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Research article

Local existence of a solution to a free boundary problem describing migration into rubber with a breaking effect

Kota Kumazaki[1,*], **Toyohiko Aiki**[2] **and Adrian Muntean**[3]

[1] Faculty of Education, Nagasaki University, Nagasaki, Japan
[2] Department of Mathematics, Faculty of Sciences, Japan Women's University, Tokyo, Japan
[3] Department of Mathematics and Computer Science, Karlstad University, Karlstad, Sweden

* **Correspondence:** Email: k.kumazaki@nagasaki-u.ac.jp.

Abstract: We consider a one-dimensional free boundary problem describing the migration of diffusants into rubber. In our setting, the free boundary represents the position of the front delimitating the diffusant region. The growth rate of this region is described by an ordinary differential equation that includes the effect of breaking the growth of the diffusant region. In this specific context, the breaking mechanism is should be perceived as a non-dissipative way of describing eventual hyperelastic response to a too fast diffusion penetration. In recent works, we considered a similar class of free boundary problems modeling diffusants penetration in rubbers, but without attempting to deal with the possibility of breaking or accelerating the occurring free boundaries. For simplified settings, we were able to show the global existence and uniqueness as well as the large time behavior of the corresponding solutions to our formulations. Since here the breaking effect is contained in the free boundary condition, our previous results are not anymore applicable. The main mathematical obstacle in ensuring the existence of a solution is the non-monotonic structure of the free boundary.
In this paper, we establish the existence and uniqueness of a weak solution to the free boundary problem with breaking effect and give explicitly the maximum value that the free boundary can reach.

Keywords: migration into rubber; free boundary problem; nonlinear initial-boundary value problem for nonlinear parabolic equations; existence of solutions; Flux boundary condition

1. Introduction

Both natural rubber and synthetic elastomers are widely used in engineering applications, such as connecting components for offshore wind farms that are nowadays intensively used to harvesting energy. Due to the growing importance of good theoretical estimations of durability properties of elastomeric parts, a well-trusted modelling of the material behaviour is an essential prerequisite so that numerical prediction is available for large times. Rubber-like materials exhibit a strongly non-linear behaviour characterized by a joint large strain and a non-linear stress-strain response [3]. In the presence of aggressive environments (like those where large bodies of water are present), the nonlinear behavior of elastomers, which can be in fact studied in controlled laboratory conditions, alters non-intuitively [2]. The main reason for this situation is that the porous structure of rubber-like materials allows small particles to diffuse inside. Ingressing particles, often ionically charged, accumulate and interact with the internal solid fabric. Such interactions lead to unwanted alterations of the originally designed mechanical behavior; see e.g., [8] for more context on this physical problem. From the modeling and mathematical upscaling point of view, the derivation of the correct model equations for such a problem setting is open.

In our recent works [1,5], and [6], we considered a class of one-dimensional free boundary problems to model the diffusants penetration in dense and foam rubbers. The use of kinetic-type free boundary conditions is meant to avoid the explicit description of the mechanics of the involved materials and how the constitutive laws (in particular, the stress tensor) are affected by the presence of diffusants. One can use such free boundary formulations to derive theoretically-grounded practical estimations of the position of the diffusants penetration front and compare them with laboratory experiments; e.g., [7] where a parameter identification exercise has been performed in this context.

The main novelty introduced in the present paper is that the growth rate of the region filled with diffusants is described by an ordinary differential equation that includes the effect of breaking its growth. The breaking mechanism should be seen here as the hyperelastic response to a too fast diffusion penetration. Such a mechanism introduces non-monotonicity in our problem formulation. This makes us wonder whether this type of problems has a chance to be well-posed and, if true, in which sense?

To address the question, we consider the following one-dimensional free boundary problem show in Eqs (1.1)–(1.5). Let $t \in [0, T]$ be the time variable for $T > 0$. The problem is to find a curve $z = s(t)$ on $[0, T]$ and a function u defined on the set $Q_s(T) := \{(t, z) | 0 < t < T,\ 0 < z < s(t)\}$ such that the following system of equations is satisfied:

$$u_t - u_{zz} = 0 \text{ for } (t, z) \in Q_s(T), \tag{1.1}$$

$$-u_z(t, 0) = \beta(b(t) - \gamma u(t, 0)) \text{ for } t \in (0, T), \tag{1.2}$$

$$-u_z(t, s(t)) = \sigma(u(t, s(t)))s_t(t) \text{ for } t \in (0, T), \tag{1.3}$$

$$s_t(t) = a_0(\sigma(u(t, s(t))) - \alpha s(t)) \text{ for } t \in (0, T), \tag{1.4}$$

$$s(0) = s_0, u(0, z) = u_0(z) \text{ for } z \in [0, s_0], \tag{1.5}$$

where β, γ, a_0 and α are given positive constants, b is a given function on $[0, T]$ and s_0 and u_0 are the initial data. In Eq (1.3) and Eq (1.4), σ is the function on $\mathbb{R}$ called the positive part, i.e. it is given by Eq (1.6),

$$\sigma(r) = \begin{cases} r & \text{if } r \geq 0, \\ 0 & \text{if } r < 0. \end{cases} \tag{1.6}$$

The problem in Eqs (1.1)–(1.5), which is denoted here by (P)(u_0, s_0, b), was proposed by Nepal S et al., in [7] as a mathematical model describing the migration of diffusants into rubber. The set $[0, s(t)]$ represents the region occupied by a solvent occupying the one-dimensional pore $[0, \infty)$, where $s = s(t)$

is the position of moving interface of the region and $u = u(t, z)$ represents the content of the diffusant at the place $z \in [0, s(t)]$ at time $t > 0$.

From the mathematical point of view, $(\mathrm{P})(u_0, s_0, b)$ is a one-phase free boundary problem with Robin-type boundary conditions at both boundaries. Such kind of problem structure has been considered in [5, 6], and [1]. In our recent work [1], as a simplified setting for $(\mathrm{P})(u_0, s_0, b)$, we consider the case $\alpha = 0$ and prove the existence and uniqueness of a globally-in-time solution. Furthermore, what concerns the large time behavior of a solution, we show that the free boundary s goes to infinity as time elapses.

In this paper, we consider the case $\alpha > 0$. As already anticipated, the difficulty in this case is the lack of the monotonicity of the boundary condition imposed on the moving boundary. Indeed, similarly to the proof of the existence result in [1], by introducing a variable $\tilde{u}(t, y) = u(t, ys(t))$ for $(t, y) \in Q(T) := (0, T) \times (0, 1)$ we transform $(\mathrm{P})(u_0, s_0, b)$ into a problem on the fixed domain $Q(T)$. Let denote this problem by $(\mathrm{PC})(\tilde{u}_0, s_0, b)$, where $\tilde{u}_0(y) = u_0(s_0 y)$ for $y \in [0, 1]$. Moreover, to find a solution $(\mathrm{PC})(\tilde{u}_0, s_0, b)$ we first consider the following auxiliary problem $(\mathrm{AP})(\tilde{u}_0, s, b)$: For a given function $s(t)$ on $[0, T]$, find $\tilde{u}(t, y)$ satisfying:

$$
\begin{aligned}
&\tilde{u}_t(t, y) - \frac{1}{s^2(t)}\tilde{u}_{yy}(t, y) = \frac{ys_t(t)}{s(t)}\tilde{u}_y(t, y) \text{ for } (t, y) \in Q(T),\\
&-\frac{1}{s(t)}\tilde{u}_y(t, 0) = \beta(b(t) - \gamma\tilde{u}(t, 0)) \text{ for } t \in (0, T),\\
&-\frac{1}{s(t)}\tilde{u}_y(t, 1) = a_0\sigma(\tilde{u}(t, 1))(\sigma(\tilde{u}(t, 1)) - \alpha s(t)) \text{ for } t \in (0, T),\\
&\tilde{u}(0, y) = \tilde{u}_0(y) \text{ for } y \in [0, 1].
\end{aligned}
$$

Here, we set $g_\alpha(r) = a_0\sigma(r)(\sigma(r) - \alpha s(t))$ for $r \in \mathbb{R}$ and $t \in (0, T)$. In the case $\alpha = 0$, due to the function σ, $g_0(r) = a_0(\sigma(r))^2$ is monotonically increasing with respect to r. Due to this fact, we can use the theory of evolution equations governed by subdifferentials of convex functions (cf. [4] and references therein) and find a solution $\tilde{u}$ of $(\mathrm{AP})(\tilde{u}_0, s, b)$ in the case $\alpha = 0$. However, looking now at the case $\alpha > 0$, since $g_\alpha(r)$ is not monotonic anymore with respect to r, we cannot apply the theory of evolution equations directly. Hence, the existence of a solution to $(\mathrm{AP})(\tilde{u}_0, s, b)$ in the case $\alpha > 0$ is not at all clear. As far as we are aware, the type of free boundary problems is novel. We refer the reader to [9] (and related references) for explanations of how and why free-boundary problems can be used to model fast transitions in materials.

The purpose of this paper is to establish a methodology to deal with the existence of locally-in-time solutions to $(\mathrm{PC})(u_0, s_0, b)$ in the case $\alpha > 0$. For this to happen, we consider weak solutions to $(\mathrm{PC})(\tilde{u}_0, s_0, b)$. The definition of our concept of weak solutions is explained in Section 2. As next step, we proceed in the following way: For a given function η on $Q(T)$ and $\varepsilon > 0$, we consider in Section 3 a smooth approximation η_ε of η and construct a solution $\tilde{u}$ to the following auxiliary problem: For given s on $[0, T]$, find $\tilde{u}(t, y)$ such that it holds

$$
\begin{aligned}
&\tilde{u}_t(t, y) - \frac{1}{s^2(t)}\tilde{u}_{yy}(t, y) = \frac{ys_t(t)}{s(t)}\eta_y(t, y) \text{ for } (t, y) \in Q(T),\\
&-\frac{1}{s(t)}\tilde{u}_y(t, 0) = \beta(b(t) - \gamma\tilde{u}(t, 0)) \text{ for } t \in (0, T),
\end{aligned}
$$

$$-\frac{1}{s(t)}\tilde{u}_y(t,1) = a_0(\sigma(\tilde{u}(t,1)))^2 - \alpha\sigma(\eta_\varepsilon(t))s(t)) \text{ for } t \in (0,T),$$
$$\tilde{u}(0,y) = \tilde{u}_0(y) \text{ for } y \in [0,1].$$

The plan is to obtain uniform estimates of solutions with respect to ε. After that, by the limiting process $\varepsilon \to 0$ and benefitting of Banach's fixed point theorem, we wish to construct a weak solution $\tilde{u}$ of (AP)($\tilde{u}_0, s, b$). In Section 4, we define a solution mapping Γ_T between s and the weak solution $\tilde{u}$ of (AP)($\tilde{u}_0, s, b$). We show that, for some $T' \le T$, the mapping $\Gamma_{T'}$ is a contraction mapping on a suitable function space. Finally, using Banach's fixed point theorem, we prove the existence and uniqueness of a weak solution to the coupled problem $(s, \tilde{u})$ of (PC)($\tilde{u}_0, s_0, b$) on $[0, T']$. Additionally, we show that the maximal length of the free boundary s is *a priori* determined by given parameters, the time derivative of the function b, and initial data s_0. Moreover, we guarantee that the solution $\tilde{u}$ is non-negative and bounded on $Q(T)$.

2. Notation, assumptions and results

In this paper, we use the following notations. We denote by $|\cdot|_X$ the norm for a Banach space X. The norm and the inner product of a Hilbert space H are denoted by $|\cdot|_H$ and $(\cdot,\cdot)_H$, respectively. Particularly, for $\Omega \subset \mathbb{R}$, we use the notation of the usual Hilbert spaces $L^2(\Omega)$, $H^1(\Omega)$, and $H^2(\Omega)$. Throughout this paper, we assume the following parameters and functions:

(A1) $a_0, \alpha, \gamma, \beta$ and T are positive constants.

(A2) $s_0 > 0$ and $u_0 \in L^\infty(0, s_0)$ such that $u_0 \ge 0$ on $[0, s_0]$.

(A3) $b \in W^{1,2}(0,T)$ with $b \ge 0$ on $(0,T)$. Also, we set

$$b^* = \max\left\{\max_{0\le t\le T} b(t), \gamma|u_0|_{L^\infty(0,s_0)}\right\}.$$

It is convenient to consider (P)(u_0, s_0, b) transformed into a non-cylindrical domain. Let $T > 0$. For given $s \in W^{1,2}(0,T)$ with $s(t) > 0$ on $[0,T]$, we introduce the following new function obtained by the change of variables and fix the moving domain:

$$\tilde{u}(t,y) = u(t, ys(t)) \text{ for } (t,y) \in Q(T) := (0,T) \times (0,1). \tag{2.1}$$

By using the function $\tilde{u}$, (P)(u_0, s_0, b) becomes the following problem (PC)($\tilde{u}_0, s_0, b$) posed on the non-cylindrical domain $Q(T)$:

$$\tilde{u}_t(t,y) - \frac{1}{s^2(t)}\tilde{u}_{yy}(t,y) = \frac{ys_t(t)}{s(t)}\tilde{u}_y(t,y) \text{ for } (t,y) \in Q(T), \tag{2.2}$$
$$-\frac{1}{s(t)}\tilde{u}_y(t,0) = \beta(b(t) - \gamma\tilde{u}(t,0)) \text{ for } t \in (0,T), \tag{2.3}$$
$$-\frac{1}{s(t)}\tilde{u}_y(t,1) = \sigma(\tilde{u}(t,1))s_t(t) \text{ for } t \in (0,T), \tag{2.4}$$
$$s_t(t) = a_0(\sigma(\tilde{u}(t,1)) - \alpha s(t)) \text{ for } t \in (0,T), \tag{2.5}$$
$$s(0) = s_0, \tag{2.6}$$
$$\tilde{u}(0,y) = u_0(ys(0))(:= \tilde{u}_0(y)) \text{ for } y \in [0,1]. \tag{2.7}$$

Here, we introduce the following function space: For $T > 0$, we put $H = L^2(0, 1)$, $X = H^1(0, 1)$, $V(T) = L^\infty(0, T; H) \cap L^2(0, T; X)$ and $|z|_{V(T)} = |z|_{L^\infty(0,T;H)} + |z_y|_{L^2(0,T;H)}$ for $z \in V(T)$. Note that $V(T)$ is a Banach space with the norm $|\cdot|_{V(T)}$. Also, we denote by X^* and $\langle \cdot, \cdot \rangle_X$ the dual space of X and the duality pairing between X and X^*, respectively.

We define now our concept of solutions to (PC)$(\tilde{u}_0, s_0, b)$ on $[0, T]$ in the following way:

Definition 2.1. For $T > 0$, let s be a function on $[0, T]$ and $\tilde{u}$ be a function on $Q(T)$, respectively. We call that a pair $(s, \tilde{u})$ is a solution of (P)$(\tilde{u}_0, s_0, b)$ on $[0, T]$ if the next conditions (S1)–(S4) hold:

(S1) $s \in W^{1,\infty}(0, T)$, $s > 0$ on $[0, T]$, $\tilde{u} \in W^{1,2}(0, T; X^*) \cap V(T)$.

