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UNIFORM STABILITY OF THE CUCKER–SMALE AND THERMODYNAMIC CUCKER–SMALE ENSEMBLES WITH SINGULAR KERNELS

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Abstract. This paper presents several sufficient frameworks for a collision avoidance and flocking dynamics of the Cucker–Smale (CS) model and thermodynamic CS (TCS) model with arbitrary dimensions and singular interaction kernels. In general, unlike regular kernels, singular kernels usually interfere with the global well-posedness of the targeted models from the perspective of the standard Cauchy–Lipschitz theory due to the possibility of a finite-in-time blow-up. Therefore, according to the intensity of the singularity of a kernel (strong or weak), we provide a detailed framework for the global well-posedness and emergent dynamics for each case. Finally, we provide an admissible set in terms of system parameters and initial data for the uniform stability of the d-dimensional TCS with a singular kernel, which can be reduced to a sufficient framework for the uniform stability of the d-dimensional CS with singular kernel if all agents have the same initial temperature.

1. **Introduction.** The emergent dynamics of interacting many-body systems are often observed in complex ecosystems. Examples include the synchronization of fireflies and pacemaker cells [7, 28, 55], aggregation of bacteria [51], flocking of birds [26], and swarming of fish [27, 50]. To briefly introduce them, we refer to [1, 6, 17, 29, 43, 47, 49, 53, 54]. We are interested in flocking dynamics in which each particle converges to a common velocity with an ordered formation by using limited information and simple laws. After the groundbreaking work [52] on the flocking model of birds proposed by Viscek et al., many mathematical models describing collective behavior have been widely investigated in the mathemathical community. Since [26], many mathematicians and physicists have been concerned with the Cucker–Smale (CS) type models derived from a Newtonian-like second-order model for *position-velocity*, governed by the following system in terms of (x_i, v_i):

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$$
\begin{cases}
\dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N] := \{1, \cdots, N\}, \\
\dfrac{dv_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{j \neq i} \psi(\|x_i - x_j\|)\,(v_j - v_i)\,, \\
(x_i(0), v_i(0)) = (x_i^0, v_i^0) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{cases} \tag{1}
$$

where N is the number of particles and κ is a strictly positive coupling strength. Many papers have studied on the CS model and its variants. This research comprises the following topics: the mean-field limit [4, 5, 20, 36, 38], kinetic model [10, 40], hydrodynamic descriptions [30, 32, 41], particle analysis [10, 17], time-delay effect [15, 19], stochastic description [11], bi-cluster flocking [21, 22], relativistic setting [3, 5, 8, 35], unit-speed constraint [13, 14, 21, 37, 48], and collision avoidance [9, 18, 20, 23, 24, 25, 42, 44, 45, 46].

However, the above literature has only addressed the CS model without the temperature field. Therefore, the authors in [39] generalized the CS model to consider the temperature settings from the system of gas mixtures with rational reductions, called the thermodynamic CS (TCS) model. Afterward, in a follow-up paper [34], the authors derived an approximated TCS model by assuming that the diffusion velocities are sufficiently small, which is given by the following second-order system for *position-velocity-temperature*, (x_i, v_i, T_i):

$$
\begin{cases}
\dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\
\dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j \neq i} \phi(\|x_i - x_j\|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i} \right), \\
\dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j \neq i} \zeta(\|x_i - x_j\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), \\
(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^{2d} \times \mathbb{R}_{>0},
\end{cases} \tag{2}
$$

where N is the number of particles and κ_1 and κ_2 are strictly positive coupling strengths. For a brief introduction to the TCS type models, we refer to papers which are derivation of the TCS model [34, 39], asymptotic behavior [34], uniform stability and uniform-in-time mean-field limit [33], hydrodynamic description [31], time-delay effect [12], unit-speed constraint [2] and collision avoidance [16].

Throughout the paper, we choose the simplest singular communication weights (or singular interaction kernels) ψ, ϕ and ζ to predict collision avoidance between each pair of particles in (1) and (2):

$$
\psi(r) := \frac{1}{r^\alpha}, \quad \phi(r) := \frac{1}{r^\beta}, \quad \zeta(r) := \frac{1}{r^\gamma}, \quad \alpha, \beta, \gamma > 0.
$$

We are only interested in the singularity when $r = 0$; thus, it is not important to determine the explicit structures of the singular weights ϕ and ζ. Indeed, we aim to observe the flocking behavior of (1) and (2) when each of the communication weights has no regularity at zero. We must first consider the global well-posedness (i.e., the noncollisional phenomenon) problem of the targeted models because the singular kernels are not well defined at $r = 0$. In addition to this motivation, collision avoidance between each pair of particles is an important issue in mechanical engineering and motion control engineering, to name a few for UAV (uncrewed aerial vehicle), drones and ACAS (airborne collision avoidance system), to name a few. Therefore, studying sufficient frameworks for collision avoidance in interacting many-body systems is of great significance. For more advanced works in this paper, we present several frameworks that differ from those in previous papers [9, 16, 18, 20, 23, 24, 25, 42, 44, 45, 46] related to collision avoidance, which can be summarized in the next paragraph. The main novelties of this paper are presented below. First of all, we prove the global well-posedness (i.e., collision avoidance) of the systems (1) and (2) in terms of quantities for L^∞-diameters, independent of the number of particles N. To do this, we employ useful functionals to derive several dissipative structures with singular kernels. Second, we present the emergent dynamics in terms of L^∞-diameters under sufficient frameworks in terms of the initial data and system parameters in (1) and (2). The flocking estimates are independent of N; therefore, it is natural to consider the uniform stability estimates of (1) and (2), which yield uniform-in-time mean-field limits from (1) and (2) to the corresponding Vlasov equation by taking $N \to \infty$, respectively. Third, we rigorously demonstrate the uniform L^2-stability result of (2) under the admissible initial data and system parameters, which can be trivially reduced to the uniform L^2-stability result of (1) by removing $(2)_3$ from Remark 1. Furthermore, we can derive the uniform stability estimate of (2) with a much simpler argument than the literature [5, 33, 36]. In summary, the ultimate goal of this paper is to extend the uniform stability independent of N of (1) on one-dimensional Euclidean space $\mathbb{R}^1$ studied in [20] to the CS and TCS models on $\mathbb{R}^d$ with arbitrary dimensions.

This paper is organized as follows. In Section 2, we briefly revisit facts regarding the temperature field $(2)_3$ and provide basic estimates for the global well-posedness of (1) and (2). In Section 3, we study several sufficient frameworks for collision avoidance, global well-posedness, and the emergent dynamics of (1) under strongly or weakly singular interaction kernels, respectively. In Section 4, we also present the global well-posedness of (2) on admissible data in terms of the initial data and system parameters by dividing the singularity of each communication weight in the targeted models into a weak case and strong case. However, unlike the case of (1) in Section 3, we provide sufficient frameworks for the emergent dynamics regardless of the intensity of the singularity. In Section 5, we provide a detailed proof for the uniform L^2-stability of (2) under appropriate admissible data using the results from Section 3 and Section 4. Finally, Section 6 is devoted to summarizing the main results and discussing the remaining issues to be investigated in future work.

Notation. Throughout the paper, we employ the following notation and abbreviations:

$$\|\cdot\| := l_2\text{-norm}, \quad \langle \cdot,\cdot\rangle := \text{standard inner product}, \quad \text{(T)CS with singular kernel}:=\text{(T)CSS},$$
$$X := (x_1, \cdots, x_N), \quad V := (v_1, \cdots, v_N), \quad T := (T_1, \cdots, T_N), \quad [N] := \{1, \cdots, N\},$$
$$D_Z := \max_{i,j\in[N]} \|z_i - z_j\|, d_X := \min_{i\neq j, i,j\in[N]} \|x_i - x_j\| \text{ for } Z = (z_1, \cdots, z_N) \in \{X, V, T\}.$$

2. Preliminaries.

In this section, we provide basic materials to guarantee the global well-posedness of the CSS model (1) and TCSS model (2). For this, we revisit previous results for the temperature field $(2)_3$ in Section 2.1 to be used throughout the paper. In Section 2.2, we provide the uniform boundedness of speed for each particle in (1) and (2).

2.1. Previous results.

In this subsection, we briefly provide facts regarding the temperature system $(2)_3$ for global well-posedness. In the literature [39], the authors proved that the total temperature sum is conserved and that the entropy principle holds, stated as follows.

Definition 2.1. [34, 39] Let $\tau \in (0,\infty]$ and (X,V,T) be a solution to the singular system (2) for $t \in (0,\tau)$. Then, the total entropy is defined as

$$\mathcal{S} := \sum_{i=1}^{N} \ln(T_i).$$

We now present the previous results on the conservation of the temperature sum and the monotonicity of the total entropy as follows.

Proposition 1. [34, 39] *For a fixed $\tau \in (0,\infty]$, suppose that (X,V,T) is a solution to the singular system (2) for $t \in (0,\tau)$. Then, the following assertions hold.*

1. *(Conserved temperature sum) The total sum $\sum_{i=1}^N T_i$ is conserved:*

$$\sum_{i=1}^{N} T_i(t) = \sum_{i=1}^{N} T_i^0 := NT^\infty, \quad \forall t \in [0,\tau).$$

2. *(Entropy principle) The total entropy $\mathcal{S}$ is monotonically increasing:*

$$\frac{d\mathcal{S}(t)}{dt} = \frac{1}{2N}\sum_{i,j=1}^{N} \zeta(\|x_j - x_i\|)\left|\frac{1}{T_i} - \frac{1}{T_j}\right|^2 \geq 0, \quad \forall t \in [0,\tau).$$

Due to the entropy principle and the simple structure of $(2)_3$, the authors in [12, 34] proved that the temperature for each particle of (2) on $t \in [0,\tau)$ is uniformly bounded.

Proposition 2. [12, 34] (Monotonocity of max-min temperatures) *Let $\tau \in (0,\infty]$. Assume that (X,V,T) is a solution to the singular system (2) for $t \in (0,\tau)$. Then, $\min_{1\leq i\leq N} T_i(t)$ is monotonically increasing and $\max_{1\leq i\leq N} T_i(t)$ is monotonically decreasing in $t \in [0,\tau)$. Hence, we have the uniform boundedness of temperature as below.*

$$0 < \min_{i\in[N]} T_i^0 := T_m^\infty \leq T_i(t) \leq \max_{i\in[N]} T_i^0 := T_M^\infty, \quad i \in [N], \quad t \in [0,\tau).$$

Remark 1. By the standard Cauchy–Lipschitz theory, the TCSS model (2) for $[0,\tau)$ can be reduced to the CSS model (1) for $[0,\tau)$ if the initial temperature data for (2) have the same positive constant, $T^0 > 0$, that is,

$$T_1^0 = \cdots T_N^0 = T^0 > 0.$$

2.2. **Basic materials.** In this subsection, we derive that the maximum speed is uniformly bounded by physical constraints in terms of the initial data in (1) to verify the global well-posedness. More concretely, we show that the maximum speed is monotonically decreasing in (1).

