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Research article

Nonlocal scalar conservation laws with discontinuous flux

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Abstract: We prove the well-posedness of entropy weak solutions for a class of space-discontinuous scalar conservation laws with nonlocal flux. We approximate the problem adding a viscosity term and we provide $\mathbf{L}^\infty$ and BV estimates for the approximate solutions. We use the doubling of variable technique to prove the stability with respect to the initial data from the entropy condition.

Keywords: conservation laws; entropy solutions; discontinuous flux; nonlocal problem

1. Introduction

The aim of this paper is to study nonlocal conservation laws characterized by a flux discontinuous in space. In particular, the nonlocality consists in the fact that the velocity function depends on a convolution term that averages the solution in space. It is worth pointing out that the discontinuity appears in the flux through a multiplicative way. We will focus on the following equation,

$$\partial_t \rho + \partial_x \left(\rho(1 - w_\eta * \rho)\mathfrak{v}(x)\right) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R},$$

where the function $\mathfrak{v} = \mathfrak{v}(x)$ is defined as follows:

$$\mathfrak{v}(x) = \begin{cases} v_l, & \text{if } x < 0, \\ v_r, & \text{if } x > 0. \end{cases}$$

The idea comes from the work in [27] in which traveling waves are studied for a nonlocal scalar space discontinuous traffic model that describes the beaviour of drivers on two consecutive roads with different speed limits. Indeed, in recent years nonlocal conservation laws have been provided to describe several phenomena, for example: flux granular flows [2], sedimentation [6], supply chains [19], conveyor belts [18], structured populations dynamics [26] and traffic flows [7, 9, 10, 16, 28]. For these reasons, we believe the matter of discontinuous nonlocal conservation laws mathematically challenging and interesting while applicable to different real-life scenarios. Here, we prove the wellposedness of a nonlocal space discontinuous problem and our approach is based on a viscous regularizing approximation of the problem and standard compactness estimates. To our knowledge, these are the first results regarding discontinuous nonlocal problem using the vanishing viscosity technique. In particular, we have been inspired by the adaptation of the classical vanishing viscosity argument for scalar conservation laws [23] to erosion models [15], scalar equations with discontinous fluxes [8, 24, 25] and triangular systems [14]. This technique is based on the approximation of the solution of the starting problem through a sequence of smooth solutions of the corresponding viscous parabolic problem. The convergence to a solution of the starting problem is obtained proving compactness estimates on the sequence of smooth solutions. The existence of the approximate smooth solutions is proved through a fixed point theorem. In [9, 10, 27] conservation laws with nonlocal flux have been applied to the traffic flow setting. In particular, in [9, 10] the authors study conservation laws with continuous flux functions and the well-posedness is obtained approximating the problem through an adapted numerical scheme and proving standard compactness estimates on the sequence of approximate solutions. In [27], travelling waves for a space-discontinuous traffic model describing two roads with rough conditions are studied. In the present work we do not need to apply an appropriate numerical discretisation of our problem due to the vanishing viscosity technique. We would like to count other more recent, noteworthy and interesting results about discontinuous nonlocal problems in [21] obtained with the fixed-point theorem technique. Our aim is to study a nonlocal equation in which the space-discontinuity occurs in the multiplicative term. It is not straightforward to deal with more general flux functions in the nonlocal setting satisfying the 'crossing condition' as in the paper [20]. Indeed, considering two different nonlocal flux functions for $x < 0$ and $x > 0$ would imply that the crossing point is not fixed but it changes position in time and this makes harder the analytical study. The paper is organized as follows. In Section 2, we describe the main results in this paper. In Section 3, we prove the existence of weak solutions of our problem, approximating it through a viscous problem and giving $\mathbf{L}^{\infty}$ and BV bounds. Finally, in Section 4, we show the uniqueness of entropy solutions, deriving an $\mathbf{L}^{1}$ contraction property using a doubling of variables argument.

2. Main results

We consider the following scalar conservation equation with discontinuous nonlocal flux coupled with an initial datum

$$\begin{cases} \partial_t \rho + \partial_x f(t, x, \rho) = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where

$$f(t, x, \rho) = \rho(1 - w_\eta * \rho)\mathfrak{v}(x),$$
$$(w_\eta * \rho)(t, x) = \int_x^{x+\eta} \rho(t, y) w_\eta(y - x) dy, \qquad \eta > 0,$$

and the velocity function $\mathfrak{v} = \mathfrak{v}(x)$ is defined as follows

$$\mathfrak{v}(x) = \begin{cases} v_l, & \text{if } x < 0, \\ v_r, & \text{if } x > 0. \end{cases}$$

In this context ρ represents the unknown function, w_η is a non-increasing kernel function whose length of the support is η. The equation in (2.1) is the space discontinuous version of the one in [7], where a nonlocal traffic model is presented.

On w_η, $\mathfrak{v}$, ρ_0 we shall assume that

$$0 < v_l < v_r; \tag{2.2}$$

$$w_\eta \in C^2([0,\eta]), \qquad w_\eta(\eta) = w'_\eta(\eta) = 0, \qquad w'_\eta \le 0 \le w_\eta, \qquad \left\|w_\eta\right\|_{L^1(0,\eta)} = 1; \tag{2.3}$$

$$0 \le \rho_0 \le 1, \qquad \rho_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R}). \tag{2.4}$$

Assumption (2.3) implies that, if ρ is continuous,

$$\begin{aligned} \partial_x(w_\eta * \rho)(t,x) &= -(w'_\eta * \rho)(t,x) - w_\eta(0)\rho(t,x), \\ \partial_x(w'_\eta * \rho)(t,x) &= -(w''_\eta * \rho)(t,x) - w'_\eta(0)\rho(t,x). \end{aligned} \tag{2.5}$$

Remark 2.1. *The assumption Eq* (2.3) *does not allow the usual choices of kernels in traffic literature, such as: the constant and the linear decreasing kernels. Our kernels are like restrictions to* $[0,\eta]$ *of cut-off functions that are equal to 1 in* $[-\eta/2,\eta/2]$ *and vanish outside* $[-\eta,\eta]$ *or, for example,*

$$w_\eta(x) := \begin{cases} e^{\frac{1}{x^2-\eta^2}} & x \in [0,\eta), \\ 0 & \textit{otherwise}, \end{cases} \tag{2.6}$$

observing that $\lim_{x\to\eta^-} w'_\eta(x) = 0$.

