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Research article

Structure-preserving scheme for one dimension and two dimension fractional KGS equations

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Abstract: In the paper, we study structure-preserving scheme to solve general fractional Klein-Gordon-Schrödinger equations, including one dimension case and two dimension case. First, the high central difference scheme and Crank-Nicolson scheme are used to one dimension fractional Klein-Gordon-Schrödinger equations. We show that the arising scheme is uniquely solvable, and approximate solutions converge to the exact solution at the rate $O(\tau^2 + h^4)$. Moreover, we prove that the resulting scheme can preserve the mass and energy conservation laws. Second, we show Crank-Nicolson scheme for two dimension fractional Klein-Gordon-Schrödinger equations, and the proposed scheme preserves the mass and energy conservation laws in discrete formulations. However, the obtained discrete system is nonlinear system. Then, we show a equivalent form of fractional Klein-Gordon-Schrödinger equations by introducing some new auxiliary variables. The new system is discretized by the high central difference scheme and scalar auxiliary variable scheme, and a linear discrete system is obtained, which can preserve the energy conservation law. Finally, the numerical experiments including one dimension and two dimension fractional Klein-Gordon-Schrödinger systems are given to verify the correctness of theoretical results.

Keywords: Fractional Klein-Gordon-Schrödinger equations; conservation law; convergence; high central difference scheme; scalar auxiliary variable scheme

1. Introduction

In the paper, we consider fractional Klein-Gordon-Schrödinger (KGS) equations with the fractional Laplacian

$$i\partial_t u - \nu(-\Delta)^{\frac{\alpha}{2}} u + u f_1(|u|^2, \phi) = 0, \tag{1.1}$$

$$\partial_{tt}\phi - \Delta\phi + \phi - f_2(|u|^2, \phi) = 0, \tag{1.2}$$

where, $f_1(|u|^2, \phi) = \dfrac{\partial f(|u|^2, \phi)}{\partial |u|^2}$, $f_2(|u|^2, \phi) = \dfrac{\partial f(|u|^2, \phi)}{\partial \phi}$, $f : R^+ \times R \longmapsto R$, $u(\cdot, t)$, $\phi(\cdot, t)$ are complex and real valued unknown functions, and the fractional Laplace operator [1–4] is defined by

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} \int_{R^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha/2}} dy.$$

Let $\phi_t = v$. Then, the fractional KGS system (1.1)–(1.2) can be expressed as the following form

$$iu_t - \nu(-\Delta)^{\frac{\alpha}{2}} u = -u f_1(|u|^2, \phi), \tag{1.3}$$

$$\phi_t = v, \tag{1.4}$$

$$v_t - \Delta\phi + \phi = f_2(|u|^2, \phi). \tag{1.5}$$

The fractional KGS system (1.3)–(1.5) has the following mass and energy conserved laws

$$\text{Mass: } M(t) = \int_{-\infty}^{+\infty} |u(x, t)|^2 dx = M(0),$$

$$\text{Energy: } E(t) = \int_{-\infty}^{+\infty} \frac{1}{2}(v^2 + |\nabla\phi|^2 + \phi^2) + \nu|(-\Delta)^{\frac{\alpha}{4}} u|^2 - f(|u|^2, \phi) dx = E(0).$$

When $\alpha = 2$, the fractional KGS system (1.1) and (1.2) reduces to the integer KGS system [5–12]. When $\alpha \neq 2$, the fractional KGS system [13–17] describes various important physical phenomena. Recently, some attentions have been paid to fractional KGS system with fractional Laplacian, which are fractional version of classical KGS equations and consider long-range interactions. In general, the analytic solution of the fractional KGS system (1.1),(1.2) can not be derived due to nonlocality and nonlinearity. Therefore, numerical method plays an important role in the study of the fractional KGS system (1.1),(1.2). Various numerical methods [13–17] such as the finite difference method, pseudo-spectral method, and symplectic method have been studied the fractional KGS system with $f(|u|^2, \phi) = |u|^2\phi$, and the stability and convergence of the numerical methods have been discussed.

It is not difficult to find that there are some deficiencies in the numerical schemes of the fractional KGS system. The first deficiencie is that the above finite difference methods are based on fractional center difference scheme, and approximate solutions converge to the exact solution at the rate $O(\tau^2 + h^2)$. To the best of the authors'knowledge, there are few research on higher-order scheme for the fractional KGS system. The second deficiencie that the above numerical schemes are only based on one dimension fractional KGS system, and high dimension fractional KGS systems are rarely studied. In addition, there exist few reports on numerical scheme for general fractional KGS system in Eqs (1.1),(1.2) with $f(|u|^2, \phi) \neq |u|^2\phi$.

The main goal of this paper is to construct structure-preserving scheme [18–22] for solving one dimension and two dimension fractional KGS equations with fractional Laplacian operator. Some numerical schemes were proposed to approximate the fractional Laplacian operator. The fractional center difference scheme based on fractional Laplace operator was first developed in [23]. Based on this work, some high-order schemes were constructed, and the results have be applied to some fractional difference equations. In addition, finite element scheme and Fourier spectral scheme of fractional Laplace operator have been studied with some special boundary conditions. In this paper, we use high-order difference scheme for fractional Laplace operator with zero boundary condition. By using some useful lemmas, we prove that the scheme can preserve the mass and energy conservation laws.

All structure-preserving schemes for the nonlinear fractional KGS system ($f(|u|^2, \phi) \neq |u|^2\phi$) face how to efficiently solve a large nonlinear system at each time step. In this paper, we consider structure-preserving scheme for general fractional KGS equations (1.3)–(1.5). The high order central differences are used for the space direction, with the Crank-Nicolson scheme applied to the time direction. However, the numerical scheme is nonlinear scheme, and it takes too much time in the numerical simulation. Recently, scalar auxiliary variable scheme was introduce for gradient flows, and the numerical scheme only requiring solving decoupled linear systems at each time step. As far as we know, there exist few reports on scalar auxiliary variable scheme for fractional KGS system (1.3)–(1.5). In this paper, we show a new scalar auxiliary variable scheme to solve two dimension fractional KGS system (1.3)–(1.5).

In this paper, we first show numerical scheme to solve one dimension fractional KGS system in Eqs (1.3)–(1.5), and discussed conservation, existence and uniqueness, stability and convergence of the numerical scheme. Second, we show numerical scheme for two dimension fractional KGS system by high order finite difference scheme in span and Crank-Nicolson scheme in time, and discussed conservation of the numerical scheme. Third, we show scalar auxiliary variable scheme to solve two dimension fractional KGS equations. Finally, some numerical examples are given, confirm theoretical results and demonstrate the efficiency of the numerical schemes.

The outline of the paper is as follows. In Section 2, the structure-preserving scheme is proposed for one dimension fractional KGS system, and convergence and stability of the numerical scheme is proved. In Section 3, the high conservative difference scheme is proposed for two dimension fractional KGS system. In Section 4, the numerical experiments are given, and the results verify the efficiency of the conservative difference scheme. Finally, a conclusion and some discussions are given in Section 5.

2. Structure-preserving scheme for one dimension fractional KGS system

In the section, we show structure-preserving scheme for one dimension fractional KGS system with $f(|u|^2, \phi) \neq |u|^2\phi$, and prove that the scheme can preserve mass and energy conservation laws. Moreover, we show that the arising scheme is uniquely solvable, and approximate solutions converge to the exact solution at the rate $O(\tau^2 + h^4)$.

2.1. Some useful lemmas

The fractional KGS system (1.1)–(1.2) contains a fractional Schrödinger equation and a classic Klein-Gordon equation, and we consider boundary condition

$$u(x,t) = 0, x \in R/\Omega; \phi(x,t) = 0, x \in \partial\Omega;\ \Omega = (a,b).$$

Let M, N. Then, choose time-step $\tau = T/N$ and mesh size $h = (b-a)/M$. Denote

$$x_j = a + jh,\ t_n = n\tau,\ j = 0,\ 1,\ 2, \cdots,\ M,\ n = 0,\ 1,\ 2,\ \cdots$$

Then

$$u_j^n = u(x_j, t_n),\ U_j^n \approx u(x_j, t_n),\ \phi_j^n = \phi(x_j, t_n),\ \Phi_j^n \approx \phi(x_j, t_n).$$

Define

$$\Omega_h = \{x_j | 1 \le j \le M-1\},\ \Omega_\tau = \{t_n | 1 \le n \le N-1\},$$
$$\bar{\Omega}_h = \{x_j | 0 \le j \le M\},\ \bar{\Omega}_\tau = \{t_n | 0 \le n \le N\}.$$

Suppose $w = \{w_j^n;\ j = 0,\ 1,\ 2\cdots\ M,\ n = 0,\ 1,\ 2\cdots\ N\}$ be a discrete function in $\bar{\Omega}_h \times \bar{\Omega}_\tau$, and

$$Z_h^0 = \{w = w_j | w_0 = w_M = 0,\ j = 0,\ 1,\ 2,\ \cdots\ M\}.$$

For convenience, we define the finite difference operators

$$(w_j^n)_t = \frac{w_j^{n+1} - w_j^n}{\tau},\ (w_j^n)_{\hat{t}} = \frac{w_j^{n+1} - w_j^{n-1}}{2\tau},\ w_j^{n+\frac{1}{2}} = \frac{w_j^{n+1} + w_j^n}{2},\ \tilde{w}_j^{n+\frac{1}{2}} = \frac{w_j^{n+1} + w_j^{n-1}}{2}.$$

For any grid function $u = \{u_j\},\ v = \{v_j\}$, define the discrete inner product, L^2-norm and L^p-norm as

$$\langle u, v\rangle = h\sum_{j=1}^{M-1} u_j \bar{v}_j,\ \|u\|^2 = \langle u, u\rangle,$$
$$\|u\|_{l_h^p}^p = h\sum_{j=1}^{M-1} |u_j|^p,\ 1 \le p < +\infty,\ \|u\|_{l_h^\infty} = \sup_{0<j<M-1} |u_j|.$$

For $0 \le \delta \le 1$, we also define the fractional Sobolev norm $\|u\|_{H^\delta}$ and semi-norm $|u|_{H^\delta}$ as

$$\|u\|_{H^\delta}^2 = \int_{-\pi/h}^{\pi/h} (1 + |k|^{2\delta})|\widehat{u}(k)|^2 dk,\ |u|_{H^\delta}^2 = \int_{-\pi/h}^{\pi/h} |k|^{2\delta}|\widehat{u}(k)|^2 dk,$$

where

$$\widehat{u}(k) = \frac{1}{\sqrt{2\pi}} h\sum_{j\in z} u_j e^{-ikx_j}, u_j = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} \widehat{u}(k)e^{ikx_j}dk.$$

Lemma 1. [24] *Let* $f(x) \in C^6[x_L, x_R],\ 2 < j < M-2$*. Then*

$$f''(x_j) = \frac{4}{3}\frac{f(x_j+h) - 2f(x_j) + f(x_j-h)}{h^2} - \frac{1}{3}\frac{f(x_j+2h) - 2f(x_j) + f(x_j-2h)}{h^2} + O(h^4).$$

When $j = 1, M-1$*,then*

$$\begin{aligned} f''(x_1) =& \frac{7}{6}\frac{f(x_2) - 2f(x_1) + f(x_0)}{h^2} - \frac{1}{6}\frac{f(x_3) - 2f(x_2) + f(x_1)}{h^2} \\ &- \frac{1}{12}f''(x_0) - \frac{1}{144}f^{(4)}(x_0) + O(h^4), \\ f''(x_{M-1}) =& \frac{7}{6}\frac{f(x_M) - 2f(x_{M-1}) + f(x_{M-2})}{h^2} - \frac{1}{6}\frac{f(x_{M-1}) - 2f(x_{M-2}) + f(x_{M-3})}{h^2} \\ &- \frac{1}{12}f''(x_M) - \frac{1}{144}f^{(4)}(x_M) + O(h^4). \end{aligned}$$

Lemma 2. [20] *Suppose that* $u \in L_1(R)$ *and*

$$u \in \mathcal{L}^{4+\alpha}(R) := \{u| \int_{-\infty}^{+\infty} (1 + |\xi|)^{4+\alpha} |\widehat{u}(\xi)| d\xi < \infty\}.$$

Then for a fixed h, we can obtain high order scheme

$$\frac{4}{3}\Delta_h^\alpha u(x) - \frac{1}{3}\Delta_{2h}^\alpha u(x) = -(-\Delta)^{\frac{\alpha}{2}} u(x) + O(h^4),$$

where

$$\Delta_h^\alpha u(x) = h^{-\alpha} \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} u(x - kh),\ \ g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}.$$

Lemma 3. *For any grid functions* $u^n \in Z_h^0$, *we can obtain*

$$Im\langle \frac{4}{3}\Delta_h^\alpha u^{n+\frac{1}{2}} - \frac{1}{3}\Delta_{2h}^\alpha u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\rangle = 0,$$
$$Re\langle \frac{4}{3}\Delta_h^\alpha u^{n+\frac{1}{2}} - \frac{1}{3}\Delta_{2h}^\alpha u^{n+\frac{1}{2}}, u_t^n\rangle = \frac{1}{2\tau}(\frac{4}{3}\|\Delta_h^\alpha U^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^\alpha U^{n+1}\|^2 - \frac{4}{3}\|\Delta_h^\alpha U^n\|^2 + \frac{1}{3}\|\Delta_{2h}^\alpha U^n\|^2),$$

where $Im(s), Re(s)$ *mean taking the imaginary part and the real part of a complex number s, respectively.*

Lemma 4. [25] *(Diserete Sobolev inequality) For every* $\frac{1}{2} \leq \delta \leq 1$, *there exist a constant* $C = C(\delta) > 0$ *independent of* $h > 0$ *such that*

$$\|u\|_{l_h^\infty} \leq C_\sigma \|u\|_{H_h^\delta},$$

for all $u \in l_h^2$.