Lemma 2.2. *Let $\tau \in (0,\infty]$ and (X,V) be a solution to the singular system (1) for $t \in (0,\tau)$. Then, it follows that*

$$\max_{i\in[N]} \|v_i\| \leq \max_{i\in[N]} \|v_i^0\|, \quad t \in [0,\tau).$$

Proof. We choose an index $M_t \in [N]$ dependent on time $t \in [0,\tau)$ such that

$$\|v_{M_t}\| := \max_{i\in[N]} \|v_i(t)\|.$$

Then, we take the inner product v_{M_t} with $\dot{v}_{M_t}$ in $(1)_2$ to obtain the following for a.e. $t \in (0, \tau)$,

$$\frac{1}{2}\frac{d\|v_{M_t}\|^2}{dt} = \frac{\kappa}{N}\sum_{j=1}^{N}\psi(\|x_{M_t} - x_j\|)\,\langle v_j - v_{M_t}, v_{M_t}\rangle \le 0.$$

Hence, we obtain

$$\frac{1}{2}\frac{d\|v_{M_t}\|^2}{dt} \le 0, \quad \text{a.e. } t \in (0,\tau) \quad \Longrightarrow \quad \|v_{M_t}\| \le \|v_{M_0}\|, \quad t \in [0,\tau),$$

implying the desired result.

$$\max_{i\in[N]}\|v_i\| \le \|v_{M_0}\| \le \max_{i\in[N]}\|v_i^0\|, \quad t \in [0,\tau).$$

□

Therefore, if we prove that (1) has a noncollisional phenomenon at any time, then we have global well-posedness with Lemma 2.2 and the Cauchy–Lipschitz theory. For detailed descriptions, we refer to Section 3. In the case of the system (2), immediately determining information about the maximum speed is challenging, so we use another method to guarantee the uniformly boundedness of the maximum speed in Section 4.

3. **Global well-posedness and emergent dynamics of CSS.** In this section, we establish sufficient frameworks in terms of the initial data and system parameters for the global well-posedness, collision avoidance, and emergent dynamics of (1), by dividing them into two cases: $\alpha \ge 1$ and $0 < \alpha < 1$, according to the singularity of ψ. To achieve this, we will employ useful functionals to derive the dissipative differential inequalities for *position-velocity* diameters D_X and D_V.

3.1. **Strongly singular kernel.** In this subsection, we study the global well-posedness of the CSS model (1) when ψ is a strongly singular kernel.

$$\psi(r) = \frac{1}{r^\alpha}, \quad r > 0, \quad \alpha \ge 1.$$

3.1.1. *Global well-posedness.* Next, we rigorously verify the global well-posedness of (1) under a strongly singular interaction kernel. It suffices to demonstrate the noncollisional state between each pair of particles on any finite time. We assume that t_0 is the first collision time of the singular system (1), and $[l]$ denotes the set of all particles that collide with the l-th particle at time t_0, i.e.,

$$[l] := \{i \in [N] \mid \|x_l(t) - x_i(t)\| \to 0 \quad \text{as} \quad t \to t_0-\}.$$

Let δ be a strictly positive real number satisfying

$$\|x_l(t) - x_i(t)\| \ge \delta > 0, \quad \forall\, t \in [0,t_0) \quad \text{and} \quad \forall i \notin [l].$$

Thus, we define the following L^∞-diameters in terms of *position-velocity* from the perspective of $[l]$ for $t \in [0, t_0)$:

$$d_{X,[l]} := \min_{i,j\in[l], i\neq j}\|x_i - x_j\|, \quad D_{X,[l]} := \max_{i,j\in[l]}\|x_i - x_j\|, \quad D_{V,[l]} := \max_{i,j\in[l]}\|v_i - v_j\|.$$

For simplicity, we use the following notation:

$$\psi_{ij} \coloneqq \psi(\|x_i - x_j\|),\ i,j \in [N],\ i \neq j, \quad \text{and} \quad \psi_{ij,[l]} \coloneqq \psi(\|x_i - x_j\|),\ i,j \in [l],\ i \neq j,$$

where $|[l]|$ is a cardinal number of the set $[l]$.

Next, we employ the following functional $\Psi_{ij,[l]}$ for $(i,j) \in [l]^2$:

$$\Psi_{ij,[l]}(t) := \frac{\psi(\|x_i - x_j\|)}{|[l]|} \quad \text{for} \quad i,j \in [l],\ i \neq j,$$
$$\Psi_{ii,[l]}(t) := \psi\left(d_{X,[l]}\right) - \frac{\sum_{j\neq i, j\in[l]} \psi(\|x_i - x_j\|)}{|[l]|}, \quad \text{for} \quad i \in [l].$$

Then, we observe that Ψ_{ij} satisfies the following three properties:

1. $\Psi_{ij,[l]} \geq \dfrac{\psi(D_{X,[l]})}{|[l]|}, \quad i,j \in [l], \quad i \neq j,$
2. $\displaystyle\sum_{j\in[l]} \Psi_{ij,[l]} = \psi\left(d_{X,[l]}\right), \quad i \in [l],$
3. $\displaystyle\sum_{j\in[l]} \Psi_{ij,[l]}\left(v_j - v_i\right) = \sum_{j\neq i, j\in[l]} \frac{\psi_{ij}}{|[l]|}\left(v_j - v_i\right), \quad i \in [l].$

Theorem 3.1. *Suppose that (X,V) is a solution to (1) with a strongly singular kernel and noncollisional position initial data, that is,*

$$\alpha \geq 1 \quad \textit{and} \quad \min_{i,j\in[N], i\neq j} \|x_i^0 - x_j^0\| > 0.$$

Then, we can obtain the global well-posedness of (1), or, equivalently, we have the global-in-time collisionless state:

$$x_i(t) \neq x_j(t), \quad (i,j) \in [N]^2, \quad i \neq j \quad \textit{and} \quad \forall t \in [0,\infty).$$

Proof. First, we use the following relation

$$\left|\frac{d\|x_i - x_j\|^2}{dt}\right| = 2|\langle x_i - x_j, v_i - v_j\rangle| \leq 2\|x_i - x_j\|\|v_i - v_j\|$$

with the Cauchy–Schwarz inequality to have that for a.e. $t \in (0,t_0)$,

$$\left|\frac{dD_{X,[l]}(t)}{dt}\right| \leq D_{V,[l]}(t). \tag{3}$$

Now, we take two indices $i_t, j_t \in [l]$ dependent on time $t \in (0,t_0)$ such that

$$D_{V,[l]}(t) := \|v_{i_t}(t) - v_{j_t}(t)\|, \quad i_t, j_t \in [l].$$

Then, it follows from $(1)_2$ that for a.e. $t \in (0,t_0)$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_{i_t}-v_{j_t}\|^2\\
&=\left\langle v_{i_t}-v_{j_t}, \frac{dv_{i_t}}{dt}-\frac{dv_{j_t}}{dt}\right\rangle\\
&=\left\langle v_{i_t}-v_{j_t}, \frac{\kappa}{N}\sum_{k=1}^{N}\psi_{i_tk}(v_k-v_{i_t})-\frac{\kappa}{N}\sum_{k=1}^{N}\psi_{j_tk}(v_k-v_{j_t})\right\rangle\\
&=\left\langle v_{i_t}-v_{j_t}, \frac{\kappa}{N}\sum_{k\notin[l]}\psi_{i_tk}(v_k-v_{i_t})-\frac{\kappa}{N}\sum_{k\notin[l]}\psi_{j_tk}(v_k-v_{j_t})\right\rangle\\
&\quad+\left\langle v_{i_t}-v_{j_t}, \frac{\kappa}{N}\sum_{k\in[l]}\psi_{i_tk}(v_k-v_{i_t})-\frac{\kappa}{N}\sum_{k\in[l]}\psi_{j_tk}(v_k-v_{j_t})\right\rangle\\
&=: \mathcal{I}_1+\mathcal{I}_2.
\end{aligned}
$$

- (Estimate of $\mathcal{I}_1$): If we apply Lemma 2.2 and the Cauchy–Schwarz inequality

to $\mathcal{I}_1$, then there exists a nonnegative constant $C(\kappa,[l],N,V(0),\delta)$ satisfying

$$
\begin{aligned}
\mathcal{I}_1 &= \left\langle v_{i_t}-v_{j_t}, \frac{\kappa}{N}\sum_{k\notin[l]}\psi_{i_tk}(v_k-v_{i_t})-\frac{\kappa}{N}\sum_{k\notin[l]}\psi_{j_tk}(v_k-v_{j_t})\right\rangle\\
&\le D_{V,[l]}\left\|\frac{\kappa}{N}\sum_{k\notin[l]}\psi_{i_tk}(v_k-v_{i_t})-\frac{\kappa}{N}\sum_{k\notin[l]}\psi_{j_tk}(v_k-v_{j_t})\right\|\\
&\le D_{V,[l]}\left(\left\|\frac{\kappa}{N}\sum_{k\notin[l]}\psi_{i_tk}(v_k-v_{i_t})\right\|+\left\|\frac{\kappa}{N}\sum_{k\notin[l]}\psi_{j_tk}(v_k-v_{j_t})\right\|\right)\\
&\le \frac{4\kappa(N-|[l]|)\psi(\delta)\max_{i\in[N]}\|v_i^0\|}{N}\cdot D_{V,[l]}\\
&=: C(\kappa,[l],N,V(0),\delta)D_{V,[l]},
\end{aligned}
\tag{4}
$$

where we used the definition of δ and monotonocity of ψ.

- (The estimate of $\mathcal{I}_2$): For this, we employ the properties of Ψ to get that

$$
\begin{aligned}
\mathcal{I}_2 &= \left\langle v_{i_t}-v_{j_t}, \frac{\kappa}{N}\sum_{k\in[l]}\psi_{i_tk}(v_k-v_{i_t})-\frac{\kappa}{N}\sum_{k\in[l]}\psi_{j_tk}(v_k-v_{j_t})\right\rangle\\
&= \left\langle v_{i_t}-v_{j_t}, \frac{\kappa|[l]|}{N}\sum_{k\in[l]}\Psi_{i_tk}(v_k-v_{i_t})-\frac{\kappa|[l]|}{N}\sum_{k\in[l]}\Psi_{j_tk}(v_k-v_{j_t})\right\rangle\\
&= -\frac{\kappa|[l]|}{N}\psi\left(d_{X,[l]}\right)\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle+\frac{\kappa|[l]|}{N}\left\langle v_{i_t}-v_{j_t}, \sum_{k\in[l]}(\Psi_{i_tk}-\Psi_{j_tk})v_k\right\rangle\\
&= -\frac{\kappa|[l]|}{N}\psi\left(d_{X,[l]}\right)\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle+\frac{\kappa|[l]|}{N}\\
&\quad\left\langle v_{i_t}-v_{j_t}, \sum_{k\in[l]}(\Psi_{i_tk}-\min(\Psi_{i_tk},\Psi_{j_tk})+\min(\Psi_{i_tk},\Psi_{j_tk})-\Psi_{j_tk})v_k\right\rangle.
\end{aligned}
\tag{5}
$$

Here, since

$$
\langle v_{i_t}-v_{j_t}, v_{j_t}\rangle \le \langle v_{i_t}-v_{j_t}, v_k\rangle \le \langle v_{i_t}-v_{j_t}, v_{i_t}\rangle,
$$

we can show that

$$
\begin{aligned}
\mathcal{I}_2 &\le -\frac{\kappa|[l]|}{N}\psi\left(d_{X,[l]}\right)\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle \\
&\quad + \frac{\kappa|[l]|}{N}\left\langle v_{i_t}-v_{j_t}, \sum_{k\in[l]}(\Psi_{i_tk}-\min(\Psi_{i_tk},\Psi_{j_tk}))v_{i_t}\right\rangle \\
&\quad + \frac{\kappa|[l]|}{N}\left\langle v_{i_t}-v_{j_t}, \sum_{k\in[l]}(\min(\Psi_{i_tk},\Psi_{j_tk})-\Psi_{j_tk})v_{j_t}\right\rangle \\
&\le -\frac{\kappa|[l]|}{N}\psi\left(d_{X,[l]}\right)\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle \\
&\quad + \frac{\kappa|[l]|}{N}\psi\left(d_{X,[l]}\right)\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle \\
&\quad - \frac{\kappa|[l]|}{N}\sum_{k\in[l]}\min(\Psi_{i_tk},\Psi_{j_tk})\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle \\
&= -\frac{\kappa|[l]|}{N}\sum_{k\in[l]}\min(\Psi_{i_tk},\Psi_{j_tk})\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle \\
&\le -\frac{\kappa|[l]|}{N}\psi(D_{X,[l]})\langle v_{i_t}-v_{j_t}, v_{i_t}-v_{j_t}\rangle = -\frac{\kappa|[l]|}{N}\psi(D_{X,[l]})D^2_{V,[l]},
\end{aligned}
\tag{6}
$$

where we used the first property of Ψ. Hence, we combine (4) with (6) to obtain for a.e. $t\in(0,t_0)$,

$$
\frac{dD_{V,[l]}}{dt} \le -\frac{\kappa|[l]|}{N}\psi(D_{X,[l]})D_{V,[l]} + C(\kappa,[l],N,V(0),\delta).
$$