Remark 2.2. *It is interesting to notice that if Eq* (2.2) *does not hold, namely* $v_l > v_r$, *we cannot say even in the local case that*

$$\|\rho\|_{L^\infty((0,\infty)\times\mathbb{R})} \le \|\rho_0\|_{L^\infty(\mathbb{R})}.$$

Let us consider this very easy example in the classical local case

$$\begin{cases} \partial_t\rho + \partial_x f(\rho) = 0, & (t,x) \in (0,\infty)\times(-\infty,0), \\ \partial_t\rho + \partial_x g(\rho) = 0, & (t,x) \in (0,\infty)\times(0,\infty), \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.7}$$

where

$$f(\rho) = 2(\rho(1-\rho)), \quad g(\rho) = \rho(1-\rho), \quad \rho_0(x) = \begin{cases} 0.25, & \textit{if } x < 0, \\ 0.77, & \textit{if } x > 0. \end{cases}$$

The entropy weak solution to the above Cauchy problem is

$$\rho = \begin{cases} \rho_l = 0.25, & \textit{if } x < \frac{f(\rho_-)-f(\rho_l)}{\rho_- - \rho_l}t, \\ \rho_- = 0.9, & \textit{if } \frac{f(\rho_-)-f(\rho_l)}{\rho_- - \rho_l}t < x < 0, \\ \rho_+ = \rho_r = 0.77, & \textit{if } x > 0. \end{cases} \tag{2.8}$$

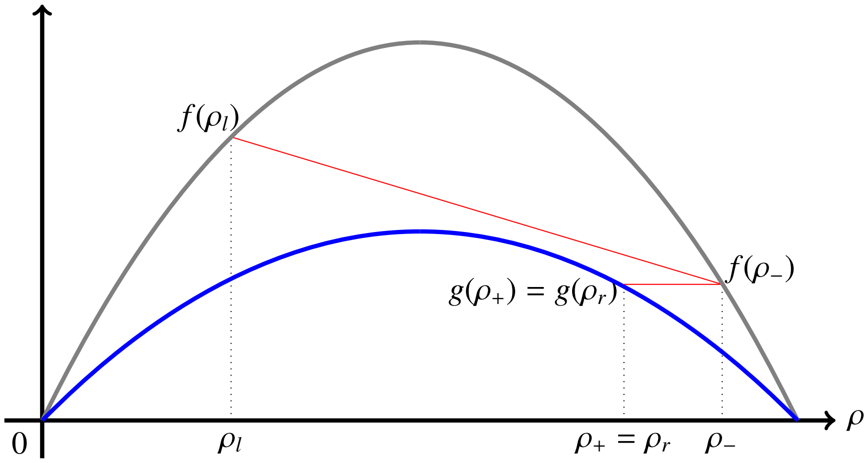

Figure 1. Fundamental diagrams relative to Eq (2.7).

A complete description of conservation laws with discontinuous flux can be found in [17, 22].

We use the following definitions of solution.

Definition 2.1. *We say that a function* $\rho : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *is a weak solution of Eq* (2.1) *if*

$$0 \leq \rho \leq 1, \qquad \|\rho(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|\rho_0\|_{L^1(\mathbb{R})}, \tag{2.9}$$

for almost every $t > 0$ *and for every test function* $\varphi \in \mathbf{C_c^1}(\mathbb{R}^2)$

$$\int_0^{\infty} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(t, x, \rho) \partial_x \varphi) \, dt dx + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) dx = 0.$$

Definition 2.2. *A function* $\rho \in (L^1 \cap L^\infty)(\mathbb{R}^+ \times \mathbb{R}; [0, \rho_{\max}])$ *is an entropy weak solution of Eq* (2.1)*, if*

(1) for all $\kappa \in \mathbb{R}$*, and any test function* $\varphi \in \mathbf{C_c^1}(\mathbb{R}^2; \mathbb{R}^+)$ *which vanishes for* $x \leq 0$*,*

$$\int_0^{+\infty} \int_{\mathbb{R}^+} |\rho - \kappa| \, \varphi_t + |\rho - \kappa| \, (1 - w_\eta * \rho) v_r \varphi_x$$
$$- \operatorname{sgn}(\rho - \kappa) \, \kappa \, \partial_x (w_\eta * \rho) v_r \varphi \, dx \, dt + \int_{\mathbb{R}^+} |\rho_0(x) - \kappa| \, \varphi(0, x) dx \geq 0;$$

(2) for all $\kappa \in \mathbb{R}$*, and any test function* $\varphi \in \mathbf{C_c^1}(\mathbb{R}^2; \mathbb{R}^+)$ *which vanishes for* $x \geq 0$*,*

$$\int_0^{+\infty} \int_{\mathbb{R}^-} |\rho - \kappa| \, \varphi_t + |\rho - \kappa| \, (1 - w_\eta * \rho) v_l \, \varphi_x$$
$$- \operatorname{sgn}(\rho - \kappa) \, \kappa \, \partial_x (w_\eta * \rho) \, v_l \, \varphi \, dx \, dt + \int_{\mathbb{R}^-} |\rho_0(x) - \kappa| \, \varphi(0, x) dx \geq 0;$$

(3) for all $\kappa \in \mathbb{R}$*, and any test function* $\varphi \in \mathbf{C_c^1}(\mathbb{R}^2; \mathbb{R}^+)$

$$\int_0^{+\infty} \int_{\mathbb{R}} |\rho - \kappa| \, \varphi_t + |\rho - \kappa| \, (1 - w_\eta * \rho) \mathfrak{v}(x) \, \varphi_x$$

$$
\begin{aligned}
&-\int_0^{+\infty}\int_{\mathbb{R}^*} \operatorname{sgn}(\rho-\kappa)\,\kappa\,\partial_x\Big((w_\eta*\rho)\,\mathfrak{v}(x)\Big)\,\varphi\,dx\,dt\\
&+\int_{\mathbb{R}} |\rho_0(x)-\kappa|\,\varphi(0,x)dx\\
&+\int_0^{+\infty}\Big|(v_r-v_l)\kappa(1-w_\eta*\rho)\Big|\,\varphi(t,0)dt\ge 0;
\end{aligned}
$$

(4) the traces are such that the jump

$$
|\rho_l-\rho_r| \tag{2.10}
$$

is the smallest possible that satisfies the Rankine-Hugoniot condition

$$
f(t,0^+,\rho_r)=f(t,0^-,\rho_l)\ \textit{i.e.}\ v_l\rho_l=v_r\rho_r,
$$

where we denoted with

$$
f(t,0^{\pm},\rho_{r,l})=\lim_{x\to0^{\pm}}\mathfrak{v}(x)\rho(t,x)\left(1-\int_x^{x+\eta}w_\eta(y-x)\rho(y,t)dy\right).
$$

Remark 2.3. *We would like to underline that the existence of strong right and left traces, respectively ρ_r and ρ_l, is ensured by the genuine non-linearity of our flux function, as it is proved in [1, 4].*

The main result of this paper is the following.

Theorem 2.1. *Assume Eq* (2.2)*, Eq* (2.3)*, and Eq* (2.4)*. Then, the initial value problem in Eq* (2.1) *possesses an unique entropy solution u in the sense of Definition 2.2. Moreover, if u and v are two entropy solutions of Eq* (2.1) *in the sense of Definition 2.2 , the following inequality holds*

$$
\|u(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{R})}\le e^{Kt}\,\|u(0,\cdot)-v(0,\cdot)\|_{L^1(\mathbb{R})}, \tag{2.11}
$$

for some suitable constant $K>0$.

3. Existence

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of Eq (2.1). We have been inspired by the viscous approximation in [11, Theorem 3.1], the sign of the term in absolute value $|1-w_\eta*\rho_\varepsilon|$ follows from Lemma 3.1 below.