(S2)

$$\int_0^T \langle \tilde{u}_t(t), z(t) \rangle_X dt + \int_{Q(T)} \frac{1}{s^2(t)} \tilde{u}_y(t) z_y(t) dy dt + \int_0^T \frac{1}{s(t)} \sigma(\tilde{u}(t, 1)) s_t(t) z(t, 1) dt$$
$$- \int_0^T \frac{1}{s(t)} \beta(b(t) - \gamma \tilde{u}(t, 0)) z(t, 0) dt = \int_{Q(T)} \frac{y s_t(t)}{s(t)} \tilde{u}_y(t) z(t) dy dt \text{ for } z \in V(T).$$

(S3) $s_t(t) = a_0(\sigma(\tilde{u}(t, 1)) - \alpha s(t))$ for a.e. $t \in (0, T)$.

(S4) $s(0) = s_0$ and $\tilde{u}(0, y) = \tilde{u}_0(y)$ for a.e. $y \in [0, 1]$.

The main result of this paper is the existence and uniqueness of a locally-in-time solution of (PC)$(\tilde{u}_0, s_0, b)$. We state this result in the next theorem.

Theorem 2.2. *Let $T > 0$. If (A1)–(A3) hold, then there exists $T^* \leq T$ such that (PC)$(\tilde{u}_0, s_0, b)$ has a unique solution $(s, \tilde{u})$ on $[0, T^*]$. Moreover, the function $\tilde{u}$ is non-negative and bounded on $Q(T)$.*

3. Auxiliary problems

In this section, we consider the following auxiliary problem (AP1)$(\tilde{u}_0, s, b)$: For $T > 0$ and $s \in W^{1,2}(0, T)$ with $s > 0$ on $[0, T]$

$$\tilde{u}_t(t, y) - \frac{1}{s^2(t)} \tilde{u}_{yy}(t, y) = \frac{y s_t(t)}{s(t)} \tilde{u}_y(t, y) \text{ for } (t, y) \in Q(T), \tag{3.1}$$

$$-\frac{1}{s(t)} \tilde{u}_y(t, 0) = \beta(b(t) - \gamma \tilde{u}(t, 0)) \text{ for } t \in (0, T), \tag{3.2}$$

$$-\frac{1}{s(t)} \tilde{u}_y(t, 1) = a_0 \sigma(\tilde{u}(t, 1))(\sigma(\tilde{u}(t, 1)) - \alpha s(t)) \text{ for } t \in (0, T), \tag{3.3}$$

$$\tilde{u}(0, y) = \tilde{u}_0(y) \text{ for } y \in [0, 1], \tag{3.4}$$

where σ is the same function as in Eq (1.6).

Definition 3.1. For $T > 0$, let $\tilde{u}$ be a function on $Q(T)$, respectively. We call that a function $\tilde{u}$ is a solution of (AP1)$(\tilde{u}_0, s, b)$ on $[0, T]$ if the conditions (S'1)–(S'3) hold:

(S'1) $\tilde{u} \in W^{1,2}(0, T; X^*) \cap V(T)$.

(S'2)

$$\int_0^T \langle \tilde{u}_t(t), z(t) \rangle_X dt + \int_{Q(T)} \frac{1}{s^2(t)} \tilde{u}_y(t) z_y(t) dy dt$$

$$+\int_0^T \frac{a_0}{s(t)}\sigma(\tilde{u}(t,1))(\sigma(\tilde{u}(t,1))-\alpha s(t))z(t,1)dt$$
$$-\int_0^T \frac{1}{s(t)}\beta(b(t)-\gamma\tilde{u}(t,0))z(t,0)dt = \int_{Q(T)} \frac{ys_t(t)}{s(t)}\tilde{u}_y(t)z(t)dydt \text{ for } z\in V(T).$$

(S'3) $\tilde{u}(0,y)=\tilde{u}_0(y)$ for $y\in[0,1]$.

Now, we introduce the following problem (AP2)$(\tilde{u}_0,s,\eta,b)$: For $s\in W^{1,\infty}(0,T)$ with $s>0$ on $[0,T]$ and $\eta\in V(T)$

$$\tilde{u}_t(t,y)-\frac{1}{s^2(t)}\tilde{u}_{yy}(t,y)=\frac{ys_t(t)}{s(t)}\eta_y(t,y) \text{ for } (t,y)\in Q(T), \tag{3.5}$$

$$-\frac{1}{s(t)}\tilde{u}_y(t,0)=\beta(b(t)-\gamma\tilde{u}(t,0)) \text{ for } t\in(0,T), \tag{3.6}$$

$$-\frac{1}{s(t)}\tilde{u}_y(t,1)=a_0((\sigma(\tilde{u}(t,1))^2-\alpha\sigma(\eta(t,1))s(t)) \text{ for } t\in(0,T), \tag{3.7}$$

$$\tilde{u}(0,y)=\tilde{u}_0(y) \text{ for } y\in[0,1], \tag{3.8}$$

The definition of solutions of (AP2)$(\tilde{u}_0,s,\eta,b)$ is Definition 3.1 with (S'2) replaced by (S"2), which now reads
(S"2):

$$\int_0^T \langle\tilde{u}_t(t),z(t)\rangle_X dt+\int_{Q(T)}\frac{1}{s^2(t)}\tilde{u}_y(t)z_y(t)dydt$$
$$+\int_0^T \frac{a_0}{s(t)}((\sigma(\tilde{u}(t,1))^2-\alpha\sigma(\eta(t,1))s(t))z(t,1)dt$$
$$-\int_0^T \frac{1}{s(t)}\beta(b(t)-\gamma\tilde{u}(t,0))z(t,0)dt=\int_{Q(T)}\frac{ys_t(t)}{s(t)}\eta_y(t)z(t)dydt \text{ for } z\in V(T).$$

First, we construct a solution $\tilde{u}$ of (AP2)$(\tilde{u}_0,s,\eta,b)$ on $[0,T]$. To do so, for each $\varepsilon>0$ we solve the following problem (AP2)$_\varepsilon(\tilde{u}_{0\varepsilon},s,\eta,b)$:

$$\tilde{u}_t(t,y)-\frac{1}{s^2(t)}\tilde{u}_{yy}(t,y)=\frac{ys_t(t)}{s(t)}\eta_y(t,y) \text{ for } (t,y)\in Q(T),$$
$$-\frac{1}{s(t)}\tilde{u}_y(t,0)=\beta(b(t)-\gamma\tilde{u}(t,0)) \text{ for } t\in(0,T),$$
$$-\frac{1}{s(t)}\tilde{u}_y(t,1)=a_0((\sigma(\tilde{u}(t,1))^2-\alpha\sigma((\rho_\varepsilon*\eta)(t,1))s(t)) \text{ for } t\in(0,T),$$
$$\tilde{u}(0,y)=\tilde{u}_{0\varepsilon}(y) \text{ for } y\in[0,1].$$

Here ρ_ε is a mollifier with support $[-\varepsilon,\varepsilon]$ in time and $\rho_\varepsilon*\eta$ is the convolution of ρ_ε with η:

$$(\rho_\varepsilon*\eta)(t,1)=\int_{-\infty}^{\infty}\rho_\varepsilon(t-s)\overline{\eta}(s,1)ds \text{ for } t\in[0,T], \tag{3.9}$$

where $\overline{\eta}(t,1)=\eta(t,1)$ for $t\in(0,T)$ and vanishes otherwise. Also, $\tilde{u}_{0\varepsilon}$ is an approximation function of $\tilde{u}_0$ such that $\{\tilde{u}_{0\varepsilon}\}\subset X$, $|\tilde{u}_{0\varepsilon}|_H\le|\tilde{u}_0|_H+1$ and $\tilde{u}_{0\varepsilon}\to\tilde{u}_0$ in H as $\varepsilon\to0$.

Now, we define a family $\{\psi^t\}_{t\in[0,T]}$ of time-dependent functionals $\psi^t : H \to \mathbb{R} \cup \{+\infty\}$ for $t \in [0,T]$ as follows:

$$\psi^t(u) := \begin{cases} \dfrac{1}{2s^2(t)}\displaystyle\int_0^1 |u_y(y)|^2 dy + \dfrac{1}{s(t)}\displaystyle\int_0^{u(1)} a_0((\sigma(\xi))^2 d\xi - a_0\alpha u(1)\sigma((\rho_\varepsilon * \eta)(t,1)) \\ -\dfrac{1}{s(t)}\displaystyle\int_0^{u(0)} \beta(b(t)-\gamma\xi)d\xi \text{ if } u \in D(\psi^t), \\ +\infty \quad \text{otherwise}, \end{cases}$$

where $D(\psi^t) = X$ for $t \in [0,T]$. Here, we show the property of ψ^t.

Lemma 3.2. *Let $s \in W^{1,2}(0,T)$ with $s > 0$ on $[0,T]$, $\eta \in V(T)$ and assume (A1)–(A3). Then the following statements hold:*

(1) There exists positive constants C_0', C_1' and C'' such that the following inequalities hold:

$$(i)\ |u(y)|^2 \le C_y'(\psi^t(u)+1) \text{ for } u \in D(\psi^t) \text{ and } y = 0,1,$$
$$(ii)\ \frac{1}{2s^2(t)}|u_y|_H^2 \le C''(\psi^t(u)+1) \text{ for } u \in D(\psi^t),$$

(2) For $t \in [0,T]$, the functional ψ^t is proper, lower semi-continuous, and convex on H.

Proof. We fix $\varepsilon > 0$ and let $t \in [0,T]$, $u \in D(\psi^t)$ and put $l = \max_{0\le t\le T} |s(t)|$ and $\eta_\varepsilon(t) = (\rho_\varepsilon * \eta)(t,1)$ for $t \in [0,T]$. If $u(1) < 0$, $\frac{1}{s(t)}\int_0^{u(1)} a_0(\sigma(\xi))^2 d\xi = 0$. If $u(1) \ge 0$, then it holds

$$\frac{1}{s(t)}\int_0^{u(1)} a_0(\sigma(\xi))^2 d\xi - a_0\alpha u(1)\sigma(\eta_\varepsilon(t)) = \frac{a_0}{s(t)}\frac{1}{3}u^3(1) - a_0\alpha u(1)\sigma(\eta_\varepsilon(t))$$
$$\ge \frac{a_0}{3s(t)}u^3(1) - \frac{1}{3}\delta^3 u^3(1) - \frac{2}{3\delta^{3/2}}(a_0\alpha\sigma(\eta_\varepsilon(t)))^{\frac{3}{2}},$$

where δ is an arbitrary positive number. By using the fact that $\sigma(r) \le |r|$ for $r \in \mathbb{R}$ and taking a suitable $\delta = \delta_0$ we have

$$\frac{1}{s(t)}\int_0^{u(1)} a_0(\sigma(\xi))^2 d\xi - a_0\alpha u(1)\eta_\varepsilon(t) \ge -\frac{2}{3\delta_0^{3/2}}(a_0\alpha\eta_\varepsilon^*)^{\frac{3}{2}}, \tag{3.10}$$

where $\eta_\varepsilon^* = \max_{0\le t\le T} |\eta_\varepsilon(t)|$. Moreover, for both cases $u(0) < 0$ and $u(0) \ge 0$, we observe that

$$-\frac{1}{s(t)}\int_0^{u(0)} \beta(b(t)-\gamma\xi)d\xi = \frac{\beta}{s(t)}\left[\frac{\gamma}{2}u^2(0) - b(t)u(0)\right]$$
$$\ge \frac{\beta\gamma}{2l}u^2(0) - \frac{\beta b^*}{a}u(0) \ge \frac{\beta\gamma}{4l}u^2(0) - \frac{\beta l}{\gamma}\left(\frac{b^*}{a}\right)^2. \tag{3.11}$$

Accordingly, if $u(1) < 0$, then we have that,

$$\psi^t(u) \ge \frac{1}{2s^2(t)}\int_0^1 |u_y(y)|^2 dy + \frac{\beta\gamma}{4l}u^2(0) - \frac{\beta l}{\gamma}\left(\frac{b^*}{a}\right)^2. \tag{3.12}$$

If $u(1) \geq 0$, then, by Eq (3.10) and Eq (3.11) we also have that:

$$\psi^t(u) \geq \frac{1}{2s^2(t)} \int_0^1 |u_y(y)|^2 dy - \frac{2}{3\delta_0^{3/2}} (a_0 \alpha \eta_\varepsilon^*)^{\frac{3}{2}} + \frac{\beta\gamma}{4l} u^2(0) - \frac{\beta l}{\gamma} \left(\frac{b^*}{a}\right)^2. \tag{3.13}$$

In Eq (3.12) and Eq (3.13), since the first term in the right-hand side is non-negative we can find a positive constant C_0' that (i) of Lemma 3.2 for $y = 0$ holds. In addition, by $\frac{\beta\gamma}{4l} u^2(0) \geq 0$ we also see that (ii) of Lemma 3.2 holds. Moreover, it holds that:

$$\begin{aligned} |u(1)|^2 &= \left| \int_0^1 u_y(y) dy + u(0) \right|^2 \leq 2 \left(\int_0^1 |u_y(y)|^2 dy + |u(0)|^2 \right) \\ &\leq 2 \left(\frac{2l^2}{2s^2(t)} \int_0^1 |u_y(y)|^2 dy + |u(0)|^2 \right). \end{aligned} \tag{3.14}$$

Therefore, by Eq (3.14) and the result for $|u(0)|^2$ and $|u_y|_H^2$ we see that there exists a positive constant C_1' such that (i) of Lemma 3.2 hold for $y = 1$.

Next, we prove statement (2). For $t \in [0, T]$ and $r \in \mathbb{R}$, put

$$\begin{aligned} g_1(s(t), \eta_\varepsilon(t), r) &= \frac{1}{s(t)} \int_0^r a_0 (\sigma(\xi))^2 d\xi - a_0 \alpha r \sigma(\eta_\varepsilon(t)), \\ g_2(s(t), b(t), r) &= -\frac{1}{s(t)} \int_0^r \beta(b(t) - \gamma\xi) d\xi. \end{aligned}$$

Then, by $g_1(s(t), \eta_\varepsilon(t), r) = -a_0 \alpha r \sigma(\eta_\varepsilon(t))$ for $r \leq 0$, $g_1(s(t), \eta_\varepsilon(t), r)$ is a linear decreasing function for $r \leq 0$. Also, by $s(t) > 0$ we see that,

$$\begin{aligned} \frac{\partial^2}{\partial r^2} g_1(s(t), \eta_\varepsilon(t), r) &= \frac{2a_0}{s(t)} r > 0 \text{ for } r > 0, \\ \frac{\partial^2}{\partial r^2} g_2(s(t), b(t), r) &= \frac{\beta\gamma}{s(t)} > 0 \text{ for } r \in \mathbb{R}. \end{aligned}$$

This means that ψ^t is convex on H. By using (i) and (ii) of Lemma 3.2 together with Sobolev's embedding $X \hookrightarrow C([0, 1])$ in one dimensional case, it is easy to prove that the level set of ψ^t is closed in H, fact which ensures to the lower semi-continuity of ψ^t. Thus, we see that statement (2) holds. □

Lemma 3.3 guarantees the existence of a solution to $(\mathrm{AP2})_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$.