We then integrate to both sides of the above inequality from s to t for $0\le s\le t<t_0$ to yield

$$
\int_s^t \psi(D_{X,[l]})D_{V,[l]}du \le \frac{N\big(D_{V,[l]}(s)+C(\kappa,[l],N,V(0),\delta)(t-s)\big)}{\kappa|[l]|}. \tag{7}
$$

Moreover, let Φ be a primitive of a strongly singular kernel ψ with $\alpha\ge 1$. Then, for fixed $t_1>0$,

$$
\Phi(t) := \Phi(t;t_1) := \int_{t_1}^t \psi(u)du = \begin{cases} \log\dfrac{t}{t_1}, & \text{if} \quad \alpha=1, \\ \dfrac{1}{1-\alpha}\left(t^{1-\alpha}-{t_1}^{1-\alpha}\right), & \text{if} \quad \alpha>1. \end{cases} \tag{8}
$$

Therefore, it follows from (7) that for $0\le s\le t<t_0$,

$$
\begin{aligned}
\left|\Phi(D_{X,[l]}(t))\right| &\le \left|\int_s^t \frac{d}{du}\Phi(D_{X,[l]}(u))du\right| + \left|\Phi(D_{X,[l]}(s))\right| \\
&= \left|\int_s^t \psi(D_{X,[l]}(u))\left(\frac{d}{du}D_{X,[l]}(u)\right)du\right| + \left|\Phi(D_{X,[l]}(s))\right| \\
&\le \int_s^t \psi(D_{X,[l]}(u))\left|\frac{d}{du}D_{X,[l]}(u)\right|du + \left|\Phi(D_{X,[l]}(s))\right| \\
&\le \int_s^t \psi(D_{X,[l]}(u))D_{V,[l]}(u)du + \left|\Phi(D_{X,[l]}(s))\right|,
\end{aligned}
$$

which implies from (7) that

$$\left|\Phi(D_{X,[l]}(t))\right| \le \left|\Phi(D_{X,[l]}(s))\right| + \frac{N\big(D_{V,[l]}(s) + C(\kappa,[l],N,V(0),\delta)(t-s)\big)}{\kappa|[l]|},$$
$$0 \le s \le t < t_0.$$

Now, we take $s = 0$ and $t \to t_0-$ for the above inequality to obtain

$$\lim_{t\to t_0-} \left|\Phi(D_{X,[l]}(t))\right| \le \left|\Phi(D_{X,[l]}(0))\right| + \frac{N\big(D_{V,[l]}(0) + C(\kappa,[l],N,V(0),\delta)t_0\big)}{\kappa|[l]|} < \infty,$$

which yields a contradiction to the definition of t_0 and (8) because

$$\lim_{t\to t_0-} \left|\Phi(D_{X,[l]}(t))\right| = \infty.$$

Consequently, we have shown the noncollisional phenomenon of (1), i.e.,

$$x_i(t) \neq x_j(t), \quad \forall (i,j) \in [N]^2 \quad \text{and} \quad \forall t \in [0,\infty).$$

Finally, one combines this with Lemma 2.2 to have the desired global well-posedness of (1) under the strongly singular kernel from the standard Cauchy–Lipschitz theory. □

Remark 2. Note that $\Phi(t)$ with $0 < \alpha < 1$ in (8) is always finite if

$$0 \le t_1 \le t < \infty.$$

Therefore, we may take a different strategy from the proof of Theorem 3.1 to guarantee the global well-posedness of (1) under a weakly singular kernel ($0 < \alpha < 1$).

3.1.2. *Emergent dynamics.* In this subsubsection, from the proof of Theorem 3.1, we study the sufficient frameworks for the emergent dynamics of (1), assuming $\alpha \ge 1$. Before we continue, we revisit the definition of the emergent dynamics of (1).

Definition 3.2. Let $Z := (X,V)$ be a global-in-time solution to the singular system (1). The configuration Z exhibits asymptotic flocking if

$$(i) \text{ (Group formation)} \iff \sup_{0\le t<\infty} \max_{i,j\in[N]} \|x_i(t) - x_j(t)\| < \infty,$$
$$(ii) \text{ (Velocity alignment)} \iff \lim_{t\to\infty} \max_{i,j\in[N]} \|v_j(t) - v_i(t)\| = 0.$$

We present sufficient frameworks for the asymptotic flocking of (1) with a strongly singular kernel using two approaches, the bootstrapping argument (or continuous argument) and the Lyapunov functional.

Theorem 3.3. (Asymptotic flocking via bootstrapping) *Let (X,V) be a global-in-time solution to (1) with*

$$\alpha \ge 1 \quad \textit{and} \quad \min_{i,j\in[N], i\neq j} \|x_i^0 - x_j^0\| > 0.$$

Further assume that there exists a positive constant D_X^∞ satisfying

$$D_X(0) + \frac{D_V(0)}{\kappa\psi(D_X^\infty)} \le D_X^\infty. \tag{9}$$

Then, we have the following asymptotic flocking result:

1. *(Group formation)* $D_X(t) < D_X^\infty$,

2. *(Velocity alignment)* $D_V(t) \le D_V(0)\exp\left(-\kappa\psi(D_X^\infty)t\right)$.

Proof. If $D_V(0) = 0$, then we have nothing to prove. Therefore, we can assume $D_V(0) > 0$. We observe from the condition (9) that the set

$$\mathcal{S} := \{t > 0 \mid D_X(s) < D_X^\infty,\ \forall s \in (0,t)\}$$

is nonempty. We define $t^* := \sup \mathcal{S} > 0$. Hence,

$$D_X(t^*) = D_X^\infty.$$

Now, we claim that

$$t^* = +\infty.$$

To prove the claim, we suppose that $t^* < +\infty$ for the proof by contradiction. Then, by replacing $[l]$ with $[N]$ in the proof of Theorem 3.1, we use the arguments of Theorem 3.1 to establish the following:

$$\left|\frac{dD_X}{dt}\right| \le D_V, \quad \frac{dD_V}{dt} \le -\kappa\psi(D_X)D_V \le -\kappa\psi(D_X^\infty)D_V, \quad \text{a.e. } t \in (0,t^*),$$

where we used the definition of $\mathcal{S}$. Then, Grönwall's lemma implies that

$$D_V(t) \le D_V(0)\exp\left(-\kappa\psi(D_X^\infty)t\right), \quad \forall t \in [0,t^*].$$

Moreover, because

$$\begin{aligned} D_X(t) &= D_X(0) + \int_0^t \frac{dD_X(s)}{ds}ds \le D_X(0) + \int_0^t D_V(s)ds \\ &\le D_X(0) + \int_0^t D_V(0)\exp\left(-\kappa\psi(D_X^\infty)s\right)ds \\ &< D_X(0) + \frac{D_V(0)}{\kappa\psi(D_X^\infty)} \le D_X^\infty, \quad \forall t \in [0,t^*], \end{aligned}$$

one can show that $D_X(t^*) < D_X^\infty$, resulting in a contradiction, i.e., $t^* = \infty$. Thus, one has

$$D_X(t) < D_X^\infty, \quad D_V(t) \le D_V(0)\exp\left(-\kappa\psi(D_X^\infty)t\right), \quad t \in [0,\infty).$$

Therefore, we achieve the desired results. $\square$

Next, we provide a different approach to achieve the asymptotic flocking of (1) via the Lyapunov functional method introduced in [38].

Theorem 3.4. (Asymptotic flocking via the Lyapunov functional) *Let (X,V) be a global-in-time solution to (1) with*

$$\alpha \ge 1, \quad \textit{and} \quad \min_{i,j\in[N], i\ne j} \|x_i^0 - x_j^0\| > 0,$$

and further assume that

$$D_V(0) \le \kappa \int_{D_X(0)}^\infty \psi(s)ds. \tag{10}$$

Then, we obtain the following asymptotic flocking estimate: there exists a strictly positive number $D_X^\infty > 0$ such that

1. *(Group formation)* $D_X(t) \le D_X^\infty$,

2. (Velocity alignment) $D_V(t) \le D_V(0)\exp\left(-\kappa\psi(D_X^\infty)t\right), \quad t \in [0,\infty).$

Proof. First, we use the same arguments employed in Theorem 3.1 and Theorem 3.3 to deduce that

$$\left|\frac{dD_X}{dt}\right| \le D_V, \quad \frac{dD_V}{dt} \le -\kappa\psi(D_X)D_V. \tag{11}$$

Now, we consider the following Lyapunov functional:

$$\mathcal{L}_\pm(D_X, D_V) := D_V \pm \kappa\Psi(D_X),$$

where $\Psi(t) := \int_0^t \psi(s)ds$. Then, we apply (11) to obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_\pm(D_X, D_V) &= \frac{dD_V}{dt} \pm \kappa\frac{dD_X}{dt}\psi(D_X) \\ &\le -\kappa\psi(D_X)D_V \pm \kappa\frac{dD_X}{dt}\psi(D_X) \\ &= \kappa\psi(D_X)\left(-D_V \pm \frac{dD_X}{dt}\right) \le 0. \end{aligned} \tag{12}$$

Next, we utilize (12) to induce $\mathcal{L}_\pm(D_X(t), D_V(t)) \le \mathcal{L}_\pm(D_X(0), D_V(0))$ and moreover,

$$\kappa\left|\int_{D_X(0)}^{D_X(t)} \psi(s)ds\right| \le D_V(t) + \kappa\left|\int_{D_X(0)}^{D_X(t)} \psi(s)ds\right| \le D_V(0).$$

Therefore, we combine this with the condition (10) to yield

$$\kappa\left|\int_{D_X(0)}^{D_X(t)} \psi(s)ds\right| \le D_V(0) \le \kappa\int_{D_X(0)}^{\infty} \psi(s)ds.$$

Hence, there is a smallest positive real number D_X^∞ satisfying

$$D_V(0) = \kappa\int_{D_X(0)}^{D_X^\infty} \psi(s)ds;$$

thus

$$D_X(t) \le D_X^\infty, \quad \forall t \in [0,\infty).$$

Due to the above estimate and (11), from Grönwall's lemma, it follows that

$$D_V(t) \le D_V(0)\exp\left(-\kappa\psi(D_X^\infty)t\right), \quad t \in [0,\infty).$$

Finally, we prove the desired asymptotic flocking results. □

Remark 3. If $\alpha > 1$, then it follows from a direct calculation that

$$\int_{D_X(0)}^{\infty} \psi(s)ds < \infty.$$

Therefore, we know that the assumption (10) can not be rejected. However, we can remove (10) in the case of $0 < \alpha \le 1$ because

$$\int_{D_X(0)}^{\infty} \psi(s)ds = \infty.$$

Ultimately, we combine the sufficient frameworks of Theorem 3.3 with Theorem 3.4 to conclude the following.

Corollary 1. (Refined asymptotic flocking) *Let (X, V) be a global-in-time solution to (1) with*

$$\alpha \geq 1 \quad \textit{and} \quad \min_{i,j \in [N], i \neq j} \|x_i^0 - x_j^0\| > 0.$$

Further assume that there are two strictly positive constants $D_{X_1}^\infty$ and $D_{X_2}^\infty$ such that

$$D_V(0) = \kappa \int_{D_X(0)}^{D_{X_1}^\infty} \psi(s)ds \leq \max\left(\kappa \int_{D_X(0)}^{\infty} \psi(s)ds,\ \kappa\psi(D_{X_2}^\infty)(D_{X_2}^\infty - D_X(0))\right).$$

Then, it follows that the following asymptotic flocking holds.