Fix a small number $\varepsilon>0$ and let $\rho_\varepsilon=\rho_\varepsilon(t,x)$ be the unique classical solution of the following problem

$$
\begin{cases}
\partial_t\rho_\varepsilon+(1-w_\eta*\rho_\varepsilon)\mathfrak{v}_\varepsilon(x)\partial_x\rho_\varepsilon+\rho_\varepsilon|1-w_\eta*\rho_\varepsilon|\mathfrak{v}_\varepsilon{}'(x)\\
\qquad\qquad+\rho_\varepsilon(w_\eta'*\rho_\varepsilon)\mathfrak{v}_\varepsilon(x)+\rho_\varepsilon^2w_\eta(0)\mathfrak{v}_\varepsilon(x)=\varepsilon\partial_{xx}^2\rho_\varepsilon, & (t,x)\in(0,\infty)\times\mathbb{R},\\
\rho_\varepsilon(0,x)=\rho_{0,\varepsilon}(x), & x\in\mathbb{R},
\end{cases} \tag{3.1}
$$

where $\rho_{0,\varepsilon}$ and $\mathfrak{v}_\varepsilon$ are $C^\infty(\mathbb{R})$ approximations of ρ_0 and $\mathfrak{v}$ such that

$$
\begin{aligned}
&\rho_{0,\varepsilon}\to\rho_0,\quad \text{a.e. and in } L^p(\mathbb{R}),\ 1\le p<\infty,\\
&0\le\rho_{0,\varepsilon}\le1,\qquad \left\|\rho_{0,\varepsilon}\right\|_{L^1(\mathbb{R})}\le\|\rho_0\|_{L^1(\mathbb{R})},\quad \left\|\partial_x\rho_{0,\varepsilon}\right\|_{L^1(\mathbb{R})}\le C_0\\
&v_l\le\mathfrak{v}_\varepsilon\le v_r,\quad \mathfrak{v}_\varepsilon{}'\ge0,\quad \mathfrak{v}_\varepsilon(x)=\begin{cases}v_l & \text{if } x<-\varepsilon,\\ v_r & \text{if } x>\varepsilon,\end{cases}
\end{aligned} \tag{3.2}
$$

for every $\varepsilon > 0$ and some positive constant C_0 independent on ε. The well-posedness of Eq (3.1) can be obtained following the same arguments of [11–13].

Let us prove some a priori estimates on ρ_ε denoting with C_0 the constants which depend only on the initial data, and $C(T)$ the constants which depend also on T.

Lemma 3.1 (L^∞ **estimate).** *Let ρ_ε be a solution of* (3.1). *We have that*

$$0 \le \rho_\varepsilon \le 1,$$

for every $\varepsilon > 0$.

Proof. Thanks to Eq (3.2), 0 is a subsolution of Eq (3.1), due to the Maximum Principle for parabolic equations we have that

$$\rho_\varepsilon \ge 0. \tag{3.3}$$

We have to prove

$$\rho_\varepsilon \le 1. \tag{3.4}$$

Assume by contradiction that Eq (3.4) does not hold.

Let us define the function $r(t,x) = e^{-\lambda t}\rho_\varepsilon(t,x)$. We can choose $\lambda > 0$ so small that

$$\|r\|_{L^\infty((0,\infty)\times\mathbb{R})} > 1. \tag{3.5}$$

Thanks to Eq (3.1), r solves the equation

$$\begin{aligned}&\partial_t r + \lambda r + (1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)\partial_x r - \varepsilon\partial_{xx}^2 r\\ &\quad = - r(w_\eta' * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x) - r|1 - w_\eta * \rho_\varepsilon|\mathfrak{v}_\varepsilon{}'(x) - e^{\lambda t} r^2 w_\eta(0)\mathfrak{v}_\varepsilon(x).\end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}(w_\eta' * \rho_\varepsilon(t,\cdot))(x) = &\int_x^{x+\eta} w_\eta'(y - x)\left(\rho_\varepsilon(t,y) - \|\rho_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})}\right) dy\\ &- \|\rho_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})}\, w_\eta(0),\end{aligned}$$

we can write

$$\begin{aligned}&\partial_t r + \lambda r + (1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)\partial_x r - \varepsilon\partial_{xx}^2 r\\ &\quad = - r\left(w_\eta' * \left(\rho_\varepsilon - \|\rho_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})}\right)\right)\mathfrak{v}_\varepsilon(x) + r w_\eta(0)\,\|\rho_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})}\,\mathfrak{v}_\varepsilon(x)\\ &\qquad - r|1 - w_\eta * \rho_\varepsilon|\mathfrak{v}_\varepsilon{}'(x) - e^{\lambda t} r^2 w_\eta(0)\mathfrak{v}_\varepsilon(x)\\ &\quad = - r\left(w_\eta' * \left(\rho_\varepsilon - \|\rho_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})}\right)\right)\mathfrak{v}_\varepsilon(x) - r|1 - w_\eta * \rho_\varepsilon|\mathfrak{v}_\varepsilon{}'(x)\\ &\qquad + r(\|\rho_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})} - \rho_\varepsilon) w_\eta(0)\mathfrak{v}_\varepsilon(x) \le 0.\end{aligned} \tag{3.7}$$

Let $(\bar{t},\bar{x})$ be such that

$$\|r\|_{L^\infty((0,\infty)\times\mathbb{R})} = r(\bar{t},\bar{x}).$$

Since, thanks to Eq (3.5),

$$\|r(0,\cdot)\|_{L^\infty(\mathbb{R})} \le 1 < r(\bar{t},\bar{x}),$$

we must have

$$\bar{t} > 0.$$

Therefore we can evaluate Eq (3.6) in $(\bar{t}, \bar{x})$ and gain

$$0 < \lambda \|r\|_{L^\infty((0,\infty)\times\mathbb{R})} \le 0.$$

Since, this cannot be, Eq (3.4) is proved. □

Using Eq (2.3) and Lemma 3.1, we know that

$$0 \le w_\eta * \rho_\varepsilon \le 1 \tag{3.8}$$

and then we can rewrite Eq (3.1) as follows

$$\begin{cases} \partial_t\rho_\varepsilon + \partial_x(\rho_\varepsilon(1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)) = \varepsilon\partial^2_{xx}\rho_\varepsilon, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ \rho_\varepsilon(0,x) = \rho_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases} \tag{3.9}$$

Lemma 3.2 (**L^1 estimate**). *Let ρ_ε be a solution of Eq* (3.1). *We have that*

$$\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le \|\rho_0\|_{L^1(\mathbb{R})}, \tag{3.10}$$

$$\left\|(w_\eta * \rho_\varepsilon)(t,\cdot)\right\|_{L^1(\mathbb{R})} \le \|\rho_0\|_{L^1(\mathbb{R})}, \tag{3.11}$$

$$\left\|\partial_x(w_\eta * \rho_\varepsilon)(t,\cdot)\right\|_{L^1(\mathbb{R})} \le 2w_\eta(0)\|\rho_0\|_{L^1(\mathbb{R})}, \tag{3.12}$$

for every $t \ge 0$ and $\varepsilon > 0$.

