Networks and Heterogeneous Media
Vol. 17, No. 6, December 2022, pp. 827–845
doi:10.3934/nhm.2022028

SMOOTH TRANSONIC FLOWS AROUND CONES

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(Communicated by Seung-Yeal Ha)

Abstract. We consider a conical body facing a supersonic stream of air at a uniform velocity. When the opening angle of the obstacle cone is small, the conical shock wave is attached to the vertex. Under the assumption of self-similarity for irrotational motions, the Euler system is transformed into the nonlinear ODE system. We reformulate the problem in a non-dimensional form and analyze the corresponding ODE system. The initial data is given on the obstacle cone and the solution is integrated until the Rankine-Hugoniot condition is satisfied on the shock cone. By applying the fundamental theory of ODE systems and technical estimates, we construct supersonic solutions and also show that no matter how small the opening angle is, a smooth transonic solution always exists as long as the speed of the incoming flow is suitably chosen for this given angle.

1. Introduction.

The supersonic flow over cones is a fundamental problem in fluid dynamics [1, 8, 14]. We consider a conical body facing a supersonic stream of air at a uniform velocity and the angle of attack is zero. Assume that the obstacle is an infinite cone with its vertex located at the origin. A shock wave is formed either as a bow shock, also called a detached shock away from the cone, or a conical shock attached to the vertex. We are interested in which case the opening angle of the conical obstacle is not too large and thus a conical shock wave with the same vertex is situated on the obstacle in three dimensional space.

The essential property of the conical flow is that all flow properties are constant along rays from a given vertex. We take the x-axis as the axis of symmetry. y is the distance from the axis. u represents the component of the flow velocity in the direction of the axis. And v is the component in the perpendicular direction

2020 *Mathematics Subject Classification.* Primary: 35L65, 35L67; Secondary: 76N10.

Key words and phrases. Euler Equations, shock waves, conical flows, supersonic flows, hyperbolic conservation laws.

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away from the axis. When the flow is isentropic and steady, the compressible Euler system is as follows:

$$\begin{aligned} \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= \frac{-1}{y}(\rho v), \\ \frac{\partial}{\partial x}(\rho u^2 + P) + \frac{\partial}{\partial y}(\rho uv) &= \frac{-1}{y}(\rho uv), \\ \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + P) &= \frac{-1}{y}(\rho v^2). \end{aligned} \tag{1.1}$$

And the pressure P depends on the density ρ and is given by $P(\rho) = \rho^\gamma, \gamma > 1$. The above system can be regarded as one-dimensional hyperbolic conservation laws with a source term. Busemann [4] first gave a graphical method to construct self-similar solutions for the irrotational flow. The flow properties depend on the self-similar variable $\sigma = x/y$. Owing to self-similarity [3, 5], he simplifies system (1.1) to obtain the nonlinear ODE.

$$\begin{aligned} \frac{dv}{d\sigma} + \sigma\frac{du}{d\sigma} = 0, \\ (1 - \frac{u^2}{c^2})\frac{du}{d\sigma} - \frac{2uv}{c^2}\frac{dv}{d\sigma} - (1 - \frac{v^2}{c^2})\sigma\frac{dv}{d\sigma} + v = 0. \end{aligned} \tag{1.2}$$

The sound speed c is a given function of u and v through Bernoulli's law. The conical flow in [4] is described as a solution of the initial value problem of the system (1.2). Through the Rankine-Hugoniot conditions, the flow velocity past the shock cone is given as the initial condition. The solution is continued so that σ increases up to an end point at which the flow and the obstacle boundary have the same direction. A given shock wave is assumed first and the particular cone that supports the given shock is then calculated. We note that a more realistic case is discussed in [11] when the obstacle is a perturbation of the infinite cone. It is shown that the perturbed 3-dimensional flow of system (1.1) exists globally in space and tends to the self-similar flow downstream. The analysis is based on an approximation scheme using local self-similar solutions as building blocks.

Also, Maccoll and Taylor [13] discuss the problem in the spherical coordinate system. The numerical solution is constructed by a direct approach. Different from Busemann's method, the initial values are given at the surface of the obstacle cone. The complete solution is then worked out by numerical integration for three cones of semi-vertical angles $\theta_b = 10°, 20°$ and $30°$. The calculated pressure at the surface is compared with observations of pressure made in a high speed wind channel and they have a good agreement.

In this paper, we study steady Euler system in the spherical coordinates. Due to the self-similarity for conical flows, we obtain the following Taylor-Maccoll equation for V_r and V_θ

$$\begin{cases} \frac{\gamma-1}{2}(V_m^2 - V_r^2 - V_\theta^2)(2V_r + V_\theta\cot\theta + \frac{dV_\theta}{d\theta}) - V_\theta\left(V_r\frac{dV_r}{d\theta} + V_\theta\frac{dV_\theta}{d\theta}\right) = 0. \\ \frac{dV_r}{d\theta} = V_\theta. \end{cases} \tag{1.3}$$

Here, the radial and normal components of the flow velocity V are denoted by V_r and V_θ respectively. V_m is the theoretical maximum speed for Bernoulli's law. We reformulate the problem in a non-dimensional form and analyze the corresponding nonlinear ODE system. The initial data is given on the obstacle cone and the solution is integrated until the Rankine-Hugoniot condition is satisfied on the shock cone. According to previous numerical results [13], the flow is either supersonic or transonic. We are thus interested in the transition between different flow patterns. The flowfield is constructed as follows. First, the obstacle cone is given with semi-vertical angle θ_b of opening and θ_b is not too large. Also, the non-dimensional speed U_0 of the incoming supersonic flow is given in the range $(\mu, 1)$, $\mu = \sqrt{(\gamma-1)/(\gamma+1)}$. We solve the initial value problem of the ODE system with the initial data given on the cone boundary. The flow velocity is chosen parallel to the boundary and the corresponding Mach number is set to be 1. Second, the solution curve of the ODE system is continued until it intersects with the shock polar for U_0. And we obtain the angle θ_S where the intersection happens. Finally, we need to verify that the surface of the shock cone is exactly $\theta = \theta_S$ when U_0 is correctly selected.

By applying the fundamental theory of ODE system, we show that no matter how small the opening angle θ_b is, smooth transonic solutions always exist when the Mach number on the cone boundary varies around 1. We note that the system is non-autonomous. To construct the solutions, we focus on a fixed region to check the Lipschitz condition. We require $\gamma \in (1, \sqrt{5})$ so that the function A of the system has a positive lower bound. Hence, the Lipschitz constant can be calculated, which is shown in Section 3. Although $\cot\theta_b$ is an important factor of the Lipschitz constant for the ODE system, it is a fixed constant since the solution is restricted to the region for which the Mach number M is around 1. We also note that θ_S changes as U_0 varies. The relation between θ_S and the angle β of the shock cone is investigated for the existence of the fixed point $\theta_S = \beta$. For technical estimates, θ_b is required to be small and satisfy the following condition:

$$\text{Condition } \mathbf{A}: \ \arctan\left(\frac{\sqrt{1-\mu^2}}{\mu}\right) \geq \ \arctan(\mu) + \theta_b.$$

At present, our main concern is the existence of smooth transonic flows. The transition line between the subsonic and supersonic regions can be clearly indicated. Analogous to the problem [11], we might ask: Does there exist a transonic flow past the obstacle which coincides at infinity with the given undisturbed flow? This is a mixed type problem and should be investigated in the context of the partial differential equations. Steady transonic flow in multidimensional space has always been an important and challenging topic. We refer to the work of Chen and Fang [6] in which they also study the conical flow. They consider the transonic shock front behind which the flow is completely subsonic. They show that the self-similar transonic shock solution is conditionally stable with respect to the conical perturbation of the cone boundary. In two dimensional space, the problem of a flow past a straight wedge for a high subsonic free stream Mach number has been investigated by several authors. Cole [7] determined the singularity required to represent the flow near the point at infinity and showed how the condition at the stagnation point must be interpreted if one uses the transonic approximation which, in principle, does not permit stagnation point. His solution can be described as resulting from the assumption of a vertical sonic line. We also refer to [2] and [10] for more knowledge of transonic flows.