Lemma 3.3. *Let $T > 0$ and $\varepsilon > 0$. If (A1)–(A3) hold, then, for given $s \in W^{1,\infty}(0, T)$ with $s > 0$ on $[0, T]$ and $\eta \in V(T)$, the problem $(AP2)_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$ admits a unique solution $\tilde{u}$ on $[0, T]$ such that $\tilde{u} \in W^{1,2}(0, T; H) \cap L^\infty(0, T; X)$. Moreover, the function $t \to \psi^t(\tilde{u}(t))$ is absolutely continuous on $[0, T]$.*

Proof. Let $\varepsilon > 0$ be arbitrarily fixed. By Lemma 3.2, for $t \in [0, T]$ ψ^t is a proper lower semi-continuous convex function on H. From the definition of the subdifferential of ψ^t, for $t \in [0, T]$, $z^* \in \partial\psi^t(u)$ is characterized by $u, z^* \in H$,

$$z^* = -\frac{1}{s^2(t)} u_{yy} \text{ on } (0, 1),$$

$$-\frac{1}{s(t)}u_y(0) = \beta(b(t) - \gamma u(0)),\ -\frac{1}{s(t)}u_y(1) = a_0((\sigma(u(1))^2 - \alpha\sigma((\rho_\varepsilon * \eta)(t,1))s(t)).$$

Namely, $\partial\psi^t$ is single-valued. Also, we see that there exists a positive constant C such that for each t_1, $t_2 \in [0, T]$ with $t_1 \leq t_2$, and for any $u \in D(\psi^{t_1})$, there exists $\bar{u} \in D(\psi^{t_2})$ satisfying the inequality

$$\begin{aligned}&|\psi^{t_2}(\bar{u}) - \psi^{t_1}(u)| \\ \leq& C(|s(t_1) - s(t_2)| + |b(t_1) - b(t_2)| + |\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)|(1 + |\psi^{t_1}(u)|),\end{aligned} \tag{3.15}$$

where $\eta_\varepsilon(t) = (\rho_\varepsilon * \eta)(t)$. Indeed, we take $\bar{u} := u$. Then, $\bar{u} \in D(\psi^{t_2})$, and by the definition of ψ^t, it holds

$$\begin{aligned}&\psi^{t_2}(\bar{u}) - \psi^{t_1}(u) \\ =&\left(\frac{1}{2s^2(t_2)} - \frac{1}{2s^2(t_1)}\right)\int_0^1 |u_y(y)|^2 dy \\ &+\left(\frac{1}{s(t_2)} - \frac{1}{s(t_1)}\right)\int_0^{u(1)} a_0(\sigma(\xi))^2 d\xi + a_0\alpha u(1)\Big(\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)\Big) \\ &-\left(\frac{1}{s(t_2)} - \frac{1}{s(t_1)}\right)\int_0^{u(0)} \beta(b(t_2) - \gamma\xi)d\xi - \frac{\beta}{s(t_1)}(b(t_2) - b(t_1))u(0).\end{aligned} \tag{3.16}$$

We denote each term in the right-hand side by I_1, I_2, I_3, I_4 and I_5. Let put $l = \max_{0\leq t\leq T} |s(t)|$ and $a = \min_{0\leq t\leq T} |s(t)|$. For other than I_2, by the fact that $|u(y)| \leq \frac{1}{2}(1 + u^2(y))$ for $y = 0, 1$ and Lemma 3.2 it easy to see that,

$$|I_1| \leq C_1|s(t_1) - s(t_2)|(|\psi^{t_1}(u)| + 1), \tag{3.17}$$

$$\sum_{k=3}^{5} |I_k| \leq C_2(|\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)| + |s(t_2) - s(t_1)| + |b(t_2) - b(t_1)|)(|\psi^{t_1}(u)| + 1), \tag{3.18}$$

where C_1 and C_2 are positive constants. For I_2 by the definition of ψ^t it holds that,

$$\begin{aligned}&\frac{1}{s(t_1)}\int_0^{u(1)} a_0(\sigma(\xi))^2 d\xi \\ =&\psi^{t_1}(u) - \frac{1}{2s^2(t_1)}\int_0^1 |u_y(y)|^2 dy + a_0\alpha u(1)\eta_\varepsilon(t_1) + \frac{1}{s(t_1)}\int_0^{u(0)} \beta(b(t_1) - \gamma\xi)d\xi.\end{aligned}$$

Here, we note that

$$\begin{aligned}&|a_0\alpha u(1)\eta_\varepsilon(t_1)| \leq \frac{a_0\alpha\eta_\varepsilon^*}{2}(1 + u^2(1)), \\ &\left|\frac{1}{s(t_1)}\int_0^{u(0)} \beta(b(t_1) - \gamma\xi)d\xi\right| \\ \leq& \frac{1}{a}\left(\beta b^*|u(0)| + \frac{\gamma}{2}u^2(0)\right) \leq \frac{1}{a}\left(\frac{\beta b^*}{2} + \left(\frac{\beta b^*}{2} + \frac{\gamma}{2}\right)u^2(0)\right).\end{aligned}$$

where $\eta_\varepsilon^* = \max_{0\leq t\leq T} |\eta_\varepsilon(t)|$. Hence, by Lemma 3.2 we have that,

$$|I_2| \leq C_3|s(t_1) - s(t_2)|(|\psi^{t_1}(u)| + 1), \tag{3.19}$$

where C_3 is a positive constant. As a consequence, by Eqs (3.17)–(3.19) we infer that there exists a positive constant C such that Eq (3.15) holds.

Now, $(AP2)_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$ can be written into the following Cauchy problem $(CP)_\varepsilon$:

$$\tilde{u}_t + \partial\psi^t(\tilde{u}(t)) = \frac{ys_t(t)}{s(t)}\eta_y(t) \text{ in } H,$$
$$\tilde{u}(0, y) = \tilde{u}_{0\varepsilon}(y) \text{ for } y \in [0, 1].$$

Since $\frac{ys_t}{s}\eta_y \in L^2(0, T; H)$, by the general theory of evolution equations governed by time dependent subdifferentials (cf. [4]) we see that $(CP)_\varepsilon$ has a solution $\tilde{u}$ on $[0, T]$ such that $\tilde{u} \in W^{1,2}(Q(T))$, $\psi^t(\tilde{u}(t)) \in L^\infty(0, T)$ and $t \to \psi^t(\tilde{u}(t))$ is absolutely continuous on $[0, T]$. This implies that $\tilde{u}$ is a unique solution of $(AP2)_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$ on $[0, T]$. □

As next step, we provide an uniform estimate with respect to ε on a solution $\tilde{u}$ of $(AP2)_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$.

Lemma 3.4. *Let $T > 0$, $s \in W^{1,\infty}(0, T)$ with $s > 0$ on $[0, T]$, $\eta \in V(T)$ and $\tilde{u}_\varepsilon$ be a solution of $(AP2)_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$ on $[0, T]$ for each $\varepsilon > 0$. Then, it holds that*

$$|\tilde{u}_\varepsilon(t)|_H^2 + \int_0^t |\tilde{u}_{\varepsilon y}(\tau)|_H^2 dy \le M(1 + |\eta|_{V(T)}^2) \text{ for } t \in [0, T] \text{ and } \varepsilon \in (0, 1], \tag{3.20}$$

where $M = M(a_0, a, \beta, b^*, T)$ is a positive constant which is independent of ε and depends on a_0, a, β, b^* and T, $a = \min_{0 \le t \le T} s(t)$.

Proof. Let $\tilde{u}_\varepsilon$ be a solution of $(AP2)_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$ on $[0, T]$ for each $\varepsilon > 0$. First, it holds that

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}_\varepsilon(t)|_H^2 - \int_0^1 \frac{1}{s^2(t)}\tilde{u}_{\varepsilon yy}(t)\tilde{u}_\varepsilon(t)dy = \int_0^1 \frac{ys_t(t)}{s(t)}\eta_y(t)\tilde{u}_\varepsilon(t)dy. \tag{3.21}$$

The second term on the left-hand side is as follows:

$$\begin{aligned}
&-\int_0^1 \frac{1}{s^2(t)}\tilde{u}_{\varepsilon yy}(t)\tilde{u}_\varepsilon(t)dy \\
=&\frac{a_0}{s(t)}((\sigma(\tilde{u}_\varepsilon(t, 1))^2 - \alpha\sigma(\eta_\varepsilon(t))s(t))\tilde{u}_\varepsilon(t, 1) \\
&- \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}_\varepsilon(t, 0))\tilde{u}_\varepsilon(t, 0) + \frac{1}{s^2(t)}\int_0^1 |\tilde{u}_{\varepsilon y}(t)|^2 dy \\
\ge& - a_0\alpha|\eta_\varepsilon(t)||\tilde{u}_\varepsilon(t, 1)| - \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}_\varepsilon(t, 0))\tilde{u}_\varepsilon(t, 0) + \frac{1}{s^2(t)}\int_0^1 |\tilde{u}_{\varepsilon y}(t)|^2 dy,
\end{aligned}$$

where $\eta_\varepsilon(t) = (\rho_\varepsilon * \eta)(t, 1)$. From the above, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}_\varepsilon(t)|_H^2 + \frac{1}{s^2(t)}\int_0^1 |\tilde{u}_{\varepsilon y}(t)|^2 dy \\
\le& \int_0^1 \frac{ys_t(t)}{s(t)}\eta_y(t)\tilde{u}_\varepsilon(t)dy + a_0\alpha|\eta_\varepsilon(t)||\tilde{u}_\varepsilon(t, 1)| \\
&+ \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}_\varepsilon(t, 0))\tilde{u}_\varepsilon(t, 0) \text{ for } t \in [0, T].
\end{aligned} \tag{3.22}$$

We estimate the right-hand side of Eq (3.22). First, by Young's inequality we have that,

$$\int_0^1 \frac{ys_t(t)}{s(t)}\eta_y(t)\tilde{u}_\varepsilon(t)dy \le \frac{1}{4s^2(t)}\int_0^1 |\eta_y(t)|^2dy + |s_t(t)|^2\int_0^1 |\tilde{u}_\varepsilon(t)|^2dy. \tag{3.23}$$

Here, by Sobolev's embedding theorem in one dimension, we note that it holds that,

$$|z(y)|^2 \le C_e|z|_X|z|_H \text{ for } z \in X \text{ and } y \in [0,1], \tag{3.24}$$

where C_e is a positive constant defined from Sobolev's embedding theorem. By Eq (3.24) and $s \ge a$ on $[0,T]$, we obtain

$$\begin{aligned}
&\frac{1}{s(t)}\beta(b(t)-\gamma\tilde{u}_\varepsilon(t,0))\tilde{u}_\varepsilon(t,0) \le \frac{\beta b^*}{s(t)}|\tilde{u}_\varepsilon(t,0)| \\
\le&\frac{\beta b^* C_e}{2s(t)}\Big(|\tilde{u}_{\varepsilon y}(t)|_H|\tilde{u}_\varepsilon(t)|_H + |\tilde{u}_\varepsilon(t)|_H^2\Big) + \frac{\beta b^*}{2s(t)} \\
\le&\frac{1}{4s^2(t)}|\tilde{u}_{\varepsilon y}(t)|_H^2 + \left(\frac{(\beta b^* C_e)^2}{4} + \frac{\beta b^* C_e}{2a}\right)|\tilde{u}_\varepsilon(t)|_H^2 + \frac{\beta b^*}{2a},
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
&a_0\alpha|\eta_\varepsilon(t)||\tilde{u}_\varepsilon(t,1)| \le \frac{a_0\alpha}{2}\Big(C_e(|\tilde{u}_{\varepsilon y}(t)|_H|\tilde{u}_\varepsilon(t)|_H + |\tilde{u}_\varepsilon(t)|_H^2) + |\eta_\varepsilon(t)|^2\Big) \\
\le& \frac{1}{4s^2(t)}|\tilde{u}_{\varepsilon y}(t)|_H^2 + \left(s^2(t)\left(\frac{a_0\alpha}{2}C_e\right)^2 + \frac{a_0\alpha}{2}C_e\right)|\tilde{u}_\varepsilon(t)|_H^2 + \frac{a_0\alpha}{2}|\eta_\varepsilon(t)|^2.
\end{aligned} \tag{3.26}$$

From Eqs (3.21)–(3.26), we have that for $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}_\varepsilon(t)|_H^2 + \frac{1}{2s^2(t)}\int_0^1 |\tilde{u}_{\varepsilon y}(\tau)|^2dy \\
\le&\left(|s_t(t)|^2 + \frac{(\beta b^* C_e)^2}{4} + \frac{\beta b^* C_e}{2a}\right)|\tilde{u}_\varepsilon(t)|_H^2 + \frac{\beta b^*}{2a} \\
&+\left(s^2(t)\left(\frac{a_0\alpha}{2}C_e\right)^2 + \frac{a_0\alpha}{2}C_e\right)|\tilde{u}_\varepsilon(t)|_H^2 + \frac{1}{4s^2(t)}|\eta_y(t)|_H^2 + \frac{a_0\alpha}{2}|\eta_\varepsilon(t)|^2.
\end{aligned} \tag{3.27}$$

Denote $F(t)$ the coefficient of $|\tilde{u}_\varepsilon|_H^2$ in the right-hand side. As $s \in W^{1,2}(0,T)$, we observe that $F \in L^1(0,T)$. Then, by Gronwall's inequality, we have for $t_1 \in [0,T]$,

$$\frac{1}{2}|\tilde{u}_\varepsilon(t_1)|_H^2 + \frac{1}{2l^2}\int_0^{t_1} |\tilde{u}_{\varepsilon y}(\tau)|_H^2dy \le \left(G_\varepsilon(T)\int_0^{t_1} F(t)dt\right)e^{\int_0^{t_1} F(t)dt} \tag{3.28}$$

where $G_\varepsilon(t) = \frac{1}{2}\Big(|\tilde{u}_0|_H^2 + 1 + \frac{\beta b^*}{a}T + \frac{1}{2a^2}\int_0^t |\eta_y(t)|_H^2dt + a_0\alpha\int_0^t |\eta_\varepsilon(t)|^2dt\Big)$ for $t \in [0,T]$. We note that it holds that

$$\begin{aligned}
&\int_0^T |\eta_\varepsilon(\tau,1)|^2d\tau \le \int_0^T |\eta(\tau,1)|^2d\tau \le C_e\int_0^T (|\eta_y(\tau)|_H|\eta(\tau)|_H + |\eta(\tau)|_H^2)dt \\
&\le C_e\left(|\eta|_{L^\infty(0,T;H)}T^{1/2}\left(\int_0^T |\eta_y(\tau)|_H^2d\tau\right)^{1/2} + T|\eta|^2_{L^\infty(0,T;H)}\right)
\end{aligned}$$

$$\leq C_e T^{1/2}(1+T^{1/2})|\eta|^2_{V(T_1)}. \tag{3.29}$$

Therefore, by Eq (3.28) and Eq (3.29) we see that there exists a positive constant $M = M(a_0, a, \beta, b^*, T)$ such that Lemma 3.4 holds. □

Lemma 3.5. *Let $T > 0$, $s \in W^{1,\infty}(0,T)$ with $s > 0$ on $[0,T]$ and $\eta \in V(T)$. If (A1)–(A3) hold, then, $(AP2)(\tilde{u}_0, s, \eta, b)$ has a unique solution $\tilde{u}$ on $[0,T]$.*

Proof. Let $s \in W^{1,\infty}(0,T)$ with $s > 0$ on $[0,T]$. Then, we already have a solution $\tilde{u}_\varepsilon$ of $(\mathrm{AP2})_\varepsilon(\tilde{u}_{0\varepsilon}, s, \eta, b)$ on $[0,T]$ for each $\varepsilon > 0$. By letting $\varepsilon \to 0$ we show the existence of a solution $\tilde{u}$ of $(\mathrm{AP2})(\tilde{u}_0, s, \eta, b)$ on $[0,T]$. First, by Lemma 3.4 we see that $\{\tilde{u}_\varepsilon\}$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;X)$. Next, for $z \in X$, it holds that,

$$\begin{aligned}
&\left|\int_0^1 \tilde{u}_{\varepsilon t}(t) z dy\right| \\
=&\left|-\frac{1}{s^2(t)}\left(\int_0^1 \tilde{u}_{\varepsilon y}(t) z_y dy\right) - \frac{a_0}{s(t)}((\sigma(\tilde{u}_\varepsilon(t,1))^2 - \alpha\sigma((\rho_\varepsilon * \eta)(t,1))s(t))z(1)\right.\\
&\left.+\frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}_\varepsilon(t,0))z(0) + \int_0^1 \frac{y s_t(t)}{s(t)}\eta_y(t) z dy\right| \\
\leq& \frac{1}{a^2}|\tilde{u}_{\varepsilon y}(t)|_H |z_y|_H + \frac{a_0}{a}|\tilde{u}_\varepsilon(t,1)|^2|z(1)| + a_0\alpha|(\rho_\varepsilon * \eta)(t,1)||z(1)| + \frac{\beta b^*}{a}|z(0)| \\
&+ \frac{\beta\gamma}{a}|\tilde{u}_\varepsilon(t,0)||z(0)| + \frac{|s_t(t)|}{a}|\eta_y(t)|_H |z|_H \text{ for a.e. } t \in [0,T].
\end{aligned} \tag{3.30}$$

By the estimate Eq (3.30) and Eq (3.24) we infer that $\{\tilde{u}_{\varepsilon t}\}$ is bounded in $L^2(0,T;X^*)$. Therefore, we take a subsequence $\{\varepsilon_i\} \subset \{\varepsilon\}$ such that for some $\tilde{u} \in W^{1,2}(0,T;X^*) \cap L^\infty(0,T;H) \cap L^2(0,T;X)$, $\tilde{u}_{\varepsilon_i} \to \tilde{u}$ weakly in $W^{1,2}(0,T;X^*) \cap L^2(0,T;X)$, weakly-* in $L^\infty(0,T;H)$ as $i \to \infty$. Also, by Aubin's compactness theorem, we see that $\tilde{u}_{\varepsilon_i} \to \tilde{u}$ in $L^2(0,T;H)$ as $i \to \infty$.