Lemma 5. [25] *(Gagliardo-Nirenberg inequality) For any* $\frac{1}{4} < \delta_0 \leq 1$, *there exist a constant* $C_{\delta_0} = C(\delta_0) > 0$ *independent of* $h > 0$ *such that*

$$\|u\|_{l_h^4} \leq C_{\delta_0} \|u\|_{H^\delta}^{\delta_0/\delta} \|u\|^{1-\delta_0/\delta}.$$

2.2. Conservative difference scheme

Applying Crank-Nicolson scheme in time and higher order difference scheme in space to the fractional KGS system (1.3)–(1.5), we can obtain the following numerical scheme

$$i(U_j^n)_t - \nu\hat{\Delta}_h^\alpha U_j^{n+\frac{1}{2}} = -\frac{1}{2}[\frac{f(|U_j^{n+1}|^2, \Phi_j^n) - f(|U_j^n|^2, \Phi_j^n)}{|U_j^{n+1}|^2 - |U_j^n|^2} + \frac{f(|U_j^{n+1}|^2, \Phi_j^{n+1}) - f(|U_j^n|^2, \Phi_j^{n+1})}{|U_j^{n+1}|^2 - |U_j^n|^2}]U_j^{n+\frac{1}{2}}, \tag{2.1}$$

$$(\Phi_j^n)_t = V_j^{n+\frac{1}{2}}, \tag{2.2}$$

$$(V_j^n)_t - \tilde{\Delta}_h^2 \Phi_j^{n+\frac{1}{2}} + \Phi_j^{n+\frac{1}{2}} = \frac{1}{2}[\frac{f(|U_j^n|^2, \Phi_j^{n+1}) - f(|U_j^n|^2, \Phi_j^n)}{\Phi_j^{n+1} - \Phi_j^n} + \frac{f(|U_j^{n+1}|^2, \Phi_j^{n+1}) - f(|U_j^{n+1}|^2, \Phi_j^n)}{\Phi_j^{n+1} - \Phi_j^n}], \tag{2.3}$$

$$U_j^0 = u_0(x_j), \Phi_j^0 = \phi_0(x_j), V_j^0 = \phi_1(x_j), \tag{2.4}$$

$$U_0^n = U_M^n, \Phi_0^n = \Phi_M^n, \tag{2.5}$$

where

$$\hat{\Delta}_h^\alpha U_j^{n+\frac{1}{2}} = \frac{4}{3}\Delta_h^\alpha U_j^{n+\frac{1}{2}} - \frac{1}{3}\Delta_{2h}^\alpha U_j^{n+\frac{1}{2}},$$

$$\tilde{\Delta}_h^2 \Phi_j^{n+\frac{1}{2}} = \begin{cases} \frac{7}{6}\Delta_h^2 \Phi_1^{n+\frac{1}{2}} - \frac{1}{12}\Delta_h^2 \Phi_2^{n+\frac{1}{2}}, j = 1, \\ \frac{4}{3}\Delta_h^2 \Phi_j^{n+\frac{1}{2}} - \frac{1}{3}\Delta_{2h}^2 \Phi_j^{n+\frac{1}{2}}, 2 < j < M-2, \\ \frac{7}{6}\Delta_h^2 \Phi_{M-1}^{n+\frac{1}{2}} - \frac{1}{12}\Delta_h^2 \Phi_{M-2}^{n+\frac{1}{2}}, j = M-1. \end{cases}$$

Theorem 1. *The scheme* (2.1)–(2.5) *is conservative in the sense*

$$\textit{Mass: } M^n = M^{n-1} \cdots = M^0,$$
$$\textit{Energy: } E^n = E^{n-1} \cdots = E^0,$$

where

$$\begin{aligned} M^n =& \|U^n\|^2, \\ E^n =& \nu(\frac{4}{3}\|\Delta_h^\alpha U^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^\alpha U^{n+1}\|^2) + \frac{1}{2}[\frac{4}{3}\|\Delta_h^2 \Phi^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^2 \Phi^{n+1}\|^2 \\ &+ \frac{h}{6}(|\Delta_h^2 \Phi_{\frac{1}{2}}^{n+1}|^2 + |\Delta_h^2 \Phi_{M-\frac{1}{2}}^{n+1}|^2) + \|\Phi^{n+1}\|^2 + \|V^{n+1}\|^2] - h\sum_{j=1}^{M-1} |f(|U_j^{n+1}|^2, \Phi_j^{n+1}). \end{aligned}$$

Proof: Multiplying $h(U_j^{n+1} + U_j^n)^*$ to Eq (2.1) and summing it up for $1 \le j \le M-1$ can yield

$$\begin{aligned} & i\langle U_t^n, 2U^{n+\frac{1}{2}}\rangle - \nu\langle \hat{\Delta}_h^\alpha U^{n+\frac{1}{2}}, 2U^{n+\frac{1}{2}}\rangle \\ =& \langle -\frac{1}{2}[\frac{f(|U^{n+1}|^2, \Phi^n) - f(|U^n|^2, \Phi^n)}{|U^{n+1}|^2 - |U^n|^2} + \frac{f(|U^{n+1}|^2, \Phi^{n+1}) - f(|U^n|^2, \Phi^{n+1})}{|U^{n+1}|^2 - |U^n|^2}]U^{n+\frac{1}{2}}, 2U^{n+\frac{1}{2}}\rangle. \end{aligned} \tag{2.6}$$

Taking the imaginary part of Eq (2.6) and noting that Lemma 3 yields $\|U^{n+1}\|^2 = \|U^n\|^2$. When $n = 0, 1, 2, \cdots$, we can obtain $M^n = M^{n-1} \cdots = M^0$.

Multiplying $2h(U_j^{n+1} - U_j^n)^*/\tau$ to Eq (2.1) and summing it up for $1 \le j \le M-1$ can yield

$$\begin{aligned} & i\langle U_t^n, 2U_t^n\rangle - \nu\langle \hat{\Delta}_h^\alpha U^{n+\frac{1}{2}}, 2U_t^n\rangle \\ =& \langle -\frac{1}{2}[\frac{f(|U^{n+1}|^2, \Phi^n) - f(|U^n|^2, \Phi^n)}{|U^{n+1}|^2 - |U^n|^2} + \frac{f(|U^{n+1}|^2, \Phi^{n+1}) - f(|U^n|^2, \Phi^{n+1})}{|U^{n+1}|^2 - |U^n|^2}]U^{n+\frac{1}{2}}, 2U_t^n\rangle. \end{aligned} \tag{2.7}$$

It follows from Lemma 3 that

$$\langle \hat{\Delta}_h^\alpha U^{n+\frac{1}{2}}, 2U_t^n\rangle = \frac{1}{\tau}(\frac{4}{3}\|\Delta_h^\alpha U^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^\alpha U^{n+1}\|^2 - \frac{4}{3}\|\Delta_h^\alpha U^n\|^2 + \frac{1}{3}\|\Delta_{2h}^\alpha U^n\|^2).$$

Taking the real part of Eq (2.7) and noting that Lemma 3 yields

$$
\begin{aligned}
&\frac{\nu}{\tau}(\frac{4}{3}\|\Delta_h^{\alpha}U^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n+1}\|^2-\frac{4}{3}\|\Delta_h^{\alpha}U^{n}\|^2+\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n}\|^2)\\
=&\frac{h}{2\tau}\sum_{j=1}^{M-1}[f(|U_j^{n+1}|^2,\Phi_j^{n+1})-f(|U_j^{n}|^2,\Phi_j^{n+1})+f(|U_j^{n+1}|^2,\Phi_j^{n})-f(|U_j^{n}|^2,\Phi_j^{n})].
\end{aligned}
\tag{2.8}
$$

Multiplying $2h(V_j^{n+1}-V_j^n)/\tau$ to Eq (2.2) and summing it up for $1\leq j\leq M-1$ can yield

$$
\langle\Phi_t^n,2V_t^n\rangle=\langle V^{n+\frac{1}{2}},2V_t^n\rangle. \tag{2.9}
$$

Multiplying $2h(\Phi_j^{n+1}-\Phi_j^n)/\tau$ to Eq (2.3) and summing it up for $1\leq j\leq M-1$ can yield

$$
\begin{aligned}
&\langle V_t^n,2\Phi_t^n\rangle-\langle\tilde{\Delta}_h^2\Phi^{n+\frac{1}{2}},2\Phi_t^n\rangle+\langle\Phi^{n+\frac{1}{2}},2\Phi_t^n\rangle\\
=&\langle\frac{1}{2}[\frac{f(|U^n|^2,\Phi^{n+1})-f(|U^n|^2,\Phi^{n})}{\Phi^{n+1}-\Phi^n}+\frac{f(|U^{n+1}|^2,\Phi^{n+1})-f(|U^{n+1}|^2,\Phi^{n})}{\Phi^{n+1}-\Phi^n}],2\Phi_t^n\rangle.
\end{aligned}
\tag{2.10}
$$

It follows from literature [24] that

$$
\begin{aligned}
&\langle\tilde{\Delta}_h^2\Phi^{n+\frac{1}{2}},2\Phi_t^n\rangle\\
=&-\frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^2\Phi^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^2\Phi^{n+1}\|^2-\frac{4}{3}\|\Delta_h^2\Phi^{n}\|^2+\frac{1}{3}\|\Delta_{2h}^2\Phi^{n}\|^2)+\frac{h}{6}(|\Delta_h^2\Phi_{\frac{1}{2}}^{n+1}|^2-|\Delta_h^2\Phi_{\frac{1}{2}}^{n}|^2)\\
&+(|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n+1}|^2-|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n}|^2)].
\end{aligned}
$$

Noting that equation $\langle\Phi_t^n,V_t^n\rangle=\langle V_t^n,\Phi_t^n\rangle$, we obtain

$$
\begin{aligned}
&\frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^2\Phi^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^2\Phi^{n+1}\|^2-\frac{4}{3}\|\Delta_h^2\Phi^{n}\|^2+\frac{1}{3}\|\Delta_{2h}^2\Phi^{n}\|^2)\\
&+\frac{h}{6}(|\Delta_h^2\Phi_{\frac{1}{2}}^{n+1}|^2-|\Delta_h^2\Phi_{\frac{1}{2}}^{n}|^2)+(|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n+1}|^2-|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n}|^2)]\\
&+\frac{1}{\tau}(\|\Phi^{n+1}\|^2-\|\Phi^n\|^2)+\frac{1}{\tau}(\|V^{n+1}\|^2-\|V^n\|^2)\\
=&\frac{h}{\tau}\sum_{j=1}^{M-1}[f(|U_j^{n+1}|^2,\Phi_j^{n+1})-f(|U_j^{n+1}|^2,\Phi_j^{n})+f(|U_j^{n}|^2,\Phi_j^{n+1})-f(|U_j^{n}|^2,\Phi_j^{n})].
\end{aligned}
\tag{2.11}
$$

Combining Eqs (2.8) and (2.11), we obtain

$$
\begin{aligned}
&\nu(\frac{4}{3}\|\Delta_h^{\alpha}U^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n+1}\|^2)+\frac{1}{2}[\frac{4}{3}\|\Delta_h^2\Phi^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^2\Phi^{n+1}\|^2\\
&+\frac{h}{6}(|\Delta_h^2\Phi_{\frac{1}{2}}^{n+1}|^2+|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n+1}|^2)+\|\Phi^{n+1}\|^2+\|V^{n+1}\|^2]-h\sum_{j=1}^{M-1}|f(|U_j^{n+1}|^2,\Phi_j^{n+1})\\
=&\nu(\frac{4}{3}\|\Delta_h^{\alpha}U^{n}\|^2-\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n}\|^2)+\frac{1}{2}[\frac{4}{3}\|\Delta_h^2\Phi^{n}\|^2-\frac{1}{3}\|\Delta_{2h}^2\Phi^{n}\|^2\\
&+\frac{h}{6}(|\Delta_h^2\Phi_{\frac{1}{2}}^{n}|^2+|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n}|^2)+\|\Phi^{n+1}\|^2+\|V^{n}\|^2]-h\sum_{j=1}^{M-1}f(|U_j^{n}|^2,\Phi_j^{n}).
\end{aligned}
$$

When $n = 0, 1, 2, \cdots$, we can obtain $E^n = E^{n-1} \cdots = E^0$. □

In [15], we show the theoretical analysis for $f(|u|^2, \phi) = |u|^2\phi + |u|^4\phi$. In similar analysis of literature [15], we can obtain the theoretical analysis for $f(|u|^2, \phi) = |u|^{2l} \cdot \phi^m$ when l, m satisfy certain conditions. For simplicity of notation, we only consider $f(|u|^2, \phi) = |u|^2\phi$ in next theoretical analysis.

Theorem 2. *The scheme* (2.1)–(2.5) *is bounded in the discrete* l_h^∞.

Proof: Using Theorem 1 can obtain $\|U^n\| = \widetilde{C}$ and

$$\begin{aligned} E^n =&\nu(\frac{4}{3}\|\Delta_h^\alpha U^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^\alpha U^{n+1}\|^2) + \frac{1}{2}[\frac{4}{3}\|\Delta_h^2 \Phi^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^2 \Phi^{n+1}\|^2 \\ &+ \frac{h}{6}(|\Delta_h^2 \Phi_{\frac{1}{2}}^{n+1}|^2 + |\Delta_h^2 \Phi_{M-\frac{1}{2}}^{n+1}|^2) + \|\Phi^{n+1}\|^2 + \|V^{n+1}\|^2] - h\sum_{j=1}^{M-1} |U_j^{n+1}|^2 \Phi_j^{n+1}. \end{aligned} \tag{2.12}$$

Noting that Young's inequality $ab \le \frac{1}{4}a^2 + b^2$, we can obtain

$$h\sum_{j=1}^{M-1} |U_j^{n+1}|^2 \Phi_j^{n+1} \le h\sum_{j=1}^{M-1} |U_j^{n+1}|^2 |\Phi_j^{n+1}| \le h\sum_{j=1}^{M-1} (|U_j^{n+1}|^4 + \frac{1}{4}|\Phi_j^{n+1}|^2) = \|U^{n+1}\|_{l_h^4}^4 + \frac{1}{4}\|\Phi^{n+1}\|^2. \tag{2.13}$$

Using Gagliardo-Nirenberg inequality and noting that $\frac{1}{4} < \sigma_0 < \frac{\alpha}{4}$ can yield

$$\|U^{n+1}\|_{l_h^4}^4 \le \hat{C}_{\delta_0} \|U^{n+1}\|_{H^{\frac{\alpha}{2}}}^{\frac{8\delta_0}{\alpha}} \|U^{n+1}\|^{4-\frac{8\delta_0}{\alpha}} \le \hat{C}_{\delta_0}(\varepsilon\|U^{n+1}\|_{H^{\frac{\alpha}{2}}}^2 + C(\varepsilon)). \tag{2.14}$$

It follows from [22] that there exists a constant $1 \le C_\alpha \le (\frac{2}{\pi})^\alpha$ such that

$$\frac{4}{3}\|\Delta_h^\alpha U^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^\alpha U^{n+1}\|^2 = C_\alpha |U^{n+1}|_{H_{\frac{\alpha}{2}}}^2, \tag{2.15}$$

$$\frac{4}{3}\|\Delta_h^2 \Phi^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^2 \Phi^{n+1}\|^2 = C|\Phi^{n+1}|_{H_1}^2. \tag{2.16}$$

Substituting Eqs (2.15),(2.16) into Eq (2.12) can yield

$$\begin{aligned} &C_\alpha \nu |U^{n+1}|_{H_{\frac{\alpha}{2}}}^2 + \frac{1}{2}[C|\Phi^{n+1}|_{H_1}^2 + \frac{h}{6}(|\Delta_h^2 \Phi_{\frac{1}{2}}^{n+1}|^2 + |\Delta_h^2 \Phi_{M-\frac{1}{2}}^{n+1}|^2) + \|\Phi^{n+1}\|^2 + \|V^{n+1}\|^2] \\ \le& E^n + \hat{C}_{\delta_0}(\varepsilon\|U^{n+1}\|_{H^{\frac{\alpha}{2}}}^2 + C(\varepsilon)) + \frac{1}{4}\|\Phi^{n+1}\|^2. \end{aligned}$$