1. *(Group formation) $D_X(t) \leq \max\left(D_{X_1}^\infty, D_{X_2}^\infty\right)$,*
2. *(Velocity alignment) $D_V(t) \leq D_V(0)\exp\left(-\kappa\psi\left(\max\left(D_{X_1}^\infty, D_{X_2}^\infty\right)\right)t\right), \quad t \in [0, \infty)$.*

Thus, we attain a larger admissible set in terms of the initial data and system parameters where asymptotic flocking occurs in (1) with a strongly singular kernel.

3.2. **Weakly singular kernel.** In this subsubsection, we present the global well-posedness of (1) with the weakly singular interaction kernel

$$\psi(r) = \frac{1}{r^\alpha}, \quad 0 < \alpha < 1 \quad \text{and} \quad r > 0.$$

For this, we first provide the following proposition regarding the existence of a collisional phenomenon of the two-particle system (1) on $\mathbb{R}^1$.

Proposition 3. [8, 16, 42] *Let (X, V) be a solution to the two-particle system (1) such that*

$$0 < \alpha < 1, \quad d = 1, \quad x_1^0 \neq x_2^0.$$

Then, there exist sufficient conditions only in terms of the initial data and system parameters satisfying a finite-in-time collision. That is, there is a strictly positive time $t_0 \in (0, \infty)$ such that

$$x_1(t_0) = x_2(t_0).$$

Thus, due to Proposition 3, we easily observe that the global well-posedness does not hold for arbitrary noncollisional position data. Hence, we deduce that the global well-posedness may be guaranteed under more restrictive sufficient conditions than those of Theorem 3.1 in the CSS model (1) with a weakly singular interaction kernel.

Theorem 3.5. (Global well-posedness and asymptotic flocking) *Let (X, V) be a solution to (1) with*

$$0 < \alpha < 1 \quad \textit{and} \quad \min_{i,j \in [N], i \neq j} \|x_i^0 - x_j^0\| > 0.$$

Further suppose that there exists a strictly positive number $D_{X_1}^\infty$ satisfying

$$D_V(0) = \kappa \int_{D_X(0)}^{D_{X_1}^\infty} \psi(s)ds < \kappa\psi(D_{X_1}^\infty) \min_{i,j \in [N], i \neq j} \|x_i^0 - x_j^0\|. \tag{13}$$

Then, we have the global well-posedness of (1) with a weakly singular kernel.

More precisely, we attain the strict positivity of the relative distance between the pairwise particles along (1) *with* $0 < \alpha < 1$*:*

$$\inf_{0\le t<\infty} \min_{i,j\in[N],i\neq j} \|x_i - x_j\| \ge \min_{i,j\in[N],i\neq j} \|x_i^0 - x_j^0\| - \frac{D_V(0)}{\kappa\psi(D_{X_1}^\infty)} > 0.$$

Furthermore, we gain the following asymptotic flocking estimate, as follows:

1. *(Group formation)* $D_X(t) \le D_{X_1}^\infty, \quad \forall t \in [0,\infty)$,
2. *(Velocity alignment)* $D_V(t) \le D_V(0)\exp\left(-\kappa\psi(D_{X_1}^\infty)t\right)$.

Proof. Suppose that the local well-posedness of (1) with $0 < \alpha < 1$ holds on $t \in (0, t_*)$ for $t_* \in (0,\infty)$. Then, it follows from Theorem 3.4 and the condition (13) that

$$\begin{aligned}
\|x_i(t) - x_j(t)\| &\ge \|x_i^0 - x_j^0\| - \int_0^t \|v_i(s) - v_j(s)\| ds \\
&\ge \|x_i^0 - x_j^0\| - \int_0^t D_V(s) ds \\
&\ge \|x_i^0 - x_j^0\| - \int_0^\infty D_V(s) ds \\
&\ge \|x_i^0 - x_j^0\| - \frac{D_V(0)}{\kappa\psi(D_{X_1}^\infty)} \\
&\ge \min_{i,j\in[N],i\neq j} \|x_i^0 - x_j^0\| - \frac{D_V(0)}{\kappa\psi(D_{X_1}^\infty)} > 0.
\end{aligned}$$

Therefore, by the standard Cauchy–Lipschitz theory and Lemma 2.2, we demonstrate that there exists a positive ϵ such that the uniqueness and existence of the solution to (1) with $0 < \alpha < 1$ on $(0, t_* + \epsilon)$ can be guaranteed. Hence, we obtain the global well-posedness and strict positivity of the relative distance between each pair of particles. Thus, the same arguments employed in Theorem 3.4 with Remark 3 can be employed to obtain the asymptotic flocking of (1) with $0 < \alpha < 1$. Consequently, we reach the desired assertions. □

4. **Global well-posedness and the emergent dynamics of TCSS.** In this section, we recall the previous results for the global well-posedness of (2) with $\beta \ge 1$, equivalently,

$$\phi(r) = \frac{1}{r^\beta}, \quad \beta \ge 1 \quad \text{and} \quad r > 0.$$

We describe the basic dissipative structures in terms of D_X, D_V, and D_T under $0 < \beta < \infty$. For this, we employ some useful functionals, as in Section 3.1, to derive several differential inequalities with respect to *position-velocity-temperature*, which are independent of the number of particles N. As the main results of this section, we present the global well-posedness and emergent dynamics of (2).

Before we describe the main results, we revisit the basic notion for the asymptotic flocking for the TCSS model (2).

Definition 4.1. Let $Z := (X, V, T)$ be a global-in-time solution to (2). The configuration Z exhibits the asymptotic flocking if

$$(i) \text{ (Group formation) } \iff \sup_{0\le t<\infty} \max_{i,j\in[N]} \|x_i(t) - x_j(t)\| < \infty,$$

$$(ii) \text{ (Velocity alignment) } \iff \lim_{t\to\infty} \max_{i,j\in[N]} \|v_j(t) - v_i(t)\| = 0,$$

$$(iii) \text{ (Temperature equilibrium) } \iff \lim_{t\to\infty} \max_{i,j\in[N]} |T_j(t) - T_i(t)| = 0.$$

To analyze the asymptotic flocking phenomenon of (2), the existence and uniqueness of the solution to (2) on a global time interval are essential. Therefore, a noncollision result is crucial to obtain the global well-posedenss of (2).

4.1. **Previous result and dissipative structures.** In this subsection, we briefly introduce the previous result related to the global well-posedness of (2) with a strongly singular kernel studied in [16].

Proposition 4. [16] (Global well-posedness of TCSS with $\beta \ge 1$) *Suppose that* (X, V, T) *is a solution to* (2) *satisfying*

$$1 \le \beta \le \frac{\gamma}{2} \quad \textit{and} \quad \min_{i,j\in[N], i\ne j} \|x_i^0 - x_j^0\| > 0.$$

Then, we have the global well-posedness (i.e., global collisionless state) of (2) *in the sense that*

$$x_i(t) \ne x_j(t), \quad (i,j) \in [N]^2, \quad i \ne j \quad \textit{and} \quad \forall t \in [0, \infty).$$

However, in this paper, we propose reasonable sufficient frameworks for the emergent dynamics in Section 4 and uniform stability independent of N in Section 5, going beyond the previous paper [16]. This independence leads to deriving a kinetic Vlasov equation corresponding to the particle model (2) via the uniform-in-time mean-field limit. This kinetic equation represents the dynamics of an infinite number of particles based on the standard BBGKY hierarchy. (For the related papers on the uniform-in-time mean-field limit and the BBGKY hierarchy, we refer to [4, 5, 10, 20, 36, 38, 40].) Furthermore, the authors of the previous literature [16] provided an inaccurate framework for the emergent dynamics of (2) when $\beta \ge 1$. Moreover, they did not provide a sufficient framework for the global well-posedness and emergent dynamics of (2) when $0 < \beta < 1$. To remedy these issues, we must first provide the following lemma concerning the dissipative differential inequalities of (2) to establish sufficient frameworks for global well-posedness when $0 < \beta < 1$ and the emergent dynamics for $0 < \beta < \infty$.

Next, we consider the following functional Φ_{ij} defined by

$$\begin{cases} \Phi_{ij}(t) := \dfrac{\phi(\|x_i - x_j\|)}{N} \quad \text{for} \quad i,j \in [N], \quad i \ne j, \\ \Phi_{ii}(t) := \phi(d_X) - \dfrac{\sum_{j\ne i} \phi(\|x_i - x_j\|)}{N}. \end{cases}$$

We can easily verify that Φ_{ij} satisfies the following properties:

1. $\Phi_{ij} \ge \dfrac{\phi_{ij}}{N} \quad \text{for} \quad i,j \in [N], \quad i \ne j, \quad \text{and} \qquad \displaystyle\sum_{j=1}^N \Phi_{ij} = \phi(d_X),$

2. $\sum_{j=1}^N \Phi_{ij}\left(\frac{v_j}{T_j}-\frac{v_i}{T_i}\right)=\sum_{j\neq i}\frac{\phi_{ij}}{N}\left(\frac{v_j}{T_j}-\frac{v_i}{T_i}\right)$, where $\phi_{ij}:=\phi(\|x_i-x_j\|)$.

Moreover, we also employ the functional Ψ defined by

$$\begin{cases} \Psi_{ij}(t) := \dfrac{\zeta(\|x_i-x_j\|)}{N} \quad \text{for} \quad i,j\in[N], \quad i\neq j, \\ \Psi_{ii}(t) := \zeta\left(d_X\right) - \dfrac{\sum_{j=1,j\neq i}^N \zeta(\|x_i-x_j\|)}{N}. \end{cases}$$

Then, we observe that Ψ_{ij} satisfies the following properties:

1. $\Psi_{ij}\geq \frac{\zeta_{ij}}{N}$ for $i,j\in[N]$, $i\neq j$, and $\sum_{j=1}^N \Psi_{ij}=\zeta\left(d_X\right)$,
2. $\sum_{j=1}^N \Psi_{ij}\left(\frac{1}{T_i}-\frac{1}{T_j}\right)=\sum_{j=1}^N\frac{\zeta_{ij}}{N}\left(\frac{1}{T_i}-\frac{1}{T_j}\right)$, where $\zeta_{ij}:=\zeta(\|x_i-x_j\|)$.

We can now derive several dissipative differential inequalities of (1.2) with $0<\beta<\infty$ using Φ and Ψ.

Lemma 4.2. *Let (X,V,T) be a solution to (2) on $[0,\tau)$ for $\tau\in(0,\infty]$ such that*

$$0<\beta<\infty \quad \textit{and} \quad \min_{i,j\in[N],i\neq j}\|x_i^0-x_j^0\|>0.$$

Then, for a.e. $t\in(0,\tau)$,

1. $\left|\frac{dD_X}{dt}\right|\leq D_V$,
2. $\frac{dD_V}{dt}\leq -\frac{\kappa_1\phi(D_X)}{T_M^\infty}D_V+\frac{2\kappa_1\phi(d_X)}{(T_m^\infty)^2}D_TD_V$,
3. $\frac{dD_T}{dt}\leq -\frac{\kappa_2\zeta(D_X)}{(T_M^\infty)^2}D_T$.