Proof. We have

$$\frac{d}{dt}\int_{\mathbb{R}} \rho_\varepsilon dx = \int_{\mathbb{R}} \partial_t\rho_\varepsilon dx = \varepsilon\int_{\mathbb{R}} \partial^2_{xx}\rho_\varepsilon dx - \int_{\mathbb{R}} \partial_x(\rho_\varepsilon(1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x))dx = 0.$$

Therefore,

$$\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} = \left\|\rho_{0,\varepsilon}\right\|_{L^1(\mathbb{R})},$$

and Eq (3.10) follows from Eq (3.2).

Using Eqs (2.3), (2.5), (3.8), and Lemma 3.1

$$\begin{aligned} \int_{\mathbb{R}} (w_\eta * \rho_\varepsilon)(t,x)dx &= \int_{\mathbb{R}}\int_x^{x+\eta} w_\eta(y-x)\rho_\varepsilon(t,y)dydx \\ &= \int_{\mathbb{R}}\int_0^{\eta} w_\eta(y)\rho_\varepsilon(t,y+x)dydx \\ &= \left\|w_\eta\right\|_{L^1(\mathbb{R})}\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} = \|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})}, \\ \int_{\mathbb{R}} |\partial_x(w_\eta * \rho_\varepsilon)(t,x)|dx &\le \int_{\mathbb{R}}\int_x^{x+\eta} |w'_\eta(y-x)|\rho_\varepsilon(t,y)dxdy + w_\eta(0)\int_{\mathbb{R}}\rho_\varepsilon dx \\ &= -\int_{\mathbb{R}}\int_0^{\eta} w'_\eta(y)\rho_\varepsilon(t,y+x)dxdy + w_\eta(0)\int_{\mathbb{R}}\rho_\varepsilon dx \\ &= 2w_\eta(0)\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})}. \end{aligned}$$

Therefore, Eq (3.10), Eq (3.11), and Eq (3.12) follow from Eq (3.2). □

Lemma 3.3 (*BV* **estimate in** *x*)**.** *Let ρ_ε be a solution of Eq* (3.1)*. We have that*

$$\|\partial_x\rho_\varepsilon(t,\cdot)\|_{L^1((-\infty,-2\delta)\cup(2\delta,\infty))} \le C_\delta,$$

for every $t \ge 0$ and ε, $\delta > 0$ where C_δ is a constant depending on δ but not on ε.

Proof. Let us consider the function

$$\chi(x) = \begin{cases} 1, & x \in (-\infty,-2\delta) \cup (2\delta,+\infty), \\ 0, & x \in (-\delta,\delta), \end{cases}$$

such that

$$\chi \in C^\infty(\mathbb{R}), \quad 0 \le \chi(x) \le 1,$$
$$\chi'(x) \ge 0 \text{ for } x \in [0,+\infty), \quad \chi'(x) \le 0 \text{ for } x \in (-\infty,0].$$

It is not restrictive to assume $\varepsilon < \delta$. In such a way we have that the supports of χ and $\mathfrak{v}_\varepsilon{}'$ are disjoint. Finally, we observe that

$$\frac{\chi'}{\chi}, \frac{\chi''}{\chi} \in L^\infty(\mathbb{R}).$$

Differentiating the equation in (3.9) w.r.t. the space variable

$$\begin{aligned}\partial^2_{tx}\rho_\varepsilon &+ \partial_x((1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)\partial_x\rho_\varepsilon) + \partial_x(\rho_\varepsilon(1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon{}'(x)) \\ &+ \partial_x(\rho_\varepsilon(w'_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)) + w_\eta(0)\partial_x(\rho^2_\varepsilon\mathfrak{v}_\varepsilon(x)) = \varepsilon\partial^3_{xxx}\rho_\varepsilon.\end{aligned}$$

Using [5, Lemma 2] and Lemmas 3.1, and 3.2

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} |\chi(x)\partial_x\rho_\varepsilon|\, dx &= \int_{\mathbb{R}} \chi(x)\partial^2_{tx}\rho_\varepsilon \,\mathrm{sgn}\, \partial_x\rho_\varepsilon dx \\
=&\varepsilon\int_{\mathbb{R}} \chi(x)\partial^3_{xxx}\rho_\varepsilon \,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx \\
&- \int_{\mathbb{R}} \chi(x)\partial_x((1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)\partial_x\rho_\varepsilon)\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx \\
&\underbrace{- \int_{\mathbb{R}} \chi(x)\partial_x(\rho_\varepsilon|1 - w_\eta * \rho_\varepsilon|\mathfrak{v}_\varepsilon{}'(x))\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx}_{=0} \\
&- \int_{\mathbb{R}} \chi(x)\partial_x(\rho_\varepsilon(w'_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x))\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx \\
&- w_\eta(0)\int_{\mathbb{R}} \chi(x)\partial_x(\rho^2_\varepsilon\mathfrak{v}_\varepsilon(x))\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx \\
=&\underbrace{-\varepsilon\int_{\mathbb{R}} \chi(x)(\partial^2_{xx}\rho_\varepsilon)^2\delta_{\{\partial_x\rho_\varepsilon=0\}}}_{\le 0} - \varepsilon\int_{\mathbb{R}} \chi'(x)\underbrace{\partial^2_{xx}\rho_\varepsilon \,\mathrm{sgn}\,\partial_x\rho_\varepsilon}_{=\partial_x|\partial_x\rho_\varepsilon|} dx \\
&+ \underbrace{\int_{\mathbb{R}} \chi(x)(1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)\partial_x\rho_\varepsilon\partial^2_{xx}\rho_\varepsilon\delta_{\{\partial_x\rho_\varepsilon=0\}}dx}_{=0}\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}} \chi'(x)(1 - w_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)|\partial_x\rho_\varepsilon|dx \\
&- \int_{\mathbb{R}} \chi(x)(w'_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)|\partial_x\rho_\varepsilon|dx \\
&\underbrace{- \int_{\mathbb{R}} \chi(x)(\rho_\varepsilon(w'_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon{}'(x)\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx}_{=0} \\
&+ \int_{\mathbb{R}} \chi(x)\rho_\varepsilon(w''_\eta * \rho_\varepsilon)\mathfrak{v}_\varepsilon(x)\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx + w'_\eta(0)\int_{\mathbb{R}} \chi(x)\rho_\varepsilon^2\mathfrak{v}_\varepsilon(x)\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx \\
&- 2w_\eta(0)\int_{\mathbb{R}} \chi(x)\rho_\varepsilon\mathfrak{v}_\varepsilon(x)|\partial_x\rho_\varepsilon|dx - w_\eta(0)\underbrace{\int_{\mathbb{R}} \chi(x)\rho_\varepsilon^2\mathfrak{v}_\varepsilon{}'(x)\,\mathrm{sgn}\,\partial_x\rho_\varepsilon dx}_{=0} \\
\le\ & c\int_{\mathbb{R}} \chi(x)|\partial_x\rho_\varepsilon|dx + c\int_{\mathbb{R}} \rho_\varepsilon dx \le c\int_{\mathbb{R}} \chi(x)|\partial_x\rho_\varepsilon|dx + c\,\|\rho_0\|_{L^1(\mathbb{R})},
\end{aligned}
$$