Now, we prove that the limit function $\tilde{u}$ is a solution of $(\mathrm{AP2})(\tilde{u}_0, s, \eta, b)$ on $[0,T]$ satisfying $\tilde{u} \in W^{1,2}(0,T;X^*) \cap V(T)$, (S"2) and (S3). Let $z \in V(T)$. Then, it holds that,

$$\begin{aligned}
&\int_0^T \int_0^1 \tilde{u}_{\varepsilon t}(t) z(t) dy dt + \int_0^T \frac{1}{s^2(t)}\left(\int_0^1 \tilde{u}_{\varepsilon y}(t) z_y(t) dy\right) dt \\
&+ \int_0^T \frac{a_0}{s(t)}((\sigma(\tilde{u}_\varepsilon(t,1))^2 - \alpha\sigma((\rho_\varepsilon * \eta)(t,1))s(t))z(t,1) dt \\
&- \int_0^T \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}_\varepsilon(t,0))z(t,0) dt = \int_0^T \int_0^1 \frac{y s_t(t)}{s(t)}\eta_y(t) z(t) dy dt.
\end{aligned} \tag{3.31}$$

From the weak convergence, it is easy to see that,

$$\begin{aligned}
&\int_0^T \int_0^1 \tilde{u}_{\varepsilon_i t}(t) z(t) dy dt \to \int_0^T \langle \tilde{u}_t(t), z(t)\rangle_X dt, \\
&\int_0^T \frac{1}{s^2(t)}\left(\int_0^1 \tilde{u}_{\varepsilon_i y}(t) z_y(t) dy\right) dt \to \int_0^T \frac{1}{s^2(t)}\left(\int_0^1 \tilde{u}_y(t) z_y(t) dy\right) dt \text{ as } i \to \infty.
\end{aligned}$$

The third term of the left-hand side of Eq (3.31) is as follows:

$$\begin{aligned}&\Big|\int_0^T \frac{a_0}{s(t)}((\sigma(\tilde u_\varepsilon(t,1))^2 - \alpha\sigma((\rho_\varepsilon * \eta)(t,1))s(t))z(t,1)dt\\&\qquad\qquad - \int_0^T \frac{a_0}{s(t)}((\sigma(\tilde u(t,1))^2 - \alpha\sigma(\eta(t,1))s(t))z(t,1)dt\Big|\\&\le \frac{a_0}{a}\Big(\int_0^T |\tilde u_\varepsilon(t,1) - \tilde u(t,1)|^2 dt\Big)^{1/2}\Big(\int_0^T 2(|\tilde u_\varepsilon(t,1)|^2 + |\tilde u(t)|^2)|z(t,1)|^2 dt\Big)^{1/2}.\\&\quad + a_0\alpha\int_0^T |(\rho_\varepsilon * \eta)(t,1) - \eta(t,1)||z(t,1)|dt\end{aligned}$$

Here, by Eq (3.24) we note that it holds that,

$$\begin{aligned}&\int_0^T |\tilde u_\varepsilon(t,z) - \tilde u(t,z)|^2 dt\\&\le C_e\Big(\int_0^T |\tilde u_\varepsilon(t) - \tilde u(t)|_X^2\Big)^{1/2}\Big(\int_0^T |\tilde u_\varepsilon(t) - \tilde u(t)|_H^2\Big)^{1/2} \text{ for } z = 0, 1,\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\int_0^T |\tilde u_\varepsilon(t,1)|^2|z(t,1)|^2 dt \le C_e^2\int_0^T |\tilde u_\varepsilon(t)|_X|\tilde u_\varepsilon(t)|_H|z(t)|_X|z(t)|_H dt\\&\le C_e^2|\tilde u_\varepsilon|_{L^\infty(0,T;H)}|z|_{L^\infty(0,T;H)}\Big(\int_0^T |\tilde u_\varepsilon(t)|_X^2\Big)^{1/2}\Big(\int_0^T |z(t)|_X^2\Big)^{1/2}.\end{aligned}$$

Since $(\rho_\varepsilon * \eta)(t,1) \to \eta(t,1)$ in $L^2(0,T)$ as $\varepsilon \to 0$, by the boundedness of $\tilde u_\varepsilon$ in $L^2(0,T;X)$, $\tilde u \in L^2(0,T;X)$ and the strong convergence in $L^2(0,T;H)$ we see that

$$\begin{aligned}&\int_0^T \frac{a_0}{s(t)}((\sigma(\tilde u_{\varepsilon_i}(t,1))^2 - \alpha\sigma((\rho_{\varepsilon_i} * \eta)(t,1))s(t))z(t,1)dt\\&\to \int_0^T \frac{a_0}{s(t)}((\sigma(\tilde u(t,1))^2 - \alpha\sigma(\eta(t,1))s(t))z(t,1)dt \text{ as } i \to \infty.\end{aligned}$$

What concerns the forth term on the left-hand side of Eq (3.31), by Eq (3.32) it follows that,

$$\int_0^T \frac{1}{s(t)}\beta(b(t) - \gamma\tilde u_{\varepsilon_i}(t,0))z(t,0)dt \to \int_0^T \frac{1}{s(t)}\beta(b(t) - \gamma\tilde u(t,0))z(t,0)dt \text{ as } i \to \infty.$$

Therefore, by the limiting process $i \to \infty$ in Eq (3.31) we see that the limit function $\tilde u$ is a solution of (AP2)$(\tilde u_0, s, \eta, b)$ on $[0,T]$. Also, the solution $\tilde u$ is unique. Indeed, let $\tilde u_1$ and $\tilde u_2$ be a solution of (AP2)$(\tilde u_0, s, \eta, b)$ on $[0,T]$ and put $\tilde u = \tilde u_1 - \tilde u_2$. Then, (S"2) implies that

$$\begin{aligned}&\langle \tilde u_t, z\rangle_X + \frac{1}{s^2}\Big(\int_0^1 \tilde u_y z_y dy\Big)\\&+ \frac{a_0}{s}\Big((\sigma(\tilde u_1(\cdot,1))^2 - \alpha\sigma(\eta(\cdot,1))s) - ((\sigma(\tilde u_2(\cdot,1))^2 - \alpha\sigma(\eta(\cdot,1))s)\Big)z(1)\end{aligned}$$

$$+\frac{1}{s}\beta\gamma(\tilde{u}_1(\cdot,0)-\tilde{u}_2(\cdot,0))z(0)=0 \text{ for } z\in X \text{ a.e. on } [0,T]. \tag{3.33}$$

We take $z=\tilde{u}$ in Eq (3.33). Then, by the monotonicity of σ we have

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2+\frac{1}{s^2(t)}\int_0^1|\tilde{u}_y(t)|^2dy\le 0 \text{ for a.e. } t\in[0,T]. \tag{3.34}$$

By Eq (3.34), we have the uniqueness of a solution $\tilde{u}$ to (AP2)$(\tilde{u}_0,s,\eta,b)$ on $[0,T]$. □

Lemma 3.6. *Let $T>0$ and $s\in W^{1,\infty}(0,T)$ with $s>0$ on $[0,T]$. Then, $(AP1)(\tilde{u}_0,s,b)$ has a unique solution $\tilde{u}$ on $[0,T]$.*

Proof. From Lemma 3.5, we see that (AP2)$(\tilde{u}_0,s,\eta,b)$ has a solution $\tilde{u}$ on $[0,T]$ such that $\tilde{u}\in W^{1,2}(0,T;X^*)\cap L^\infty(0,T;H)\cap L^2(0,T;X)$. Define a solution operator $\delta_T(\eta)=\tilde{u}$, where $\tilde{u}$ is a unique solution of (AP2)$(\tilde{u}_0,s,\eta,b)$ for given $\eta\in V(T)$. Let put $\delta_T(\eta_i)=\tilde{u}_i$ for $i=1,2$ and $\eta=\eta_1-\eta_2$ and $\tilde{u}=\tilde{u}_1-\tilde{u}_2$. Then, by (S"2) it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2+\frac{1}{s^2}\int_0^1|\tilde{u}_y(t)|^2dy\\&+\frac{a_0}{s(t)}\Big(((\sigma(\tilde{u}_1(t,1))^2-\alpha\sigma(\eta_1(t,1))s(t))-((\sigma(\tilde{u}_2(t,1))^2-\alpha\sigma(\eta_2(t,1))s(t))\Big)\tilde{u}(t,1)\\&+\frac{\beta\gamma}{s(t)}|\tilde{u}(t,0)|^2=\int_0^1\frac{ys_t(t)}{s(t)}\eta_y(t)\tilde{u}(t)dy \text{ for a.e. } t\in[0,T].\end{aligned} \tag{3.35}$$

For the third term I_3 in the left-hand side of Eq (3.35), by the monotonicity of σ we have that $I_3\ge -a_0\alpha|\eta(t,1)||\tilde{u}(t,1)|$. From Eq (3.24) it holds that,

$$|\eta(t,1)||\tilde{u}(t,1)|\le C_e^{1/2}|\eta(t,1)|(|\tilde{u}_y(t)|_H^{1/2}|\tilde{u}(t)|_H^{1/2}+|\tilde{u}(t)|_H)$$

and

$$\begin{aligned}&\int_0^t|\eta(\tau,1)||\tilde{u}(\tau,1)|d\tau\\ \le&C_e^{1/2}\Big(|\tilde{u}|_{L^\infty(0,T_1;H)}^{1/2}\int_0^t|\eta(t,1)||\tilde{u}_y(\tau)|_H^{1/2}d\tau+|\tilde{u}|_{L^\infty(0,T_1;H)}\int_0^t|\eta(t,1)|d\tau\Big)\\ \le&C_e^{1/2}\Big(|\tilde{u}|_{L^\infty(0,T_1;H)}^{1/2}T_1^{1/4}\Big(\int_0^t|\tilde{u}_y(\tau)|_H^2\Big)^{1/4}+|\tilde{u}|_{L^\infty(0,T_1;H)}T_1^{1/2}\Big)\Big(\int_0^t|\eta(t,1)|^2\Big)^{1/2}.\end{aligned} \tag{3.36}$$

Let $T_1\in(0,T]$ and we integrate Eq (3.35) over $[0,t]$ for any $t\in[0,T_1]$. Then, by Eq (3.36) we obtain,

$$\begin{aligned}&\delta\Big(|\tilde{u}(t)|_H^2+\int_0^t|\tilde{u}_y(\tau)|_H^2d\tau\Big)\\ \le&a_0\alpha C_e^{1/2}T_1^{1/4}(1+T_1^{1/4})\Big(\int_0^t|\eta(\tau,1)|^2d\tau\Big)^{1/2}|\tilde{u}|_{V(T_1)}\\&+\frac{|s_t|_{L^\infty(0,T_1)}}{a}T_1^{1/2}\Big(\int_0^t|\eta_y(\tau)|_H^2d\tau\Big)^{1/2}|\tilde{u}|_{V(T_1)}\end{aligned} \tag{3.37}$$

and $\delta = \min\{1/2, 1/l^2\}$, where $l = \max_{0\le t\le T} |s(t)|$. Finally, by Eq (3.37) we have

$$\delta|\tilde{u}|_{V(T_1)} \le \Big[a_0\alpha C_e^{1/2}T_1^{1/4}(1+T_1^{1/4}) + \frac{|s_t|_{L^\infty(0,T_1)}}{a}T_1^{1/2}\Big]|\eta|_{V(T_1)}. \tag{3.38}$$

From Eq (3.38) we see that there exists $T_1 \le T$ such that δ_{T_1} is a contraction on $V(T_1)$. Hence, Banach's fixed point theorem guarantees that there exists $\tilde{u} \in V(T_1)$ such that $\delta_{T_1}(\tilde{u}) = \tilde{u}$. Thus, we see that (AP1)$(\tilde{u}_0, s, b)$ has a solution $\tilde{u}$ on $[0, T_1]$. Here, T_1 is independent of the choice of the initial data. Therefore, by repeating the local existence argument, we have a unique solution $\tilde{u}$ of (AP1)$(\tilde{u}_0, s, b)$ on the whole interval $[0, T]$. Thus, we see that Lemma 3.6 holds. □

Here, for given $s \in W^{1,\infty}(0, T)$ with $s > 0$ on $[0, T]$ we show that a solution $\tilde{u}$ of (AP1)$(\tilde{u}_0, s, b)$ is non-negative and bounded on $Q(T)$.

Lemma 3.7. *Let $T > 0$, $s \in W^{1,\infty}(0, T)$ with $s > 0$ on $[0, T]$ and $\tilde{u}$ be a solution of (AP1)$(\tilde{u}_0, s, b)$ on $[0, T]$. Then, it holds that*

$$0 \le \tilde{u}(t) \le u^*(T) := \max\{\alpha l(T), \frac{b^*}{\gamma}\} \text{ on } [0,1] \text{ for } t \in [0, T],$$

where $l(T) = \max_{0\le t\le T} |s(t)|$.