Proof. For the first assertion, we note that

$$\left|\frac{d\|x_i-x_j\|^2}{dt}\right| = 2|\langle x_i-x_j, v_i-v_j\rangle| \leq 2\|x_i-x_j\|\|v_i-v_j\|,$$

which combines with the Cauchy–Schwarz inequality, for a.e. $t\in(0,\tau)$, to yield

$$\left|\frac{dD_X(t)}{dt}\right|\leq D_V(t).$$

Next, we prove the third assertion with respect to the L^∞-diameter for temperature. For this, we select two indices M_t and m_t, depending on time t, satisfying

$$D_T(t)=T_{M_t}(t)-T_{m_t}(t), \quad m_t, M_t\in[N].$$

Then, it follows from the properties of Ψ_{ij} and $(2)_3$ that, for a.e. $t\in(0,\tau)$,

$$\begin{aligned} \frac{dD_T}{dt} &= \dot{T}_{M_t}-\dot{T}_{m_t} \\ &= \frac{\kappa_2}{N}\sum_{k=1}^N \zeta_{M_tk}\left(\frac{1}{T_{M_t}}-\frac{1}{T_k}\right)-\frac{\kappa_2}{N}\sum_{k=1}^N\zeta_{m_tk}\left(\frac{1}{T_{m_t}}-\frac{1}{T_k}\right) \\ &= \kappa_2\sum_{k=1}^N\Psi_{M_tk}\left(\frac{1}{T_{M_t}}-\frac{1}{T_k}\right)-\kappa_2\sum_{k=1}^N\Psi_{m_tk}\left(\frac{1}{T_{m_t}}-\frac{1}{T_k}\right) \end{aligned}$$

$$
\begin{aligned}
&= \kappa_2 \zeta \left(\min_{1 \le i,j \le N} \|x_i - x_j\| \right) \left(\frac{1}{T_{M_t}} - \frac{1}{T_{m_t}} \right) - \kappa_2 \sum_{k=1}^N \frac{1}{T_k} \left(\Psi_{M_t k} - \Psi_{m_t k} \right) \\
&= \kappa_2 \zeta \left(\min_{1 \le i,j \le N} \|x_i - x_j\| \right) \left(\frac{1}{T_{M_t}} - \frac{1}{T_{m_t}} \right) \\
&\quad - \kappa_2 \sum_{k=1}^N \frac{1}{T_k} \left(\Psi_{M_t k} - \min(\Psi_{M_t k}, \Psi_{m_t k}) + \min(\Psi_{M_t k}, \Psi_{m_t k}) - \Psi_{m_t k} \right) \\
&\le \kappa_2 \zeta \left(\min_{1 \le i,j \le N} \|x_i - x_j\| \right) \left(\frac{1}{T_{M_t}} - \frac{1}{T_{m_t}} \right) + \frac{\kappa_2}{T_{m_t}} \sum_{k=1}^N (\Psi_{m_t k} - \min(\Psi_{M_t k}, \Psi_{m_t k})) \\
&\quad - \frac{\kappa_2}{T_{M_t}} \sum_{k=1}^N (\Psi_{M_t k} - \min(\Psi_{M_t k}, \Psi_{m_t k})) \\
&= -\kappa_2 \left(\frac{1}{T_{m_t}} - \frac{1}{T_{M_t}} \right) \sum_{k=1}^N (\min(\Psi_{M_t k}, \Psi_{m_t k})) \le -\frac{\kappa_2 D_T}{(T_M^\infty)^2} \sum_{k=1}^N (\min(\Psi_{M_t k}, \Psi_{m_t k})) \\
&\le -\frac{\kappa_2 \zeta(D_X)}{(T_M^\infty)^2} D_T.
\end{aligned}
$$

For the second assertion, we take two indices i_t and j_t, depending on time $t \in (0, \tau)$, satisfying

$$
D_V(t) := \|v_{i_t}(t) - v_{j_t}(t)\|, \quad i_t, j_t \in [N],
$$

and assume that $\sum_{i=1}^N v_i = 0$ without loss of generality. Then, we use $(2)_2$ to obtain that for a.e. $t \in (0, \tau)$,

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|v_{i_t} - v_{j_t}\|^2 = \left\langle v_{i_t} - v_{j_t}, \frac{dv_{i_t}}{dt} - \frac{dv_{j_t}}{dt} \right\rangle \\
&\quad = \left\langle v_{i_t} - v_{j_t}, \frac{\kappa_1}{N} \sum_{k=1}^N \phi_{i_t k} \left(\frac{v_k}{T_k} - \frac{v_{i_t}}{T_{i_t}} \right) - \frac{\kappa_1}{N} \sum_{k=1}^N \phi_{j_t k} \left(\frac{v_k}{T_k} - \frac{v_{j_t}}{T_{j_t}} \right) \right\rangle \\
&\quad = \left\langle v_{i_t} - v_{j_t}, \kappa_1 \sum_{k=1}^N \Phi_{i_t k} \left(\frac{v_k}{T_k} - \frac{v_{i_t}}{T_{i_t}} \right) - \kappa_1 \sum_{k=1}^N \Phi_{j_t k} \left(\frac{v_k}{T_k} - \frac{v_{j_t}}{T_{j_t}} \right) \right\rangle \\
&\quad = -\kappa_1 \phi\left(d_X\right) \left\langle v_{i_t} - v_{j_t}, \frac{v_{i_t}}{T_{i_t}} - \frac{v_{j_t}}{T_{j_t}} \right\rangle + \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N (\Phi_{i_t k} - \Phi_{j_t k}) \frac{v_k}{T_k} \right\rangle.
\end{aligned}
$$

We apply the properties of Φ and the following relation

$$
\Phi_{i_t k} - \Phi_{j_t k} = \Phi_{i_t k} - \min(\Phi_{i_t k}, \Phi_{j_t k}) + \min(\Phi_{i_t k}, \Phi_{j_t k}) - \Phi_{j_t k}
$$

to have

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|v_{i_t} - v_{j_t}\|^2 \\
&= -\kappa_1 \phi\left(d_X\right) \left\langle v_{i_t} - v_{j_t}, \frac{v_{i_t}}{T_{i_t}} - \frac{v_{j_t}}{T_{j_t}} \right\rangle \\
&\quad + \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N (\Phi_{i_t k} - \min(\Phi_{i_t k}, \Psi_{j_t k}) + \min(\Phi_{i_t k}, \Phi_{j_t k}) - \Phi_{j_t k}) \frac{v_k}{T_k} \right\rangle \\
&:= \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
$$

•(Estimate of $\mathcal{I}_2$) For this, we apply the following inequality

$$\langle v_{i_t} - v_{j_t}, v_{j_t} \rangle \le \langle v_{i_t} - v_{j_t}, v_k \rangle \le \langle v_{i_t} - v_{j_t}, v_{i_t} \rangle$$

to $\mathcal{I}_2$ to show that

$$\begin{aligned}
\mathcal{I}_2 &\le \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N (\Phi_{i_t k} - \min(\Phi_{i_t k}, \Phi_{j_t k})) \frac{v_{i_t}}{T_k} \right\rangle \\
&\quad + \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N (\min(\Phi_{i_t k}, \Phi_{j_t k}) - \Phi_{j_t k}) \frac{v_{j_t}}{T_k} \right\rangle \\
&= -\kappa_1 \sum_{k=1}^N \min(\Phi_{i_t k}, \Phi_{j_t k}) \frac{\|v_{i_t} - v_{j_t}\|^2}{T_k} \\
&\quad + \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N \Phi_{i_t k} \frac{v_{i_t}}{T_k} \right\rangle - \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N \Phi_{j_t k} \frac{v_{j_t}}{T_k} \right\rangle .
\end{aligned} \tag{14}$$

where we used the nonnegativity of Φ. Then, it follows from (14), Proposition 2 and the properties of Φ that for a.e. $t \in (0, \tau)$,

$$\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 &\le -\kappa_1 \sum_{k=1}^N \min(\Phi_{i_t k}, \Phi_{j_t k}) \frac{\|v_{i_t} - v_{j_t}\|^2}{T_k} \\
&\quad + \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N \Phi_{i_t k} v_{i_t} \left(\frac{1}{T_k} - \frac{1}{T_{i_t}} \right) \right\rangle \\
&\quad - \kappa_1 \left\langle v_{i_t} - v_{j_t}, \sum_{k=1}^N \Phi_{j_t k} v_{j_t} \left(\frac{1}{T_k} - \frac{1}{T_{j_t}} \right) \right\rangle \\
&\le -\kappa_1 \phi(D_X) \frac{\|v_{i_t} - v_{j_t}\|^2}{T_M^\infty} + \kappa_1 \|v_{i_t} - v_{j_t}\| \|v_{i_t}\| \sum_{k=1}^N \Phi_{i_t k} \left| \frac{1}{T_k} - \frac{1}{T_{i_t}} \right| \\
&\quad + \kappa_1 \|v_{i_t} - v_{j_t}\| \|v_{j_t}\| \sum_{k=1}^N \Phi_{j_t k} \left| \frac{1}{T_k} - \frac{1}{T_{j_t}} \right| .
\end{aligned}$$

Therefore, using Proposition 2, the property of Φ and the following relations:

$$\left| \frac{1}{T_i} - \frac{1}{T_j} \right| \le \frac{D_T}{(T_m^\infty)^2}, \quad \|v_i\| = \left| v_i - \frac{\sum_{j=1}^N v_j}{N} \right| \le D_V, \quad i, j \in [N],$$

one can show that

$$\frac{1}{2} \frac{dD_V^2}{dt} = \mathcal{I}_1 + \mathcal{I}_2 \le -\frac{\kappa_1 \phi(D_X)}{T_M^\infty} D_V^2 + \frac{2\kappa_1 \phi(d_X)}{(T_m^\infty)^2} D_T D_V^2,$$

which implies that for a.e. $t \in (0, \tau)$,

$$\frac{dD_V}{dt} \le -\frac{\kappa_1 \phi(D_X)}{T_M^\infty} D_V + \frac{2\kappa_1 \phi(d_X)}{(T_m^\infty)^2} D_T D_V.$$

Finally, we get the desired second assertion. □

4.2. **Collision avoidance and asymptotic flocking.** In this subsection, we study the sufficient framework for the global well-posedness and emergent dynamics of (2) under $0 < \beta < \infty$ using Lemma 4.2 and the bootstrapping argument. More concretely, we present a sufficient framework for the strict positivity of each relative distance for all particles in (2), assuming $0 < \beta < \infty$.