where $\delta_{\{\partial_x\rho_\varepsilon=0\}}$ is the Dirac delta concentrated on the set $\{\partial_x\rho_\varepsilon = 0\}$ and c is a constant that depends on δ and does not depend on ε. Thanks to the Gronwall Lemma we get

$$
\|\chi\partial_x\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le e^{ct}\left\|\chi\partial_x\rho_{0,\varepsilon}\right\|_{L^1(\mathbb{R})} + c(e^{ct} - 1),
$$

and using (3.2) we get the claim. □

Lemma 3.4 (**Compactness**). *There exists a function $\rho : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and a subsequence $\{\varepsilon_k\}_k \subset (0,\infty)$, $\varepsilon_k \to 0$, such that*

$$
\begin{aligned}
&0 \le \rho \le 1, \qquad \rho \in BV((0,\infty) \times ((-\infty,-\delta) \cup (\delta,\infty))), \qquad \delta > 0, \\
&\rho_{\varepsilon_k} \to \rho \qquad \text{a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty.
\end{aligned}
$$

Proof. Thanks to Lemma 3.3 the sequence $\{\rho_\varepsilon\chi_{I_\delta}\}_{\varepsilon,\delta>0}$ of approximate solutions to Eq (2.1) constructed by vanishing viscosity has uniformly bounded variation on each interval of the type $I_\delta = (-\infty,-\delta) \cup (\delta,+\infty)$, $\delta > 0$. Moreover, thanks to Lemma 3.1 the L^∞−norm of the sequence $\{\rho_\varepsilon\chi_{I_\delta}\}_{\varepsilon,\delta>0}$ is bounded by 1. Thus, applying Helly's Theorem and by a diagonal procedure, we can extract a subsequence $\{\rho_{\varepsilon_k}\chi_{I_{\delta_k}}\}_{k\in\mathbb{N}}$ that converges to a function $\rho : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ that satisfies the following conditions: $\rho \in BV((0,\infty) \times ((-\infty,-\delta) \cup (\delta,\infty)))$ and $0 \le \rho \le 1$,

$$
\rho_{\varepsilon_k}\chi_{I_{\delta_k}} \to \rho \qquad \text{a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty.
$$

Thus, we obtain the compactness of the sequence $\{\rho_{\varepsilon_k}\}_{k\in\mathbb{N}}$ a.e. in $(0,\infty)\times\mathbb{R}$ and for this reason we get the claim. It is worth remarking that being δ as small as we want we get the convergence on the whole space $\mathbb{R}$. □

4. Uniqueness and Stability

We are now ready to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. The existence of entropy solutions follows using the same arguments of [3] and Lemma 3.4. In particular, the nature of entropy solution of our limit function is related to the equivalence between [3, Definition 3] and [3, Definition 4] based on the germs theory, being our solution obtained through the vanishing viscosity technique. Moreover, one can observe that the points 1 and 2 of Definition 2.2 are directly satisfied multiplying equation (3.9) times the $\text{sgn}(\rho - k)$, integrating with respect to time and space, and passing to the limit as $\varepsilon \to 0$. The sketch of this proof is the following: we start from an L^1 contraction property proved using the doubling of variables technique. After that we choose appropriate test functions in order to deal with the discontinuity in 0. We apply some limit procedures on the test functions and the classical Rankine-Hugoniot condition. At the end the Gronwall's inequality gives us the statement.

Let us prove the inequality Eq (2.11). In Lemma 4.1 we prove the following inequality through the doubling of variables technique. For any two entropy solutions u and v we derive the L^1 contraction property:

$$\begin{aligned}&\iint_{\mathbb{R}^+\times\mathbb{R}} \left(|u-v|\,\phi_t + \text{sgn}(u-v)(f(t,x,u)-f(t,x,v))\phi_x\right) dxdt\\ &\leq K \iint_{\mathbb{R}^+\times\mathbb{R}} |u-v|\,\phi dxdt, \end{aligned} \tag{4.1}$$

for any $0 \leq \phi \in \mathbf{C}_\mathbf{c}^\infty\left(\mathbb{R}^+ \times \mathbb{R} \setminus \{0\}\right)$. We remove the assumption in Eq (4.1) that ϕ vanishes near 0, by introducing the following Lipschitz function for $h > 0$

$$\mu_h(x) = \begin{cases} \frac{1}{h}(x+2h), & x \in [-2h,-h],\\ 1, & x \in [-h,h],\\ \frac{1}{h}(2h-x), & x \in [h,2h],\\ 0, & |x| \geq 2h. \end{cases}$$

Now we can define $\Psi_h(x) = 1 - \mu_h(x)$, noticing that $\Psi_h \to 1$ in L^1 as $h \to 0$. Moreover, Ψ_h vanishes in a neighborhood of 0. For any $0 \leq \Phi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}^+ \times \mathbb{R})$, we can check that $\phi = \Phi\Psi_h$ is an admissible test function for Eq (4.1). Using ϕ in Eq (4.1) and integrating by parts we get

$$\begin{aligned}&\iint_{\mathbb{R}^+\times\mathbb{R}} \left(|u-v|\,\Phi_t\Psi_h + \text{sgn}(u-v)(f(t,x,u)-f(t,x,v))\Phi_x\Psi_h\right) dxdt\\ &\underbrace{-\iint_{\mathbb{R}^+\times\mathbb{R}} \text{sgn}(u-v)(f(t,x,u)-f(t,x,v))\Phi(t,x)\Psi_h'(x)dxdt}_{J(h)}\\ &\leq K \iint_{\mathbb{R}^+\times\mathbb{R}} |u-v|\,\Phi\Psi_h dxdt.\end{aligned}$$

Sending $h \to 0$ we end up with

$$\begin{aligned}&\iint_{\mathbb{R}^+\times\mathbb{R}} \left(|u-v|\,\Phi_t + \text{sgn}(u-v)(f(t,x,u)-f(t,x,v))\Phi_x\right) dxdt\\ &\leq K \iint_{\mathbb{R}^+\times\mathbb{R}} |u-v|\,\Phi dxdt + \lim_{h\to 0} J(h).\end{aligned}$$