The paper is organized as follows. In Section 2, we describe the self-similar flow in the spherical coordinate system. The basic properties and equations are summarized in Section 2.1. Due to the self-similarity and the conservation laws, the Taylor-Maccoll equation is derived as a 2nd order ordinary differential equation. In Section 2.2, we introduce the non-dimensional physical quantities and rewrite the equation as a 1st order system for the self-similar solutions. In Section 3, we analyze the nonlinear ODE system. By the fundamental theory and some technical estimates, we construct the supersonic solutions. Further analysis using Gronwall's inequality reveals the phenomenon of the transition from the supersonic to subsonic region. Finally, the numerical simulations are presented in Section 4. In the Appendix, we derive the Rankine-Hugoniot condition and the equation of the shock polar in the spherical coordinates.

2. **Self-similar flows in the spherical coordinate system.**

2.1. **The basic properties and equations.** The essential property of the conical flow is that all flow properties are constant along rays from a given vertex. When the opening angle is not too large, an oblique shock wave S is attached at the vertex. The shape of the shock wave is also conical. This aspect of conical flow has been proven experimentally. We take the x_3-axis as the axis of symmetry and use the spherical coordinates:

$$x_1 = r\sin\theta\cos\phi,\ x_2 = r\sin\theta\sin\phi,\ x_3 = r\cos\theta,$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The corresponding components of the flow velocity $\vec{V}$ in the spherical coordinates are denoted by V_r, V_ϕ, and V_θ respectively.([1, Ch10.3]) The Euler equations for the steady flow are as follows.

Conservation of Mass:

$$\frac{\partial\rho}{\partial r} = \frac{-1}{V_r}\left[\rho\frac{\partial V_r}{\partial r} + \frac{2\rho V_r}{r} + \frac{1}{r\sin\theta}\frac{\partial(\rho V_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho V_\phi)}{\partial\phi}\right] \tag{2.1}$$

Momentum in r direction:

$$\frac{\partial V_r}{\partial r} = \frac{-1}{V_r}\left[\frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{V_\theta^2 + V_\phi^2}{r} + \frac{1}{\rho}\frac{\partial P}{\partial r}\right] \tag{2.2}$$

Momentum in θ direction:

$$\frac{\partial V_\theta}{\partial r} = \frac{-1}{V_r}\left[\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial V_\theta}{\partial\phi} + \frac{V_rV_\theta}{r} - \frac{V_\phi^2\cot\theta}{r} + \frac{1}{r\rho}\frac{\partial P}{\partial\theta}\right] \tag{2.3}$$

Momentum in ϕ direction:

$$\frac{\partial V_\phi}{\partial r} = \frac{-1}{V_r}\left[\frac{V_\theta}{r}\frac{\partial V_\phi}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} + \frac{V_rV_\phi}{r} + \frac{V_\theta V_\phi\cot\theta}{r} + \frac{1}{r\rho\sin\theta}\frac{\partial P}{\partial\phi}\right] \tag{2.4}$$

Here we consider the polytropic gases: $P(\rho) = \rho^\gamma$, $\gamma > 1$.

The PDE system (2.1)-(2.4) is complicated and thus we focus on the typical model with the self-similarity property [3, 5]. We know that for the conical flow, $\frac{\partial}{\partial\phi} \equiv 0$ due to the axisymmetry. That is, $\frac{\partial}{\partial\phi}\rho = 0$, $\frac{\partial}{\partial\phi}V_r = 0$, $\frac{\partial}{\partial\phi}V_\phi = 0$, and $\frac{\partial}{\partial\phi}V_\theta = 0$. And the flow properties are constant along a ray from the vertex, which implies that $\frac{\partial}{\partial r}\rho = 0$ and $\frac{\partial}{\partial r}V_r = 0$. Hence, the conservation law of mass is as

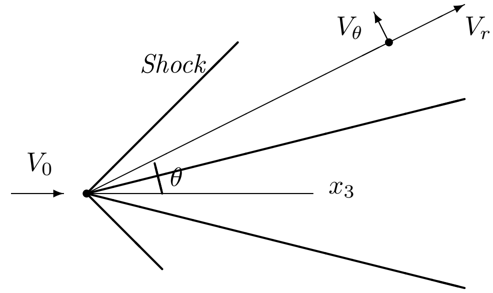

FIGURE 1. Spherical coordinate system

follows:

$$2\rho V_r + \rho V_\theta \cot\theta + \rho \frac{\partial V_\theta}{\partial \theta} + V_\theta \frac{\partial \rho}{\partial \theta} = 0. \tag{2.5}$$

Since the conical flow is irrotational: curl $\vec{V} = \nabla \times \vec{V} = 0$, we obtain that in spherical coordinates,

$$V_\theta = \frac{\partial V_r}{\partial \theta}. \tag{2.6}$$

From the conservation laws, Bernoulli's law for steady flow holds across a shock front:

$$\frac{c^2}{\gamma - 1} + \frac{V^2}{2} = \frac{V_m^2}{2}, \tag{2.7}$$

where $c = \sqrt{P'(\rho)}$ is the sound speed and V_m is the theoretical maximum velocity. Note that V_m can be served as the characteristic velocity. Here we use the notation $V^2 = V_r^2 + V_\theta^2$ since $V_\phi = 0$. We note that the flow is everywhere isentropic behind the shock front. Thus, V_m is a constant for the flow. ([8, Ch 6.B], or [1, 13]). Since V_r, V_θ and ρ depend only on one variable θ (see Figure 1); thus, the PDE system can be transformed into the ODE system.

The Bernoulli principle [1, Ch8.2] for irrotational flows can be expressed as follows.

$$dP = -\frac{1}{2}\rho \, d(V^2).$$

Hence,

$$dP = -\rho(V_r dV_r + V_\theta dV_\theta). \tag{2.8}$$

Since

$$\frac{dP}{d\rho} = c^2,$$

by (2.7) and (2.8), we obtain

$$\frac{d\rho}{\rho} = \frac{-2}{\gamma - 1}\left(\frac{V_r dV_r + V_\theta dV_\theta}{V_m^2 - V_r^2 - V_\theta^2}\right). \tag{2.9}$$

By (2.5), (2.6) and (2.9), we obtain the Taylor-Maccoll equation:

$$\tfrac{\gamma-1}{2}\left[V_m^2 - V_r^2 - \left(\tfrac{dV_r}{d\theta}\right)^2\right]\left[2V_r + \tfrac{dV_r}{d\theta}\cot\theta + \tfrac{d^2V_r}{d\theta^2}\right] - \tfrac{dV_r}{d\theta}\left[V_r\tfrac{dV_r}{d\theta} + \tfrac{dV_r}{d\theta}\tfrac{d^2V_r}{d\theta^2}\right] = 0. \tag{2.10}$$

We note that ρ can be represented by V_r and V_θ through (2.7).