Theorem 4.3. (Global well-posedness and asymptotic flocking) *Suppose that the initial data and system parameters satisfy*

$$0 < \beta < \infty \quad \textit{and} \quad \min_{i,j\in[N],i\neq j} \|x_i^0 - x_j^0\| > 0.$$

Further assume that there exist two positive constants d_X^∞ and D_X^∞ such that

$$\begin{aligned} &d_X(0) - \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right)\frac{D_V(0)T_M^\infty}{\kappa_1\phi(D_X^\infty)} \geq d_X^\infty, \\ &D_X(0) + \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right)\frac{D_V(0)T_M^\infty}{\kappa_1\phi(D_X^\infty)} \leq D_X^\infty. \end{aligned} \tag{15}$$

Then, we have the following global well-posedness and asymptotic flocking results on $(0,\infty)$:

1. *(Collision avoidance and Group formation)*
$$D_X(t) < D_X^\infty \quad \textit{and} \quad d_X(t) > d_X^\infty,$$

2. *(Velocity alignment)*
$$D_V(t) \leq \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right) D_V(0)\exp\left(-\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty}t\right),$$

3. *(Temperature equilibrium)*
$$D_T(t) \leq D_T(0)\exp\left(-\kappa_2\zeta\left(\frac{D_X^\infty}{(T_M^\infty)^2}\right)t\right).$$

Proof. First, suppose that $[0,\tau)$ is a maximal interval for which an unique solution of (2) under $0 < \beta < \infty$ exists and $\tau < \infty$ for the proof by contradiction. When $D_V(0) = 0$, then we have nothing to prove. Therefore, we now assume $D_V(0) > 0$. We note from (15) that the following set

$$\mathcal{S} := \{t > 0 \mid d_X(s) > d_X^\infty, \quad D_X(s) < D_X^\infty, \quad \forall s \in (0,t) \quad \text{and} \quad t \leq \tau\}$$

is nonempty. Here, we define $t^* := \sup \mathcal{S} > 0$, which implies that

$$D_X(t^*) = D_X^\infty \quad \text{or} \quad d_X(t^*) = d_X^\infty.$$

Thus, it suffices to demonstrate that

$$t^* = \tau.$$

Suppose that $t^* < \tau$ for the proof by contradiction. Then, we utilize the third assertion of Lemma 4.2 with $\mathcal{S}$ to attain that

$$\left|\frac{dD_X}{dt}\right| \leq D_V, \quad \frac{dD_T}{dt} \leq -\frac{\kappa_2\zeta(D_X)}{(T_M^\infty)^2}D_T \leq -\frac{\kappa_2\zeta(D_X^\infty)}{(T_M^\infty)^2}D_T, \quad \text{a.e. } t \in (0,t^*),$$

where, Grönwall's lemma yields

$$D_T(t) \le D_T(0)\exp\left(-\frac{\kappa_2\zeta(D_X^\infty)}{(T_M^\infty)^2}t\right), \quad \forall t \in [0, t^*]. \tag{16}$$

Next, we use the second assertion of Lemma 4.2 together with $\mathcal{S}$ and (16) to get that for a.e. $t \in (0, t^*)$,

$$\begin{aligned}\frac{dD_V}{dt} &\le -\frac{\kappa_1\phi(D_X)}{T_M^\infty}D_V + \frac{2\kappa_1\phi(d_X)}{(T_m^\infty)^2}D_T D_V\\ &\le -\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty}D_V + \frac{2\kappa_1\phi(d_X^\infty)}{(T_m^\infty)^2}D_T D_V\\ &\le -\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty}D_V + \frac{2\kappa_1\phi(d_X^\infty)D_T(0)}{(T_m^\infty)^2}\exp\left(-\frac{\kappa_2\zeta(D_X^\infty)}{(T_M^\infty)^2}t\right)D_V.\end{aligned} \tag{17}$$

Hence, we apply the comparison principle for ODE to (17) to obtain

$$D_V(t) \le \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right)D_V(0)\exp\left(-\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty}t\right). \tag{18}$$

Therefore, it follows from (18) and the first assertion of Lemma 4.2 that

$$\begin{aligned}D_X(t) &= D_X(0) + \int_0^t \frac{dD_X(s)}{ds}ds \le D_X(0) + \int_0^t D_V(s)ds\\ &\le D_X(0) + \int_0^t \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right)D_V(0)\exp\left(-\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty}s\right)ds\\ &< D_X(0) + \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right)\frac{D_V(0)T_M^\infty}{\kappa_1\phi(D_X^\infty)} \le D_X^\infty, \forall t \in [0, t^*],\end{aligned} \tag{19}$$

which means that $D_X(t^*) < D_X^\infty$. Moreover, we again employ the first assertion of Lemma 4.2 to deduce that

$$\begin{aligned}d_X(t) &= d_X(0) - \int_0^t \frac{d(d_X(s))}{ds}ds \le d_X(0) - \int_0^t D_V(s)ds\\ &\ge d_X(0) - \int_0^t \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right)D_V(0)\exp\left(-\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty}s\right)ds\\ &> d_X(0) - \exp\left(\frac{2\kappa_1 D_T(0)\phi(d_X^\infty)(T_M^\infty)^2}{\kappa_2\zeta(D_X^\infty)(T_m^\infty)^2}\right)\frac{D_V(0)T_M^\infty}{\kappa_1\phi(D_X^\infty)} \ge d_X^\infty, \quad \forall t \in [0, t^*],\end{aligned} \tag{20}$$

which leads to $d_X(t^*) > d_X^\infty$. Thus, we have $t^* = \tau$. However, from the standard Cauchy–Lipschitz theory with Proposition 2, (18), (20), and the conservation of momentum (i.e., $\sum_{k=1}^N v_i$ = constant), there exists a positive ϵ such that the uniqueness and existence of solution to (2) with $0 < \beta < \infty$ on $(0, \tau + \epsilon)$ can be guaranteed. In conclusion, we obtain the global well-posedness (i.e., $\tau = \infty$). Consequently, it follows from the set $\mathcal{S}$ with $t^* = \infty$, (16), (18), (19), and (20) that the desired results hold. □

5. **Uniform L^2-stability of TCSS.** In this section, we define a sufficient framework for the uniform L^2-stability estimate of (2) with an arbitrary singular interaction kernel on $0 < \beta < \infty$ and dimension $d \in \mathbb{N}$. With Remark 1, if we set $T_1^0 = \cdots = T_N^0 = T^\infty > 0$, it follows that the TCSS model (2) can be reduced to the CSS model (1). Therefore, it suffices to construct a sufficient framework for the L^2-uniform stability of (2). To do this, we revisit the TCSS model (2) which is governed by the following system in terms of (X, V, T):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_i - x_j\|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i} \right), \\ \dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), \\ (x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^{2d} \times \mathbb{R}_{>0}, \\ \displaystyle\sum_{i=1}^{N} v_i^0 = N v^\infty = 0, \quad \displaystyle\sum_{i=1}^{N} T_i^0 = N T^\infty. \end{cases} \tag{21}$$

The definition of the uniform L^2-stability estimate of (2) is as follows.

Definition 5.1. *(Uniform L^2-stability)*: For the two solutions (X, V, T) and $(\bar{X}, \bar{V}, \bar{T})$ to (2) with the initial data (X^0, V^0, T^0) and $(\bar{X}^0, \bar{V}^0, \bar{T}^0)$, respectively, if there exists a positive constant G^∞ independent of t such that

$$\sup_{0 \le t < \infty} \left(\|X(t) - \bar{X}(t)\| + \|V(t) - \bar{V}(t)\| \right) \le G^\infty \left(\|X^0 - \bar{X}^0\| + \|V^0 - \bar{V}^0\| \right),$$

then we say that the equation (2) satisfies the uniform L^2-stability.

In particular, we estimate G^∞ defined in Definition 5.1 so that this is independent of the number of particle N in (2) because the independence of N is very important for deriving uniform-in-time mean-field limits based on the standard BBGKY method. Assume that there exist two global-in-time solutions (X, V, T) and $(\bar{X}, \bar{V}, \bar{T})$ of (2) with the initial data (X^0, V^0, T^0) and $(\bar{X}^0, \bar{V}^0, \bar{T}^0)$, respectively. Then, without loss of generality, we further suppose that

$$v^\infty = \bar{v}^\infty = 0,$$

where the assumption $v^\infty = \bar{v}^\infty$ for two average momentums v^∞ and $\bar{v}^\infty$ is very crucial to derive the uniform L^2-stability of (2). Indeed, under appropriate conditions for asymptotic flocking, the uniform stability between two clusters with different average velocities cannot be established. However, we verify the uniform L^2-stability estimate when the averages of the sum of the initial temperatures are different from each other, that is,

$$T^\infty = \bar{T}^\infty \quad \text{or} \quad T^\infty \neq \bar{T}^\infty.$$

We use the following simple notation:

- $\phi(\|x_i - x_j\|) := \phi_{ij}, \zeta(\|x_i - x_j\|) := \zeta_{ij}, \phi(\|\bar{x}_i - \bar{x}_j\|) := \bar{\phi}_{ij}, \zeta(\|\bar{x}_i - \bar{x}_j\|) := \bar{\zeta}_{ij}$
- $D_{\bar{X}},\ D_{\bar{V}},\ D_{\bar{T}},\ \bar{T}_m^\infty,\ \bar{T}_M^\infty$ is defined similarly as before.

We study the system of differential inequalities in terms of $\|X - \bar{X}\|$, $\|V - \bar{V}\|$, and $\|T - \bar{T}\|$ to deduce a sufficient framework for the uniform L^2-stability of (21).

Lemma 5.2. *Suppose that (X, V, T) and $(\bar{X}, \bar{V}, \bar{T})$ are two global-in-time solutions to (21) such that*

$$0 < \beta < \infty, \quad \min\left(\min_{i,j \in [N], i \neq j} \|\bar{x}_i^0 - \bar{x}_j^0\|, \min_{i,j \in [N], i \neq j} \|x_i^0 - x_j^0\| \right) > 0.$$

and (15) hold, respectively. Then, it follows that for a.e. $t \in (0, \infty)$,

1. *(Differentiation of the difference between X and $\bar{X}$)*

$$\left|\frac{d\|X-\bar{X}\|}{dt}\right| \le \|V-\bar{V}\|,$$

2. *(Differentiation of the difference between V and $\bar{V}$)*

$$\frac{d\|V-\bar{V}\|}{dt} \le -\left(\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty} - \frac{\kappa_1\phi(d_X^\infty)D_T}{(T_m^\infty)^2}\right)\|V-\bar{V}\| + \frac{2\kappa_1\phi(d_X^\infty)D_{\bar{V}}}{T_m^\infty\bar{T}_m^\infty}\|T-\bar{T}\| + \frac{2\beta\kappa_1}{\min(d_X,d_{\bar{X}})^{\beta+1}}\left(\frac{D_{\bar{V}}}{\bar{T}_m^\infty} + \frac{D_{\bar{V}}D_{\bar{T}}}{(\bar{T}_m^\infty)^2}\right)\|X-\bar{X}\|,$$

3. *(Differentiation of the difference between T and $\bar{T}$)*

$$\frac{d\|T-\bar{T}\|}{dt} \le \frac{2\gamma\kappa_2 D_{\bar{T}}}{(\bar{T}_m^\infty)^2(\min(d_X^\infty,d_{\bar{X}}^\infty))^{\beta+1}}\|X-\bar{X}\| + \kappa_2\zeta(d_X^\infty)\left(\frac{D_{\bar{T}}}{T_m^\infty(\bar{T}_m^\infty)^2} + \frac{D_T}{\bar{T}_m^\infty(T_m^\infty)^2}\right)\|T-\bar{T}\|.$$

Proof. • (Proof of (1)) For the first assertion, we apply the Cauchy–Schwarz inequality to show that

$$\frac{1}{2}\left|\frac{d\|X-\bar{X}\|^2}{dt}\right| = |\langle X-\bar{X}, V-\bar{V}\rangle| \le \|X-\bar{X}\|\|V-\bar{V}\|,$$

which implies that for a.e. $t\in(0,\infty)$,

$$\left|\frac{d\|X-\bar{X}\|}{dt}\right| \le \|V-\bar{V}\|.$$

• (Proof of (3)) For the third assertion, we use $(21)_3$ for a.e. $t\in(0,\infty)$ to obtain the following:

$$\begin{aligned}\frac{1}{2}\frac{d\|T-\bar{T}\|^2}{dt} &= \frac{\kappa_2}{N}\sum_{i=1}^N (T_i-\bar{T}_i)\left(\sum_{j=1}^N \zeta_{ij}\left(\frac{1}{T_i}-\frac{1}{T_j}\right) - \sum_{j=1}^N \bar{\zeta}_{ij}\left(\frac{1}{\bar{T}_i}-\frac{1}{\bar{T}_j}\right)\right)\\ &= \frac{\kappa_2}{N}\sum_{i,j=1}^N \left(\zeta_{ij}-\bar{\zeta}_{ij}\right)(T_i-\bar{T}_i)\left(\frac{1}{\bar{T}_i}-\frac{1}{\bar{T}_j}\right)\\ &\quad + \frac{\kappa_2}{N}\sum_{i,j=1}^N \zeta_{ij}(T_i-\bar{T}_i)\left(\frac{1}{T_i}-\frac{1}{T_j}-\frac{1}{\bar{T}_i}+\frac{1}{\bar{T}_j}\right)\\ &:= \mathcal{I}_1+\mathcal{I}_2.\end{aligned}$$