We can write

$$\begin{aligned}\lim_{h\to 0} J(h) &= \lim_{h\to 0}\frac{1}{h}\int_0^{+\infty}\int_h^{2h}\operatorname{sgn}(u-v)(f(t,x,u)-f(t,x,v))\Phi(t,x)dxdt\\ &-\lim_{h\to 0}\frac{1}{h}\int_0^{+\infty}\int_{-2h}^{-h}\operatorname{sgn}(u-v)(f(t,x,u)-f(t,x,v))\Phi(t,x)dxdt\\ &=\int_0^{+\infty}[\operatorname{sgn}(u-v)(f(t,x,u)-f(t,x,v))]_{x=0^-}^{x=0^+}\Phi(t,0)dt,\end{aligned}$$

where we indicate the limits from the right and left at $x = 0$. The aim is to prove that the limit $\lim_{h\to 0} J(h) \le 0$, then it is sufficient to prove

$$S := [\operatorname{sgn}(u-v)(f(t,x,u)-f(t,x,v))]_{x=0^-}^{x=0^+} \le 0.$$

In particular, denoting the right and left traces of u and v with $u_\pm$ and $v_\pm$, we can write

$$\begin{aligned}S =& v_r\operatorname{sgn}(u_+-v_+)\left(u_+\left(1-\int_0^\eta u(t,y)w_\eta(y)dy\right)-v_+\left(1-\int_0^\eta v(t,y)w_\eta(y)dy\right)\right)\\ &- v_l\operatorname{sgn}(u_--v_-)\left(u_-\left(1-\int_0^\eta u(t,y)w_\eta(y)dy\right)-v_-\left(1-\int_0^\eta v(t,y)w_\eta(y)dy\right)\right)\\ =& v_r\operatorname{sgn}(u_+-v_+)(v_+-u_+)\int_0^\eta u(t,y)w_\eta(y)dy\\ &- v_r\operatorname{sgn}(u_+-v_+)v_+\int_0^\eta (u(t,y)-v(t,y))w_\eta(y)dy\\ &+ v_r|u_+-v_+|\\ &- v_l\operatorname{sgn}(u_--v_-)(v_--u_-)\int_0^\eta u(t,y)w_\eta(y)dy\\ &+ v_l\operatorname{sgn}(u_--v_-)v_-\int_0^\eta (u(t,y)-v(t,y))w_\eta(y)dy\\ &- v_l|u_--v_-|\\ =& \underbrace{(v_r|u_+-v_+|-v_l|u_--v_-|)}_{=0}\left(1-\int_0^\eta u(y,t)w_\eta(y)dy\right)\\ &+\underbrace{(v_rv_+-v_lv_-)}_{=0}\operatorname{sgn}(u_--v_-)\int_0^\eta (v(t,y)-u(t,y))w_\eta(y)dy.\end{aligned}$$

A simple application of the Rankine-Hugoniot condition yields $S = 0$, being $u_+ = \frac{v_l}{v_r}u_-$ and $v_+ = \frac{v_l}{v_r}v_-$. In this way we know that (4.1) holds for any $0 \le \phi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}^+\times\mathbb{R})$. For $r > 1$, let $\gamma_r : \mathbb{R} \to \mathbb{R}$ be a C^∞ function which takes values in $[0, 1]$ and satisfies

$$\gamma_r(x) = \begin{cases}1, & |x| \le r,\\ 0, & |x| \ge r+1.\end{cases}$$

Fix s_0 and s such that $0 < s_0 < s$. For any $\tau > 0$ and $k > 0$ with $0 < s_0+\tau < s+k$, let $\beta_{\tau,k} : [0,+\infty] \to \mathbb{R}$ be a Lipschitz function that is linear on $[s_0, s_0 + \tau[\cup[s, s + k]$ and satisfies

$$\beta_{\tau,k}(t) = \begin{cases} 0, & t \in [0, s_0] \cup [s + k, +\infty], \\ 1, & t \in [s_0 + \tau, s]. \end{cases}$$

We can take the admissible test function via a standard regularization argument $\phi = \gamma_r(x)\beta_{\tau,k}(t)$. Using this test function in Eq (4.1) we obtain

$$\begin{aligned} &\frac{1}{k}\int_s^{s+k}\int_{\mathbb{R}} |u(t,x) - v(t,x)|\,\gamma_r(x)dxdt - \frac{1}{\tau}\int_{s_0}^{s_0+k}\int_{\mathbb{R}} |u(t,x) - v(t,x)|\,\gamma_r(x)dxdt \\ &\quad\le K\int_{s_0}^{s_0+k}\int_{\mathbb{R}} |u - v|\,\gamma_r(x)dxdt \\ &\qquad + \left\|\gamma_r'\right\|_\infty \int_{s_0}^{s+k}\int_{r\le|x|\le r+1} \operatorname{sgn}(u - v)(f(t,x,u) - f(t,x,v))dxdt. \end{aligned}$$

Sending $s_0 \to 0$, we get

$$\begin{aligned} &\frac{1}{k}\int_s^{s+k}\int_{-r}^{r} |u(t,x) - v(t,x)|\,\gamma_r(x)dx\,dt \\ &\le \int_{-r}^{r} |u_0(x) - v_0(x)|\,dx + \frac{1}{\tau}\int_0^{\tau}\int_{-r}^{r} |v(t,x) - v_0(x)|\,dxdt \\ &\quad + \frac{1}{\tau}\int_0^{\tau}\int_{-r}^{r} |u(t,x) - u_0(x)|\,dx\,dt + K\int_0^{t+\tau}\int_{\mathbb{R}} |u - v|\,\gamma_r(x)dx\,dt + o\left(\frac{1}{r}\right). \end{aligned}$$

Observe that the second and the third terms on the right-hand side of the inequality tends to zero as $\tau \to 0$ following the same argument in [20, Lemma B.1], because our initial condition is satisfied in the "weak" sense of the definition of our entropy condition. Sending $\tau \to 0$ and $r \to \infty$, we have

$$\begin{aligned} \frac{1}{k}\int_s^{s+k}\int_{\mathbb{R}} |u(t,x) - v(t,x)|\,dx\,dt \le &\int_{\mathbb{R}} |u_0(x) - v_0(x)|\,dx \\ &+ K\int_0^{s+k}\int_{\mathbb{R}} |u(t,x) - v(t,x)|\,dx\,dt. \end{aligned}$$

Sending $k \to 0$ and an application of Gronwall's inequality gives us the statement. □

Lemma 4.1 (**A Kružkov-type integral inequality**). *For any two entropy solutions* $u = u(t,x)$ *and* $v = v(t,x)$ *the integral inequality of Eq* (4.1) *holds for any* $0 \le \phi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}^+ \times \mathbb{R} \setminus \{0\})$.

Proof. The proof follows [20]. Let $0 \le \phi \in \mathbf{C}_\mathbf{c}^\infty\left((\mathbb{R}^+ \times \mathbb{R} \setminus \{0\})^2\right)$, $\phi = \phi(t,x,s,y)$, $u = u(t,x)$ and $v = v(s,y)$. From the definition of entropy solution for $u = u(t,x)$ with $\kappa = v(s,y)$ we get

$$\begin{aligned} &-\iint_{\mathbb{R}^+\times\mathbb{R}} \left(|u - v|\,\phi_t + \operatorname{sgn}(u - v)\left(f(t,x,u) - f(t,x,v)\right)\phi_x\right)\,dt\,dx \\ &+ \iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}} \operatorname{sgn}(u - v) f(t,x,v)_x \phi\,dt\,dx \le 0. \end{aligned}$$

Integrating over $(s, y) \in \mathbb{R}^+ \times \mathbb{R}$, we find

$$\begin{aligned} &- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \left(|u - v|\,\phi_t + \operatorname{sgn}(u - v)\,(f(t, x, u) - f(t, x, v))\,\phi_x\right) dt\,dx\,ds\,dy \\ &+ \iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(u - v) f(t, x, v)_x \phi\,dt\,dx\,ds\,dy \leq 0. \end{aligned} \tag{4.2}$$