2.2. **The ODE system of the self-similar flow.** Because of (2.9), (2.5) can be written as a system of first order ODEs. Together with (2.6), we obtain the following system:

$$\begin{cases} \frac{\gamma-1}{2}(V_m^2 - V_r^2 - V_\theta^2)(2V_r + V_\theta \cot\theta + \frac{dV_\theta}{d\theta}) - V_\theta\left(V_r\frac{dV_r}{d\theta} + V_\theta\frac{dV_\theta}{d\theta}\right) = 0. \\ \frac{dV_r}{d\theta} = V_\theta. \end{cases} \tag{2.11}$$

We introduce the non-dimensional physical quantities as follows.

$$U \equiv \frac{V}{V_m},\ U_r \equiv \frac{V_r}{V_m},\ U_\theta \equiv \frac{V_\theta}{V_m}.$$

After direct calculations, we obtain

$$\begin{cases} -U_\theta U_r\frac{dU_r}{d\theta} + A\frac{dU_\theta}{d\theta} = B, \\ \frac{dU_r}{d\theta} = U_\theta, \end{cases} \tag{2.12}$$

where

$$\begin{cases} A &= & -U_\theta^2 + \left(\frac{\gamma-1}{2}\right)(1-U_r^2-U_\theta^2) \\ B &= & \left(\frac{1-\gamma}{2}\right)(1-U_r^2-U_\theta^2)(2U_r + U_\theta\cot\theta). \end{cases}$$

Hence, we have the ODE system:

$$\begin{cases} \frac{dU_\theta}{d\theta} = \frac{U_\theta^2 U_r + B}{A}, \\ \frac{dU_r}{d\theta} = U_\theta. \end{cases} \tag{2.13}$$

By direct calculations, we obtain the following lemma.

Lemma 2.1. *Let* $\gamma > 1$. $A = -U_\theta^2 + (\frac{\gamma-1}{2})(1-U_r^2-U_\theta^2)$.

1. $A \geq 0$ *if and only if*

$$\left(\frac{\gamma+1}{\gamma-1}\right)U_\theta^2 + U_r^2 \leq 1.$$

2. *Let* $U^2 = U_\theta^2 + U_r^2$. *Suppose that* (U_θ, U_r) *is a solution of the system* (2.13). *Then*

$$\frac{d}{d\theta}\left(U^2\right) = \frac{1}{A}\left[(1-\gamma)(1-U^2)U_\theta(U_r + \cot\theta\cdot U_\theta)\right].$$

Proof. (1) can be proved by direct calculations.
(2): Substituting (2.13) gives

$$\begin{aligned} \frac{d}{d\theta}\left(U^2\right) &= \frac{dU_\theta}{d\theta}\cdot 2U_\theta + \frac{dU_r}{d\theta}\cdot 2U_r \\ &= \frac{1}{A}\left[2U_\theta(\frac{1-\gamma}{2})(1-U^2)(2U_r + \cot\theta\cdot U_\theta) + 2U_rU_\theta(\frac{\gamma-1}{2})(1-U^2)\right] \\ &= \frac{1}{A}\left[(1-\gamma)(1-U^2)U_\theta(U_r + \cot\theta\cdot U_\theta)\right]. \end{aligned}$$

□

Remark 1. By (2.12), we have that

$$\begin{aligned} A\cdot\frac{dU_\theta}{d\theta} &= U_\theta^2U_r+B,\\ &= U_r(-A)+(\frac{1-\gamma}{2})(1-U_r^2-U_\theta^2)(U_r+U_\theta\cot\theta). \end{aligned}$$

If $A=0$ at some point $(U_\theta(\tilde\theta),U_r(\tilde\theta))$, then $U_r+U_\theta\cot\theta=0$ at $(U_\theta(\tilde\theta),U_r(\tilde\theta))$.

Remark 2. Since V_m is determined by (2.7)

$$\frac{1}{2}V^2+\frac{c^2}{\gamma-1}=\frac{1}{2}V_m^2,$$

it follows that

$$\frac{U^2}{2}+\frac{U^2}{(\gamma-1)(V/c)^2}=\frac{1}{2}.$$

Here, $U^2=U_r^2+U_\theta^2$. The relationship between the Mach number $M=V/c$ and the non-dimensional speed $|U|$ is as follows

$$|U|=\left[1+\frac{2}{(\gamma-1)M^2}\right]^{-1/2}. \tag{2.14}$$

3. **The existence and stability analysis.**

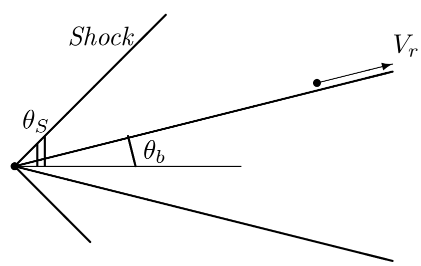

FIGURE 2. The shock cone

3.1. **The supersonic solution.** In this section, we present the existence and stability of solutions of the system (2.13). Let the solution be denoted by

$$\mathbf{U}=(U_\theta,U_r).$$

We rewrite the system as follows:

$$\begin{cases} \dfrac{dU_\theta}{d\theta}=F(U_\theta,U_r,\theta)=\dfrac{U_\theta^2U_r+B}{A},\\ \dfrac{dU_r}{d\theta}=G(U_\theta,U_r,\theta)=U_\theta, \end{cases} \tag{3.1}$$

where $A=-U_\theta^2+(\frac{\gamma-1}{2})(1-U_r^2-U_\theta^2)$ and $B=(\frac{1-\gamma}{2})(1-U_r^2-U_\theta^2)(2U_r+U_\theta\cot\theta)$. Since the flow is parallel to the obstacle cone on the cone boundary $\theta=\theta_b$ (Fig 2), we have $U_\theta(\theta_b)=0$. Hence, the initial condition of system (3.1) is always given on the boundary of the obstacle cone as follows:

$$\mathbf{U}(\theta_b)=(0,u_b). \tag{3.2}$$

Here $0 < u_b < 1$ and $A(\theta_b) > 0$ according to Lemma 2.1. The ODE system is non-autonomous. We need to prove that the functions F and G are Lipschitz in $\mathbf{U} = (U_\theta, U_r)$. Thus, by the fundamental existence and uniqueness theorem of differential equations [9], the solution exists and is continuous with respect to the initial conditions. Furthermore, the boundary condition of the solution is given by the shock conditions on the shock cone, which is derived in the Appendix. That is, the solution of the ODE system is continued until it intersects with the shock polar.

The procedure of the construction is as follows.

Step 1. (Initial Condition)

The initial condition is chosen as

$$\mathbf{U}(\theta_b) = \left(0, \sqrt{\frac{\gamma-1}{\gamma+1}}\right).$$

That is, $M = 1$ at $\theta = \theta_b$. We first solve the initial value problem of the ODE system (3.1) and such solution is denoted by $\mathbf{U}_1(\theta)$.