⋄ (Estimate of $\mathcal{I}_1$) To estimate $\mathcal{I}_1$, we first observe from Proposition 2 that for a.e. $t\in(0,\infty)$,

$$\begin{aligned}\mathcal{I}_1 &= \frac{\kappa_2}{N}\sum_{i,j=1}^N \left(\zeta_{ij}-\bar{\zeta}_{ij}\right)(T_i-\bar{T}_i)\left(\frac{1}{\bar{T}_i}-\frac{1}{\bar{T}_j}\right)\\ &\le \frac{\kappa_2}{N}\sum_{i,j=1}^N \left|\zeta_{ij}-\bar{\zeta}_{ij}\right||T_i-\bar{T}_i|\left|\frac{1}{\bar{T}_i}-\frac{1}{\bar{T}_j}\right|\\ &\le \frac{\kappa_2 D_{\bar{T}}}{N(\bar{T}_m^\infty)^2}\sum_{i,j=1}^N \left|\zeta_{ij}-\bar{\zeta}_{ij}\right||T_i-\bar{T}_i|\end{aligned}$$

$$\leq \frac{\gamma\kappa_2 D_{\bar{T}}}{N(\bar{T}_m^\infty)^2(\min(d_X^\infty, d_{\bar{X}}^\infty))^{\gamma+1}} \sum_{i,j=1}^N (\|x_i - \bar{x}_i\| + \|x_j - \bar{x}_j\|)|T_i - \bar{T}_i|$$
$$\leq \frac{2\gamma\kappa_2 D_{\bar{T}}}{(\bar{T}_m^\infty)^2(\min(d_X^\infty, d_{\bar{X}}^\infty))^{\gamma+1}} \|X - \bar{X}\|\|T - \bar{T}\|,$$

where we used 1) the uniform boundedness of the Lipschitz norm of the ζ with triangle inequality and Theorem 4.1 to estimate the third inequality, and 2) the Cauchy–Schwarz inequality to estimate the last inequality.

$\diamond$ (The estimate of $\mathcal{I}_2$) (Estimate of $\mathcal{I}_2$) We apply the standard technique of interchanging i and j and dividing by 2 to yield

$$\begin{aligned}
\mathcal{I}_2 &= \frac{\kappa_2}{N} \sum_{i,j=1}^N \zeta_{ij}(T_i - \bar{T}_i) \left(\frac{1}{T_i} - \frac{1}{T_j} - \frac{1}{\bar{T}_i} + \frac{1}{\bar{T}_j} \right) \\
&= \frac{\kappa_2}{2N} \sum_{i,j=1}^N \zeta_{ij}((T_i - \bar{T}_i) - (T_j - \bar{T}_j)) \left(\frac{1}{T_i} - \frac{1}{T_j} - \frac{1}{\bar{T}_i} + \frac{1}{\bar{T}_j} \right) \\
&= \frac{\kappa_2}{2N} \sum_{i,j=1}^N \zeta_{ij}((T_i - \bar{T}_i) - (T_j - \bar{T}_j)) \left(\frac{1}{T_i} - \frac{1}{T_j} - \frac{1}{\bar{T}_i} + \frac{1}{\bar{T}_j} \right) \\
&= \frac{\kappa_2}{2N} \sum_{i,j=1}^N \zeta_{ij}((T_i - \bar{T}_i) - (T_j - \bar{T}_j)) \left(\frac{-((T_i - \bar{T}_i) - (T_j - \bar{T}_j))}{T_i\bar{T}_i} \right) \\
&\quad + \frac{\kappa_2}{2N} \sum_{i,j=1}^N \zeta_{ij}((T_i - \bar{T}_i) - (T_j - \bar{T}_j)) \left(T_j - \bar{T}_j\right) \left(\frac{1}{T_j\bar{T}_j} - \frac{1}{T_i\bar{T}_i} \right) \\
&\leq \frac{\kappa_2}{2N} \sum_{i,j=1}^N \zeta_{ij}((T_i - \bar{T}_i) - (T_j - \bar{T}_j)) \left(T_j - \bar{T}_j\right) \left(\frac{1}{T_j\bar{T}_j} - \frac{1}{T_i\bar{T}_i} \right) \\
&\leq \frac{\kappa_2}{2N} \sum_{i,j=1}^N \zeta_{ij}(|T_i - \bar{T}_i| + |T_j - \bar{T}_j|) \left|T_j - \bar{T}_j\right| \left| \frac{1}{T_j\bar{T}_j} - \frac{1}{T_i\bar{T}_i} \right| \\
&\leq \frac{\kappa_2\zeta(d_X^\infty)}{2N} \left(\frac{D_{\bar{T}}}{T_m^\infty(\bar{T}_m^\infty)^2} + \frac{D_T}{\bar{T}_m^\infty(T_m^\infty)^2} \right) \sum_{i,j=1}^N (|T_i - \bar{T}_i| + |T_j - \bar{T}_j|) \left|T_j - \bar{T}_j\right| \\
&\leq \kappa_2\zeta(d_X^\infty) \left(\frac{D_{\bar{T}}}{T_m^\infty(\bar{T}_m^\infty)^2} + \frac{D_T}{\bar{T}_m^\infty(T_m^\infty)^2} \right) \|T - \bar{T}\|^2,
\end{aligned}$$

since Theorem 4.3 holds and

$$\left| \frac{1}{T_j\bar{T}_j} - \frac{1}{T_i\bar{T}_i} \right| \leq \frac{1}{T_i} \left| \frac{1}{\bar{T}_i} - \frac{1}{\bar{T}_j} \right| + \frac{1}{\bar{T}_j} \left| \frac{1}{T_i} - \frac{1}{T_j} \right| \leq \frac{D_{\bar{T}}}{T_m^\infty(\bar{T}_m^\infty)^2} + \frac{D_T}{\bar{T}_m^\infty(T_m^\infty)^2}.$$

Therefore, we combine $\mathcal{I}_1$ with $\mathcal{I}_2$ for a.e. $t \in (0, \infty)$ to attain the following:

$$\begin{aligned}
\frac{d\|T - \bar{T}\|}{dt} &\leq \frac{2\gamma\kappa_2 D_{\bar{T}}}{(\bar{T}_m^\infty)^2(\min(d_X^\infty, d_{\bar{X}}^\infty))^{\beta+1}} \|X - \bar{X}\| \\
&\quad + \kappa_2\zeta(d_X^\infty) \left(\frac{D_{\bar{T}}}{T_m^\infty(\bar{T}_m^\infty)^2} + \frac{D_T}{\bar{T}_m^\infty(T_m^\infty)^2} \right) \|T - \bar{T}\|,
\end{aligned}$$

which gives the desired second assertion of the lemma.

• (Proof of (3)) For this, by using the second equation $(21)_2$, one can easily check that

$$\begin{aligned}\frac{1}{2}\frac{d\|V-\bar{V}\|^2}{dt} &= \frac{\kappa_1}{N}\sum_{i=1}^N \left\langle v_i-\bar{v}_i, \sum_{j=1}^N \phi_{ij}\left(\frac{v_j}{T_j}-\frac{v_i}{T_i}\right) - \sum_{j=1}^N \bar{\phi}_{ij}\left(\frac{\bar{v}_j}{\bar{T}_j}-\frac{\bar{v}_i}{\bar{T}_i}\right)\right\rangle \\ &= \frac{\kappa_1}{N}\sum_{i,j=1}^N \left(\phi_{ij}-\bar{\phi}_{ij}\right)\left\langle v_i-\bar{v}_i, \frac{\bar{v}_j}{\bar{T}_j}-\frac{\bar{v}_i}{\bar{T}_i}\right\rangle \\ &\quad + \frac{\kappa_1}{N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar{v}_i, \frac{v_j}{T_j}-\frac{v_i}{T_i}-\frac{\bar{v}_j}{\bar{T}_j}+\frac{\bar{v}_i}{\bar{T}_i}\right\rangle \\ &:= \mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

⋄ (Estimate of $\mathcal{J}_1$) We employ Theorem 4.1 and the following relation

$$\|\bar{v}_i\| = \left\|\bar{v}_i - \frac{\sum_{i=1}^N \bar{v}_i}{N}\right\| \le \frac{N-1}{N}\cdot D_{\bar{V}} \le D_{\bar{V}}$$

to deduce that

$$\begin{aligned}\mathcal{J}_1 &= \frac{\kappa_1}{N}\sum_{i,j=1}^N \left(\phi_{ij}-\bar{\phi}_{ij}\right)\left\langle v_i-\bar{v}_i, \frac{\bar{v}_j}{\bar{T}_j}-\frac{\bar{v}_i}{\bar{T}_i}\right\rangle \\ &\le \frac{\kappa_1}{N}\sum_{i,j=1}^N \left|\phi_{ij}-\bar{\phi}_{ij}\right| \|v_i-\bar{v}_i\| \left\|\frac{\bar{v}_j}{\bar{T}_j}-\frac{\bar{v}_i}{\bar{T}_i}\right\| \\ &\le \frac{\beta\kappa_1}{N\min(d_X,d_{\bar{X}})^{\beta+1}}\sum_{i,j=1}^N \left(\|x_i-\bar{x}_i\|+\|x_j-\bar{x}_j\|\right)\|v_i-\bar{v}_i\|\left\|\frac{\bar{v}_j}{\bar{T}_j}-\frac{\bar{v}_i}{\bar{T}_i}\right\| \\ &\le \frac{\beta\kappa_1}{N\min(d_X,d_{\bar{X}})^{\beta+1}} \\ &\qquad \times \sum_{i,j=1}^N \left(\|x_i-\bar{x}_i\|+\|x_j-\bar{x}_j\|\right)\|v_i-\bar{v}_i\|\left(\left\|\frac{\bar{v}_j-\bar{v}_i}{\bar{T}_j}\right\|+\|\bar{v}_i\|\left|\frac{1}{\bar{T}_j}-\frac{1}{\bar{T}_i}\right|\right) \\ &\le \frac{\beta\kappa_1}{N\min(d_X,d_{\bar{X}})^{\beta+1}}\sum_{i,j=1}^N \left(\|x_i-\bar{x}_i\|+\|x_j-\bar{x}_j\|\right)\|v_i-\bar{v}_i\|\left(\frac{D_{\bar{V}}}{\bar{T}_m^\infty}+\frac{D_{\bar{V}}D_{\bar{T}}}{(\bar{T}_m^\infty)^2}\right) \\ &\le \frac{2\beta\kappa_1}{\min(d_X,d_{\bar{X}})^{\beta+1}}\cdot\left(\frac{D_{\bar{V}}}{\bar{T}_m^\infty}+\frac{D_{\bar{V}}D_{\bar{T}}}{(\bar{T}_m^\infty)^2}\right)\|X-\bar{X}\|\,\|V-\bar{V}\|,\end{aligned}$$

where we used the Lipschitz norm of $f(r)=\frac{1}{r^\beta},\ r>\min(d_X^\infty,d_{\bar{X}}^\infty)$ in the second inequality and utilized Theorem 4.3 with Proposition 2 in the fourth inequality and used the Cauchy–Schwarz inequality to estimate the last inequality.