Proof. By (S'2), we note that it holds that,

$$\begin{aligned}&\langle \tilde{u}_t, z\rangle_X + \int_0^1 \frac{1}{s^2}\tilde{u}_y z_y dy + \frac{a_0}{s}\sigma(\tilde{u}(\cdot,1))(\sigma(\tilde{u}(\cdot,1)) - \alpha s)z(1)\\ &- \frac{1}{s}\beta(b(\cdot) - \gamma\tilde{u}(\cdot,0))z(0) = \int_0^1 \frac{y s_t}{s}\tilde{u}_y z dy \text{ for } z \in X \text{ a.e. on } [0, T].\end{aligned} \tag{3.39}$$

First, we prove that $\tilde{u}(t) \ge 0$ on $[0, 1]$ for $t \in [0, T]$. By taking $z = -[-\tilde{u}]^+$ in Eq (3.39) we have,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|[-\tilde{u}(t)]^+|_H^2 + \frac{1}{s^2(t)}\int_0^1 |[-\tilde{u}(t)]_y^+|^2 dy\\ &- \frac{a_0}{s(t)}\sigma(\tilde{u}(\cdot,1))(\sigma(\tilde{u}(\cdot,1)) - \alpha s(t))[-\tilde{u}(t,1)]^+ + \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}(\cdot,0))[-\tilde{u}(t,0)]^+\\ &= -\int_0^1 \frac{y s_t(t)}{s(t)}\tilde{u}_y(t)[-\tilde{u}(t)]^+ dy \text{ a.e. on } [0, T].\end{aligned} \tag{3.40}$$

Here, the third term in the left-hand side of Eq (3.40) is equal to 0 and the forth term in the left-hand side of Eq (3.40) is non-negative. Also, we obtain that

$$\int_0^1 \frac{y s_t(t)}{s(t)}[-\tilde{u}(t)]_y^+[-\tilde{u}(t)]^+ dy \le \frac{1}{2s^2(t)}|[-\tilde{u}(t)]_y^+|_H^2 + \frac{(s_t(t))^2}{2}|[-\tilde{u}(t)]^+|_H^2,$$

Then, we have

$$\frac{1}{2}\frac{d}{dt}|[-\tilde{u}(t)]^+|_H^2 + \frac{1}{2s^2(t)}\int_0^1 |[-\tilde{u}(t)]_y^+|^2 dy \le \frac{(s_t(t))^2}{2}|[-\tilde{u}(t)]^+|_H^2 \text{ for a.e. } t \in [0, T].$$

Therefore, by Gronwall's inequality and the assumption that $\tilde{u}_0 \geq 0$ on $[0, 1]$, we conclude that $\tilde{u}(t) \geq 0$ on $[0, 1]$ for $t \in [0, T]$.

Next, we show that a solution $\tilde{u}$ of (AP1)($\tilde{u}_0, s, b$) has a upper bound $u^*(T)$. Put $U(t, y) = [\tilde{u}(t, y) - u^*(T)]^+$ for $y \in [0, 1]$ and $t \in [0, T]$. Then, it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|U(t)|_H^2 + \frac{1}{s^2(t)}\int_0^1 |U_y(t)|^2 dy + \frac{a_0}{s(t)}\sigma(\tilde{u}(\cdot, 1))(\sigma(\tilde{u}(\cdot, 1)) - \alpha s(t))U(t, 1)\\&- \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}(\cdot, 0))U(t, 0) = \int_0^1 \frac{y s_t(t)}{s(t)}\tilde{u}_y(t)U(t)dy \text{ for a.e. } t \in [0, T].\end{aligned} \tag{3.41}$$

Here, by $\tilde{u}(t) \geq 0$ on $[0, 1]$ for $t \in [0, T]$ we note that $a_0\sigma(\tilde{u}(\cdot, 1))(\sigma(\tilde{u}(t, 1)) - \alpha s(t)) = a_0\tilde{u}(t, 1)(\tilde{u}(t, 1) - \alpha s(t))$. Then, by $u^*(T) \geq \alpha l(T) \geq \alpha s(t)$ for $t \in [0, T]$, it holds that

$$\frac{a_0}{s(t)}\tilde{u}(t, 1)(\tilde{u}(t, 1) - \alpha s(t))U(t, 1) \geq \frac{a_0}{s(t)}\tilde{u}(t, 1)(u^*(T) - \alpha s(t))U(t, 1) \geq 0.$$

Also, by Eq (1.3) and $b \leq b^*$, we observe that,

$$\begin{aligned}&- \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}(t, 0))U(t, 0) = \frac{1}{s(t)}\beta(\gamma\tilde{u}(t, 0) - b^* + b^* - b(t))U(t, 0)\\&\geq \frac{\beta\gamma}{s(t)}|U(t, 0)|^2 + \frac{\beta}{s(t)}(b^* - b(t))U(t, 0) \geq 0.\end{aligned} \tag{3.42}$$

By applying the above two results to Eq (3.41) we obtain that,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |U(t)|^2 dy + \frac{1}{2s^2(t)}\int_0^1 |U_y(t)|^2 dy \leq \frac{(s_t(t))^2}{2}|U(t)|_H^2 \text{ for a.e. } t \in [0, T].$$

This result and the assumption that $\tilde{u}_0 \leq b^*/\gamma$ on $[0, 1]$ implies that $\tilde{u}(t) \leq u^*(T)$ on $[0, 1]$ for $t \in [0, T]$. Thus, Lemma 3.7 is proven. □

At the end of this section, we relax the condition $s \in W^{1,\infty}(0, T)$, namely, for given $s \in W^{1,2}(0, T)$ with $s > 0$ on $[0, T]$, we construct a solution to (AP1)($\tilde{u}_0, s, b$).

Lemma 3.8. *Let $T > 0$ and $s \in W^{1,2}(0, T)$ with $s > 0$ on $[0, T]$. If (A1)–(A3) hold, then, $(AP1)(\tilde{u}_0, s, b)$ has a unique solution $\tilde{u}$ on $[0, T]$.*

Proof. For given $s \in W^{1,2}(0, T)$ with $s > 0$ on $[0, T]$, we choose a sequence $\{s_n\} \subset W^{1,\infty}(0, T)$ and l, $a > 0$ satisfying $a \leq s_n \leq l$ on $[0, T]$ for each $n \in \mathbb{N}$, $s_n \to s$ in $W^{1,2}(0, T)$ as $n \to \infty$. By Lemma 3.6 we can take a sequence $\{\tilde{u}_n\}$ of solutions to (AP1)($\tilde{u}_0, s_n, b$) on $[0, T]$. Let $z \in X$. Then, it holds that

$$\begin{aligned}&\langle\tilde{u}_{nt}, z\rangle_X + \frac{1}{s_n^2}\left(\int_0^1 \tilde{u}_{ny}z_y dy\right)\\&+ \frac{a_0}{s_n}\sigma(\tilde{u}_n(\cdot, 1))(\sigma(\tilde{u}_n(\cdot, 1)) - \alpha s_n)z(1)\\&- \frac{1}{s_n}\beta(b - \gamma\tilde{u}_n(\cdot, 0))z(0) = \int_0^1 \frac{y s_{nt}}{s_n}\tilde{u}_{ny}z dy \text{ a.e. on } [0, T].\end{aligned} \tag{3.43}$$

We take $z = \tilde{u}_n$ in Eq (3.43). Then, similarly to the proof of Lemma 3.4 we derive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\tilde{u}_n(t)|_H^2 + \frac{1}{4s_n^2(t)}\int_0^1 |\tilde{u}_{ny}(\tau)|^2 dy\\ \leq &\left(|s_{nt}(t)|^2 + \frac{(\beta b^* C_e)^2}{4} + \frac{\beta b^* C_e}{2a}\right)|\tilde{u}_n(t)|_H^2 + \frac{\beta b^*}{2a}\\ &+ \left(s_n^2(t)\,(a_0\alpha C_e)^2 + a_0\alpha C_e\right)|\tilde{u}_n(t)|_H^2 \text{ for a.e. } t \in [0,T].\end{aligned}$$

From this, we infer that $\{\tilde{u}_n\}$ is bounded in $L^\infty(0,T;H)\cap L^2(0,T;X)$. Also, by referring to the derivation of Eq (3.30) we obtain from Eq (3.43) that

$$\begin{aligned}&\left|\int_0^1 \tilde{u}_{nt}(t) z dy\right|\\ \leq &\frac{1}{a^2}|\tilde{u}_{ny}(t)|_H|z_y|_H + \frac{a_0}{a}|\tilde{u}_n(t,1)|^2|z(1)| + a_0\alpha|\tilde{u}_n(t,1)||z(1)| + \frac{\beta b^*}{a}|z(0)|\\ &+ \frac{\beta\gamma}{a}|\tilde{u}_n(t,0)||z(0)| + \frac{|s_{nt}(t)|}{a}\Big(|\tilde{u}_n(t,1)||z(1)| + |\tilde{u}_n(t)|_H(|z_y|_H + |z|_H)\Big).\end{aligned}$$

Here, we note that

$$\int_0^1 \frac{y s_{nt}(t)}{s_n(t)}\tilde{u}_{ny}(t) z dy = \frac{s_{nt}(t)}{s_n(t)}\Big(\tilde{u}_n(t,1)z(1) - \int_0^1 \tilde{u}_n(t)(y z_y + z)dy\Big).$$

Here, by Lemma 3.7, we note that $0 \leq \tilde{u}_n \leq u^*$ on $Q(T)$ for each $n \in \mathbb{N}$, where $u^* = \max\{\alpha l, \frac{b^*}{\gamma}\}$. Using this result and the boundedness of $\{\tilde{u}_n\}$ in $V(T)$, we see that $\{\tilde{u}_{nt}\}$ is bounded in $L^2(0,T;X^*)$. Therefore, we take a subsequence $\{n_j\} \subset \{n\}$ such that for some $\tilde{u} \in W^{1,2}(0,T;X^*) \cap V(T)$, $\tilde{u}_{nj} \to \tilde{u}$ strongly in $L^2(0,T;H)$, weakly in $W^{1,2}(0,T;X^*) \cap L^2(0,T;X)$, weakly-* in $L^\infty(0,T;H)$, and $\tilde{u}_{nj}(\cdot,x) \to \tilde{u}(\cdot,x)$ in $L^2(0,T)$ at $x = 0, 1$ as $j \to \infty$.

Now, we consider the limiting process $j \to \infty$ in the following way:

$$\begin{aligned}&\int_0^T \langle \tilde{u}_{nt}(t), z(t)\rangle_X dt + \int_0^T \frac{1}{s_n^2(t)}\Big(\int_0^1 \tilde{u}_{ny}(t) z_y(t) dy\Big)dt\\ &+ \int_0^T \frac{a_0}{s_n(t)}\sigma(\tilde{u}_n(t,1))(\sigma(\tilde{u}_n(t,1)) - \alpha s_n(t))z(t,1)dt\\ &- \int_0^T \frac{1}{s_n(t)}\beta(b(t) - \gamma\tilde{u}_n(t,0))z(t,0)dt\\ &= \int_0^T\int_0^1 \frac{y s_{nt}(t)}{s_n(t)}\tilde{u}_{ny}(t) z(t) dy dt \text{ for } z \in V(T).\end{aligned} \tag{3.44}$$

Note that by $s_{n_j} \to s$ in $W^{1,2}(0,T)$ as $j \to \infty$, it holds that $s_{n_j} \to s$ in $C([0,T])$ as $j \to \infty$. From the convergence of $\tilde{u}_{n_j}$ and s_{nj}, Eq (3.32) and $a \leq s_{n_j} \leq l$ on $[0,T]$, it is clear that

$$\int_0^T \langle \tilde{u}_{n_j t}(t), z(t)\rangle_X dt \to \int_0^T \langle \tilde{u}_t(t), z(t)\rangle_X dt,$$

$$\int_0^T \frac{a_0}{s_{n_j}(t)}\sigma(\tilde{u}_{n_j}(t,1))(\sigma(\tilde{u}_{n_j}(t,1)) - \alpha s_{n_j}(t))z(t,1)dt$$
$$\to \int_0^T \frac{a_0}{s(t)}\sigma(\tilde{u}(t,1))(\sigma(\tilde{u}(t,1)) - \alpha s(t))z(t,1)dt$$

and

$$\int_0^T \frac{1}{s_{n_j}(t)}\beta(b(t) - \gamma\tilde{u}_{n_j}(t,0))z(t,0)dt \to \int_0^T \frac{1}{s(t)}\beta(b(t) - \gamma\tilde{u}(t,0))z(t,0)dt$$

as $j \to \infty$. For the second term in the left-hand side of Eq (3.44), it follows that,

$$\begin{aligned}
&\Big|\int_0^T \frac{1}{s_{n_j}^2(t)}\Big(\int_0^1 \tilde{u}_{n_jy}(t)z_y(t)dy\Big)dt - \int_0^T \frac{1}{s^2(t)}\Big(\int_0^1 \tilde{u}_y(t)z_y(t)dy\Big)dt\Big|\\
\le &\int_0^T \Big|\frac{1}{s_{n_j}^2(t)} - \frac{1}{s^2(t)}\Big||\tilde{u}_{n_jy}(t)|_H|z_y(t)|_H dt\\
&+ \Big|\int_0^T \frac{1}{s^2(t)}\Big(\int_0^1 (\tilde{u}_{n_jy}(t) - \tilde{u}_y(t))z_y(t)dy\Big)dt\Big|\\
\le &\frac{2l}{a^2}|s_{n_j} - s|_{C([0,T])}|\tilde{u}_{n_jy}|_{L^2(0,T;H)}|z_y|_{L^2(0,T;H)}\\
&+ \Big|\int_0^T \Big(\tilde{u}_{n_jy}(t) - \tilde{u}_y(t), \frac{1}{s^2(t)}z_y(t)\Big)_H dt\Big|
\end{aligned}$$

Hence, we observe that

$$\int_0^T \frac{1}{s_{n_j}^2(t)}\Big(\int_0^1 \tilde{u}_{n_jy}(t)z_y(t)dy\Big)dt \to \int_0^T \frac{1}{s^2(t)}\Big(\int_0^1 \tilde{u}_y(t)z_y(t)dy\Big)dt \text{ as } j \to \infty.$$

Also, the right-hand side of Eq (3.44) is as follows:

$$\begin{aligned}
&\Big|\int_0^T \int_0^1 \frac{ys_{n_jt}(t)}{s_{n_j}(t)}\tilde{u}_{n_jy}(t)z(t)dydt - \int_0^T \int_0^1 \frac{ys_t(t)}{s(t)}\tilde{u}_y(t)z(t)dydt\Big|\\
\le &\int_0^T \Big|\frac{s_{n_jt}(t)}{s_{n_j}(t)} - \frac{s_t(t)}{s(t)}\Big||\tilde{u}_{n_jy}(t)|_H|z(t)|_H dt + \Big|\int_0^T \Big(\tilde{u}_{n_jy}(t) - \tilde{u}_y(t), \frac{ys_t(t)}{s(t)}z(t)\Big)_H dt\Big|\\
\le &\frac{1}{a}|s_{n_jt} - s_t|_{L^2(0,T)}|\tilde{u}_{n_jy}|_{L^2(0,T;H)}|z|_{L^\infty(0,T;H)}\\
&+ \frac{1}{a^2}|s_{n_j} - s|_{C([0,T])}|s_t|_{L^2(0,T)}|\tilde{u}_{n_jy}|_{L^2(0,T;H)}|z|_{L^\infty(0,T;H)}\\
&+ \Big|\int_0^T \Big(\tilde{u}_{n_jy}(t) - \tilde{u}_y(t), \frac{ys_t(t)}{s(t)}z(t)\Big)_H dt\Big|
\end{aligned}$$

From this, we have that

$$\int_0^T \int_0^1 \frac{ys_{n_jt}(t)}{s_{n_j}(t)}\tilde{u}_{n_jy}(t)z(t)dydt \to \int_0^T \int_0^1 \frac{ys_t(t)}{s(t)}\tilde{u}_y(t)z(t)dydt \text{ as } j \to \infty.$$

Finally, by letting $j \to \infty$ we see that $\tilde{u}$ is a solution of (AP)$(\tilde{u}_0, s, b)$ on $[0,T]$. Uniqueness is proved by the same argument of the proof of Lemma 3.5. □

4. Proof of Theorem 2.2

In this section, using the results obtained in Section 3, we establish the existence of a locally-in-time solution (PC)$(\tilde{u}_0, s_0, b)$. Throughout of this section, we assume (A1)–(A3). First, for $T > 0$, $l > 0$ and $a > 0$ such that $a < s_0 < l$ we set

$$M(T, a, l) := \{s \in W^{1,2}(0, T) | a \le s \le l \text{ on } [0, T], s(0) = s_0\}.$$

Also, for given $s \in M(T, a, l)$, we define two solution mappings as follows: $\Psi : M(T, a, l) \to W^{1,2}(0, T; X^*) \cap L^\infty(0, T; H) \cap L^2(0, T; X)$ by $\Psi(s) = \tilde{u}$, where $\tilde{u}$ is a unique solution of (AP1)$(\tilde{u}_0, s, b)$ on $[0, T]$ and $\Gamma_T : M(T, a, l) \to W^{1,2}(0, T)$ by $\Gamma_T(s) = s_0 + \int_0^t a_0(\sigma(\Psi(s)(\tau, 1)) - \alpha s(\tau))d\tau$ for $t \in [0, T]$. Moreover, for any $K > 0$ we put

$$M_K(T) := \{s \in M(T, a, l) |\, |s|_{W^{1,2}(0,T)} \le K\}.$$

Now, we show that for some $T > 0$, Γ_T is a contraction mapping on the closed set of $M_K(T)$ for any $K > 0$.