Step 2. (Boundary Condition)

The non-dimensional speed U_0 of the incoming supersonic flow is given in the range $(\mu, 1)$. The solution curve $\mathbf{U}_1(\theta)$ of the ODE system is continued until it intersects with the corresponding shock polar for U_0. And we obtain the angle θ_S and the 1-state $(U_\theta(\theta_S), U_r(\theta_S))$ where the intersection happens.

Step 3. (The Shock Condition)

The angle the shock makes with the upstream flow is denoted by β and it is calculated by the formula (in the Appendix)

$$\tan\beta = \frac{\sqrt{U_0^2 - U_r^2}}{U_r}$$

Here the 1-state (U_θ, U_r) is obtained in Step 2. Finally, we need to verify whether or not

$$\theta_S = \beta$$

for the given U_0. The shock cone is shown in Figure 2.

We first focus on the region Ω in $U_\theta - U_r$ plane, where

$$\Omega = \left\{ (U_\theta, U_r) : U_r \in \left(0, \sqrt{\frac{\gamma-1}{\gamma+1}}\,\right],\ U_\theta \in \left[-\sqrt{\frac{\gamma-1}{\gamma+3}}, 0\right] \right\}$$

According to (A.8) in the Appendix, the non-dimensional speed U_0 of the incoming supersonic flow is chosen to satisfy that

$$1 > U_0 > \sqrt{\frac{\gamma-1}{\gamma+1}}. \tag{3.3}$$

The intersection point of the shock polar with the U_θ axis is denoted by $(p_0, 0)$. By the Remark A.1 and direct calculations, $0 < -p_0 < \sqrt{\frac{\gamma-1}{\gamma+3}}$ for $U_0 > \sqrt{\frac{\gamma}{\gamma+1}}$. Hence, the corresponding shock polar which we discuss is inside the region Ω. We also note that the curve of $M = 1$ (or $U_r^2 + U_\theta^2 = \frac{\gamma-1}{\gamma+1}$) intersects with the shock polar in Ω by straightforward calculations. The local solutions are in the region Ω when the incoming speed satisfies $U_0 > \sqrt{\frac{\gamma}{\gamma+1}}$.

Lemma 3.1. *Let $\gamma \in (1, \sqrt{5})$. The function A defined in Lemma 2.1 has a positive lower bound $a_0(\gamma)$ for$U_r \in \left(0, \sqrt{(\gamma-1)/(\gamma+1)}\,\right]$ and $|U_\theta| \in \left[0, \sqrt{(\gamma-1)/(\gamma+3)}\right]$.*

Proof. By direct calculations, we obtain

$$A = \frac{1}{2}(\gamma - 1)\left[1 - U_r^2 - \frac{\gamma+1}{\gamma-1}U_\theta^2\right] \geq \frac{(\gamma-1)(5-\gamma^2)}{2(\gamma+1)(\gamma+3)} = a_0(\gamma)$$

for $U_r \in \left(0, \sqrt{(\gamma-1)/(\gamma+1)}\,\right]$ and $|U_\theta| \in \left[0, \sqrt{(\gamma-1)/(\gamma+3)}\,\right]$. And $a_0(\gamma) > 0$ for $\gamma \in (1, \sqrt{5})$. □

Lemma 3.2. *For any given opening angle of the obstacle cone $\theta_b \in (0, \pi/2)$ and $d \equiv \cot\theta_b > 0$, there exist some constants $C_1 = C_1(\gamma)$ and $C_2 = C_2(\gamma)$ such that*

$$\left|\frac{\partial F}{\partial U_\theta}\right| + \left|\frac{\partial F}{\partial U_r}\right| + \left|\frac{\partial G}{\partial U_\theta}\right| + \left|\frac{\partial G}{\partial U_r}\right| \leq C_1 d + C_2$$

for $U_r \in \left(0, \sqrt{(\gamma-1)/(\gamma+1)}\,\right]$ and $|U_\theta| \in \left[0, \sqrt{(\gamma-1)/(\gamma+3)}\,\right]$.

Proof. We know that $\cot\theta$ is decreasing with respect to θ and $0 < \cot\theta \leq d$ for $\theta \in [\theta_b, \pi/2)$. By calculating the partial derivatives of B, where

$$B = \frac{1}{2}(1-\gamma)(1 - U_r^2 - U_\theta^2)(2U_r + U_\theta \cot\theta),$$

we have

$$|B| + \left|\frac{\partial B}{\partial U_r}\right| + \left|\frac{\partial B}{\partial U_\theta}\right| \leq c_1 d + c_2.$$

Here c_i are some constants depending on γ.

Furthermore, $G = U_\theta$ and $F = \frac{1}{A}(U_\theta^2 U_r + B)$.

$$\begin{aligned} \frac{\partial F}{\partial U_\theta} &= \frac{1}{A^2}\left[A\left(2U_rU_\theta + \frac{\partial B}{\partial U_\theta}\right) - \frac{\partial A}{\partial U_\theta}\left(U_\theta^2 U_r + B\right)\right] \\ \frac{\partial F}{\partial U_r} &= \frac{1}{A^2}\left[A\left(U_\theta^2 + \frac{\partial B}{\partial U_r}\right) - \frac{\partial A}{\partial U_r}\left(U_\theta^2 U_r + B\right)\right] \end{aligned}$$

It is easy to check by calculations and Lemma 3.1 that

$$a_0(\gamma) < A + \left|\frac{\partial A}{\partial U_r}\right| + \left|\frac{\partial A}{\partial U_\theta}\right| \leq \hat{c},$$

where $\hat{c}$ is a constant depending on γ. By combining the above estimates, we obtain the lemma. □

By Lemma 3.2, the functions F and G are Lipschitz in $\mathbf{U}$. By the fundamental existence and uniqueness theorem of differential equations [9], the solution $\mathbf{U}_1(\theta)$ exists and is continuous with respect to the initial condition. We write

$$\mathbf{U}_1(\theta) = (U_\theta(\theta), U_r(\theta))$$

and the corresponding A of $\mathbf{U}_1$ is

$$A(\theta) = -U_\theta^2(\theta) + (\frac{\gamma-1}{2})(1 - U_r^2(\theta) - U_\theta^2(\theta)).$$

We note that there exists $\bar{\theta} > \theta_b$ such that $A(\theta) > 0$ for all $\theta \in [\theta_b, \bar{\theta})$ and $A(\bar{\theta}) = 0$.

Lemma 3.3. *Let $g(\theta) = U_r(\theta) + \cot\theta \cdot U_\theta(\theta)$. Then $g(\theta) > 0$ for all $\theta \in [\theta_b, \bar{\theta})$.*

Proof. Due to the system (3.1) and direct calculations, we have

$$\begin{aligned} g'(\theta) &= U_r' + (-\csc^2\theta)U_\theta + \cot\theta \cdot U_\theta' \\ &= -\cot^2\theta \cdot U_\theta - \cot\theta \cdot U_r - (\cot\theta \cdot C(\theta)g(\theta)) \\ &= -\cot\theta\,(1 + C(\theta))\,g(\theta), \end{aligned}$$

where

$$C(\theta) = \left(\frac{\gamma-1}{2}\right) \cdot \frac{(1 - U_r^2 - U_\theta^2)}{A}$$

We note that $C(\theta) > 0$. And thus

$$g(\theta) = \left(e^{\int_{\theta_b}^{\theta} -\cot\tau(1+C(\tau))d\tau}\right) g(\theta_b).$$

Since $g(\theta_b) = \mu > 0$, we obtain that $g(\theta) > 0$ for all $\theta \in \left[\theta_b, \bar{\theta}\right)$.