Noting that $\|U^n\|_{H^{\frac{\alpha}{2}}}^2 = |U^n|_{H_{\frac{\alpha}{2}}}^2 + \|U^n\|^2, E^n = E^0$, we have

$$(\nu C_\alpha - \hat{C}_{\delta_0}\varepsilon)|U^{n+1}|_{H_{\frac{\alpha}{2}}}^2 + \frac{1}{2}[C|\Phi^{n+1}|_{H_1}^2 + \frac{h}{6}(|\Delta_h^2 \Phi_{\frac{1}{2}}^{n+1}|^2 + |\Delta_h^2 \Phi_{M-\frac{1}{2}}^{n+1}|^2) + \frac{1}{2}\|\Phi^{n+1}\|^2 + \|V^{n+1}\|^2] \le \hat{C}_{\delta_0} C(\varepsilon) + E^0.$$

When $\varepsilon < \frac{\nu C_\alpha}{\hat{C}_{\delta_0}}$, there exist a constant C, such that

$$|\Phi^{n+1}|_{H_1}^2 \le C, \|\Phi^n\|^2 \le C, \|U^n\|_{H_{\frac{\alpha}{2}}}^2 \le C.$$

It follows from Lemma 4 that $\|U^n\|_{l_h^\infty} \le C, \|\Phi^n\|_{l_h^\infty} \le C$. □

2.3. Existence and Unique

Theorem 3. *The solution of the numerical scheme* (2.1)–(2.5) *exists and unique.*

Proof: Let $P^n = U^n - \widehat{U}^n, R^n = V^n - \widehat{V}^n, Q^n = \Phi^n - \widehat{\Phi}^n$. Then, it follows from Eqs (2.1)–(2.15) that

$$i(P_j^n)_t - \nu\hat{\Delta}_h^\alpha P_j^{n+\frac{1}{2}} = -(U_j^{n+\frac{1}{2}}\Phi_j^{n+\frac{1}{2}} - \widehat{U}_j^{n+\frac{1}{2}}\widehat{\Phi}_j^{n+\frac{1}{2}}), 1 < j < M-1, 1 < n < N, \tag{2.17}$$

$$(Q_j^n)_t = R_j^{n+\frac{1}{2}}, 1 < j < M-1, 1 < n < N, \tag{2.18}$$

$$(R_j^n)_t - \tilde{\Delta}_h^2 Q_j + Q_j^{n+\frac{1}{2}} = \frac{1}{2}(|U_j^n|^2 + |U_j^{n+1}|^2) - \frac{1}{2}(|\widehat{U}_j^n|^2 + |\widehat{U}_j^{n+1}|^2), 1 < j < M-1, 1 < n < N. \tag{2.19}$$

Multiplying $h(P_j^{n+1} + P_j^n)^*$ to Eq (2.17) and summing it up for $1 < j < M-1$ can yield

$$i\langle P_t^n - \nu\hat{\Delta}_h^\alpha P^{n+\frac{1}{2}}, P^{n+\frac{1}{2}}\rangle = -\langle U^{n+\frac{1}{2}}\Phi^{n+\frac{1}{2}} - \widehat{U}^{n+\frac{1}{2}}\widehat{\Phi}^{n+\frac{1}{2}}, P^{n+\frac{1}{2}}\rangle. \tag{2.20}$$

Taking the imaginary part of Eq (2.20) can obtain

$$\frac{1}{2\tau}(\|P^{n+1}\|^2 - \|P^n\|^2) = \mathrm{Im}\langle -(U^{n+\frac{1}{2}}\Phi^{n+\frac{1}{2}} - \widehat{U}^{n+\frac{1}{2}}\widehat{\Phi}^{n+\frac{1}{2}}), P^{n+\frac{1}{2}}\rangle. \tag{2.21}$$

Noting that Theorem 2, then we can obtain

$$\frac{1}{2\tau}(\|P^{n+1}\|^2 - \|P^n\|^2) \le C(\|P^{n+1}\|^2 + \|P^n\|^2 + \|Q^{n+1}\|^2 + \|Q^n\|^2). \tag{2.22}$$

Multiplying $(R_j^{n+1} - R_j^n)/\tau$ to Eq (2.18) and summing it up for $1 < j < M-1$ can yield

$$\langle Q_t^n, R_t^n\rangle = \langle R^{n+\frac{1}{2}}, R_t^n\rangle. \tag{2.23}$$

Multiplying $(Q_j^{n+1} - Q_j^n)/\tau$ to Eq (2.19) and summing it up for $1 < j < M-1$ can yield

$$\begin{aligned}
&\langle R_t^n, Q_t^n\rangle + [(\frac{4}{3}\|\Delta_h^2 Q^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^2 Q^{n+1}\|^2 - \frac{4}{3}\|\Delta_h^2 Q^n\|^2 + \frac{1}{3}\|\Delta_{2h}^2 Q^n\|^2)\\
&+ \frac{h}{6}(|\Delta_h^2 Q_{\frac{1}{2}}^{n+1}|^2 - |\Delta_h^2 Q_{\frac{1}{2}}^n|^2) + \frac{h}{6}(|\Delta_h^2 Q_{M-\frac{1}{2}}^{n+1}|^2 - |\Delta_h^2 Q_{M-\frac{1}{2}}^n|^2)] + \langle Q^{n+1}, Q_t^n\rangle\\
=&\langle \frac{1}{2}(|U^n|^2 + |U^{n+1}|^2) - \frac{1}{2}(|\widehat{U}^n|^2 + |\widehat{U}^{n+1}|^2), Q_t^n\rangle.
\end{aligned} \tag{2.24}$$

It follows from Theorem 2 that the Eq (2.24) can be expressed as the following form

$$\begin{aligned}
&\langle R_t^n, Q_t^n\rangle + \frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^2 Q^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^2 Q^{n+1}\|^2 - \frac{4}{3}\|\Delta_h^2 Q^n\|^2 + \frac{1}{3}\|\Delta_{2h}^2 Q^n\|^2)\\
&+ \frac{h}{6}(|\Delta_h^2 Q_{\frac{1}{2}}^{n+1}|^2 - |\Delta_h^2 Q_{\frac{1}{2}}^n|^2) + \frac{h}{6}(|\Delta_h^2 Q_{M-\frac{1}{2}}^{n+1}|^2 - |\Delta_h^2 Q_{M-\frac{1}{2}}^n|^2)] + \frac{1}{\tau}(\|Q^{n+1}\| - \|Q^n\|)\\
\le&\frac{C}{2}(\|P^n\|^2 + \|R^{n+1}\|^2 + \|R^n\|).
\end{aligned}$$

Combining Eq (2.23) can yield

$$\frac{1}{\tau}(\|R^{n+1}\|^2 - \|R^n\|) + \frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^2 Q^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^2 Q^{n+1}\|^2 - \frac{4}{3}\|\Delta_h^2 Q^n\|^2 + \frac{1}{3}\|\Delta_{2h}^2 Q^n\|^2)$$

$$+\frac{h}{6}(|\Delta_h^2 Q_{\frac{1}{2}}^{n+1}|^2-|\Delta_h^2 Q_{\frac{1}{2}}^{n}|^2)+\frac{h}{6}(|\Delta_h^2 Q_{M-\frac{1}{2}}^{n+1}|^2-|\Delta_h^2 Q_{M-\frac{1}{2}}^{n}|^2)]+\frac{1}{\tau}(\|Q^{n+1}\|-\|Q^n\|) \tag{2.25}$$
$$\leq\frac{C}{2}(\|P^n\|^2+\|R^{n+1}\|^2+\|R^n\|).$$

Adding Eqs (2.22) and (2.25) can yield

$$\frac{1}{\tau}(\|R^{n+1}\|^2-\|R^n\|)+\frac{1}{\tau}(\|P^{n+1}\|^2-\|P^n\|)+\frac{1}{\tau}(\|Q^{n+1}\|-\|Q^n\|)$$
$$+\frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^2 Q^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^2 Q^{n+1}\|^2-\frac{4}{3}\|\Delta_h^2 Q^{n}\|^2+\frac{1}{3}\|\Delta_{2h}^2 Q^{n}\|^2)$$
$$+\frac{h}{6}(|\Delta_h^2 Q_{\frac{1}{2}}^{n+1}|^2-|\Delta_h^2 Q_{\frac{1}{2}}^{n}|^2)+\frac{h}{6}(|\Delta_h^2 Q_{M-\frac{1}{2}}^{n+1}|^2-|\Delta_h^2 Q_{M-\frac{1}{2}}^{n}|^2)]$$
$$\leq\frac{C}{2}(\|P^n\|^2+\|P^{n+1}\|^2+\|Q^n\|^2+\|Q^{n+1}\|^2+\|R^{n+1}\|^2+\|R^n\|).$$

Let

$$B^n=\|R^{n+1}\|^2+\|P^{n+1}\|^2+\|Q^{n+1}\|+\frac{4}{3}\|\Delta_h^\alpha Q^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^\alpha Q^{n+1}\|^2+\frac{h}{6}|\Delta_h^2 Q_{\frac{1}{2}}^{n+1}|^2-\frac{h}{6}|\Delta_h^2 Q_{M-\frac{1}{2}}^{n+1}|^2.$$

Then, we can obtain

$$B^n-B^{n-1}\leq C\tau(B^n+B^{n-1}).$$

It follows from Gronwall's inequality that

$$\max_{1\leq n\leq N}(\|B^n\|^2)\leq e^{4CT}\|B^0\|^2=0.$$

Noting that $\|P^{n+1}\|^2+\|Q^{n+1}\|^2\leq B^n$, we can obtain $\|P^{n+1}\|^2=0, \|Q^{n+1}\|^2=0$. □

2.4. Convergence and error estimates

Let $u_j^n=u(x_j,t_n)$, $v_j^n=v(x_j,t_n)$, $\phi_j^n=\phi(x_j,t_n)$. Then, we define the local truncation error as

$$R_1^n=i(u_j^n)_t-\nu\hat{\Delta}_h^\alpha u_j^{n+\frac{1}{2}}+u_j^{n+\frac{1}{2}}\phi_j^{n+\frac{1}{2}}, 1<j<M-1, 1<n<N, \tag{2.26}$$
$$R_2^n=(\phi_j^n)_t-v_j^{n+\frac{1}{2}}, 1<j<M-1, 1<n<N, \tag{2.27}$$
$$R_3^n=(v_j^n)_t-\tilde{\Delta}_h^2\phi_j+\phi_j^{n+\frac{1}{2}}-\frac{1}{2}(|u_j^n|^2+|u_j^{n+1}|^2), 1<j<M-1, 1<n<N. \tag{2.28}$$

According to Taylor expansion, we obtain the following result.

Theorem 4. $|R_j^n|\leq\widehat{C}(\tau^2+h^4)$ *holds as* $\tau, h\to 0$.

Theorem 5. *Suppose that the problem* (1.1), (1.2) *has a smooth solution, then the solution* U^n, Φ^n *of difference scheme* (2.1)–(2.5) *converges to the true solution* u, ϕ *with order* $O(\tau^2+h^4)$ *by the* $\|\cdot\|_{l_h^\infty}$ *norm.*

Proof: Let $e_j^n = U_j^n - u_j^n, \eta_j^n = \Phi_j^n - \phi_j^n,\ \xi_j = V_j^n - v_j^n$. Then, we obtain

$$R_1^n = i(e_j^n)_t - \nu\hat{\Delta}_h^{\alpha} e_j^{n+\frac{1}{2}} + F_j^{n+\frac{1}{2}}, 1 < j < M-1, 1 < n < N, \tag{2.29}$$

$$R_2^n = (\eta_j^n)_t - \xi_j^{n+\frac{1}{2}}, 1 < j < M-1, 1 < n < N, \tag{2.30}$$

$$R_3^n = (\xi_j^n)_t - \tilde{\Delta}_h^2 \eta_j + \eta_j^{n+\frac{1}{2}} - G_j^n, 1 < j < M-1, 1 < n < N, \tag{2.31}$$

where

$$\begin{aligned} F^{n+\frac{1}{2}} =& U^{n+\frac{1}{2}}\Phi^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\phi^{n+\frac{1}{2}}, \\ G^n =& \frac{1}{2}(|U^n|^2 + |U^{n+1}|^2) - \frac{1}{2}(|u^n|^2 + |u^{n+1}|^2) \\ =& \frac{1}{2}(U^n\overline{U}^n + U^{n+1}\overline{U}^{n+1}) - \frac{1}{2}(u^n\overline{u}^n + u^{n+1}\overline{u}^{n+1}) \\ =& \frac{1}{2}(e^n\overline{U^n} + u^n\overline{e^n} + e^{n+1}\overline{U^{n+1}} + u^{n+1}\overline{e^{n+1}}). \end{aligned}$$

Multiplying $h(e_j^{n+1} + e_j^n)^*$ to Eq (2.29) and summing it up for $1 < j < M-1$ can yield

$$\langle R_1^n, 2e^{n+\frac{1}{2}}\rangle = i\langle e_t^n, 2e^{n+\frac{1}{2}}\rangle - \nu\langle\hat{\Delta}_h^{\alpha} e^{n+\frac{1}{2}}, 2e^{n+\frac{1}{2}}\rangle + \langle F^{n+\frac{1}{2}}, 2e^{n+\frac{1}{2}}\rangle. \tag{2.32}$$

Taking the imaginary part of Eq (2.32) can yield

$$\frac{1}{\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) \leq \|R_1^n\|^2 + C(\|e^{n+1}\|^2 + \|e^n\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2). \tag{2.33}$$

Multiplying $h(\xi_j^{n+1} - \xi_j^n)/\tau$ to Eq (2.30) and summing it up for $1 < j < M-1$ can yield

$$\langle R_2^n, \xi_t^n\rangle = \langle \eta_t^n - \xi_j^{n+\frac{1}{2}}, \xi_t^n\rangle. \tag{2.34}$$

Multiplying $h(\eta_j^{n+1} - \eta_j^n)/\tau$ to Eq (2.31) and summing it up for $1 < j < M-1$ can yield

$$\begin{aligned} \langle R_3^n, \eta_t^n\rangle =& \langle \xi_t^n, \eta_t^n\rangle + \frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^2\eta^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^2\eta^{n+1}\|^2 - \frac{4}{3}\|\Delta_h^2\eta^n\|^2 + \frac{1}{3}\|\Delta_{2h}^2\eta^n\|^2) \\ &+ \frac{h}{6}(|\Delta_h^2\eta_{\frac{1}{2}}^{n+1}|^2 - |\Delta_h^2\eta_{\frac{1}{2}}^n|^2) + (|\Delta_h^2\eta_{M-\frac{1}{2}}^{n+1}|^2 - |\Delta_h^2\eta_{M-\frac{1}{2}}^n|^2)] + \frac{1}{\tau}(\|\eta^{n+1}\| - \|\eta^n\|) + \langle G^n, \eta_t^n\rangle. \end{aligned} \tag{2.35}$$