Lemma 4.1. *Let $K > 0$. Then, there exists a positive constant $T^* \le T$ such that the mapping Γ_{T^*} is a contraction on the closed set $M_K(T^*)$ in $W^{1,2}(0, T^*)$.*

Proof. For $T > 0$, $a > 0$ and $l > 0$ such that $a < s_0 < l$, let $s \in M(T, a, l)$ and $\tilde{u} = \Psi(s)$. First, from the proof of Lemma 3.8 we note that it holds

$$|\Psi(s)|_{W^{1,2}(0,T;X^*)} + |\Psi(s)|_{L^\infty(0,T;H)} + |\Psi(s)|_{L^2(0,T;X)} \le C \text{ for } s \in M_K(T), \tag{4.1}$$

where $C = C(T, \tilde{u}_0, K, l, b^*, \beta, s_0)$ is a positive constant depending on T, $\tilde{u}_0$, K, l, b^*, β and s_0.

First we show that there exists $T_0 \le T$ such that $\Gamma_{T_0} : M_K(T_0) \to M_K(T_0)$ is well-defined. Let $K > 0$ and $s \in M_K(T)$. By the definition of σ and $\Psi(s) = \tilde{u}$ is a solution of (AP1)$(\tilde{u}_0, s, b)$, we observe that

$$\begin{aligned}\Gamma_T(s)(t) &= s_0 + \int_0^t a_0(\sigma(\Psi(s)(\tau, 1)) - \alpha s(\tau))d\tau \\ &\ge s_0 - a_0\alpha l t \text{ for } t \in [0, T].\end{aligned} \tag{4.2}$$

Also, by Eq (3.24) and Eq (4.1), it holds that

$$\begin{aligned}&\Gamma_T(s)(t) \le s_0 + a_0 T^{1/2}(\int_0^t |\tilde{u}(\tau, 1)|^2 d\tau)^{1/2} \\ &\le s_0 + a_0 T^{1/2}(C_e|\tilde{u}|_{L^\infty(0,t;H)} \int_0^t |\tilde{u}(\tau)|_X)^{1/2} \le s_0 + a_0 T^{1/2}(T^{1/4} C_e^{1/2} C), \\ &\int_0^t |\Gamma_T(s)(\tau)|^2 d\tau \le 2s_0^2 T + 4a_0^2 T \int_0^t (|\tilde{u}(\tau, 1)|^2 + (\alpha s(t))^2) d\tau \\ &\le 2s_0^2 T + 4a_0^2 T\Big(C_e|\tilde{u}|_{L^\infty(0,t;H)} \int_0^t |\tilde{u}(\tau)|_X + (\alpha l)^2 T\Big) \\ &\le 2s_0^2 T + 4a_0^2 T(T^{1/2} C_e C^2 + (\alpha l)^2 T),\end{aligned} \tag{4.3}$$

and

$$\int_0^t |\Gamma'_T(s)(\tau)|^2 d\tau \le a_0^2 \int_0^t |\sigma(\Psi(s)(\tau,1)) - \alpha s(\tau)|^2 d\tau \le 2a_0^2(T^{1/2}C_e C^2 + (\alpha l)^2 T), \tag{4.4}$$

where C is the same positive constant as in (4.1). Therefore, by Eqs (4.2)–(4.4) we see that there exists $T_0 \le T$ such that $\Gamma_{T_0}(s) \in M_K(T_0)$.

Next, for s_1 and $s_2 \in M_K(T_0)$, let $\tilde{u}_1 = \Psi(s_1)$ and $\tilde{u}_2 = \Psi(s_2)$ and set $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, $s = s_1 - s_2$. Then, it holds that

$$\begin{aligned}
&\langle \tilde{u}_t, z\rangle_X + \int_0^1 (\frac{1}{s_1^2}\tilde{u}_{1y} - \frac{1}{s_2^2}\tilde{u}_{2y})z_y dy \\
&+ a_0\Big(\frac{1}{s_1}\sigma(\tilde{u}_1(\cdot,1))(\sigma(\tilde{u}_1(\cdot,1)) - \alpha s_1) - \frac{1}{s_2}\sigma(\tilde{u}_2(\cdot,1))(\sigma(\tilde{u}_2(\cdot,1)) - \alpha s_2)\Big)z(1) \\
&- \Big(\frac{1}{s_1}\beta(b(t) - \gamma\tilde{u}_1(\cdot,0)) - \frac{1}{s_2}\beta(b(t) - \gamma\tilde{u}_2(\cdot,0))\Big)z(0) \\
&= \int_0^1 (\frac{ys_{1t}}{s_1}\tilde{u}_{1y} - \frac{ys_{2t}}{s_2}\tilde{u}_{2y})z dy \text{ for } z \in X \text{ a.e. on } [0,T].
\end{aligned} \tag{4.5}$$

By taking $z = \tilde{u}$ in Eq (4.5) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}|_H^2 + \int_0^1 (\frac{1}{s_1^2}\tilde{u}_{1y} - \frac{1}{s_2^2}\tilde{u}_{2y})\tilde{u}_y dy \\
&+ a_0\Big(\frac{1}{s_1}\sigma(\tilde{u}_1(\cdot,1))(\sigma(\tilde{u}_1(\cdot,1)) - \alpha s_1)) - \frac{1}{s_2}\sigma(\tilde{u}_2(\cdot,1))(\sigma(\tilde{u}_2(\cdot,1)) - \alpha s_2)\Big)\tilde{u}(\cdot,1) \\
&- \Big(\frac{1}{s_1}\beta(b(t) - \gamma\tilde{u}_1(\cdot,0)) - \frac{1}{s_2}\beta(b(t) - \gamma\tilde{u}_2(\cdot,0)\Big)\tilde{u}(\cdot,0) \\
&= \int_0^1 (\frac{ys_{1t}}{s_1}\tilde{u}_{1y} - \frac{ys_{2t}}{s_2}\tilde{u}_{2y})\tilde{u} dy \text{ a.e. on } [0,T].
\end{aligned} \tag{4.6}$$

For the second term of the left-hand side of Eq (4.6), we observe that

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{s_1^2(t)}\tilde{u}_{1y}(t) - \frac{1}{s_2^2(t)}\tilde{u}_{2y}(t)\right)\tilde{u}_y(t)dy \\
=&\frac{1}{s_1^2(t)}|\tilde{u}_y(t)|_H^2 + \int_0^1 \left(\frac{1}{s_1^2(t)} - \frac{1}{s_2^2(t)}\right)\tilde{u}_{2y}(t)\tilde{u}_y(t)dy \\
\ge&\frac{1}{s_1^2(t)}|\tilde{u}_y(t)|_H^2 - \frac{2l|s(t)|}{a^3 s_1(t)}|\tilde{u}_{2y}(t)|_H|\tilde{u}_y(t)|_H \\
\ge&\left(1 - \frac{\eta}{2}\right)\frac{1}{s_1^2(t)}|\tilde{u}_y(t)|_H^2 - \frac{1}{2\eta}\left(\frac{2l}{a^3}\right)^2|s(t)|^2|\tilde{u}_{2y}(t)|_H^2,
\end{aligned} \tag{4.7}$$

where η is arbitrary positive number. Next, the third term in the left-hand side of Eq (4.6) is as follows:

$$a_0\Big(\frac{\sigma(\tilde{u}_1(t,1))(\sigma(\tilde{u}_1(t,1)) - \alpha s_1(t))}{s_1(t)} - \frac{\sigma(\tilde{u}_2(t,1))(\sigma(\tilde{u}_2(t,1)) - \alpha s_2(t))}{s_2(t)}\Big)\tilde{u}(t,1)$$

$$
\begin{aligned}
=&a_0\Big[\frac{1}{s_1(t)}\Big(\sigma(\tilde{u}_1(t,1))(\sigma(\tilde{u}_1(t,1))-\alpha s_1(t))-\sigma(\tilde{u}_2(t,1))(\sigma(\tilde{u}_2(t,1))-\alpha s_2(t))\Big)\\
&+\left(\frac{1}{s_1(t)}-\frac{1}{s_2(t)}\right)\sigma(\tilde{u}_2(t,1))(\sigma(\tilde{u}_2(t,1))-\alpha s_2(t))\Big]\tilde{u}(t,1)\\
=&a_0\Big[\frac{1}{s_1(t)}(\sigma(\tilde{u}_1(t,1))-\sigma(\tilde{u}_2(t,1)))(\sigma(\tilde{u}_1(t,1))-\alpha s_1(t))\Big]\tilde{u}(t,1)\\
&+\frac{\sigma(\tilde{u}_2(t,1))}{s_1(t)}\Big(\sigma(\tilde{u}_1(t,1))-\alpha s_1(t)-(\sigma(\tilde{u}_2(t,1))-\alpha s_2(t))\Big)\tilde{u}(t,1)\\
&+\left(\frac{1}{s_1(t)}-\frac{1}{s_2(t)}\right)\sigma(\tilde{u}_2(t,1))(\sigma(\tilde{u}_2(t,1))-\alpha s_2(t))\tilde{u}(t,1)\\
:=&I_1+I_2+I_3.
\end{aligned}
$$

By the monotonicity of $\sigma(r)$ and Eq (3.24) we have that

$$
I_1 \geq -\frac{a_0}{s_1(t)}(|\tilde{u}_1(t,1)|+\alpha l)|C_e|\tilde{u}(t)|_X|\tilde{u}(t)|_H \tag{4.8}
$$

and

$$
\begin{aligned}
I_2 =& \frac{a_0}{s_1(t)}\sigma(\tilde{u}_2(t,1))\Big(\sigma(\tilde{u}_1(t,1))-\alpha s_1(t)-(\sigma(\tilde{u}_2(t,1))-\alpha s_2(t))\Big)\tilde{u}(t,1)\\
\geq& -\frac{a_0|\tilde{u}_2(t,1)|}{s_1(t)}\alpha|s(t)||\tilde{u}(t,1)|\\
\geq& -\frac{C_e(a_0\alpha\tilde{u}_2(t,1))^2}{2s_1^2(t)}|\tilde{u}(t)|_X|\tilde{u}(t)|_H-\frac{1}{2}|s(t)|^2.
\end{aligned} \tag{4.9}
$$

Also, using the fact that $\sigma(r)\leq|r|$ for $r\in\mathbb{R}$ and Eq (3.24), we have the following estimate:

$$
\begin{aligned}
|I_3| \leq& \left(\frac{|s(t)|}{s_1(t)s_2(t)}\right)a_0\sigma(\tilde{u}_2(t,1))(\sigma(\tilde{u}_2(t,1))+\alpha l)|\tilde{u}(t,1)|\\
\leq& \frac{C_e(a_0\tilde{u}_2(t,1))^2}{2a^2s_1^2(t)}|\tilde{u}(t)|_X|\tilde{u}(t)|_H+\frac{1}{2}\tilde{u}_2^2(t,1)|s(t)|^2\\
&+\frac{C_e(a_0\alpha l\tilde{u}_2(t,1))^2}{2a^2s_1^2(t)}|\tilde{u}(t)|_X|\tilde{u}(t)|_H+\frac{1}{2}|s(t)|^2.
\end{aligned} \tag{4.10}
$$

Here, we put $L_{s_1}^{(1)}(t) = a_0(|\tilde{u}_1(t,1)|+\alpha l)C_e$ and $L_{s_2}^{(1)}(t) = C_e(a_0\alpha\tilde{u}_2(t,1))^2/2 + C_e(a_0\tilde{u}_2(t,1))^2/2a^2 + C_e(a_0\alpha l\tilde{u}_2(t,1))^2/2a^2$. Next, by using Eq (3.24) and (A3), we have

$$
\begin{aligned}
&-\beta\left(\frac{1}{s_1(t)}-\frac{1}{s_2(t)}\right)b(t)\tilde{u}(t,0)+\beta\gamma\left(\frac{1}{s_1(t)}\tilde{u}_1(t,0)-\frac{1}{s_2(t)}\tilde{u}_2(t,0)\right)\tilde{u}(t,0)\\
=&-\beta\left(\frac{1}{s_1(t)}-\frac{1}{s_2(t)}\right)b(t)\tilde{u}(t,0)+\beta\gamma\frac{1}{s_1(t)}|\tilde{u}(t,0)|^2\\
&+\beta\gamma\left(\frac{1}{s_1(t)}-\frac{1}{s_2(t)}\right)\tilde{u}_2(t,0)\tilde{u}(t,0)
\end{aligned}
$$

$$\geq -\left(\frac{(\beta b^*)^2 C_e}{2a^2 s_1^2(t)} + \frac{(\beta\gamma|\tilde{u}_2(t,0)|)^2 C_e}{2a^2 s_1^2(t)}\right)|\tilde{u}(t)|_X|\tilde{u}(t)|_H - |s(t)|^2 \text{ for } t \in [0, T_0]. \tag{4.11}$$

For the right-hand side of Eq (4.6), we separate as follows:

$$\begin{aligned}
&\int_0^1 \left(\frac{ys_{1t}(t)}{s_1(t)}\tilde{u}_{1y}(t) - \frac{ys_{2t}(t)}{s_2(t)}\tilde{u}_{2y}(t)\right)\tilde{u}(t)dy \\
=&\int_0^1 \frac{ys_{1t}(t)}{s_1(t)}\tilde{u}_y(t)\tilde{u}(t)dy + \int_0^1 \frac{ys_t(t)}{s_1(t)}\tilde{u}_{2y}(t)\tilde{u}(t)dy \\
&+ \int_0^1 \left(\frac{1}{s_1(t)} - \frac{1}{s_2(t)}\right) ys_{2t}(t)\tilde{u}_{2y}(t)\tilde{u}(t)dy \\
:=&I_4 + I_5 + I_6.
\end{aligned}$$

Then, the three terms are estimated in the following way:

$$\begin{aligned}
I_4 &\leq \frac{\eta}{2s_1^2(t)}|\tilde{u}_y(t)|_H^2 + \frac{1}{2\eta}|s_{1t}(t)|^2|\tilde{u}(t)|_H^2, \\
I_5 &\leq \frac{1}{2a}\left(|s_t(t)|^2 + |\tilde{u}_{2y}(t)|_H^2|\tilde{u}(t)|_H^2\right), \\
I_6 &\leq \frac{1}{2a^2}\left(|s(t)|^2|\tilde{u}_{2y}(t)|_H^2 + |s_{2t}(t)|^2|\tilde{u}(t)|_H^2\right).
\end{aligned}$$