□

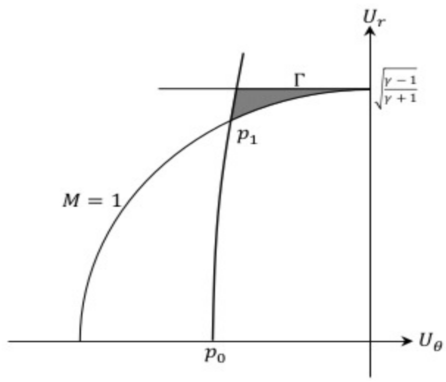

FIGURE 3. The Γ region

Theorem 3.4. *Let the non-dimensional speed U_0 of the incoming flow be given in $(\sqrt{\frac{\gamma}{\gamma+1}}, 1)$. Then, the solution $\mathbf{U}_1(\theta)$ intersects with the corresponding shock polar at some point $\mathbf{U}_1(\theta_S)$ and $\mathbf{U}_1(\theta)$ is supersonic for $\theta \in (\theta_b, \theta_S]$.*

Proof. We first solve system (3.1) with (3.2) given by $u_b = \sqrt{\frac{\gamma-1}{\gamma+1}}$; that is, $M = 1$ at $\theta = \theta_b$. According to system (3.1), $\dfrac{dU_r}{d\theta}(\theta_b) = 0$ and $\dfrac{dU_\theta}{d\theta}(\theta_b) = -2u_b$. Such solution is denoted by $\mathbf{U}_1(\theta)$. By Lemma 2.1 and Lemma 3.3,

$$\frac{d}{d\theta}\left(U^2\right) = \frac{1}{A}\left[(1-\gamma)(1-U^2)U_\theta(U_r + \cot\theta \cdot U_\theta)\right] > 0.$$

$|\mathbf{U}_1(\theta)|$ is increasing in θ. Thus M is increasing with respect to θ by (A.8)

$$|U| = \left[1 + \frac{2}{(\gamma-1)M^2}\right]^{-1/2}.$$

According to the ODE system, $U_\theta < 0$ for $\theta > \theta_b$ and U_r is decreasing in θ. Since M is increasing in θ, $\mathbf{U}_1(\theta)$ is a supersonic solution for $\theta > \theta_b$. Hence, the curve of

$\mathbf{U}_1(\theta)$ lies above the curve $M = 1$ (or $U_r^2 + U_\theta^2 = \frac{\gamma-1}{\gamma+1}$) and below the horizontal line $U_r = \sqrt{\frac{\gamma-1}{\gamma+1}}$. Let Γ denote the region enclosed by the shock polar for the given U_0, the curve $M = 1$ and the horizontal line $U_r = \sqrt{\frac{\gamma-1}{\gamma+1}}$ as shown in Figure 3. $\mathbf{U}_1(\theta)$ is inside the region Γ. By Remark A.1, the shock polar is given by

$$-U_\theta = \frac{\mu^2(1-U_r^2)}{\sqrt{U_0^2 - U_r^2}}$$

and U_θ is increasing with respect to U_r. By direct calculations, the shock polar intersects with the curve $M = 1$ at p_1 and also with the horizontal line $U_r = \sqrt{\frac{\gamma-1}{\gamma+1}}$. Thus, $\mathbf{U}_1(\theta)$, which is in between the curves $M = 1$ and $U_r = \sqrt{\frac{\gamma-1}{\gamma+1}}$, intersects with the shock polar in the region Γ. □

3.2. **Technical estimates.** In Section 3.1, we just show that the supersonic solution exists in the Γ region when $U_0 \in (\sqrt{\frac{\gamma}{\gamma-1}}, 1)$. Actually, $\mathbf{U_1}(\theta)$ can be extended until the fixed point is reached for the shock condition to be satisfied. In this subsection, we prove the existence of the fixed point $\theta_S = \beta$ under the following condition:

$$\text{Condition } \mathbf{A}: \ \arctan\left(\frac{\sqrt{1-\mu^2}}{\mu}\right) \geq \ \arctan(\mu) + \theta_b. \tag{3.4}$$

Let ϕ denote the angle between $\overline{O\mathbf{U_1}(\theta)}$ and the positive U_r-axis as shown in Figure 4. That is,

$$\phi(\theta) = \arctan\left(\frac{-U_\theta(\theta)}{U_r(\theta)}\right), \ \theta \in [\theta_b, \bar{\theta}).$$

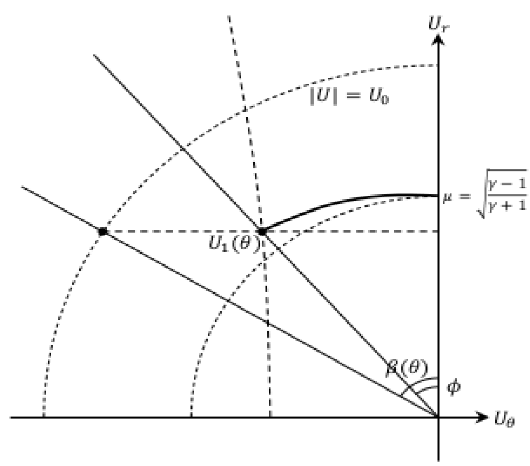

FIGURE 4. ϕ and β

Lemma 3.5. $\phi(\theta_b) = 0$, *and* $\phi'(\theta) \geq 1$ *for* $\theta \in [\theta_b, \bar{\theta})$.

Proof. Substituting $\mathbf{U_1}(\theta_b) = (U_\theta(\theta_b), U_r(\theta_b)) = (0, u_b)$ on $\phi(\theta)$ yields

$$\phi(\theta_b) = \arctan\left(\frac{-U_\theta(\theta_b)}{U_r(\theta_b)}\right) = \arctan\left(\frac{0}{\mu}\right) = 0.$$

By direct calculations, we have

$$\begin{aligned}\phi'(\theta) &= \left(\frac{1}{1+\left(\frac{U_\theta}{U_r}\right)^2}\right)\cdot\left(\frac{-U_\theta' U_r-(-U_\theta)U_r'}{U_r^2}\right)\\ &= \frac{U_r^2+U_\theta^2+U_r C(\theta)g(\theta)}{U_r^2+U_\theta^2},\end{aligned}$$

where

$$C(\theta)=\left(\frac{\gamma-1}{2}\right)\cdot\frac{(1-U_r^2-U_\theta^2)}{A}\ >0$$

and by Lemma 3.3

$$g(\theta)=U_r(\theta)+\cot\theta\cdot U_\theta(\theta)\ >0.$$

Since $U_r>0$, $C(\theta)>0$ and $g(\theta)>0$, we obtain $\phi'(\theta)\geq 1$. □

To investigate the relation between θ_S and the shock angle β, we need to calculate the following function along the solution $\mathbf{U}_1(\theta)$:

$$\tan\beta(\theta)=\frac{\sqrt{U_0^2-U_r^2(\theta)}}{U_r(\theta)},$$

where U_0 is given by the formula

$$U_\theta^2(\theta)=\frac{\mu^4(1-U_r^2(\theta))^2}{U_0^2-U_r^2(\theta)}. \tag{3.5}$$

We then compare these functions $y=\beta(\theta)$, $y=\theta$ and $y=\phi(\theta)$ in the following lemmas to show the existence of the fixed point (see Figure 5).