It follows from Cauchy-Schwarz inequality that

$$\langle G, \eta_t^n\rangle = \langle G, R_2^n - \eta^{n+\frac{1}{2}}\rangle \leq C(\|\eta^{n+1}\|^2 + \|\eta^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2), \tag{2.36}$$

$$\langle R_3^n, \eta_t^n\rangle = \langle R_3^n, R_2^n - \eta^{n+\frac{1}{2}}\rangle \leq C(\|R_3^n\|^2 + \|R_2^n\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2). \tag{2.37}$$

Substituting Eqs. (2.36) and (2.37) into Eq (2.35) and noting that Eqs (2.33) and (2.34) yields

$$\begin{aligned} &\frac{1}{\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) + \frac{1}{\tau}(\|\xi^{n+1}\|^2 - \|\eta^n\|^2) + \frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^{\alpha}\eta^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^{\alpha}\eta^{n+1}\|^2 - \frac{4}{3}\|\Delta_h^{\alpha}\eta^n\|^2 + \frac{1}{3}\|\Delta_{2h}^{\alpha}\eta^n\|^2) \\ &+ \frac{h}{6}(|\Delta_h^{\alpha}\eta_{\frac{1}{2}}^{n+1}|^2 - |\Delta_h^{\alpha}\eta_{\frac{1}{2}}^n|^2) + (|\Delta_h^{\alpha}\eta_{M-\frac{1}{2}}^{n+1}|^2 - |\Delta_h^{\alpha}\eta_{M-\frac{1}{2}}^n|^2)] + \frac{1}{\tau}(\|\eta^{n+1}\|^2 - \|\eta^n\|^2) \end{aligned}$$

$$\leq C(\|R_1^n\|^2 + \|R_2^n\|^2 + \|R_3^n\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2).$$

Let

$$B^n = \|e^{n+1}\|^2 + \|\xi^{n+1}\|^2 + \frac{4}{3}\|\Delta_h^\alpha \eta^{n+1}\|^2 - \frac{1}{3}\|\Delta_{2h}^\alpha \eta^{n+1}\|^2 + \frac{h}{6}(|\Delta_h^\alpha \eta_{\frac{1}{2}}^{n+1}|^2 + |\Delta_h^\alpha \eta_{M-\frac{1}{2}}^{n+1}|^2) + \|\eta^{n+1}\|^2.$$

Then,we can obtain

$$B^n - B^{n-1} \leq \tau(|R_1^n\|^2 + |R_2^n|^2 + |R_3^n\|^2) + C\tau(B^n + B^{n-1}).$$

It follows from Gronwall's inequality that

$$\begin{aligned}\max_{1\leq n\leq N} B^n \leq &(B^0 + \tau\sum_{l=1}^{N}(|R_1^l\|^2 + |R_2^l\|^2))e^{8CT}\\ \leq &(B^0 + \widehat{C}T(\tau^2 + h^4)^2)e^{8CT}.\end{aligned}$$

Noting that $B^0 = O(\tau^2 + h^4)$, we can obtain

$$\|e^n\| \leq C(\tau^2 + h^4), \|\eta^n\|_{H_1} \leq C(\tau^2 + h^4), \|\eta^n\| \leq C(\tau^2 + h^4). \tag{2.38}$$

Multiplying $h(e_j^{n+1} - e_j^n)^*/\tau$ to Eq (2.29) and summing it up for $1 < j < M-1$ can yield

$$\langle R_1^n, e^{n+1} - e^n\rangle = i\langle e_t^n, e^{n+1} - e^n\rangle - \nu\langle \hat{\Delta}_h^\alpha e^{n+\frac{1}{2}}, e^{n+1} - e^n\rangle + \langle F^{n+\frac{1}{2}}, e^{n+1} - e^n\rangle. \tag{2.39}$$

It follows from Eq (2.32) that can obtain

$$\begin{aligned}\mathrm{Re}\langle F^{n+\frac{1}{2}}, e^{n+1} - e^n\rangle =&\tau\mathrm{Re}\langle F^{n+\frac{1}{2}}, -i\nu\hat{\Delta}_h^\alpha e^{n+\frac{1}{2}} + iF^{n+\frac{1}{2}} - iR_1^n\rangle\\ =&\tau\mathrm{Im}\langle F^{n+\frac{1}{2}}, -\nu\hat{\Delta}_h^\alpha e^{n+\frac{1}{2}}\rangle - \tau\langle F^{n+\frac{1}{2}}, R_1^n\rangle,\end{aligned}$$

and

$$\langle \hat{\Delta}_h^\alpha e^{n+\frac{1}{2}}, e^{n+1} - e^n\rangle = \widehat{C}_\alpha\tau(|e^{n+1}|_{H^{\alpha/2}}^2 - |e^n|_{H^{\alpha/2}}^2).$$

Noting that

$$|\mathrm{Im}\langle F^{n+\frac{1}{2}}, -\nu\hat{\Delta}_h^\alpha e^{n+\frac{1}{2}}\rangle| \leq C(|F^{n+\frac{1}{2}}|^2 + |e|_{H^{\frac{\alpha}{2}}}^n + |e|_{H^{\frac{\alpha}{2}}}^{n+1}),$$

we can yield

$$|e^{n+1}|_{H^{\alpha/2}}^2 - |e^n|_{H^{\alpha/2}}^2 \leq\tau(|e^{n+1}|_{H^{\alpha/2}}^2 + |e^n|_{H^{\alpha/2}}^2 + |F^{n+\frac{1}{2}}|_{H^{\alpha/2}}^2 + \mathrm{Re}\langle R_1^n, e^{n+1} - e^n\rangle). \tag{2.40}$$

It follows from [15] that can obtain

$$|F^{n+\frac{1}{2}}|_{H^{\alpha/2}}^2 \leq C(|e^{n+1}|_{H^{\alpha/2}}^2 + |e^n|_{H^{\alpha/2}}^2 + (\tau^2 + h^4)^2), \;\|F^{n+\frac{1}{2}}\|^2 \leq C(\tau^2 + h^4)^2.$$

Thus, Eq (2.40) can be expressed as

$$|e^{n+1}|_{H^{\alpha/2}}^2 - |e^n|_{H^{\alpha/2}}^2 \leq C\tau(|e^{n+1}|_{H^{\alpha/2}}^2 + |e^n|_{H^{\alpha/2}}^2 + (\tau^2 + h^2)^2) + \mathrm{Re}\langle R_1^n, e^{n+1} - e^n\rangle. \tag{2.41}$$

Summing up the superscript n to N and then replacing N by n, we get

$$|e^{n+1}|_{H^{\alpha/2}}^2 \leq C\sum_{l=0}^{n}|e^l|_{H^{\alpha/2}}^2 + CT(\tau^2 + h^4)^2.$$

It follows from Gronwall Inequality that

$$|e^{n+1}|_{H^{\alpha/2}} \leq C(\tau^2 + h^4).$$

Noting that Eq (2.38) can yield $\|e\|_{l_h^\infty} \leq C(\tau^2 + h^4),\; \|\eta\|_{l_h^\infty} \leq C(\tau^2 + h^4)$. □

2.5. Iterative algorithm

Let $U = [U_1, U_2, \cdots, U_{M-1}]$, $V = [V_1, V_2, \cdots, V_{M-1}]$, $\Phi = [\Phi_1, \Phi_2, \cdots, \Phi_{M-1}]$. Then, we rewrite the numerical scheme (2.1)–(2.5) as the following vector form

$$
\begin{aligned}
& iU_t^n - \nu\widetilde{A}U^{n+\frac{1}{2}} = -\frac{1}{2}[\frac{f(|U^{n+1}|^2, \Phi^n) - f(|U^n|^2, \Phi^n)}{|U^{n+1}|^2 - |U^n|^2} + \frac{f(|U^{n+1}|^2, \Phi^{n+1}) - f(|U^n|^2, \Phi^{n+1})}{|U^{n+1}|^2 - |U^n|^2}]U^{n+\frac{1}{2}}, \\
& \Phi_t^n = V^{n+\frac{1}{2}}, \\
& V_t^n - \widetilde{B}\Phi^{n+\frac{1}{2}} + \Phi^{n+\frac{1}{2}} = \frac{1}{2}[\frac{f(|U^n|^2, \Phi^{n+1}) - f(|U^n|^2, \Phi^n)}{\Phi^{n+1} - \Phi^n} + \frac{f(|U_j^{n+1}|^2, \Phi^{n+1}) - f(|U^{n+1}|^2, \Phi^n)}{\Phi^{n+1} - \Phi^n}], \\
& U^0 = u_0, \Phi^0 = \phi_0, V^0 = \phi_1,
\end{aligned}
$$

where, matrices $\widetilde{A}$, $\widetilde{B}$ represent differential matrix of fractional Laplacian operator and classical Laplacian operator, respectively. In order to solve above nonlinear numerical scheme , we will use the following iterative algorithm

$$
\begin{aligned}
& i\frac{U^{n+1(s+1)} - U^n}{\tau} - \nu\widetilde{A}\frac{U^{n+1(s+1)} + U^n}{2} \\
= & -\frac{1}{2}[\frac{f(|U^{n+1(s)}|^2, \Phi^n) - f(|U^n|^2, \Phi^n)}{|U^{n+1(s)}|^2 - |U^n|^2} + \frac{f(|U^{n+1(s)}|^2, \Phi^{n+1(s)}) - f(|U^n|^2, \Phi^{n+1(s)})}{|U^{n+1(s)}|^2 - |U^n|^2}]\frac{U^{n+1(s)} + U^n}{2}, \\
& \frac{\Phi^{n+1(s+1)} - \Phi^n}{\tau} = \frac{V^{n+1(s)} + V^n}{2}, \\
& \frac{V^{n+1(s+1)} - V^n}{\tau} - \widetilde{B}\frac{\Phi^{n+1(s+1)} + \Phi^n}{2} + \frac{\Phi^{n+1(s+1)} + \Phi^n}{2} \\
= & \frac{1}{2}[\frac{f(|U^n|^2, \Phi^{n+1(s+1)}) - f(|U^n|^2, \Phi^n)}{\Phi^{n+1(s+1)} - \Phi^n} + \frac{f(|U^{n+1(s+1)}|^2, \Phi^{n+1(s+1)}) - f(|U^{n+1(s+1)}|^2, \Phi^n)}{\Phi^{n+1(s+1)} - \Phi^n}].
\end{aligned}
$$

3. Structure-preserving scheme for two dimension fractional KGS system

In this section, we first show Crank-Nicolson scheme in time and high central difference scheme in space, and the obtained scheme preserves mass and energy conservation laws. However, the obtained discrete system is nonlinear system, and it takes too much time in the numerical simulation for two dimension case. Then, we show a equivalent form of two dimension fractional KGS system by introducing some new auxiliary variables. The new system is discretized by the scalar auxiliary variable scheme, and a linear discrete system is obtained, which can preserve energy conservation law.

Now, we consider boundary condition

$$u(x, y, t) = 0, \ (x, y) \in R^2/\Omega; \ \phi(x, y, t) = 0, \ (x, y) \in \partial\Omega, \ \Omega = (x_L, x_R) \times (y_L, y_R).$$

Let

$$h_x = (x_R - x_L)/M, \ h_y = (y_R - y_L)/M, \ \tau = T/N,$$

where M, N be positive integers. Then,

$$x_j = x_L + jh_x, \ y_k = y_L + kh_y, \ t_n = n\tau.$$

Denote

$$\Omega_{h_x,h_y} = \{(x_j, y_k)|1 \le j \le M-1,\ 1 \le k \le M-1\},\ \Omega_\tau = \{t_n|1 \le n \le N-1\},$$
$$\bar{\Omega}_{h_x,h_y} = \{(x_j, y_k)|0 \le j \le M,\ 0 \le k \le M\},\ \bar{\Omega}_\tau = \{t_n|0 \le n \le N\}.$$

Then, the grid function can be defined by $U^n_{j,k} \approx u(x_j, y_k, t_n)$, $\Phi^n_{j,k} \approx \phi(x_j, y_k, t_n)$, where $u^n_{j,k} = u(x_j, y_k, t_n)$, $\phi^n_{j,k} = \phi(x_j, y_k, t_n)$. For any grid $u = \{u_{j,k}\}$, $\phi = \{\phi_{j,k}\}$, we can define

$$\langle u, v\rangle = h_x h_y \sum_{j=1}^{M-1}\sum_{k=1}^{M-1} u_{j,k}\bar{v}_{j,k},\ \|u\|^2 = \langle u, u\rangle.$$

Lemma 6. [26, 27] *Suppose that $u \in L_1(R^2)$ and*

$$u \in \mathcal{L}^{4+\alpha}(R^2) := \{u| \int_{-\infty}^{+\infty} (1 + |\xi|)^{4+\alpha}|\widehat{u}(\xi)|d\xi < \infty\}.$$

Then, for a fixed $h = h_x = h_y$, we can obtain high order scheme

$$\frac{4}{3}\Delta_h^\alpha u(x, y) - \frac{1}{3}\Delta_{2h}^\alpha u(x, y) = \Delta^{\frac{\alpha}{2}} u(x, y) + O(h^4), \tag{3.1}$$

where

$$\Delta_h^\alpha u(x, y) = h^{-2\alpha} \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g_{k,l}^{(\alpha)} u(x - kh, y - lh),$$

and $g^{(\alpha)}_{k,l}$ are Fourier expansion coefficients of generation function

$$\rho(x, y) = [4\sin^2(\frac{x}{2}) + 4\sin^2(\frac{y}{2})]^{\frac{\alpha}{2}},$$

which can be calculated as

$$g_{k,l}^{(\alpha)} = \frac{1}{(2\pi)^2}\int\int_{[-\pi,\pi]^2} \rho(x, y)e^{-i(kx+ly)}dxdy.$$

Lemma 7. *Let*

$$\tilde{\Delta}^2_{h_x,h_y}\Phi(x_j, y_k) = \tilde{\Delta}^2_{h_x}\Phi(x_j, y_k) + \tilde{\Delta}^2_{h_y}\Phi(x_j, y_k),$$

where

$$\tilde{\Delta}^2_{h_x}\Phi(x_j, y_k) = \begin{cases} \frac{7}{6}\Delta_h^2\Phi(x_1, y_k)^{n+\frac{1}{2}} - \frac{1}{12}\Delta_h^2\Phi(x_2, y_k)^{n+\frac{1}{2}}, j = 1, \\ \frac{4}{3}\Delta_h^2\Phi(x_j, y_k)^{n+\frac{1}{2}} - \frac{1}{3}\Delta_{2h}^2\Phi(x_j, y_k)^{n+\frac{1}{2}}, 2 < j < M-2, \\ \frac{7}{6}\Delta_h^2\Phi(x_{M-1}, y_k)^{n+\frac{1}{2}} - \frac{1}{12}\Delta_h^2\Phi(x_{M-2}, y_k)^{n+\frac{1}{2}}, j = M-1, \end{cases}$$

$$\tilde{\Delta}^2_{h_y}\Phi(x_j, y_k) = \begin{cases} \frac{7}{6}\Delta_h^2\Phi(x_j, y_1)^{n+\frac{1}{2}} - \frac{1}{12}\Delta_h^2\Phi(x_j, y_2)^{n+\frac{1}{2}}, k = 1, \\ \frac{4}{3}\Delta_h^2\Phi(x_j, y_k)^{n+\frac{1}{2}} - \frac{1}{3}\Delta_{2h}^2\Phi(x_j, y_k)^{n+\frac{1}{2}}, 2 < k < M-2, \\ \frac{7}{6}\Delta_h^2\Phi(x_j, y_{M-1})^{n+\frac{1}{2}} - \frac{1}{12}\Delta_h^2\Phi(x_j, y_{M-2})^{n+\frac{1}{2}}, k = M-1. \end{cases}$$