Similarly, for the entropy solution $v = v(s, y)$ with $\alpha(y) = u(t, x)$

$$\begin{aligned} &- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \left(|v - u|\,\phi_s + \operatorname{sgn}(v - u)\,(f(s, y, v) - f(s, y, u))\,\phi_x\right) dt\,dx\,ds\,dy \\ &+ \iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(u - v) f(t, x, v)_x \phi\,dt\,dx\,ds\,dy \leq 0. \end{aligned} \tag{4.3}$$

Note that we can write, for each $(t, x) \in \mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} &\operatorname{sgn}(u - v)(f(t, x, u) - f(t, x, v))\phi_x - \operatorname{sgn}(u - v) f(t, x, v)_x \phi \\ &= \operatorname{sgn}(u - v)(f(t, x, u) - f(s, y, v))\phi_x - \operatorname{sgn}(u - v)\left[(f(t, x, v) - f(s, y, v))\phi\right]_x, \end{aligned}$$

so that

$$\begin{aligned} &- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(u - v)(f(t, x, u) - f(t, x, v))\phi_x\,dt\,dx\,ds\,dy \\ &+ \iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(u - v) f(t, x, v)_x \phi\,dt\,dx\,ds\,dy \\ = &- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(u - v)(f(t, x, u) - f(s, y, v))\phi_x\,dt\,dx\,ds\,dy \\ &+ \iiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(u - v)\left[(f(t, x, v) - f(s, y, v))\phi\right]_x\,dt\,dx\,ds\,dy. \end{aligned}$$

Similarly, writing, for each $(y, s) \in \mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$

$$\begin{aligned} &\operatorname{sgn}(v - u)(f(s, y, v) - f(s, y, u))\phi_y - \operatorname{sgn}(v - u) f(s, y, u)_y \phi \\ &= \operatorname{sgn}(u - v)(f(s, y, v) - f(s, y, u))\phi_y - \operatorname{sgn}(u - v)\left[(f(t, x, u) - f(s, y, u))\phi\right]_x, \end{aligned}$$

so that

$$\begin{aligned} &- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(u - v)(f(s, y, v) - f(s, y, u))\phi_y\,dt\,dx\,ds\,dy \\ &+ \iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(u - v) f(s, y, u)_y \phi\,dt\,dx\,ds\,dy \\ = &- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(u - v)(f(t, x, v) - f(s, y, u))\phi_x\,dt\,dx\,ds\,dy \\ &+ \iiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(u - v)\left[(f(t, x, u) - f(s, y, u))\phi\right]_y\,dt\,dx\,ds\,dy. \end{aligned}$$

Let us introduce the notations

$$\partial_{t+s} = \partial_t + \partial_s, \quad \partial_{x+y} = \partial_x + \partial_y,$$
$$\partial^2_{x+y} = (\partial_x + \partial_y)^2 = \partial^2_x + 2\partial_x\partial_y + \partial^2_y.$$

Adding Eq (4.2) and Eq (4.3) we obtain

$$\begin{aligned}
&- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \Big(|u-v|\,\partial_{t+s}\phi + \operatorname{sgn}(u-v)\,(f(t,x,u) - f(s,y,v))\,\partial_{x+y}\phi\Big)\, dt\,dx\,ds\,dy \\
&+ \iiiint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}} \operatorname{sgn}(u-v)\,(\partial_x\,[(f(t,x,v) - f(s,y,v))\phi] \\
&\qquad\qquad + \partial_y\,[(f(t,x,u) - f(s,y,u))\phi]\Big)\, dt\,dx\,ds\,dy \le 0.
\end{aligned} \tag{4.4}$$

We introduce a non-negative function $\delta \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R})$, satisfying $\delta(\sigma) = \delta(-\sigma)$, $\delta(\sigma) = 0$ for $|\sigma| \ge 1$, and $\int_\mathbb{R} \delta(\sigma)d\sigma = 1$. For $u > 0$ and $z \in \mathbb{R}$, let $\delta_p(z) = \frac{1}{p}\delta(\frac{z}{p})$. We take our test function $\phi = \phi(t,x,s,y)$ to be of the form

$$\Phi(t,x,s,y) = \phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),$$

where $0 \le \phi \in \mathbf{C}^\infty_\mathbf{c}\,(\mathbb{R}^+ \times \mathbb{R}\setminus\{0\})$ satisfies

$$\phi(t,x) = 0, \quad \forall (t,x) \in [0,T]\times[-h,h],$$

for small $h > 0$. By making sure that

$$p < h,$$

one can check that Φ belongs to $\mathbf{C}^\infty_\mathbf{c}\left((\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2\right)$. We have

$$\partial_{t+s}\Phi(t,x,s,y) = \partial_{t+s}\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),$$
$$\partial_{x+y}\Phi(t,x,s,y) = \partial_{x+y}\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),$$

Using Φ as test function in Eq (4.4)

$$\begin{aligned}
&- \iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} (I_1(t,x,s,y) + I_2(t,x,s,y))\,\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right) dt\,dx\,ds\,dy \\
&\le \iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} (I_3(t,x,s,y) + I_4(t,x,s,y) + I_5(t,x,s,y))\; dt\,dx\,ds\,dy,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= |u(t,x) - v(s,y)|\,\partial_{t+s}\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right), \\
I_2 &= \operatorname{sgn}(u(t,x) - v(s,y))(f(t,x,u) - f(s,y,v))\partial_{x+y}\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right), \\
I_3 &= -\operatorname{sgn}(u(t,x) - v(s,y))\left(\partial_x f(t,x,v) - \partial_y f(s,y,u)\right)
\end{aligned}$$

$$\phi\left(\frac{t+s}{2},\frac{x+y}{2},\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),$$

$$\begin{aligned}I_4 = &-\operatorname{sgn}(u(t,x)-v(s,y))\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right)\\&\left[\partial_x\phi\left(\frac{t+s}{2},\frac{x+y}{2},\right)(f(t,x,v)-f(s,y,v))\right.\\&\left.\qquad\partial_y\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right)(f(t,x,u)-f(s,y,u))\right],\\I_5 = &\,(F(x,u(t,x),v(s,y))-F(y,u(t,x),v(s,y)))\\&\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\partial_x\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),\end{aligned}$$

where $F(x,u,c) := \operatorname{sgn}(u-c)\,(f(t,x,u)-f(t,x,c))$.
We now use the change of variables

$$\tilde{x} = \frac{x+y}{2},\quad \tilde{t} = \frac{t+s}{2},\quad z = \frac{x-y}{2},\quad \tau = \frac{t-s}{2},$$

which maps $(\mathbb{R}^+\times\mathbb{R})^2$ in $\Omega\subset\mathbb{R}^4$ and $(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2$ in $\Omega_0\subset\mathbb{R}^4$, where

$$\begin{aligned}\Omega &= \{(\tilde{x},\tilde{t},z,\tau)\in\mathbb{R}^4 :\ 0<\tilde{t}\pm\tau<T\},\\\Omega_0 &= \{(\tilde{x},\tilde{t},z,\tau)\in\Omega : \tilde{x}\pm z\neq 0\},\end{aligned}$$

resepectively. With this changes of variables,

$$\begin{aligned}\partial_{t+s}\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right) &= \partial_{\tilde{t}}\phi(\tilde{t},\tilde{x}),\\\partial_{x+y}\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right) &= \partial_{\tilde{x}}\phi(\tilde{t},\tilde{x}).\end{aligned}$$