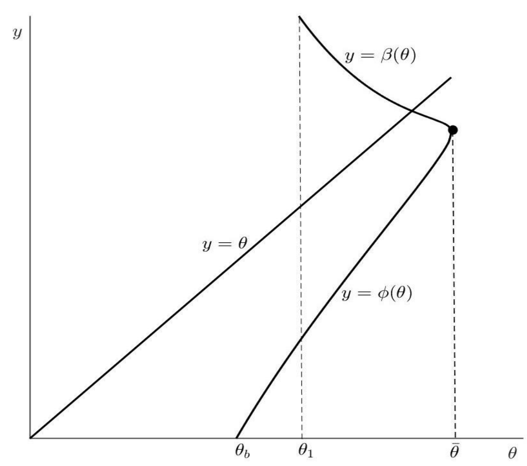

FIGURE 5. The fixed point

Lemma 3.6. *$\beta(\theta) > \phi(\theta)$ for all $\theta \in [\theta_b, \bar{\theta})$.*

Proof. By (3.5), we have

$$\tan\beta(\theta) = \frac{\sqrt{U_0^2 - U_r^2}}{U_r} = \frac{1}{U_r|U_\theta|}\frac{\gamma-1}{\gamma+1}(1-U_r^2).$$

Since $A > 0$,

$$U_\theta^2 < \left(\frac{\gamma-1}{\gamma+1}\right)(1-U_r^2).$$

Hence,

$$\tan\beta(\theta) = \frac{1}{U_r|U_\theta|}\frac{\gamma-1}{\gamma+1}(1-U_r^2) > \frac{U_\theta^2}{U_r|U_\theta|} = \frac{|U_\theta|}{U_r} = \tan\phi(\theta).$$

We thus obtain that $\beta(\theta) > \phi(\theta)$ for all $\theta \in [\theta_b, \bar{\theta})$. □

Remark 3. We also obtain that $\beta(\bar{\theta}) = \phi(\bar{\theta})$ by following the proof of the above lemma when $A = 0$.

Lemma 3.7. *$\theta > \phi(\theta)$ for all $\theta \in [\theta_b, \bar{\theta})$.*

Proof. It is clear that $\theta_b > \phi(\theta_b)$. If there exists $\hat{\theta} \in (\theta_b, \bar{\theta})$ such that $\phi(\hat{\theta}) = \hat{\theta}$, i.e. $\tan(\hat{\theta}) = \frac{-U_\theta(\hat{\theta})}{U_r(\hat{\theta})}$, then

$$g(\hat{\theta}) = U_r(\hat{\theta}) + \cot(\hat{\theta})U_\theta(\hat{\theta}) = 0,$$

which is a contradiction to Lemma 3.3. □

When $U_0 = 1$, the formula of the corresponding shock polar is given by (A.5):

$$U_\theta^2 = \mu^4(1-U_r^2). \tag{3.6}$$

Since $\phi'(\theta) \geq 1$, we know that $\mathbf{U}_1(\theta)$ intersects with the shock polar (3.6) and there exists uniquely $\theta_1 \in (\theta_b, \bar{\theta})$ such that

$$\frac{U_\theta^2(\theta_1)}{\mu^4} + U_r^2(\theta_1) = 1.$$

Lemma 3.8. *For the solution $\mathbf{U}_1(\theta)$,*

$$\frac{1}{\mu} > \tan\beta(\theta_1) > \frac{\sqrt{1-\mu^2}}{\mu}$$

Proof. We focus on the Γ region and set $U_0 = 1$. We know that $\mathbf{U}_1(\theta)$ intersects with the shock polar at $\theta = \theta_1$. The shock polar intersects with the horizontal line $U_r = \mu$ at the point $(-\mu^2\sqrt{1-\mu^2}, \mu)$ and it intersects with the curve $M = 1$ at the point $p_1 = (-\mu^2/\sqrt{1+\mu^2},\ \mu/\sqrt{1+\mu^2})$ by direct calculations. Thus,

$$\mu > U_r(\theta_1) > \frac{\mu}{\sqrt{1+\mu^2}}.$$

Consider the function

$$f(x) = \frac{\sqrt{1-x^2}}{x},\ x \in (0,1).$$

It is easy to check that $f'(x) < 0$. Hence,

$$f(\mu) < f(U_r(\theta_1)) < f\left(\frac{\mu}{\sqrt{1+\mu^2}}\right).$$

That is,

$$\frac{\sqrt{1-\mu^2}}{\mu} < \tan\beta(\theta_1) < \frac{1}{\mu}.$$

□

Lemma 3.9. $\beta(\theta_1) > \theta_1$ *when* (3.4) *is true.*

Proof. We focus on the Γ region and follow the proof of Lemma 3.7. By considering the point p_1 and $\mathbf{U}_1(\theta_1)$, we have that

$$\phi(\theta_1) < \arctan\left(\frac{\frac{\mu^2}{\sqrt{1+\mu^2}}}{\frac{\mu}{\sqrt{1+\mu^2}}}\right) = \arctan(\mu). \tag{3.7}$$

By mean value theorem, there exists $\theta_c \in (\theta_b, \theta_1)$ so that

$$\phi(\theta_1) - \phi(\theta_b) = \phi'(\theta_c)(\theta_1 - \theta_b).$$

By Lemma 3.4 and (3.7),

$$\arctan(\mu) > \phi(\theta_1) \geq \theta_1 - \theta_b.$$

Since we have that $\arctan\left(\frac{\sqrt{1-\mu^2}}{\mu}\right) \geq \arctan(\mu) + \theta_b$, by Lemma 3.7,

$$\beta(\theta_1) > \arctan\left(\frac{\sqrt{1-\mu^2}}{\mu}\right) > \theta_1.$$

□

Theorem 3.10. *For the solution* $\mathbf{U}_1(\theta)$*, there exists* $\theta_S \in [\theta_b, \bar{\theta}]$ *such that* $\theta_S = \beta(\theta_S)$ *when* (3.4) *is true.*

Proof. $\beta(\theta)$ is a continuous function defined along the solution $\mathbf{U}_1(\theta)$. By Lemma 3.8, $\beta(\theta_1) > \theta_1$. By Lemma 3.6 and Remark 3.1, $\beta(\bar{\theta}) \leq \bar{\theta}$. Thus, there exists $\theta_S \in [\theta_b, \bar{\theta}]$ such that $\theta_S = \beta(\theta_S)$. □

Remark 4. We note that $\beta(\theta)$ may not be a decreasing function when θ_b is very small according to our numerical results.