Then, we obtain

$$\Delta\Phi(x_j, y_k) = \tilde{\Delta}^2_{h_x,h_y}\Phi(x_j, y_k) + O(h_x^4 + h_y^4).$$

3.1. Crank-Nicolson scheme

Applying Crank-Nicolson scheme in time and higher order difference scheme in space to two dimension fractional KGS system (1.3)–(1.5), we can obtain the following numerical scheme

$$i(U^n_{j,k})_t - \nu\hat{\Delta}^\alpha_h U^{n+\frac{1}{2}}_{j,k} = -\frac{1}{2}\Big[\frac{f(|U^{n+1}_{j,k}|^2,\Phi^n_{j,k}) - f(|U^n_{j,k}|^2,\Phi^n_{j,k})}{|U^{n+1}_{j,k}|^2 - |U^n_{j,k}|^2} + \frac{f(|U^{n+1}_{j,k}|^2,\Phi^{n+1}_{j,k}) - f(|U^n_{j,k}|^2,\Phi^{n+1}_{j,k})}{|U^{n+1}_{j,k}|^2 - |U^n_{j,k}|^2}\Big]U^{n+\frac{1}{2}}_{j,k}, \tag{3.2}$$

$$j,k = 1,2,\cdots,M-1, n = 0,1,\cdots,$$

$$(\Phi^n_{j,k})_t = V^{n+\frac{1}{2}}_{j,k}, j,k = 1,2,\cdots,M-1, n = 0,1,\cdots, \tag{3.3}$$

$$(V^n_{j,k})_t - \tilde{\Delta}^2_{h_x,h_y}\Phi^{n+\frac{1}{2}}_{j,k} + \Phi^{n+\frac{1}{2}}_{j,k} = \frac{1}{2}\Big[\frac{f(|U^n_{j,k}|^2,\Phi^{n+1}_{j,k}) - f(|U^n_{j,k}|^2,\Phi^n_{j,k})}{\Phi^{n+1}_{j,k} - \Phi^n_{j,k}} + \frac{f(|U^{n+1}_{j,k}|^2,\Phi^{n+1}_{j,k}) - f(|U^{n+1}_{j,k}|^2,\Phi^n_{j,k})}{\Phi^{n+1}_{j,k} - \Phi^n_{j,k}}\Big], \tag{3.4}$$

$$j,k = 1,\cdots,M-1, n = 0,1,\cdots,$$

$$U^0_{j,k} = u_0(x_j,y_k), \Phi^0_{j,k} = \phi_0(x_j,y_k), V^0_{j,k} = \phi_1(x_j,y_k), j,k = 1,2,\cdots,M-1, \tag{3.5}$$

$$U^n_{0,k} = U^n_{M,k}, \Phi^n_{0,k} = \Phi^n_{M,k},\ U^n_{j,0} = U^n_{j,M}, \Phi^n_{j,M} = \Phi^n_{j,M},\ n = 0,1,\cdots, \tag{3.6}$$

where

$$\hat{\Delta}^\alpha_h U^{n+\frac{1}{2}}_{j,k} = \frac{4}{3}\Delta^\alpha_h U^{n+\frac{1}{2}}_{j,k} - \frac{1}{3}\Delta^\alpha_{2h} U^{n+\frac{1}{2}}_{j,k}.$$

Let

$$\tilde{\Delta}^2_{h_x,h_y}\Phi_{j,k} = (D_x \otimes I + I \otimes D_y)\Phi,\ \frac{4}{3}\Delta^\alpha_h U_{j,k} - \frac{1}{3}\Delta^\alpha_{2h} U_{j,k} = AU.$$

Then, we can obtain

$$iU_t - \nu AU^{n+\frac{1}{2}} = -\frac{1}{2}\Big[\frac{f(|U^{n+1}|^2,\Phi^n) - f(|U^n|^2,\Phi^n)}{|U^{n+1}|^2 - |U^n|^2} + \frac{f(|U^{n+1}|^2,\Phi^{n+1}) - f(|U^n|^2,\Phi^{n+1})}{|U^{n+1}|^2 - |U^n|^2}\Big]U^{n+\frac{1}{2}}, n = 0,1,\cdots, \tag{3.7}$$

$$\Phi^n_t = V^{n+\frac{1}{2}}, n = 0,1,\cdots, \tag{3.8}$$

$$V^n_t - (D_x \otimes I + I \otimes D_y)\Phi^{n+\frac{1}{2}} + \Phi^{n+\frac{1}{2}} = \frac{1}{2}\Big[\frac{f(|U^n|^2,\Phi^{n+1}) - f(|U^n|^2,\Phi^n)}{\Phi^{n+1} - \Phi^n} + \frac{f(|U^{n+1}|^2,\Phi^{n+1}) - f(|U^{n+1}|^2,\Phi^n)}{\Phi^{n+1} - \Phi^n}\Big], \tag{3.9}$$

$$n = 0,1,\cdots,$$

where, $\otimes$ represents kronecker product of matrices, D_x, D_y are differential matrix of x, y direction, and I is identity matrix.

Lemma 8. [19] *Let $A \in R^{n\times n}$ have eigenvalues $\{\lambda_j\}^n_{j=1}$, and let $B \in R^{m\times m}$ have eigenvalues$\{\mu_j\}^m_{j=1}$. Then the $m \times n$ eigenvalues of $A \otimes B$ are*

$$\lambda_1\mu_1,\cdots,\lambda_1\mu_m,\lambda_2\mu_1,\cdots,\lambda_2\mu_m,\cdots,\lambda_n\mu_1,\cdots,\lambda_n\mu_m.$$

Lemma 9. [19] *For matrixs A and B,* $(A \otimes B)^T = A^T \otimes B^T$.

Let $\Lambda_h = D_x \otimes I + I \otimes D_y$. Then, it follows from Lemmas 8 and 9 that the matrix Λ_h and A are symmetric positive matrixs. Moreover, there exists fractional symmetric positive difference quotient operator denoted by $\Lambda_h^{\frac{1}{2}}$ and $A^{\frac{1}{2}}$ such that

$$-\langle \Lambda_h v, v\rangle = \langle \Lambda_h^{\frac{1}{2}} v, \Lambda_h^{\frac{1}{2}} v\rangle, \ -\langle A v, v\rangle = \langle A^{\frac{1}{2}} v, A^{\frac{1}{2}} v\rangle.$$

It follows from Lemmas 8 and 9 that the numerical scheme (3.7)–(3.9) preserves mass and energy conservation laws.

3.2. Scalar auxiliary variable scheme

For $f(|u|^2, \phi) = |u|^2\phi$, it follows from Eqs (3.2)–(3.6) that we can obtain the following numerical scheme

$$i(U_{j,k}^n)_t - \nu\hat{\Delta}_h^{\alpha} U_{j,k}^{n+\frac{1}{2}} = -U_{j,k}^{n+\frac{1}{2}} \Phi_{j,k}^{n+\frac{1}{2}}, j,k = 1,2,\cdots, M-1, n = 0,1,\cdots, \tag{3.10}$$

$$(\Phi_{j,k}^n)_t = V_{j,k}^{n+\frac{1}{2}}, j,k = 1,2,\cdots, M-1, n = 0,1,\cdots, \tag{3.11}$$

$$(V_{j,k}^n)_t - \tilde{\Delta}_{h_x,h_y}^2 \Phi_{j,k}^{n+\frac{1}{2}} + \Phi_{j,k}^{n+\frac{1}{2}} = \frac{1}{2}(|U_{j,k}^n|^2 + |U_{j,k}^{n+1}|^2), j,k = 1,\cdots, M-1, n = 0,1,\cdots, \tag{3.12}$$

$$U_{j,k}^0 = u_0(x_j, y_k), \Phi_{j,k}^0 = \phi_0(x_j, y_k), V_{j,k}^0 = \phi_1(x_j, y_k), j,k = 1,2,\cdots, M-1, \tag{3.13}$$

$$U_{0,k}^n = U_{M,k}^n, \Phi_{0,k}^n = \Phi_{M,k}^n, \ U_{j,0}^n = U_{j,M}^n, \Phi_{j,M}^n = \Phi_{j,M}^n, \ n = 0,1,\cdots. \tag{3.14}$$

We can prove that the resulting scheme (3.10)–(3.14) can preserve the mass and energy conservation laws. However, the above numerical scheme is nonlinear scheme. In order to construct linear scheme, we consider also the following finite difference scheme for fractional KGS system (1.3)–(1.5) with $f(|u|^2, \phi) = |u|^2\phi$

$$i(U_{j,k}^n)_t - \nu\hat{\Delta}_h^{\alpha} U_{j,k}^{n+\frac{1}{2}} = -U_{j,k}^{n+\frac{1}{2}} \Phi_{j,k}^{n+\frac{1}{2}}, j,k = 1,2,\cdots, M-1, n = 0,1,\cdots, \tag{3.15}$$

$$(\Phi_{j,k}^n)_{\hat{t}} = \widetilde{V}_{j,k}^n, j,k = 1,2,\cdots, M-1, n = 0,1,\cdots, \tag{3.16}$$

$$(V_{j,k}^n)_{\hat{t}} - \tilde{\Delta}_{h_x,h_y}^2 \widetilde{\Phi}_{j,k}^n + \widetilde{\Phi}_{j,k}^n = |U_{j,k}^n|^2, j,k = 2,\cdots, M-2, n = 0,1,\cdots, \tag{3.17}$$

$$U_{j,k}^0 = u_0(x_j, y_k), \Phi_{j,k}^0 = \phi_0(x_j, y_k), V_{j,k}^0 = \phi_1(x_j, y_k), j,k = 1,2,\cdots, M-1, \tag{3.18}$$

$$U_{0,k}^n = U_{M,k}^n, \Phi_{0,k}^n = \Phi_{M,k}^n, \ U_{j,0}^n = U_{j,M}^n, \Phi_{j,M}^n = \Phi_{j,M}^n, \ n = 0,1,\cdots. \tag{3.19}$$

We can also prove that the resulting scheme (3.15)–(3.19) can preserve the mass and energy conservation laws. However, the scheme is only conservation for $f(|u|^2, \phi) = |u|^2\phi$. For $f(|u|^2, \phi) \neq |u|^2\phi$, we use scalar auxiliary variable scheme to obtain linearly implicit scheme. Let $q = \sqrt{\langle f(|u|^2, \phi), 1\rangle + C_0}$. Then,

$$iu_t - \nu(-\Delta^{\frac{\alpha}{2}})u = -u\frac{\partial f(|u|^2, \phi)}{\partial |u|^2} \frac{q}{\sqrt{\langle f(|u|^2, \phi), 1\rangle + C_0}}, \tag{3.20}$$

$$\phi_t = v, \tag{3.21}$$

$$v_t - \Delta\phi + \phi = \frac{\partial f(|u|^2,\phi)}{\partial\phi}\frac{q}{\sqrt{\langle f(|u|^2,\phi),1\rangle + C_0}}, \tag{3.22}$$

$$q_t = \frac{\langle \frac{\partial f(|u|^2,\phi)}{\partial |u|^2}, 2Re(u\cdot u_t)\rangle + \langle \frac{\partial f(|u|^2,\phi)}{\partial\phi}, \frac{\partial\phi}{\partial t}\rangle}{2\sqrt{\langle f(|u|^2,\phi),1\rangle + C_0}}. \tag{3.23}$$

Lemma 10. *The fractional KGS system* (3.20)–(3.23) *has the following energy conserved laws*

$$E(t) = \int_{-\infty}^{+\infty} v^2 + |\nabla\phi|^2 + \phi^2 + 2|(-\Delta)^{\frac{\alpha}{4}}u|^2 + q^2 dxdy = E(0).$$

Applying scalar auxiliary variable scheme in time and higher order difference scheme in space to the fractional KGS system (1.3)–(1.5), we can obtain the following numerical scheme

$$i(U^n_{j,k})_t - \nu\hat{\Delta}^\alpha_h U^{n+\frac{1}{2}}_{j,k} = -\widetilde{U_{j,k}}^{n+\frac{1}{2}}\frac{\partial f(|\widetilde{U_{j,k}}^{n+\frac{1}{2}}|^2,\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}})}{\partial|\widetilde{U_{j,k}}^{n+\frac{1}{2}}|^2}\frac{Q^{n+\frac{1}{2}}_{j,k}}{\sqrt{\langle f(|\widetilde{U_{j,k}}^{n+\frac{1}{2}}|^2,\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}}),1\rangle + C_0}}, \tag{3.24}$$
$$j,k = 1,2,\cdots,M-1, n = 0,1,\cdots,$$

$$(\Phi^n_{j,k})_t = V^{n+\frac{1}{2}}_{j,k}, j = 1,2,\cdots,M-1, n = 0,1,\cdots, \tag{3.25}$$

$$(V^n_{j,k})_t - \tilde{\Delta}^2_{h_x,h_y}\Phi^{n+\frac{1}{2}}_{j,k} + \Phi^{n+\frac{1}{2}}_{j,k} = \frac{\partial f(|\widetilde{U_{j,k}}^{n+\frac{1}{2}}|^2,\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}}}\frac{Q^{n+\frac{1}{2}}_{j,k}}{\sqrt{\langle f(|\widetilde{U_{j,k}}^{n+\frac{1}{2}}|^2,\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}}),1\rangle + C_0}}, \tag{3.26}$$
$$j,k = 1,2,\cdots,M-1, n = 0,1,\cdots,$$

$$(Q^n_{j,k})_t = \frac{\langle \frac{\partial f(|\widetilde{U_{j,k}}^{n+\frac{1}{2}}|^2,\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}})}{\partial|\widetilde{U^{n+\frac{1}{2}}_{j,k}}|^2}, 2Re(\widetilde{U_{j,k}}^{n+\frac{1}{2}}\cdot(U^n_{j,k})_t)\rangle + \langle \frac{\partial f(|\widetilde{U_{j,k}}|^2,\widetilde{\Phi_{ij}}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}}}, (\Phi^n_{j,k})_t\rangle}{2\sqrt{\langle f(|\widetilde{U_{j,k}}^{n+\frac{1}{2}}|^2,\widetilde{\Phi_{j,k}}^{n+\frac{1}{2}}),1\rangle + C_0}}, j,k = 1,\cdots,M-1, n = 0,1,\cdots, \tag{3.27}$$

where $\widetilde{U_{j,k}}^{n+\frac{1}{2}} = \dfrac{3U^n_{j,k} - U^{n-1}_{j,k}}{2}, \widetilde{\Phi_{j,k}}^{n+\frac{1}{2}} = \dfrac{3\Phi^n_{j,k} - \Phi^{n-1}_{j,k}}{2}$.