From Eq (4.6) and all estimates we derive the following inequality:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 + (1-\eta)\frac{1}{s_1^2(t)}|\tilde{u}_y(t)|_H^2 \\
\leq&\frac{1}{s_1(t)}L_{s_1}^{(1)}(t)|\tilde{u}(t)|_X|\tilde{u}(t)|_H \\
&+ \frac{1}{s_1^2(t)}\left(L_{s_2}^{(1)}(t) + \frac{(\beta\gamma|\tilde{u}_2(t,0)|)^2 C_e}{2a^2} + \frac{(\beta b^*)^2 C_e}{2a^2}\right)|\tilde{u}(t)|_X|\tilde{u}(t)|_H \\
&+ \left(\frac{1}{2\eta}|s_{1t}(t)|^2 + \frac{1}{2a}|\tilde{u}_{2y}(t)|_H^2 + \frac{1}{2a^2}|s_{2t}(t)|^2\right)|\tilde{u}(t)|_H^2 \\
&+ \left(\frac{1}{2a^2}|\tilde{u}_{2y}(t)|_H^2 + \frac{1}{2\eta}\left(\frac{2l}{a^3}\right)^2|\tilde{u}_{2y}(t)|_H^2 + \frac{1}{2}\tilde{u}_2^2(t,1) + 2\right)|s(t)|^2 + \frac{1}{2a}|s_t(t)|^2.
\end{aligned} \tag{4.12}$$

We put $C_5(t) = ((\beta\gamma|\tilde{u}_2(t,0)|)^2 C_e)/2a^2 + ((\beta b^*)^2 C_e)/2a^2$. By Young's inequality we have that

$$\begin{aligned}
&\frac{1}{s_1(t)}L_{s_1}^{(1)}(t)|\tilde{u}(t)|_X|\tilde{u}(t)|_H \\
\leq&\frac{1}{s_1(t)}L_{s_1}^{(1)}(t)\left(|\tilde{u}_y(t)|_H|\tilde{u}(t)|_H + |\tilde{u}(t)|_H^2\right) \\
\leq&\frac{\eta}{2s_1^2(t)}|\tilde{u}_y(t)|_H^2 + \left(\frac{(L_{s_1}^{(1)}(t))^2}{2\eta} + \frac{1}{a}L_{s_1}^{(1)}(t)\right)|\tilde{u}(t)|_H^2,
\end{aligned}$$

and

$$
\begin{aligned}
&(L_{s_2}^{(1)}(t) + C_5(t))\frac{1}{s_1^2(t)}|\tilde{u}(t)|_X|\tilde{u}(t)|_H \\
\leq &(L_{s_2}^{(1)}(t) + C_5(t))\frac{1}{s_1^2(t)}(|\tilde{u}_y(t)|_H|\tilde{u}(t)|_H + |\tilde{u}(t)|_H^2) \\
\leq &\frac{1}{s_1^2(t)}\frac{\eta}{2}|\tilde{u}_y(t)|_H^2 + \frac{1}{a^2}\left(\frac{(L_{s_2}^{(1)}(t) + C_5(t))^2}{2\eta} + (L_{s_2}^{(1)}(t) + C_5(t))\right)|\tilde{u}(t)|_H^2.
\end{aligned}
$$

Hence, by applying these results to Eq (4.12) and taking a suitable $\eta = \eta_0$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 + \frac{1}{2}\frac{1}{s_1^2(t)}|\tilde{u}_y(t)|_H^2 \\
\leq &\left(\frac{(L_{s_1}^{(1)}(t))^2}{2\eta_0} + \frac{1}{a}L_{s_1}^{(1)}(t)\right)|\tilde{u}(t)|_H^2 \\
&+ \frac{1}{a^2}\left(\frac{(L_{s_2}^{(1)}(t) + C_5(t))^2}{2\eta_0} + (L_{s_2}^{(1)}(t) + C_5(t))\right)|\tilde{u}(t)|_H^2. \\
&+ \left(\frac{1}{2\eta_0}|s_{1t}(t)|^2 + \frac{1}{2a}|\tilde{u}_{2y}(t)|_H^2 + \frac{1}{2a^2}|s_{2t}(t)|^2\right)|\tilde{u}(t)|_H^2 \\
&+ \left(\frac{1}{2a^2}|\tilde{u}_{2y}(t)|_H^2 + \frac{1}{2\eta_0}\left(\frac{2l}{a^3}\right)^2|\tilde{u}_{2y}(t)|_H^2 + \frac{1}{2}\tilde{u}_2^2(t,1) + 2\right)|s(t)|^2 + \frac{1}{2a}|s_t(t)|^2.
\end{aligned} \tag{4.13}
$$

Now, we put the summation of all coefficients of $|\tilde{u}(t)|_H^2$ by $L_s^{(2)}(t)$ for $t \in [0, T_0]$ and $L_{s_2}^{(3)}(t) = |\tilde{u}_{2y}(t)|_H^2/2a^2 + (4l^2|\tilde{u}_{2y}(t)|_H^2)/2\eta_0 a^6 + \tilde{u}_2^2(t,1)/2 + 2$. Then, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 + \frac{1}{2}\frac{1}{s_1^2(t)}|\tilde{u}_y(\tau)|_H^2 \\
\leq &L_s^{(2)}(t)|\tilde{u}(t)|_H^2 + L_{s_2}^{(3)}(t)|s(t)|^2 + \frac{1}{2a}|s_t(t)|^2 \text{ for } t \in [0, T_0].
\end{aligned} \tag{4.14}
$$

By Eq (3.24) and Eq (4.1) we note that $\tilde{u}_i^2(\cdot,1), \tilde{u}_i^4(\cdot,1) \in L^1(0, T_0)$ for $i = 1, 2$. From this and the fact that $s_i \in M_K(T_0)$ for $i = 1, 2$, we see that $L_s^{(2)} \in L^1(0, T_0)$ and $L_{s_2}^{(3)} \in L^1(0, T_0)$. Also, it holds that

$$
L_{s_2}^{(3)}(t)|s(t)|^2 \leq L_{s_2}^{(3)}(t)T_0|s_t|_{L^2(0,t)}^2 \text{ for } t \in [0, T_0].
$$

Therefore, Gronwall's inequality guarantees that

$$
\begin{aligned}
&\frac{1}{2}|\tilde{u}(t)|_H^2 + \frac{1}{2}\frac{1}{l^2}\int_0^t |\tilde{u}_y(\tau)|_H^2 d\tau \\
\leq &\left[\left(2|L_{s_2}^{(3)}|_{L^1(0,T_0)}T_0 + \frac{1}{a}\right)|s_t|_{L^2(0,t)}^2\right]e^{2\int_0^t L_s^{(2)}(\tau)d\tau} \text{ for } t \in [0, T_0].
\end{aligned} \tag{4.15}
$$

By using Eq (4.15) we show that there exists $T^* \leq T_0$ such that Γ_{T^*} is a contraction mapping on the closed subset of $M_K(T^*)$. To do so, from the subtraction of the time derivatives of $\Gamma_{T_0}(s_1)$ and $\Gamma_{T_0}(s_2)$ and relying on Eq (3.24) and Eq (4.15), we have for $T_1 \leq T_0$ the following estimate:

$$
|(\Gamma_{T_1}(s_1))_t - (\Gamma_{T_1}(s_2))_t|_{L^2(0,T_1)}
$$

$$\leq a_0\Big(|\sigma(\tilde{u}_1(\cdot,1))-\sigma(\tilde{u}_2(\cdot,1))|_{L^2(0,T_1)}+\alpha|s_1(t)-s_2(t)|_{L^2(0,T_1)}\Big)$$
$$\leq a_0\sqrt{C_e}\Big(\int_0^{T_1}(|\tilde{u}_y(t)|_H|\tilde{u}(t)|_H+|\tilde{u}(t)|_H^2)dt\Big)^{1/2}+a_0\alpha T_1|s|_{W^{1,2}(0,T_1)}$$
$$\leq a_0\sqrt{C_e}\left(|\tilde{u}|^{\frac{1}{2}}_{L^\infty(0,T_1;H)}\left(\int_0^{T_1}|\tilde{u}_y(t)|_H dt\right)^{\frac{1}{2}}+\sqrt{T_1}|\tilde{u}|_{L^\infty(0,T_1;H)}\right)$$
$$+a_0\alpha T_1|s|_{W^{1,2}(0,T_1)}. \tag{4.16}$$

Using Eq (4.15) and Eq (4.16), we obtain

$$|\Gamma_{T_1}(s_1)-\Gamma_{T_1}(s_2)|_{L^2(0,T_1)}$$
$$\leq T_1C_6\Big(T_1^{\frac{1}{4}}|s|_{W^{1,2}(0,T_1)}+\sqrt{T_1}|s|_{W^{1,2}(0,T_1)}+T_1|s|_{W^{1,2}(0,T_1)}\Big), \tag{4.17}$$

where C_6 is a positive constant obtained by Eq (4.15). Therefore, by Eq (4.16) and Eq (4.17) we see that there exists $T^* \leq T_0$ such that Γ_{T^*} is a contraction mapping on a closed subset of $M_K(T^*)$. □

From Lemma 4.1, by applying Banach's fixed point theorem, there exists $s \in M_K(T^*)$, where T^* is the same as in Lemma 4.1 such that $\Gamma_{T^*}(s) = s$. This implies that (PC)$(\tilde{u}_0, s_0, b)$ has a unique solution $(s, \tilde{u})$ on $[0, T^*]$.

At the end of this section, we show the boundedness of a solution $\tilde{u}$ to (PC)$(\tilde{u}_0, s_0, b)$ which completes Theorem 2.2. By (S2) it holds that

$$\langle\tilde{u}_t, z\rangle_X+\int_0^1\frac{1}{s^2}\tilde{u}_yz_ydy+\frac{1}{s}\sigma(\tilde{u}(\cdot,1))s_tz(1)$$
$$-\frac{1}{s}\beta(b(\cdot)-\gamma\tilde{u}(\cdot,0))z(0)=\int_0^1\frac{ys_t}{s}\tilde{u}_yzdy \text{ for } z\in X \text{ a.e. on } [0,T]. \tag{4.18}$$

First, we note that the solution $\tilde{u}$ of (PC)$(\tilde{u}_0, s_0, b)$ is non-negative.

Lemma 4.2. *Let $T > 0$ and $(s, \tilde{u})$ be a solution of (PC)$(\tilde{u}_0, s_0, b)$ on $[0, T]$. Then, $\tilde{u}(t) \geq 0$ on $[0, 1]$ for $t \in [0, T]$.*

Proof. Lemma 4.2 is proved by taking $z = -[-\tilde{u}]^+$ in Eq (4.18) and using the argument of the proof of Lemma 3.7. Here, by $s_t(t) = a_0(\sigma(\tilde{u}(t, 1)) - \alpha s(t))$ we note that it holds that

$$\int_0^1\frac{ys_t(t)}{s(t)}[-\tilde{u}(t)]^+_y[-\tilde{u}(t)]^+dy=\frac{s_t(t)}{s(t)}\int_0^1\frac{1}{2}\Big(\frac{d}{dy}(y|[-\tilde{u}(t)]^+|^2)-|[-\tilde{u}(t)]^+|^2\Big)$$
$$=\frac{s_t(t)}{2s(t)}|[-\tilde{u}(t,1)]^+|^2-\frac{s_t(t)}{2s(t)}|[-\tilde{u}(t)]^+|_H^2$$
$$=\frac{a_0(\sigma(\tilde{u}(t,1))-\alpha s(t))}{2s(t)}|[-\tilde{u}(t,1)]^+|^2-\frac{a_0(\sigma(\tilde{u}(t,1))-\alpha s(t))}{2s(t)}|[-\tilde{u}(t)]^+|_H^2$$
$$\leq\frac{a_0\alpha}{2}|[-\tilde{u}(t)]^+|_H^2.$$

From this, we derive

$$\frac{1}{2}\frac{d}{dt}|[-\tilde{u}(t)]^+|_H^2+\frac{1}{2s(t)}\int_0^1|[-\tilde{u}(t)]^+_y|^2dy\leq\frac{a_0\alpha}{2}|[-\tilde{u}(t)]^+|_H^2 \text{ for a.e. } t\in[0,T].$$

This implies that $\tilde{u}(t) \geq 0$ on $[0, 1]$ for $t \in [0, T]$. □

Next, we show the boundedness of the solution $(s, \tilde{u})$ of (PC)$(\tilde{u}_0, s_0, b)$.

Lemma 4.3. *Let $T > 0$ and $(s, \tilde{u})$ be a solution of (PC)$(\tilde{u}_0, s_0, b)$ on $[0, T]$. Then, it holds that*

$$(i) \quad s(t) \le M \text{ for } t \in [0, T],$$
$$(ii) \quad 0 \le \tilde{u}(t) \le u^* := \max\{\alpha M, \frac{b^*}{\gamma}\} \text{ on } [0,1] \text{ for } t \in [0, T],$$

where M is a positive constant which depends on β, γ, α, b^*, $|\tilde{u}_0|_H$, s_0, $|b_t|_{L^2(0,T)}$ and $|b_t|_{L^1(0,T)}$.

Proof. First, we prove (i). By taking $z = s(\tilde{u} - \frac{b}{\gamma})$ in Eq (4.18) it holds that:

$$\begin{aligned}
&\frac{s(t)}{2}\frac{d}{dt}\left|\tilde{u}(t) - \frac{b(t)}{\gamma}\right|_H^2 + \left(\frac{b_t(t)}{\gamma}, s(t)(\tilde{u}(t) - \frac{b(t)}{\gamma})\right)_H \\
&+ \frac{1}{s(t)}\int_0^1 |\tilde{u}_y(t)|^2 dy + \sigma(\tilde{u}(t,1))s_t(t)\left(\tilde{u}(t,1) - \frac{b(t)}{\gamma}\right) \\
&- \beta(b(t) - \gamma\tilde{u}(t,0))\left(\tilde{u}(t,0) - \frac{b(t)}{\gamma}\right) = \int_0^1 y s_t(t)\tilde{u}_y(t)\left(\tilde{u}(t) - \frac{b(t)}{\gamma}\right)dy \\
&\text{for a.e. } t \in [0, T].
\end{aligned} \tag{4.19}$$

The second term of the left-hand side of Eq (4.19) is follows:

$$\begin{aligned}
&\left|\left(\frac{b_t(t)}{\gamma}, s(t)(\tilde{u}(t) - \frac{b(t)}{\gamma})\right)_H\right| = \left|\frac{b_t(t)}{\gamma}s(t)\int_0^1 (\tilde{u}(t,y) - \tilde{u}(t,0) + \tilde{u}(t,0) - \frac{b(t)}{\gamma})dy\right| \\
\le&\frac{1}{\gamma}|b_t(t)||s(t)|\left(|\tilde{u}_y(t)|_H + \left|\tilde{u}(t,0) - \frac{b(t)}{\gamma}\right|\right).
\end{aligned} \tag{4.20}$$

Also, by Lemma 4.2 we note that $\sigma(\tilde{u}(t,1))s_t(t) = \tilde{u}(t,1)s_t(t)$. Moreover, we observe that

$$\begin{aligned}
&\int_0^1 y s_t(t)\tilde{u}_y(t)\left(\tilde{u}(t) - \frac{b(t)}{\gamma}\right)dy \\
&= s_t(t)\int_0^1 \frac{1}{2}\left(\frac{\partial}{\partial y}\left(y(\tilde{u}(t) - \frac{b(t)}{\gamma})^2\right) - (\tilde{u}(t) - \frac{b(t)}{\gamma})^2\right)dy \\
&= \frac{s_t(t)}{2}\left(\tilde{u}(t,1) - \frac{b(t)}{\gamma}\right)^2 - \frac{s_t(t)}{2}\int_0^1\left(\tilde{u}(t) - \frac{b(t)}{\gamma}\right)^2 dy.
\end{aligned} \tag{4.21}$$