Now we can write

$$\begin{aligned}&-\iiiint_{\Omega}(I_1(\tilde{t},\tilde{x},\tau,z)+I_2(\tilde{t},\tilde{x},\tau,z))\,\delta_p(z)\,\delta_p(\tau)\ d\tilde{t}\,d\tilde{x}\,d\tau\,dz\\&\qquad\le\iiiint_{\Omega_0}\big(I_3(\tilde{t},\tilde{x},\tau,z)+I_4(\tilde{t},\tilde{x},\tau,z)+I_5(\tilde{t},\tilde{x},\tau,z)\big)\ d\tilde{t}\,d\tilde{x}\,d\tau\,dz,\end{aligned}$$

where

$$\begin{aligned}I_1(\tilde{t},\tilde{x},\tau,z) &= \left|u(\tilde{t}+\tau,\tilde{x}+z)-v(\tilde{t}-\tau,\tilde{x}-z)\right|\partial_{\tilde{t}}\phi(\tilde{t},\tilde{x}),\\I_2(\tilde{t},\tilde{x},\tau,z) &= \operatorname{sgn}(u(\tilde{t}+\tau,\tilde{x}+z)-v(\tilde{t}-\tau,\tilde{x}-z))\\&\qquad(f(\tilde{t}+\tau,\tilde{x}+z,u)-f(\tilde{t}-\tau,\tilde{x}-z,v))\partial_{\tilde{x}}\phi(\tilde{t},\tilde{x}),\\I_3(\tilde{t},\tilde{x},\tau,z) &= -\operatorname{sgn}(u(\tilde{t}+\tau,\tilde{x}+z)-v(\tilde{t}-\tau,\tilde{x}-z))\\&\qquad(\partial_{\tilde{x}+z}f(\tilde{t}+\tau,\tilde{x}+z,v)-\partial_{\tilde{x}-z}f(\tilde{t}-\tau,\tilde{x}-z,u))\,\phi(\tilde{t},\tilde{x})\,\delta_p(z)\,\delta_p(\tau),\\I_4(\tilde{t},\tilde{x},\tau,z) &= -\operatorname{sgn}(u(\tilde{t}+\tau,\tilde{x}+z)-v(\tilde{t}-\tau,\tilde{x}-z))\\&\qquad\partial_{\tilde{x}}\phi(\tilde{t},\tilde{x})\,\delta_p(z)\,\delta_p(\tau)\,[(f(\tilde{t}+\tau,\tilde{x}+z,v)-f(\tilde{t}-\tau,\tilde{x}-z,v))\\&\qquad\qquad+(f(\tilde{t}+\tau,\tilde{x}+z,u)-f(\tilde{t}-\tau,\tilde{x}-z,u))],\end{aligned}$$

$$I_5(\tilde t,\tilde x,\tau,z) = \big(F(\tilde x+z,u(\tilde t+\tau,\tilde x+z),v(\tilde t-\tau,\tilde x-z)) \\ -F(\tilde x-z,u(\tilde t+\tau,\tilde x+z),v(\tilde t-\tau,\tilde x-z))\big)\,\phi\,(\tilde t,\tilde x)\,\partial_z\delta_p\,(z)\,\delta_p\,(\tau)\,.$$

Employing Lebesgue's differentiation theorem, to obtain the following limits

$$\lim_{p\to 0}\iiiint_{\Omega} I_1(\tilde t,\tilde x,\tau,z)\delta_p(z)\delta_p(\tau)\,d\tilde t\,d\tilde x\,d\tau\,dz \\ = \iint_{\mathbb{R}^+\times\mathbb{R}} |u(t,x)-v(t,x)|\,\partial_t\phi(t,x)dtdx,$$

$$\lim_{p\to 0}\iiiint_{\Omega} I_2(\tilde t,\tilde x,\tau,z)\delta_p(z)\delta_p(\tau)\,d\tilde t\,d\tilde x\,d\tau\,dz \\ = \iint_{\mathbb{R}^+\times\mathbb{R}} \operatorname{sgn}(u(t,x)-v(t,x))(f(t,x,u)-f(t,x,v))\partial_x\phi(t,x)dtdx.$$

Let us consider the term I_3. Note that $I_3(\tilde t,\tilde x,\tau,z)=0$, if $\tilde x\in[-h,h]$, since then $\phi(\tilde t,\tilde x)=0$ for any $\tilde t$, or if $|z|\geq p$. On the other hand, if $\tilde x\notin[-h,h]$, then $\tilde x\pm z<0$ or $\tilde x\pm z>0$, at least when $|z|<p$ and $p<h$. Defining $U(t,x)=1-w_\eta * u$ and $V(t,x)=1-w_\eta * v$, and sending $p\to 0$:

$$\lim_{p\to 0}\iiiint_{\Omega_0} I_3(\tilde t,\tilde x,\tau,z)\,d\tilde t\,d\tilde x\,d\tau\,dz \\ = \iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}} \operatorname{sgn}(u(t,x)-v(t,x))\mathfrak{v}(x)\,(v\partial_x V-u\partial_x U)\,\phi\,(t,x)\;dt\,dx \\ \leq v_r\|\partial_x V\|\iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}}|u-v|\,\phi(t,x)\,dt\,dx + v_r\iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}}|\rho|\,|\partial_x V-\partial_x U|\;dt\,dx \\ \leq K_1\iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}}|u-v|\,\phi(t,x)\,dt\,dx.$$

In fact,

$$|\partial_x V-\partial_x U| \leq \left\|\omega_\eta'\right\|\,\|u(t,\cdot)-v(t,\cdot)\|_{L^1} \\ + w_\eta(0)\,(|u-v|\,(t,x+\eta)+|u-v|\,(t,x))\,.$$

The term I_4 converges to zero as $p\to 0$. Finally, the term I_5

$$\lim_{p\to 0}\iiiint_{\Omega_0} I_5(\tilde t,\tilde x,\tau,z)\,d\tilde t\,d\tilde x\,d\tau\,dz \leq K_2\iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}}|u-v|\,\phi(t,x)\,dt\,dx.$$

□

5. Conclusions and open problems

In this paper we proved the well-posedness of a Cauchy problem characterized by a nonlocal conservation law with space-discontinuous flux using the vanishing viscosity technique. This kind of equations can be applied to describe different real phenomena, such as: traffic flow, sedimentation, conveyor belts and others. It is worth noticing that the discontinuity appears in a multiplicative way. For this reason, one can think to consider more general nonlocal flux functions satisfying proper 'crossing conditions' in a future work and to study nonlocal-to-local limit in this space-discontinuous setting.

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Conflict of interest

The authors declare there is no conflict of interest.

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