Theorem 6. *The scheme* (3.24)–(3.27) *is conservative in the sense*

$$\textit{Energy: } E^n = E^{n-1}\cdots = E^0,$$

where

$$\begin{aligned} E^n =& \frac{4}{3}\|\Delta^\alpha_h U^{n+1}\|^2 - \frac{1}{3}\|\Delta^\alpha_{2h}U^{n+1}\|^2 + \frac{4}{3}\|\Delta^2_h\Phi^{n+1}\|^2 - \frac{1}{3}\|\Delta^2_{2h}\Phi^{n+1}\|^2 \\ &+ \frac{h}{6}(|\Delta^2_h\Phi^{n+1}_{\frac{1}{2}}|^2 + |\Delta^2_h\Phi^{n+1}_{M-\frac{1}{2}}|^2) + \|\Phi^{n+1}\|^2 + \|V^{n+1}\|^2 - \|Q^{n+1}\|^2. \end{aligned}$$

Proof: Multiplying $2h(U^{n+1}_{j,k} - U^n_{j,k})^*/\tau$ to Eq (3.24) and summing it up for $1 \le j,k \le M-1$ can yield

$$i\langle U^n_t - \nu\hat{\Delta}^\alpha_h U^{n+\frac{1}{2}}, 2U^{n+\frac{1}{2}}_t\rangle = -\langle\widetilde{U}^{n+\frac{1}{2}}\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U}^{n+\frac{1}{2}}|^2}\frac{Q^{n+\frac{1}{2}}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle + C_0}}, 2U^{n+\frac{1}{2}}_t\rangle. \tag{3.28}$$

Taking the real part of Eq (3.28) yields

$$\begin{aligned}&\frac{1}{\tau}(\frac{4}{3}\|\Delta_h^{\alpha}U^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n+1}\|^2-\frac{4}{3}\|\Delta_h^{\alpha}U^{n}\|^2+\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n}\|^2)\\ =&Re\langle\widetilde{U}^{n+\frac{1}{2}}\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U}^{n+\frac{1}{2}}|^2}\frac{Q^{n+\frac{1}{2}}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle+C_0}},2U_t^{n+\frac{1}{2}}\rangle.\end{aligned}\tag{3.29}$$

Multiplying $2h(V_j^{n+1}-V_j^n)/\tau$ to Eq (3.25) and summing it up for $1\leq j,k\leq M-1$ can yield

$$\langle\Phi_t^n,2V_t^n\rangle=\langle V^{n+\frac{1}{2}},2V_t^n\rangle.\tag{3.30}$$

Multiplying $2h(\Phi_j^{n+1}-\Phi_j^n)/\tau$ to Eq (3.26) and summing it up for $1\leq j,k\leq M-1$ can yield

$$\langle V_t^n-\tilde{\Delta}_{h_x,h_y}^2\Phi^{n+\frac{1}{2}}+\Phi^{n+\frac{1}{2}},2\Phi_t^n\rangle=\langle\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}\frac{Q^{n+\frac{1}{2}}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle+C_0}},2\Phi_t^n\rangle.\tag{3.31}$$

It is easy to check that

$$\langle\Phi^{n+\frac{1}{2}},\Phi_t^n\rangle=\frac{1}{\tau}(\|\Phi^{n+1}\|^2-\|\Phi^n\|^2).$$

Noting that equation $\langle\Phi_t^n,V_t^n\rangle=\langle V_t^n,\Phi_t^n\rangle$, we obtain

$$\begin{aligned}&\frac{1}{\tau}[(\frac{4}{3}\|\Delta_h^2\Phi^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^2\Phi^{n+1}\|^2-\frac{4}{3}\|\Delta_h^2\Phi^{n}\|^2+\frac{1}{3}\|\Delta_{2h}^2\Phi^{n}\|^2)\\ &+\frac{h}{6}(|\Delta_h^2\Phi_{\frac{1}{2}}^{n+1}|^2-|\Delta_h^2\Phi_{\frac{1}{2}}^{n}|^2)+(|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n+1}|^2-|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n}|^2)]\\ &+\frac{1}{\tau}(\|\Phi^{n+1}\|^2-\|\Phi^n\|^2)+\frac{1}{\tau}(\|V^{n+1}\|^2-\|V^n\|^2)\\ =&\langle\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}\frac{Q^{n+\frac{1}{2}}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle+C_0}},\Phi_t^n\rangle.\end{aligned}\tag{3.32}$$

Multiplying $2hQ_{j,k}^{n+\frac{1}{2}}$ to Eq (3.27) and summing it up for $1\leq j,k\leq M-1$ can yield

$$\langle Q_t^n,2Q^{n+\frac{1}{2}}\rangle=\langle\frac{\langle\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U^{n+\frac{1}{2}}}^{n+\frac{1}{2}}|^2},2Re(\widetilde{U}^{n+\frac{1}{2}}\cdot U_t^n)\rangle+\langle\frac{\partial f(|\widetilde{U}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}},\Phi_t^n\rangle}{2\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle+C_0}},2Q^{n+\frac{1}{2}}\rangle.\tag{3.33}$$

Combining Eqs (3.29),(3.32),(3.33), we obtain

$$\begin{aligned}&\frac{4}{3}\|\Delta_h^{\alpha}U^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n+1}\|^2+\frac{4}{3}\|\Delta_h^2\Phi^{n+1}\|^2-\frac{1}{3}\|\Delta_{2h}^2\Phi^{n+1}\|^2\\ &+\frac{h}{6}(|\Delta_h^2\Phi_{\frac{1}{2}}^{n+1}|^2+|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n+1}|^2)+\|\Phi^{n+1}\|^2+\|V^{n+1}\|^2-\|Q^{n+1}\|^2\\ =&\frac{4}{3}\|\Delta_h^{\alpha}U^{n}\|^2-\frac{1}{3}\|\Delta_{2h}^{\alpha}U^{n}\|^2+\frac{4}{3}\|\Delta_h^2\Phi^{n}\|^2-\frac{1}{3}\|\Delta_{2h}^2\Phi^{n}\|^2\\ &+\frac{h}{6}(|\Delta_h^2\Phi_{\frac{1}{2}}^{n}|^2+|\Delta_h^2\Phi_{M-\frac{1}{2}}^{n}|^2)+\|\Phi^{n}\|^2+\|V^{n}\|^2-\|Q^{n}\|^2.\end{aligned}$$

When $n=0,1,2,\cdots,n$, we can obtain $E^n=E^{n-1}\cdots=E^0$. □

3.3. Iterative algorithm

In above subsection, we construct some structure-preserving schemes to solve two dimension fractional KGS equations. In similar method of one dimension case, we can obtain the iterative algorithms of numerical scheme (3.2)–(3.6) and numerical scheme (3.10)–(3.14). Now, we consider the iterative algorithm of numerical scheme (3.15)–(3.19) and numerical scheme (3.24)–(3.27). Let $U = [U_{11}, U_{12}, \cdots, U_{M-1M-1}]$, $V = [V_{11}, V_{12}, \cdots, V_{M-1M-1}]$, $\Phi = [\Phi_{11}, \Phi_{12}, \cdots, \Phi_{M-1M-1}]$. Then, we rewrite the numerical scheme (3.15)–(3.19) and numerical scheme (3.24)–(3.27) as the following vector form

$$i\frac{U^{n+1}-U^n}{\tau} - \nu A\frac{U^{n+1}+U^n}{2} = -\frac{U^{n+1}+U^n}{2}\frac{\Phi^{n+1}+\Phi^n}{2}, \tag{3.34}$$

$$\frac{\Phi^{n+1}-\Phi^{n-1}}{2\tau} = \frac{V^{n+1}+V^{n-1}}{2}, \tag{3.35}$$

$$\frac{V^{n+1}-V^{n-1}}{2\tau} - B\frac{\Phi^{n+1}+\Phi^{n-1}}{2} = |U^n|^2, \tag{3.36}$$

and

$$iU_t^n - \nu AU^{n+\frac{1}{2}} = -\widetilde{U}^{n+\frac{1}{2}}\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U}^{n+\frac{1}{2}}|^2}\frac{Q^{n+\frac{1}{2}}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}}), 1\rangle + C_0}}, \tag{3.37}$$

$$\Phi_t^n = V^{n+\frac{1}{2}}, \tag{3.38}$$

$$V_t^n - B\Phi^{n+\frac{1}{2}} + \Phi^{n+\frac{1}{2}} = \frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}\frac{Q^{n+\frac{1}{2}}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}}), 1\rangle + C_0}}, \tag{3.39}$$

$$Q_t^n = \frac{\langle \frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U^{n+\frac{1}{2}}}^{n+\frac{1}{2}}|^2}, 2Re(\widetilde{U}^{n+\frac{1}{2}}\cdot(U^n)_t)\rangle + \langle\frac{\partial f(|\widetilde{U}|^2, \widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}, (\Phi^n)_t\rangle}{2\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}}), 1\rangle + C_0}}, \tag{3.40}$$

where, matrice B represents differential matrix of two dimension Laplacian operator.

Consider numerical scheme (3.34)–(3.36), if (U^n, Φ^n, V^n), $n = 0, 1, 2, \cdots$ are known, then Φ^{n+1}, V^{n+1} of numerical scheme (3.34)–(3.36) is solve by the following linear equations

$$\begin{aligned} &\Phi^{n+1} - \tau V^{n+1} = \Phi^{n-1} + \tau V^{n-1}, \\ &V^{n+1} - \tau B\Phi^{n+1} = \tau B\Phi^{n-1} + V^{n-1} + 2\tau|U^n|^2. \end{aligned}$$

Then, we can obtain U^{n+1} of numerical scheme (3.34)–(3.36) by solve linear equations

$$iU^{n+1} - \frac{\nu\tau}{2}AU^{n+1} + \frac{\tau}{4}U^{n+1}(\Phi^{n+1}+\Phi^n) = iU^n + \frac{\nu\tau}{2}AU^n - \frac{\tau}{4}U^n(\Phi^{n+1}+\Phi^n).$$

Consider numerical scheme (3.37)–(3.40), the numerical solution U^{n+1}, Φ^{n+1}, V^{n+1} of numerical scheme (3.37)–(3.40) is solve by the following linear equations

$$iU^{n+1} - \frac{\nu\tau}{2}AU^{n+1} - \frac{\tau}{2}\widetilde{U}^{n+\frac{1}{2}}\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U}^{n+\frac{1}{2}}|^2}\frac{Q^{n+1}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2, \widetilde{\Phi}^{n+\frac{1}{2}}), 1\rangle + C_0}}$$

$$
\begin{aligned}
&= iU^n + \frac{\nu\tau}{2}AU^n - \frac{\tau}{2}\widetilde{U}^{n+\frac{1}{2}}\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U}^{n+\frac{1}{2}}|^2}\frac{Q^n}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle + C_0}},\\
&\Phi^{n+1} - \tau V^{n+1} = \Phi^n - \tau V^n,\\
&V^{n+1} - \frac{\tau}{2}B\Phi^{n+1} + \frac{\tau}{2}\Phi^{n+1} - \frac{\tau}{2}\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}\frac{Q^{n+1}}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle + C_0}}\\
&= V^n + \frac{\tau}{2}B\Phi^n - \frac{\tau}{2}\Phi^n - \frac{\tau}{2}\frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}\frac{Q^n}{\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle + C_0}},\\
&Q^{n+1} - \frac{\langle \frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U^{n+\frac{1}{2}}}^{n+\frac{1}{2}}|^2}, 2Re(\widetilde{U}^{n+\frac{1}{2}}\cdot(U^n)_t)\rangle + \langle \frac{\partial f(|\widetilde{U}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}, (\Phi^{n+1})\rangle}{2\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle + C_0}}\\
&= Q^n + \frac{\langle \frac{\partial f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial|\widetilde{U^{n+\frac{1}{2}}}^{n+\frac{1}{2}}|^2}, 2Re(\widetilde{U}^{n+\frac{1}{2}}\cdot(U^n)_t)\rangle + \langle \frac{\partial f(|\widetilde{U}|^2,\widetilde{\Phi}^{n+\frac{1}{2}})}{\partial\widetilde{\Phi}^{n+\frac{1}{2}}}, (\Phi^{n})\rangle}{2\sqrt{\langle f(|\widetilde{U}^{n+\frac{1}{2}}|^2,\widetilde{\Phi}^{n+\frac{1}{2}}),1\rangle + C_0}}.
\end{aligned}
$$

4. Numerical experiments

In above sections, we study some numerical schemes to solve following one dimension and two dimension fractional KGS systems:

$$
iu_t - \nu(-\Delta)^{\frac{\alpha}{2}}u = -u\frac{\partial f(|u|^2,\phi)}{\partial|u|^2}, \tag{4.1}
$$

$$
\phi_{tt} - \Delta\phi + \phi = \frac{\partial f(|u|^2,\phi)}{\partial\phi}, \tag{4.2}
$$

and

$$
iu_t - \nu(-\Delta)^{\frac{\alpha}{2}}u = -u\phi, \tag{4.3}
$$

$$
\phi_{tt} - \Delta\phi + \phi = |u|^2. \tag{4.4}
$$

Recently, some structure-preserving schemes such as linearly implicit conservative scheme, symplectic scheme and multi-symplectic scheme have been designed and investigated for solving classical and fractional KGS system. However, in these works main system (4.3),(4.4) have been considered. As far as we know, there exist few studies on system (4.1),(4.2). In this paper, we consider not noly structure-preserving scheme of fractional KGS system (4.3),(4.4) but also fractional KGS system (4.1),(4.2). For two dimension case, we show linearly implicit conservative scheme (3.15)–(3.19) and fully implicit conservative scheme (3.10)–(3.14) to solve fractional KGS system (4.3),(4.4), and the fully implicit conservative scheme (3.10)–(3.14) is can obtain numerical result by above iterative algorithm. Moreover, we show also linearly implicit conservative scheme (3.24)–(3.27) and fully implicit conservative scheme (3.2)–(3.6) to solve fractional KGS system (4.1),(4.2). For one dimension case, we only give a fully implicit conservative scheme (2.1)–(2.5), in fact, linearly implicit(3.15)–(3.19) and (3.24)–(3.27) are still applicable to one dimensional case.

In this section, we give some numerical experiments to show the efficiency of the structure-preserving schemes. The first numerical example shows the numerical errors and convergence rates of the structure-preserving scheme, and check conservation property of the schemes. The second numerical example shows the numerical result for the general fractional KGS system. The third numerical example shows the numerical result for two dimension fractional KGS system.