Hence, by Eq (4.20) and Eq (4.21) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(s(t)\left|\tilde{u}(t) - \frac{b(t)}{\gamma}\right|_H^2\right) + \frac{1}{s(t)}\int_0^1 |\tilde{u}_y(t)|^2 dy \\
&+ \tilde{u}(t,1)s_t(t)\left(\tilde{u}(t,1) - \frac{b(t)}{\gamma}\right) - \frac{s_t(t)}{2}\left(\tilde{u}(t,1) - \frac{b(t)}{\gamma}\right)^2 + \beta\gamma|\tilde{u}(t,0) - \frac{b(t)}{\gamma}|^2 \\
\le&\frac{1}{\gamma}|b_t(t)||s(t)|\left(|\tilde{u}_y(t)|_H + \left|\tilde{u}(t,0) - \frac{b(t)}{\gamma}\right|\right) \text{ for a.e. } t \in [0, T].
\end{aligned} \tag{4.22}$$

Here, the third and forth terms of the left-hand side of Eq (4.22) is as follows:

$$
\begin{aligned}
&\tilde{u}(t,1)s_t(t)\Big(\tilde{u}(t,1)-\frac{b(t)}{\gamma}\Big)-\frac{s_t(t)}{2}\Big(\tilde{u}(t,1)-\frac{b(t)}{\gamma}\Big)^2\\
=&s_t(t)\tilde{u}^2(t,1)-s_t(t)\tilde{u}(t,1)\frac{b(t)}{\gamma}-\frac{s_t(t)}{2}\Big(\tilde{u}^2(t,1)-2\tilde{u}(t,1)\frac{b(t)}{\gamma}+\Big(\frac{b(t)}{\gamma}\Big)^2\Big)\\
=&\frac{s_t(t)}{2}\tilde{u}^2(t,1)-\frac{s_t(t)}{2}\Big(\frac{b(t)}{\gamma}\Big)^2. \qquad (4.23)
\end{aligned}
$$

Since $s_t(t)=a_0(\tilde{u}(t,1)-\alpha s(t))$, it holds that

$$
\begin{aligned}
\frac{s_t(t)}{2}\tilde{u}^2(t,1)=&\frac{\tilde{u}(t,1)}{2}\left(\frac{|s_t(t)|^2}{a_0}+\alpha s(t)s_t(t)\right)\\
=&\frac{1}{2a_0}\tilde{u}(t,1)|s_t(t)|^2+\frac{\alpha s(t)}{2}\left(\frac{|s_t(t)|^2}{a_0}+\alpha s(t)s_t(t)\right)\\
=&\frac{1}{2a_0}\tilde{u}(t,1)|s_t(t)|^2+\frac{\alpha s(t)}{2}\frac{|s_t(t)|^2}{a_0}+\frac{\alpha^2}{6}\frac{d}{dt}s^3(t). \qquad (4.24)
\end{aligned}
$$

Note that the first and second terms in the last of Eq (4.24) are non-negative. Accordingly, by Eq (4.23) and Eq (4.24) we have that

$$
\begin{aligned}
&\tilde{u}(t,1)s_t(t)\Big(\tilde{u}(t,1)-\frac{b(t)}{\gamma}\Big)-\frac{s_t(t)}{2}\Big(\tilde{u}(t,1)-\frac{b(t)}{\gamma}\Big)^2\\
\geq&\frac{\alpha^2}{6}\frac{d}{dt}s^3(t)-\frac{s_t(t)}{2}\Big(\frac{b(t)}{\gamma}\Big)^2\\
=&\frac{\alpha^2}{6}\frac{d}{dt}s^3(t)-\frac{1}{2\gamma^2}\Big(\frac{d}{dt}(s(t)b^2(t))-2s(t)b(t)b_t(t)\Big). \qquad (4.25)
\end{aligned}
$$

Also, for the right-hand side of Eq (4.22) we have

$$
\begin{aligned}
&\frac{1}{\gamma}|b_t(t)||s(t)|\Big(|\tilde{u}_y(t)|_H+|\tilde{u}(t,0)-\frac{b(t)}{\gamma}|\Big)\\
\leq&\frac{1}{2s(t)}|\tilde{u}_y(t)|_H^2+\frac{1}{2}s^3(t)\Big(\frac{|b_t(t)|}{\gamma}\Big)^2+\frac{\beta\gamma}{2}\Big|\tilde{u}(t,0)-\frac{b(t)}{\gamma}\Big|^2+\frac{1}{2\beta\gamma}|s(t)|^2\Big(\frac{|b_t(t)|}{\gamma}\Big)^2. \qquad (4.26)
\end{aligned}
$$

By combining Eq (4.25) and Eq (4.26) with Eq (4.22) we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(s(t)\Big|\tilde{u}(t)-\frac{b(t)}{\gamma}\Big|_H^2\Big)+\frac{1}{2s(t)}\int_0^1|\tilde{u}_y(t)|^2dy\\
&+\frac{\alpha^2}{6}\frac{d}{dt}s^3(t)-\frac{1}{2\gamma^2}\frac{d}{dt}(s(t)b^2(t))+\frac{\beta\gamma}{2}|\tilde{u}(t,0)-\frac{b(t)}{\gamma}|^2\\
\leq&\frac{1}{\gamma^2}s(t)b(t)|b_t(t)|+\frac{1}{2}s^3(t)\Big(\frac{|b_t(t)|}{\gamma}\Big)^2+\frac{1}{2\beta\gamma}|s(t)|^2\Big(\frac{|b_t(t)|}{\gamma}\Big)^2 \text{ for a.e. } t\in[0,T]. \qquad (4.27)
\end{aligned}
$$

Then, by integrating Eq (4.27) over $[0,t]$ for $t\in[0,T]$ we obtain that

$$
\frac{1}{2}s(t)\Big|\tilde{u}(t)-\frac{b(t)}{\gamma}\Big|_H^2+\int_0^t\frac{1}{2s(\tau)}|\tilde{u}_y(\tau)|_H^2d\tau+\frac{\alpha^2}{6}s^3(t)-\frac{1}{2\gamma^2}(s(t)b^2(t))
$$

$$\leq \frac{1}{2} s_0 \left| \tilde{u}_0 - \frac{b(0)}{\gamma} \right|_H^2 + \frac{\alpha^2}{6} s_0^3 + \frac{1}{2\gamma^2} \int_0^t s^3(\tau) |b_t(\tau)|^2 d\tau$$
$$+ \frac{b^*}{\gamma^2} \int_0^t s(\tau) |b_t(\tau)| d\tau + \frac{1}{2\beta\gamma^3} \int_0^t |s(\tau)|^2 |b_t(\tau)|^2 d\tau \text{ for } t \in [0, T]. \tag{4.28}$$

Here, by Young's inequality it follows that

$$\frac{\alpha^2}{6} s^3(t) - \frac{1}{2\gamma^2} (s(t) b^2(t))$$
$$\geq \frac{\alpha^2}{6} s^3(t) - \left(\frac{\eta^3}{3} s^3(t) + \frac{2}{3\eta^{3/2}} \left(\frac{1}{2\gamma^2} (b^*)^2 \right)^{3/2} \right), \tag{4.29}$$

where η is an arbitrary positive number. Therefore, by Eq (4.28) and Eq (4.29) we obtain

$$\left(\frac{\alpha^2}{6} - \frac{\eta^3}{3} \right) s^3(t) \leq \frac{2}{3\eta^{3/2}} \left(\frac{1}{2\gamma^2} (b^*)^2 \right)^{3/2}$$
$$+ \frac{1}{2} s_0 \left| \tilde{u}_0 - \frac{b(0)}{\gamma} \right|_H^2 + \frac{\alpha^2}{6} s_0^3 + \frac{1}{2\gamma^2} \int_0^t s^3(\tau) |b_t(\tau)|^2 d\tau$$
$$+ \frac{b^*}{\gamma^2} \int_0^t s(\tau) |b_t(\tau)| d\tau + \frac{1}{2\beta\gamma^3} \int_0^t |s(\tau)|^2 |b_t(\tau)|^2 d\tau \text{ for } t \in [0, T]. \tag{4.30}$$

We put $M(\eta) = (s_0 |\tilde{u}_0 - \frac{b(0)}{\gamma}|_H^2)/2 + (\alpha^2 s_0^3)/6 + (2(\frac{1}{2\gamma^2}(b^*)^2)^{3/2}))/3\eta^{3/2}$. Then, by taking a suitable $\eta = \eta_0$ and $J_1(t) = \int_0^t s^3(\tau) |b_t(\tau)|^2 d\tau$, $J_2(t) = \int_0^t s(\tau) |b_t(\tau)| d\tau$ and $J_3(t) = \int_0^t |s(\tau)|^2 |b_t(\tau)|^2 d\tau$ for $t \in [0, T]$ we derive that

$$\frac{\alpha^2}{12} s^3(t) \leq M(\eta_0) + \frac{1}{2\gamma^2} J_1(t) + \frac{b^*}{\gamma^2} J_2(t) + \frac{1}{2\beta\gamma^3} J_3(t) \text{ for } t \in [0, T]. \tag{4.31}$$

Then, we have that

$$J_1'(t) \leq \frac{12 M(\eta_0)}{\alpha^2} |b_t(t)|^2 + \frac{6}{(\gamma\alpha)^2} |b_t(t)|^2 J_1(t)$$
$$+ \frac{12}{\alpha^2} \left(\frac{b^*}{\gamma^2} J_2(t) + \frac{1}{2\beta\gamma^3} J_3(t) \right) |b_t(t)|^2 \text{ for } t \in [0, T].$$

Hence, by Gronwall's inequality we obtain that

$$J_1(t) \leq \left[\frac{12 M(\eta_0)}{\alpha^2} |b_t|_{L^2(0,T)}^2 \right.$$
$$\left. + \frac{12}{\alpha^2} \left(\frac{b^*}{\gamma^2} J_2(t) + \frac{1}{2\beta\gamma^3} J_3(t) \right) |b_t|_{L^2(0,T)}^2 \right] e^{\frac{6}{(\gamma\alpha)^2} |b_t|_{L^2(0,T)}^2} \text{ for } t \in [0, T]. \tag{4.32}$$

Here, we put $N(T) = \frac{12}{\alpha^2} |b_t|_{L^2(0,T)}^2 e^{\frac{6}{(\gamma\alpha)^2} |b_t|_{L^2(0,T)}^2}$. Then, by Eq (4.31) and Eq (4.32) we obtain that

$$\frac{\alpha^2}{12} s^3(t) \leq M(\eta_0) + \frac{1}{2\gamma^2} M(\eta_0) N(T)$$

$$+\frac{b^*}{\gamma^2}(1+\frac{N(T)}{2\gamma^2})J_2(t)+\frac{1}{2\beta\gamma^3}(1+\frac{N(T)}{2\gamma^2})J_3(t) \text{ for } t\in[0,T]. \tag{4.33}$$

Now, we put $l(T)=\max_{0\le t\le T}|s(t)|$. Then, we have

$$\frac{b^*}{\gamma^2}(1+\frac{N(T)}{2\gamma^2})J_2(t)\le\frac{b^*}{\gamma^2}(1+\frac{N(T)}{2\gamma^2})l(T)|b_t|_{L^1(0,T)}$$
$$\le\frac{\eta^3}{3}l^3(T)+\frac{2}{3\eta^{3/2}}\Big(\frac{b^*}{\gamma^2}(1+\frac{N(T)}{2\gamma^2})|b_t|_{L^1(0,T)}\Big)^{3/2},$$

and

$$\frac{1}{2\beta\gamma^3}(1+\frac{N(T)}{2\gamma^2})J_3(t)\le\frac{1}{2\beta\gamma^3}(1+\frac{N(T)}{2\gamma^2})l^2(T)|b_t|^2_{L^2(0,T)}$$
$$\le\frac{2\eta^{3/2}}{3}l^3(T)+\frac{1}{3\eta^3}\Big(\frac{1}{2\beta\gamma^3}(1+\frac{N(T)}{2\gamma^2})|b_t|^2_{L^2(0,T)}\Big)^3.$$

Hence, by adding these estimates to Eq (4.33) and taking a suitable $\eta=\eta_0$ we see that there exists a positive constant M which depends on β, γ, α, b^*, s_0, $|b_t|_{L^2(0,T)}$ and $|b_t|_{L^1(0,T)}$ such that $s(t)\le M$ for $t\in[0,T]$.

Next, we show (ii). Put $U(t,y)=[\tilde{u}(t,y)-u^*]^+$ for $y\in[0,1]$ and $t\in[0,T]$, and then take $z=U(t,y)$ in Eq (4.18). Here, by $\tilde{u}(t)\ge 0$ on $[0,1]$ for $t\in[0,T]$ we note that $s_t(t)=a_0(\sigma(\tilde{u}(t,1))-\alpha s(t))=a_0(\tilde{u}(t,1)-\alpha s(t))$. Then, we observe that:

$$\int_0^1\frac{ys_t(t)}{s(t)}\tilde{u}_y(t)U(t)dy=\frac{s_t(t)}{2s(t)}\int_0^1\Big(\frac{d}{dy}(yU^2(t))-U^2(t)\Big)dy$$
$$=\frac{s_t(t)}{2s(t)}|U(t,1)|^2-\frac{s_t(t)}{2s(t)}|U(t)|^2_H\le\frac{s_t(t)}{2s(t)}|U(t,1)|^2+\frac{a_0\alpha}{2}|U(t)|^2_H.$$

Then, by using the argument of the proof of Lemma 3.7 we derive

$$\frac{1}{2}\frac{d}{dt}|U(t)|^2_H+\frac{1}{s^2(t)}\int_0^1|U_y(t)|^2dy+\frac{1}{s(t)}\tilde{u}(t,1)s_t(t)U(t,1)-\frac{s_t(t)}{2s(t)}|U(t,1)|^2$$
$$\le\frac{a_0\alpha}{2}|U(t)|^2_H \text{ for a.e. } t\in[0,T]. \tag{4.34}$$

Here, by $u^*\ge\alpha M\ge\alpha s(t)$ for $t\in[0,T]$, it holds that

$$\frac{s_t(t)}{s(t)}\tilde{u}(t,1)U(t,1)-\frac{s_t(t)}{2s(t)}|U(t,1)|^2=\frac{s_t(t)}{2s(t)}|U(t,1)|^2+\frac{s_t(t)}{s(t)}u^*U(t,1)$$
$$\ge\frac{a_0(u^*-\alpha s(t))}{s(t)}(\frac{|U(t,1)|^2}{2}+u^*U(t,1))\ge 0.$$

By applying the result to Eq (4.34) we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_0^1|U(t)|^2dy+\frac{1}{2s^2(t)}\int_0^1|U_y(t)|^2dy\le\frac{a_0\alpha}{2}|U(t)|^2_H \text{ for a.e. } t\in[0,T].$$

This implies that $\tilde{u}(t)\le u^*$ on $[0,1]$ for $t\in[0,T]$. Thus, Lemma 4.3 is now proven. □

Combining the statements of Lemma 4.2 and Lemma 4.3, we can conclude that Theorem 2.2 holds.

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Conflict of interest

The authors declare there is no conflict of interest.

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