - v_{\alpha i}\|^2}{2T_{\alpha j}} + \frac{\kappa_1}{N}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i} - x_{\beta j}\|)\left(v_{\beta j} - v_{\alpha i} + \frac{v_{\alpha i}\|v_{\beta k} - v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}}.$$

If we denote v_α^∞ as

$$\begin{aligned} v_\alpha^\infty &:= \lim_{t\to\infty} v_\alpha^{cen}(t) \\ &= v_\alpha^{cen}(0) + \frac{\kappa_1}{NN_\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\alpha}\int_0^\infty \phi(\|x_{\alpha i} - x_{\alpha j}\|)\frac{v_{\alpha i}\|v_{\alpha j} - v_{\alpha i}\|^2}{2T_{\alpha j}} \\ &\quad + \frac{\kappa_1}{NN_\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_0^\infty \phi(\|x_{\alpha i} - x_{\beta j}\|)\left(v_{\beta j} - v_{\alpha i} + \frac{v_{\alpha i}\|v_{\beta j} - v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}}, \end{aligned}$$

then we have that

$$\|v_\alpha^{cen}(t) - v_\alpha^\infty\| \le \frac{\kappa_1}{NN_\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\alpha}\int_t^\infty \phi(\|x_{\alpha i} - x_{\alpha j}\|)\frac{v_{\alpha i}\|v_{\alpha j} - v_{\alpha i}\|^2}{2T_{\alpha j}} + \frac{\kappa_1}{NN_\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_t^\infty \phi(\|x_{\alpha i} - x_{\beta j}\|)\left(v_{\beta j} - v_{\alpha i} + \frac{v_{\alpha i}\|v_{\beta j} - v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}}$$

because

$$v_\alpha^{cen} = v_\alpha^{cen}(0) + \frac{\kappa_1}{NN_\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\alpha}\int_0^t \phi(\|x_{\alpha i} - x_{\alpha j}\|)\frac{v_{\alpha i}\|v_{\alpha j} - v_{\alpha i}\|^2}{2T_{\alpha j}}ds + \frac{\kappa_1}{NN_\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_0^t \phi(\|x_{\alpha i} - x_{\beta j}\|)\left(v_{\beta j} - v_{\alpha i} + \frac{v_{\alpha i}\|v_{\beta j} - v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}}ds.$$

Then, the multi-flocking estimate studied in Theorem 4.1 and the monotonicity and non-negativity of ϕ imply that

$$\|v_\alpha^{cen}(t) - v_\alpha^\infty\| = O\left(\exp(-\Lambda_0 t) + \int_t^\infty \phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) ds\right).$$

Subsequently, we recall from Theorem 4.1 that

$$\|v_{\alpha i}(t) - v_\alpha^{cen}(t)\| = \mathcal{O}\left(\exp\left(-\frac{\Lambda_0}{2}t\right) + \phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right)\right).$$

We combine the above estimates to derive that for $\alpha = 1, \cdots, n$,

$$\begin{aligned}\|v_{\alpha i}(t) - v_\alpha^\infty\| = \mathcal{O}\Bigg(&\exp\left(-\frac{\Lambda_0}{2}t\right) + \phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) \\ &+ \int_t^\infty \phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) ds\Bigg).\end{aligned}$$

Similar to the above, there exists some positive value T_α^∞ such that for $\alpha = 1, \cdots, n$,

$$\begin{aligned}|T_{\alpha i}(t) - T_\alpha^\infty| = \mathcal{O}\Bigg(&\exp\left(-\frac{\bar{\Lambda}_0}{2}t\right) + \zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) \\ &+ \int_t^\infty \zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0}{2}\right) ds\Bigg).\end{aligned}$$

We conclude the desired results.

5. Conclusion

In this paper, we have demonstrated various sufficient frameworks regarding the mono-cluster flocking, the non-emergence of mono-cluster flocking, and multi-cluster flocking of the TCSUS system. First, we presented the admissible data for the mono-cluster flocking of TCSUS to occur. From the result, we observed that the mono-cluster flocking occurs when the coupling strength is large enough, and then we were interested in how small the coupling strength must be to avoid mono-cluster flocking emerging. Second, we verified that if the coupling strength is smaller than some appropriate value in the TCSUS model with an integrable communication weight ϕ, then the mixed configuration gradually becomes separated after some time, and then each sub-ensemble simultaneously moves away linearly as the time increases. Hence, this showed the non-emergence of the mono-cluster flocking to the system. However, when ϕ is non-integrable, we did not provide a suitable sufficient framework for the non-emergence of the mono-cluster flocking and we only gave a sufficient condition independent of the coupling strength for mono-cluster flocking to occur. Third, employing the spatial separation r_0 and velocity separations I_α's, when the initial configuration is well separated given similar to multi-cluster, we proved that the multi-cluster flocking holds in the system with an integrable ϕ. The novelty of this paper is that we have extended the multi-cluster flocking of system (1.2) (see [29]) to a temperature field and generalize the bi-cluster flocking of system (1.5) (see [2]) to the multi-cluster flocking. We were unable to demonstrate a sufficient framework where the multi-cluster flocking emerges in a mixed initial configuration (not well separated) rather than from the multi-cluster flocking under the conditions such that the initial configuration is well separated could be an interesting research topic. This issue is left for future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work of H. Ahn was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (2022R1C12007321).

Conflict of interest

The authors declare there is no conflict of interest.

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