4.1. Experiment A

When $\alpha = 2$, the KGS system (4.3),(4.4) has the following solitary wave solutions [5]

$$u(x,t,v) = \frac{3\sqrt{2}}{4\sqrt{1-v^2}}\mathrm{sech}^2\frac{1}{2\sqrt{1-v^2}}(x-vt-x_0)\exp(\mathrm{i}(vx+\frac{1-v^2+v^4}{2(1-v^2)t})),$$
$$\phi(x,t,v) = \frac{3}{4(1-v^2)}\mathrm{sech}^2\frac{1}{2\sqrt{1-v^2}}(x-vt-x_0),$$

where v is the propagating velocity of the wave and x_0 is the initial phase. We consider initial-value $(v = 0.8, x_0 = -10)$

$$u_0 = \frac{3\sqrt{2}}{4\sqrt{1-v^2}}\mathrm{sech}^2\frac{1}{2\sqrt{1-v^2}}(x-x_0)\exp(\mathrm{i}(vx)),$$
$$\phi_0 = \frac{3}{4(1-v^2)}\mathrm{sech}^2\frac{1}{2\sqrt{1-v^2}}(x-x_0).$$

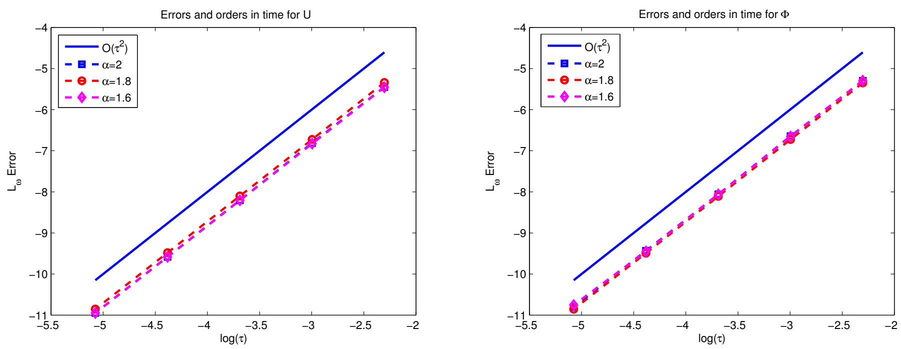

Figure 1. Errors and convergence orders in time for $h = 0.00001,\ t = 1$.

In this example, we text errors, convergence orders and conservation of mass and energy of one dimension KGS system by numerical scheme (2.1)–(2.5). First, we show errors and convergence orders of numerical scheme (2.1)–(2.5) at time $t = 1$. For $\alpha \neq 2$, the numerical exact solutions are obtained by a very fine mesh and a small time step. Then, we fix the space mesh $h = 0.00001$ and time step $\tau = 0.00001$ to test time convergence orders and space convergence orders by numerical scheme (2.1)–(2.5). The Figures 1–2 show time convergence orders and space convergence orders for different α, and it is found that the scheme is of order 2 in time, order 4 in space. From Figures 1–2, we can draw the observations: the approximate solution converge to the exact solution at the rate $O(\tau^2 + h^4)$,

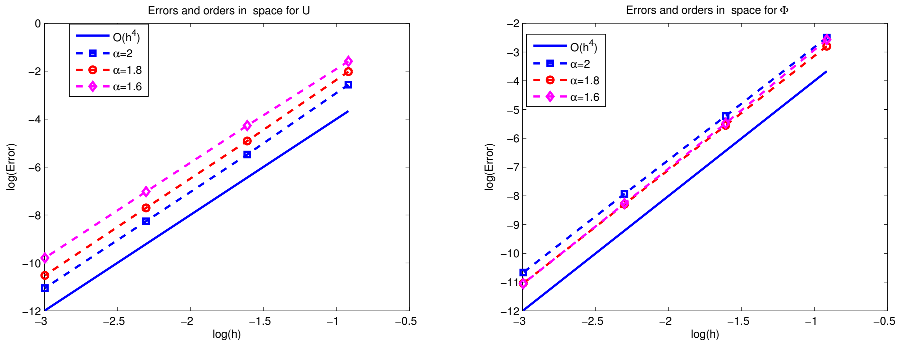

Figure 2. Errors and convergence orders in time for $\tau = 0.00001,\ t = 1$.

and consistent with the theoretical estimates of Theorems 5–6. Second, we examine the conservation of mass and energy with $x \in [-20, 20], t \in [0, 100], \tau = 0.01, h = 0.1$. Figure 3 shows relative residuals on the mass and energy errors for different values of α by numerical scheme (2.1)–(2.5). It is found that the numerical scheme preserves mass and energy conservation very well although energy varies with α.

4.2. *Experiment B*

Consider following one dimension fractional KGS system

$$i\partial_t u - \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}} u = u\phi + \gamma|u|^2 u, \tag{4.5}$$

$$\partial_{tt}\phi + \phi_{xx} + \phi = |u|^2, \tag{4.6}$$

$$u_0 = \frac{3\sqrt{2}}{4\sqrt{1-v^2}} \text{sech}^2 \frac{1}{2\sqrt{1-v^2}}(x - x_0)\exp(i(vx)), \tag{4.7}$$

$$\phi_0 = \frac{3}{4(1-v^2)} \text{sech}^2 \frac{1}{2\sqrt{1-v^2}}(x - x_0). \tag{4.8}$$

In this example, we simulate solitary wave and collisions of two solitary waves of fractional KGS system (4.5)–(4.8) by numerical scheme (2.1)–(2.5). First, we simulate solitary wave of numerical solutions for different orders α and difference parameter γ by the numerical scheme (2.1)–(2.5).

Figures 4–6 display solitary wave of the numerical solutions for difference value of $\gamma = 0.8,\ 1.5,\ 2$ and same value of $\alpha = 2$. It is found that the parameter γ affects the propagation velocity of the solitary wave, and larger γ, the propagation of the soliton got slower.

Figures 7–9 display solitary wave of the numerical solutions for difference value of $\alpha = 2,\ 1.8,\ 1.5$ and same value of $\gamma = 1$. It is found that the parameter α affects also the propagation velocity of the solitary wave, and smaller α, the propagation of the soliton got slower.

Second, we consider collisions of two solitary waves with $x \in [-20, 20],\ t \in [0, 30],\ \tau = 0.01, h = 0.1$, and the initial data are chosen as $(p_1 = 10, p_2 = 10, v_1 = 0.8, v_2 = -0.8)$

$$u_0 = u(x - p_1, 0, v_1) + u(x - p_2, 0, v_2),$$
$$\phi_0 = \phi(x - p_1, 0, v_1) + \phi(x - p_2, 0, v_2).$$

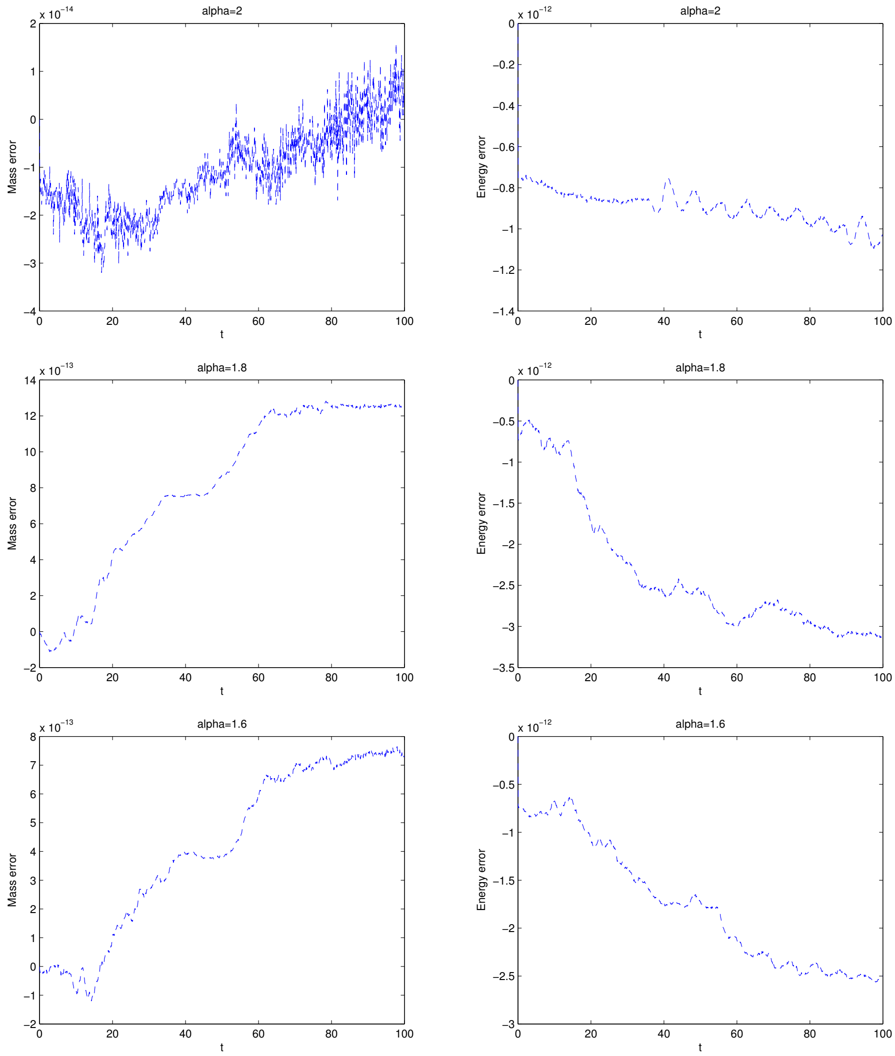

Figure 3. Conservation of mass (left) energy (right) of fractional KGS system by numerical scheme (2.1)–(2.5).

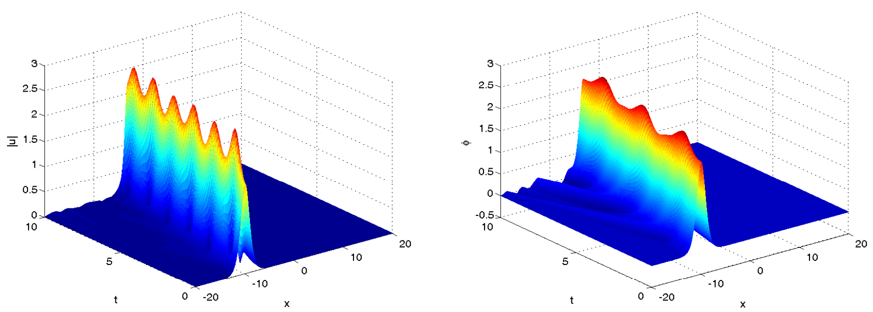

Figure 4. The wave forms of the numerical solution, $\gamma = 0.8, \alpha = 2$.

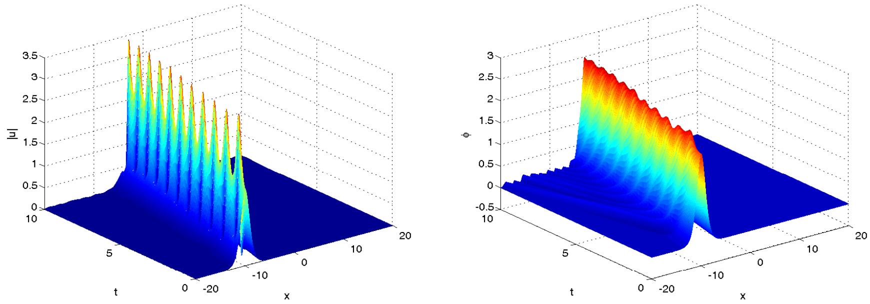

Figure 5. The wave forms of the numerical solution,$\gamma = 1.5, \alpha = 2$.

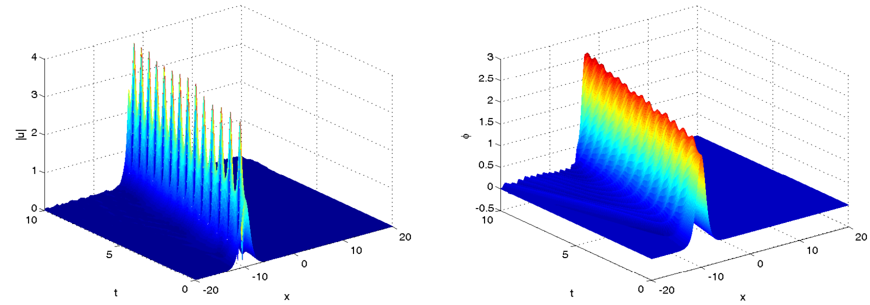

Figure 6. The wave forms of the numerical solution, $\gamma = 2, \alpha = 2$.

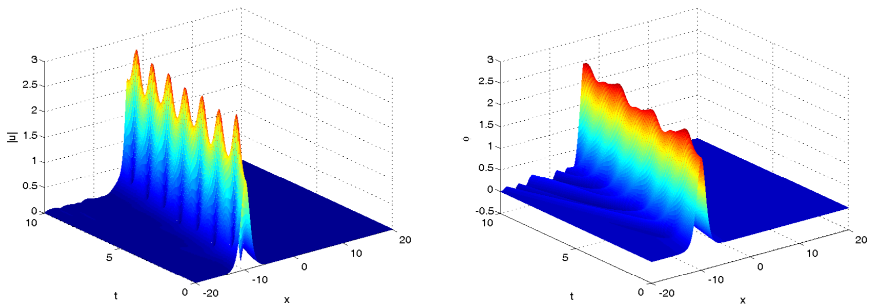

Figure 7. The wave forms of the numerical solution, $\gamma = 1, \alpha = 2$.

Figures 10–12 display solitary wave of the numerical solutions for difference value of $\gamma = 0.8,\ 1.5,\ 2$ and same value of $\alpha = 2$. Figures 13–15 display solitary wave of the numerical solutions for difference value of $\alpha = 2,\ 1.8,\ 1.5$ and same value of $\gamma = 1$. It is found that the parameters $\alpha,\ \gamma$ affect also the propagation velocity of the solitary wave. When smaller α and larger γ, the propagation of the soliton got slower, the soliton changes faster and even a high oscillation appears.

In the two example, we show some numerical results of fractional KGS system (4.5)–(4.8) for solitary wave case and collisions of two solitary waves case. In the process of time evolution, the solitary wave moves towards the boundary gradually, and produces some small waves around the solitary wave. In this paper, we only consider spatial range $[-20, 20]$. If we want to get a better numerical result, we

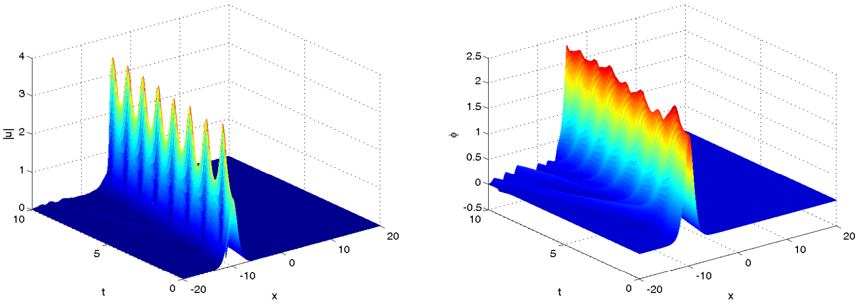

Figure 8. The wave forms of the numerical solution, $\gamma = 1, \alpha = 1.8$.