3.3. **The transonic solution.** We now construct the transonic solution $\mathbf{U}_*(\theta)$ as follows. Let $\epsilon_* > 0$ be sufficiently small and the initial data of $\mathbf{U}_*(\theta)$ given by

$$\mathbf{U}_*(\theta_b) = \left(0, \sqrt{\frac{\gamma-1}{\gamma+1}} - \epsilon_*\right).$$

Thus, $M < 1$ at $\theta = \theta_b$. By Lemma 3.1 and Lemma 3.2, $K = C_1 d + C_2$ is a Lipschitz constant in $\mathbf{U}$ for F and G. K depends only on γ and d. By the fundamental theory [9], the solution $\mathbf{U}_*(\theta)$ exists and is continuous with respect to the initial conditions. According to the system and following the proofs in Section 3.1 and 3.2, $U_r(\theta)$ is decreasing with respect to θ.

The solution $\mathbf{U}_1(\theta)$ can be continued until it intersects with the shock polar. According to Theorem 3.2, $\mathbf{U}_1(\theta)$ intersect with the shock polar at $\theta = \theta_S^1$, where θ_S^1 is the angle of the shock cone. We can choose some $\theta_* \in (\theta_b, \theta_S^1)$ such that $|\mathbf{U}_1(\theta_*)| > \mu$ and $A(\mathbf{U}_*(\theta_*)) > a_0(\gamma) > 0$. Gronwall's inequality gives that

$$|\mathbf{U}_1(\theta_*) - \mathbf{U}_*(\theta_*)| \leq e^{(K|\theta_* - \theta_b|)} \left|\mathbf{U}_1(\theta_b) - \mathbf{U}_*(\theta_b)\right|.$$

Since $\mathbf{U}_1$ is supersonic and $|\theta_S^1 - \theta_b| < \pi/2$, we then choose ϵ_* small enough so that $|\mathbf{U}_*(\theta_*)| > \mu$ and $\mathbf{U}_*$ is supersonic in the neighborhood of θ_*. Hence, $\mathbf{U}_*(\theta)$ is a smooth transonic solution of the initial value problem for the system (3.1) and (3.2). We state the main results in this section as follows.

Theorem 3.11. *Let $\gamma \in (1, \sqrt{5})$. For any small opening semi-vertical angle θ_b of the obstacle cone satisfying (3.4), the solution of system (3.1) exists with the initial data given in the neighborhood of $u_b = \sqrt{\frac{\gamma-1}{\gamma+1}}$. The solution can be continued until it intersects with the shock polar.*

Most importantly,

1. *The solution $\mathbf{U}_1(\theta)$ exists with the initial data $\mathbf{U}_1(\theta_b) = \left(0, \sqrt{\frac{\gamma-1}{\gamma+1}}\right)$. And $\mathbf{U}_1(\theta)$ is supersonic for $\theta \in (\theta_b, \theta_S^1]$. Here, θ_S^1 is the angle of the shock cone.*
2. *We can choose $\epsilon_* > 0$ sufficiently small such that the solution $\mathbf{U}_*(\theta)$ is transonic with the initial data given by $\mathbf{U}_*(\theta_b) = \left(0, \sqrt{\frac{\gamma-1}{\gamma+1}} - \epsilon_*\right)$.*

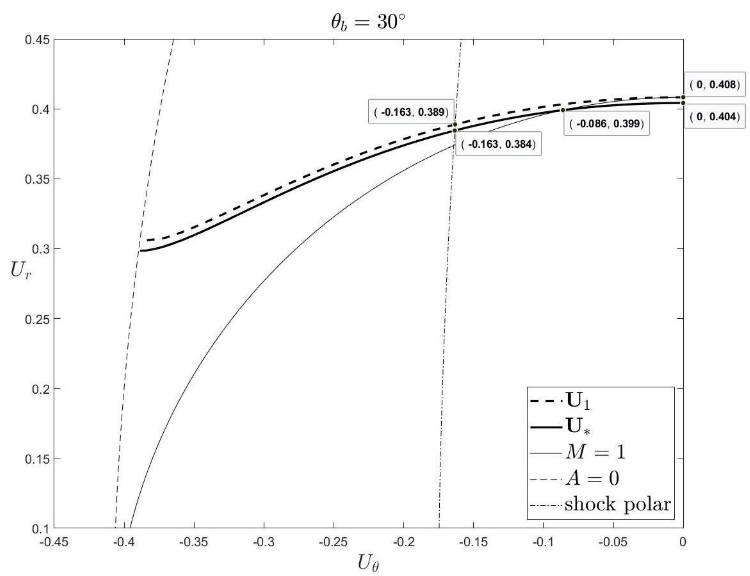

FIGURE 6. The numerical solution for $\gamma = 1.4$

4. **Numerical results.** In summary, we construct smooth transonic solutions when the opening angle of the conical obstacle is small and a conical shock wave is situated on the obstacle with the same vertex. Under the assumption of self-similarity for irrotational fluid, the Euler system (1.1) of steady flows is transformed into the nonlinear ODE system (3.1). We analyze the ODE system in Section 3 and present the numerical solutions in this section.

We choose $U_0 = 0.95$ and $\gamma = 1.4$ in Figure 6. Since the numerical results are similar for different angles $\theta_b = 5°, 10°$ and $30°$, we only show the result of $\theta_b = 30°$. When $\gamma = 1.4$, Condition **A** is true if $\theta_b < 43°$. And the numerical result for $\beta - \theta_S - U_0$ relation is shown in Figure 7. We note that the fixed point for $\theta_S = \beta$ is found when U_0 is around 0.6.

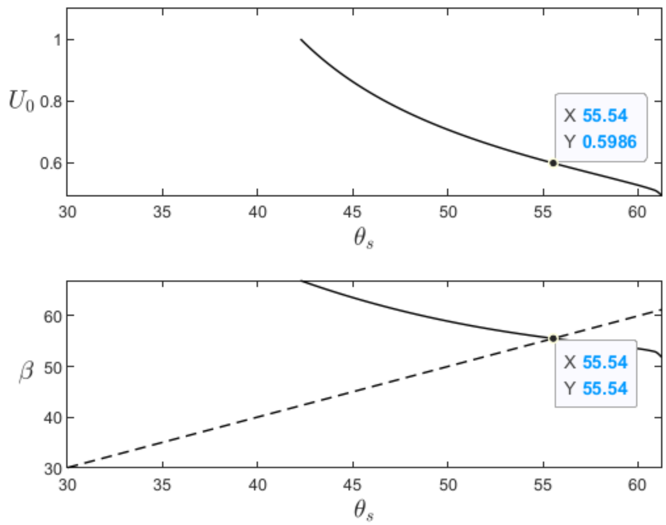

FIGURE 7. The numerical result for $\beta - \theta_S - U_0$ relation

Appendix A. Shock polars. We briefly review the quantitative analysis of shock polars for polytropic gases. We refer the readers to Courant and Friedrichs [8, Ch4.C: Section 121] and the references therein for more details.

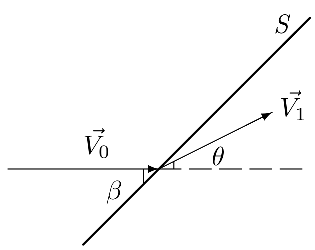

FIGURE 8. List of the symbols

Consider a shock S in the (x, y)-plane with the upstream state of velocity $\vec{V}_0 = (V_0, 0)$ and the downstream state of velocity $\vec{V}_1$, which makes angle θ with the upstream flow. The angle the shock makes with the upstream flow is denoted by β as shown in Figure 8. For the given velocity $\vec{V}_0$ and density ρ_0, there is a one-parametric family of possible states, with velocity $\vec{V}_1$ and density ρ_1, which can be reached through a shock. These possible states are given by the Rankine-Hugoniot conditions of the conservation of mass, momentum and energy. On the phase space of the velocity $\vec{V}$, the states $\vec{V}_1$ lie on a curve, called the shock polar. Let N and L be the normal and tangential components of the velocity $\vec{V}$ to the shock line S respectively. We have

$$N_0 = V_0 \sin\beta \tag{A.1}$$

$$L_1 = L_0 = V_0 \cos\beta \text{ (continuity of tangential component)} \tag{A.2}$$

Furthermore, Bernoulli's law for steady flows holds across a shock front:

$$\frac{1}{2}V_0^2 + \frac{c_0^2}{\gamma - 1} = \frac{1}{2}V_1^2 + \frac{c_1^2}{\gamma - 1} = \frac{1}{2}V_m^2. \tag{A.3}$$

Also, we have the Prandtl relation for the polytropic gases.

$$N_0 N_1 = c_p^2 - \mu^2 L_0^2, \tag{A.4}$$

where $\mu^2 = \dfrac{\gamma - 1}{\gamma + 1}$. $c_p = \mu V_m$ is the critical speed. In the present case, we consider steady and irrotational flows. The flow is everywhere isentropic behind the shock front. Thus, the maximum speed V_m is the same throughout the flow field. [8, CH6.B]

By (A.1) and (A.2), the angle β satisfies

$$\sin\beta = \frac{1}{V_0}\sqrt{V_0^2 - L_1^2}\ .$$

It thus follows from (A.3) and (A.4) that

$$N_1 = \frac{c_p^2 - \mu^2 L_1^2}{V_0 \sin\beta} = \frac{c_p^2 - \mu^2 L_1^2}{\sqrt{V_0^2 - L_1^2}}\ .$$

In the spherical coordinates, the radial and normal components of the flow velocity V are denoted by V_r and V_θ respectively. That is, $N_1 = |V_\theta|$ and $L_1 = |V_r|$. We then obtain the equation of the shock polar:

$$|V_\theta| = \frac{c_p^2 - \mu^2 V_r^2}{\sqrt{V_0^2 - V_r^2}}\ .$$

By using the non-dimensional velocity, $U_0 = V_0/V_m$, $U_r = V_r/V_m$ and $U_\theta = V_\theta/V_m$, we have the following equation

$$U_\theta^2 = \frac{\mu^4(1 - U_r^2)^2}{U_0^2 - U_r^2}\ . \tag{A.5}$$

Hence, for a given speed U_0, the possible 1-states (U_θ, U_r) satisfy the following equation:

$$-U_\theta = \frac{\mu^2(1 - U_r^2)}{\sqrt{U_0^2 - U_r^2}}\ . \tag{A.6}$$

And for the 1-state (U_θ, U_r), β can be calculated by

$$\tan\beta = \frac{\sqrt{U_0^2 - U_r^2}}{U_r}. \tag{A.7}$$

For the present problem, the incoming flow is supersonic with the given density ρ_0 and the velocity $\vec{V_0}$, i.e., $V_0 > c_0$. Then, the flow continues as a conical flow with constant entropy after crossing the shock. Density and pressure rise across the shock front. Thus, $|V_0| > |V_1|$. We note that at the 0 state,

$$N_0^2 = \mu^2(V_m^2 - L_0^2),\ V_0^2 = N_0^2 + L_0^2.$$

Hence at the 0 state,

$$U_\theta^2 = [1 - U_0^2]\mu^2/(1 - \mu^2),\ U_r^2 = [U_0^2 - \mu^2]/(1 - \mu^2).$$

And $U_0 \in [\mu, 1]$. In the following figure, we choose $\gamma = 1.4$. The circle denotes the 0 state. The thick line branch is all the possible 1 states for which $|V_0| > |V_1|$.

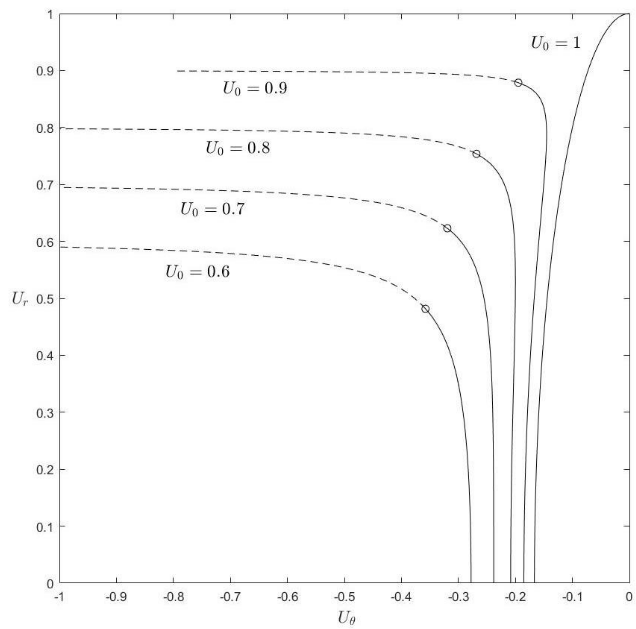

FIGURE 9. The simulations of shock polars

Since V_m is determined by (A.3)

$$\frac{1}{2}V_0^2 + \frac{c_0^2}{\gamma - 1} = \frac{1}{2}V_m^2,$$

it follows that

$$\frac{U_0^2}{2} + \frac{U_0^2}{(\gamma - 1)(V_0/c_0)^2} = \frac{1}{2}.$$

The relationship between the Mach number $M = V/c$ and the non-dimensional speed $|U|$ is as follows

$$|U| = \left[1 + \frac{2}{(\gamma - 1)M^2}\right]^{-1/2}. \tag{A.8}$$

When $U_0 = 0.8$, M_0 is around 3.

Remark 5. For the present problem, $U_\theta \in (-1, 0]$ and $U_r \in \left(0, \sqrt{\frac{\gamma-1}{\gamma+1}}\,\right]$. $\gamma > 1$ is given. Considering equation (A.5), we can regard U_θ as a function depending on U_r. By careful calculations, if $U_0 > \sqrt{\frac{\gamma}{\gamma+1}}$, then U_θ increases as U_r increases.

Remark 6. At the shock angle β, we need to request $A(\beta) > 0$ for the ODE system (2.13). By Lemma 2.1 and (A.5), it is equivalent to the following condition:

$$U_0^2 > \left(\frac{\gamma - 1}{\gamma + 1}\right) + U_r^2\left(\frac{2}{\gamma + 1}\right). \tag{A.9}$$

Acknowledgments. The authors would like to thank Prof. M-H. Chen for his helpful discussions. Lien is supported in part by MOST Grant 107-2115-M-006-012. Liu is supported in part by MOST Grant 107-2115-M-006-017.

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Received for publication March 2022; early access June 2022.

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