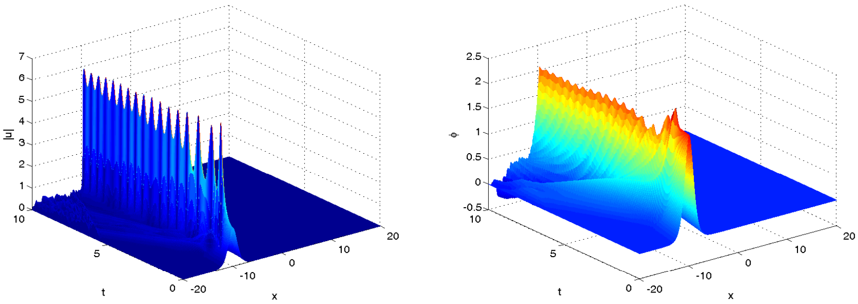

Figure 9. The wave forms of the numerical solution, $\gamma = 1, \alpha = 1.5$.

can expand the space a bit, but it will take more calculation time.

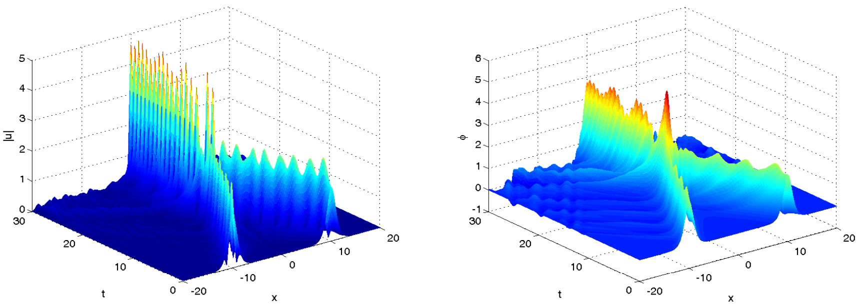

Figure 10. The wave forms of the numerical solution, $\gamma = 0.8, \alpha = 2$.

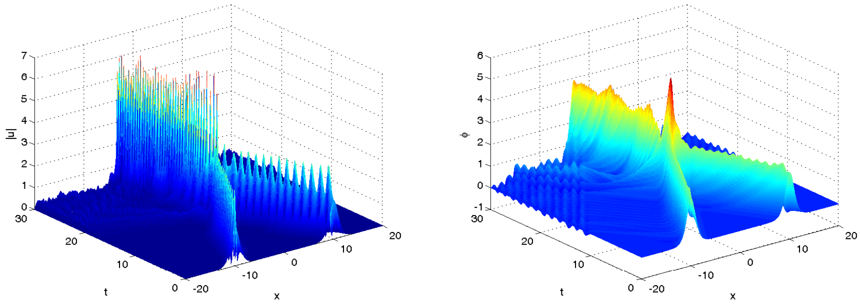

Figure 11. The wave forms of the numerical solution, $\gamma = 1.5, \alpha = 2$.

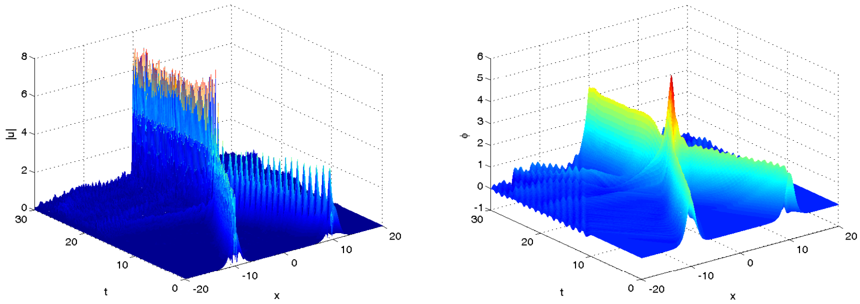

Figure 12. The wave forms of the numerical solution, $\gamma = 2, \alpha = 2$.

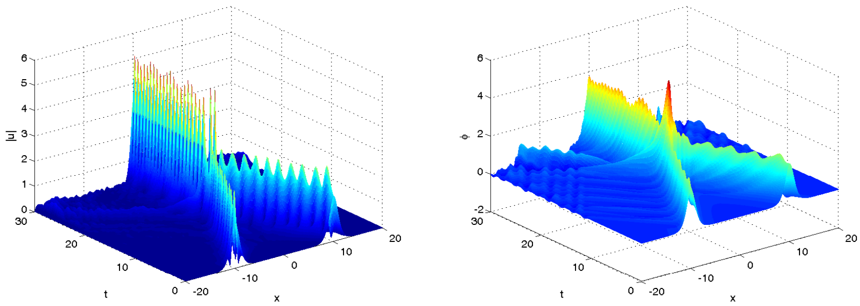

Figure 13. The wave forms of the numerical solution, $\gamma = 1, \alpha = 2$.

4.3. Experiment C

Consider two dimension fractional KGS system

$$i\partial_t u - \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}} u = -u\phi, \tag{4.9}$$

$$\partial_{tt}\phi + \Delta\phi + \phi = |u|^2, \tag{4.10}$$

$$u(x, y, 0) = \frac{2}{e^{x^2+2y^2} + e^{-(x^2+2y^2)}} \cdot e^{i5/\cosh(\sqrt{4x^2+y^2})}, \tag{4.11}$$

$$\phi(x, y, 0) = e^{-(x^2+y^2)}, \phi_t(x, 0) = e^{-(x^2+y^2)}/2. \tag{4.12}$$

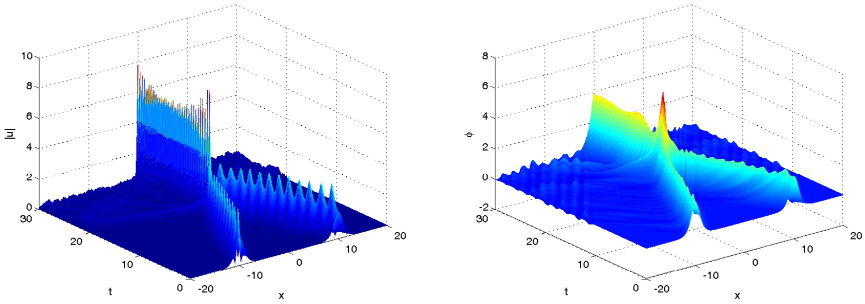

Figure 14. The wave forms of the numerical solution, $\gamma = 1, \alpha = 1.8$.

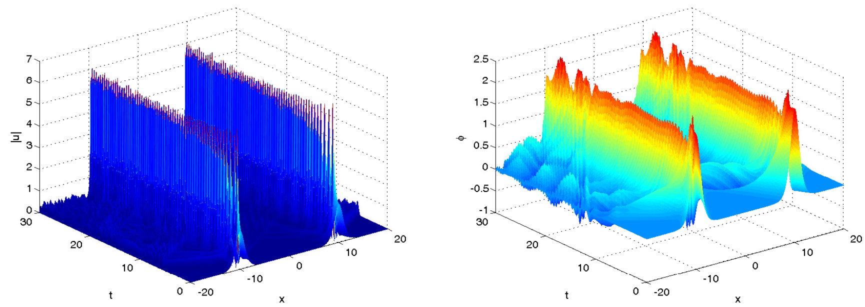

Figure 15. The wave forms of the numerical solution, $\gamma = 1, \alpha = 1.5$.

In the example, we simulate solitary wave numerical solutions for different orders α by the scalar auxiliary variable scheme (3.24)–(3.27). Figures 16–18 show the surface plots of the nucleon density $|u|^2$ and meson field ϕ for different time $t = 3, 5, 8$ and same value of $\alpha = 2$, respectively.

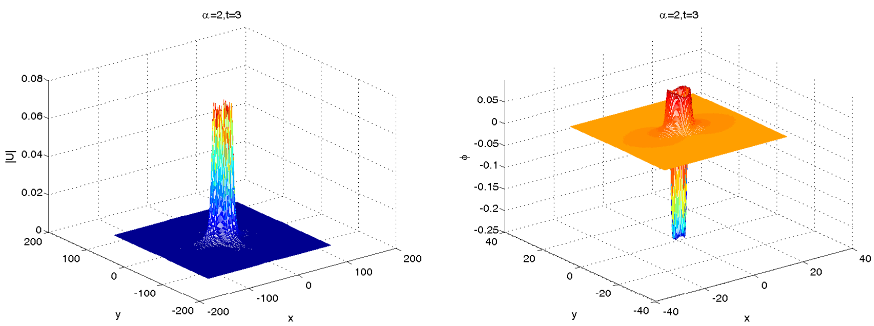

Figure 16. Surface plots of the nucleon density $|u|^2$ (left column) and meson field ϕ (right column).

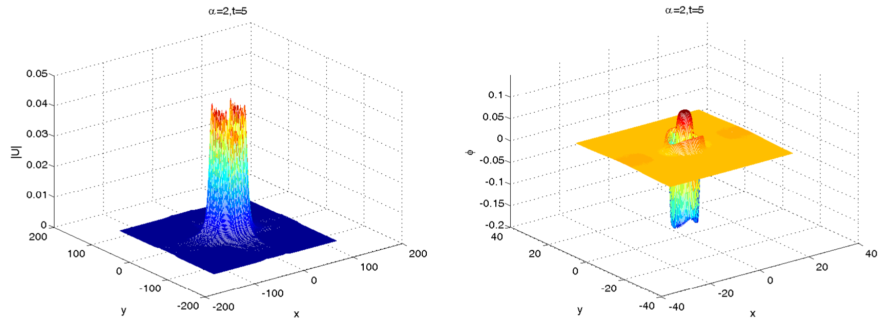

Figure 17. Surface plots of the nucleon density $|u|^2$ (left column) and meson field ϕ (right column).

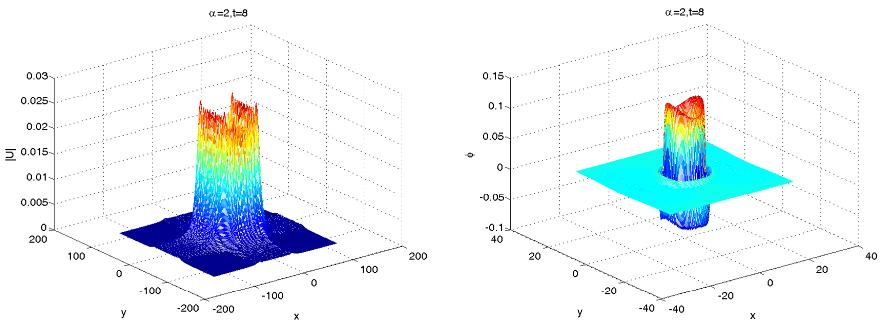

Figure 18. Surface plots of the nucleon density $|u|^2$ (left column) and meson field ϕ (right column).

Figures 19–21 show the surface plots of the nucleon density $|u|^2$ and meson field ϕ for different time $t = 3, 5, 8$ and same value of $\alpha = 1.8$, respectively. From Figures 16–21, we fine that the meson field change periodically, and the order α affects the shape of nucleon field. They also show that α affects the propagation velocity of the solitary wave.

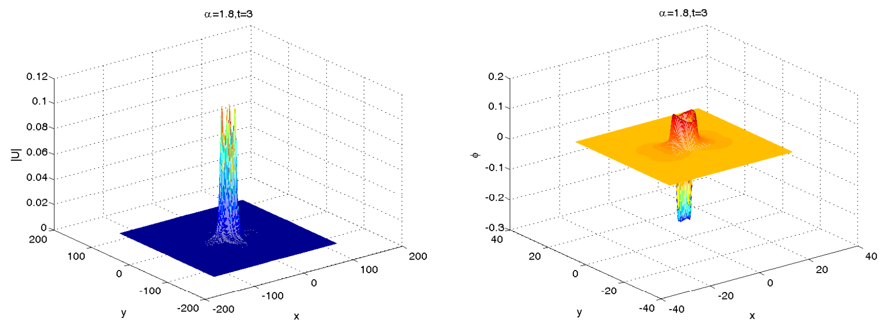

Figure 19. Surface plots of the nucleon density $|u|^2$ (left column) and meson field ϕ (right column).

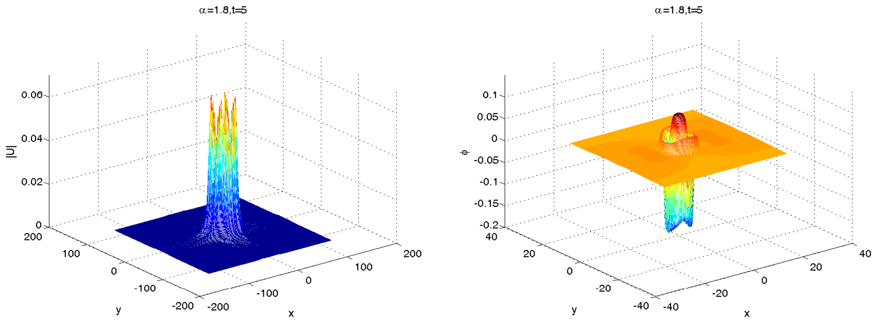

Figure 20. Surface plots of the nucleon density $|u|^2$ (left column) and meson field ϕ (right column).

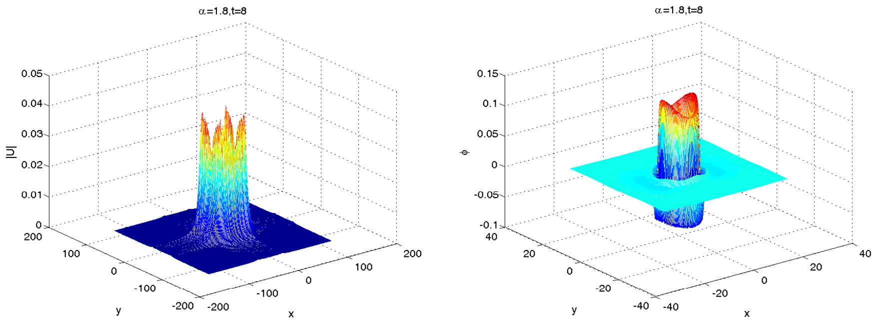

Figure 21. Surface plots of the nucleon density $|u|^2$ (left column) and meson field ϕ (right column).

5. Conclusion

In the paper, we study structure-preserving scheme to solve one dimension and two dimension space fractional KGS equations. First, we use the high central differences scheme in space and Crank-Nicolson scheme in time to discrete one dimension fractional KGS equations, which preserve mass and energy conservation laws of the fractional system. Then, we show that the arising scheme is uniquely solvable and approximate solutions converge to the exact solution at the rate $O(\tau^2 + h^4)$. Second, we give the high central differences scheme in space, Crank-Nicolson scheme and scalar auxiliary variable scheme in time for two dimension fractional KGS equations, which preserve one or more analytical properties of the fractional system. Finally, the numerical experiments including some one dimensional and two dimensional fractional KGS systems are given to verify the correctness of theoretical results.

Conflict of interest

The authors declare there is no conflict of interest.

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