⋄ (Estimate of $\mathcal{J}_2$) Again, from the standard technique of interchanging i and j and dividing by 2,

$$\mathcal{J}_2 = \frac{\kappa_1}{N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar{v}_i, \frac{v_j}{T_j}-\frac{v_i}{T_i}-\frac{\bar{v}_j}{\bar{T}_j}+\frac{\bar{v}_i}{\bar{T}_i}\right\rangle$$

$$
\begin{aligned}
&= \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{v_j}{T_j}-\frac{v_i}{T_i}-\frac{\bar v_j}{\bar T_j}+\frac{\bar v_i}{\bar T_i}\right\rangle \\
&= \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{v_j}{T_i}-\frac{v_i}{T_i}-\frac{\bar v_j}{T_i}+\frac{\bar v_i}{T_i}\right\rangle \\
&\quad + \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{v_j}{T_j}-\frac{v_j}{T_i}+\frac{\bar v_j}{T_i}-\frac{\bar v_i}{T_i}-\frac{\bar v_j}{\bar T_j}+\frac{\bar v_i}{\bar T_i}\right\rangle \\
&:= \mathcal{J}_{21}+\mathcal{J}_{22}.
\end{aligned}
$$

⋆ (Estimate of $\mathcal{J}_{21}$) We note from Proposition 2 and Theorem 4.3 that

$$
\begin{aligned}
\mathcal{J}_{21} &= -\frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\frac{\|v_i-\bar v_i-v_j+\bar v_j\|^2}{T_i} \\
&\le -\frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi(D_X^\infty)\frac{\|v_i-\bar v_i-v_j+\bar v_j\|^2}{T_M^\infty} \\
&= -\frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi(D_X^\infty)\left(\frac{\|v_i-\bar v_i\|^2+\|v_j-\bar v_j\|^2}{T_M^\infty}\right) \\
&= -\frac{\kappa_1\phi(D_X^\infty)}{T_M^\infty}\|V-\bar V\|^2,
\end{aligned}
$$

because we assumed that $v^\infty=\bar v^\infty=0$.

⋆ (Estimate of $\mathcal{J}_{22}$) Moreover, due to Theorem 4.3,

$$
\begin{aligned}
\mathcal{J}_{22} &= \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{v_j}{T_j}-\frac{v_j}{T_i}+\frac{\bar v_j}{T_i}-\frac{\bar v_i}{T_i}-\frac{\bar v_j}{\bar T_j}+\frac{\bar v_i}{\bar T_i}\right\rangle \\
&= \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{v_j}{T_j}-\frac{\bar v_j}{T_j}+\frac{\bar v_j}{T_j}-\frac{v_j}{T_i}+\frac{\bar v_j}{T_i}-\frac{\bar v_i}{T_i}-\frac{\bar v_j}{\bar T_j}+\frac{\bar v_i}{\bar T_i}\right\rangle \\
&= \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{\bar v_j}{T_j}-\frac{\bar v_i}{T_i}-\frac{\bar v_j}{\bar T_j}+\frac{\bar v_i}{\bar T_i}\right\rangle \\
&\quad + \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{v_j}{T_j}-\frac{\bar v_j}{T_j}-\frac{v_j}{T_i}+\frac{\bar v_j}{T_i}\right\rangle \\
&:= \mathcal{J}_{221}+\mathcal{J}_{222}.
\end{aligned}
$$

∘ (Estimate of $\mathcal{J}_{221}$) We employ the Cauchy–Schwarz inequality, Proposition 2, Theorem 4.3, and $v^\infty=\bar v^\infty=0$ to deduce that

$$
\begin{aligned}
\mathcal{J}_{221} &= \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}\left\langle v_i-\bar v_i - v_j+\bar v_j, \frac{\bar v_j}{T_j}-\frac{\bar v_i}{T_i}-\frac{\bar v_j}{\bar T_j}+\frac{\bar v_i}{\bar T_i}\right\rangle \\
&\le \frac{\kappa_1}{2N}\sum_{i,j=1}^N \phi_{ij}(\|v_i-\bar v_i\|+\|v_j-\bar v_j\|)\left(\|\bar v_j\|\left|\frac{1}{T_j}-\frac{1}{\bar T_j}\right|+\|\bar v_i\|\left|\frac{1}{T_i}-\frac{1}{\bar T_i}\right|\right)
\end{aligned}
$$

$$\le \frac{\kappa_1 \phi(d_X^\infty) D_{\bar V}}{2N T_m^\infty \bar T_m^\infty} \sum_{i,j=1}^N (\|v_i - \bar v_i\| + \|v_j - \bar v_j\|)(|T_i - \bar T_i| + |T_j - \bar T_j|)$$
$$\le \frac{2\kappa_1 \phi(d_X^\infty) D_{\bar V}}{T_m^\infty \bar T_m^\infty} \|V - \bar V\| \|T - \bar T\|.$$

∘ (The estimate of $\mathcal{J}_{222}$) Likewise, it is easy to verify that

$$\begin{aligned}
\mathcal{J}_{222} &= \frac{\kappa_1}{2N} \sum_{i,j=1}^N \phi_{ij} \left\langle v_i - \bar v_i - v_j + \bar v_j, \frac{v_j}{T_j} - \frac{\bar v_j}{T_j} - \frac{v_j}{T_i} + \frac{\bar v_j}{T_i} \right\rangle \\
&\le \frac{\kappa_1}{2N} \sum_{i,j=1}^N \phi_{ij} (\|v_i - \bar v_i\| + \|v_j - \bar v_j\|) \left(\|v_j - \bar v_j\| \left| \frac{1}{T_j} - \frac{1}{T_i} \right| \right) \\
&\le \frac{\kappa_1 \phi(d_X^\infty) D_T}{2N (T_m^\infty)^2} \sum_{i,j=1}^N (\|v_i - \bar v_i\| + \|v_j - \bar v_j\|) \|v_j - \bar v_j\| \\
&\le \frac{\kappa_1 \phi(d_X^\infty) D_T}{(T_m^\infty)^2} \|V - \bar V\|^2.
\end{aligned}$$

Finally, we combine $\mathcal{J}_1$ with $\mathcal{J}_{21}$, $\mathcal{J}_{221}$ and $\mathcal{J}_{222}$ to concldue that for a.e. $t \in (0, \infty)$,

$$\begin{aligned}
\frac{d\|V - \bar V\|}{dt} \le &- \left(\frac{\kappa_1 \phi(D_X^\infty)}{T_M^\infty} - \frac{\kappa_1 \phi(d_X^\infty) D_T}{(T_m^\infty)^2} \right) \|V - \bar V\| + \frac{2\kappa_1 \phi(d_X^\infty) D_{\bar V}}{T_m^\infty \bar T_m^\infty} \|T - \bar T\| \\
&+ \frac{2\beta\kappa_1}{\min(d_X, d_{\bar X})^{\beta+1}} \cdot \left(\frac{D_{\bar V}}{\bar T_m^\infty} + \frac{D_{\bar V} D_{\bar T}}{(\bar T_m^\infty)^2} \right) \left\| X - \bar X \right\|,
\end{aligned}$$

which yields the desired second assertion. □

Remark 4. The results in Lemma 5.2 are similar to those in the previous paper [33]; however, we proved Lemma 5.2 in a much more concise way. The authors of [33] verified Lemma 5.2_2 and the result is similar to Lemma 5.2_3 by dividing N particles into two sets for technical calculations, and too long estimates were made for each of them.

Next, we are ready to prove the uniform L^2-stability of (21) under a sufficient framework in terms of the initial data and system parameters, which can be used to derive the uniform-in-time mean-field limit from (21) to the corresponding kinetic Vlasov equation. Thus, we must estimate G^∞ defined in Definition 5.1 so that it is independent of the initial data (X^0, V^0, T^0) as well as the number of particle N for the sake of an uniqueness of measure-valued solution to the kinetic equation. For papers related to the uniform-in-time mean-field limit and the measure-valued solution framework, refer to [4, 5, 20, 33, 36, 38].

Next, we set the following simple notation to simply express the three differential inequalities of Lemma 5.2.

$$\|X - \bar X\| =: \mathcal{X}, \quad \|V - \bar V\| =: \mathcal{V}, \quad \|T - \bar T\| =: \mathcal{T}.$$

Theorem 5.3. *Let* (X, V, T) *and* $(\bar X, \bar V, \bar T)$ *be two global-in-time solutions of* (21) *such that*

$$0 < \beta < \infty, \quad \min \left(\min_{i,j \in [N], i \ne j} \|x_i^0 - x_j^0\|, \min_{i,j \in [N], i \ne j} \|\bar x_i^0 - \bar x_j^0\| \right) > 0$$

and (15) *hold, respectively. Then, the uniform* L^2*-stability estimate holds for* (21). *More precisely, there exist* $G^\infty > 0$ *and* $C_1 > 0$ *such that for* $\epsilon \in (0, C_1)$,

1. (Uniform stability for $\mathcal{X}$)
$$\mathcal{X}(t) \le G^\infty \left(\mathcal{X}(0) + \mathcal{V}(0) + \mathcal{T}(0)\right),$$
2. (Uniform stability for $\mathcal{V}$)
$$\mathcal{V}(t) \le G^\infty \left(\mathcal{X}(0) + \mathcal{V}(0) + \mathcal{T}(0)\right) \exp\left(-(C_1 - \epsilon)t\right),$$
3. (Uniform stability for $\mathcal{T}$)
$$\mathcal{T}(t) \le G^\infty \left(\mathcal{X}(0) + \mathcal{V}(0) + \mathcal{T}(0)\right), \quad t \in (0, \infty).$$

Proof. First, it follows from Theorem 4.3 and Lemma 5.2 that there exist a strictly positive constant C_1 and nonnegative constants $\{\tilde{C}_k\}_{k=1}^5$ independent of t, N and the initial data such that

$$\begin{aligned}
&\bullet \left|\frac{d\mathcal{X}}{dt}\right| \le \mathcal{V}, \quad \text{a.e. } t \in (0, \infty), \\
&\bullet \frac{d\mathcal{V}}{dt} \le -C_1\mathcal{V} + \tilde{C}_1 \exp(-C_1 t)\mathcal{V} + \tilde{C}_2 \exp(-C_1 t)\mathcal{X} + \tilde{C}_3 \exp(-C_1 t)\mathcal{T}, \\
&\bullet \frac{d\mathcal{T}}{dt} \le \tilde{C}_4 \exp(-C_1 t)\mathcal{X} + \tilde{C}_5 \exp(-C_1 t)\mathcal{T}.
\end{aligned} \tag{22}$$

Now, for an arbitrarily given $\epsilon \in (0, C_1)$, we set
$$\mathcal{W}(t) := \mathcal{V}(t) \exp\left((C_1 - \epsilon)t\right).$$
Then, (22) can be converted to the following inequalities:

$$\begin{aligned}
&\bullet \left|\frac{d\mathcal{X}}{dt}\right| \le \mathcal{W} \exp(-(C_1 - \epsilon)t), \quad \text{a.e. } t \in (0, \infty), \\
&\bullet \frac{d\mathcal{W}}{dt} \le -\epsilon C_1\mathcal{W} + \tilde{C}_1 \exp(-\epsilon t)\mathcal{W} + \tilde{C}_2 \exp(-\epsilon t)\mathcal{X} + \tilde{C}_3 \exp(-\epsilon t)\mathcal{T}, \\
&\bullet \frac{d\mathcal{T}}{dt} \le \tilde{C}_4 \exp(-C_1 t)\mathcal{X} + \tilde{C}_5 \exp(-C_1 t)\mathcal{T}.
\end{aligned} \tag{23}$$

By defining $\bar{C}_1 = \min(\epsilon, C_1 - \epsilon)$, we estimate (23) as follows:

$$\begin{aligned}
&\bullet \left|\frac{d\mathcal{X}}{dt}\right| \le \mathcal{W} \exp(-\bar{C}_1 t), \quad \text{a.e. } t \in (0, \infty), \\
&\bullet \frac{d\mathcal{W}}{dt} \le \tilde{C}_1 \exp(-\bar{C}_1 t)\mathcal{W} + \tilde{C}_2 \exp(-\bar{C}_1 t)\mathcal{X} + \tilde{C}_3 \exp(-\bar{C}_1 t)\mathcal{T}, \\
&\bullet \frac{d\mathcal{T}}{dt} \le \tilde{C}_4 \exp(-\bar{C}_1 t)\mathcal{X} + \tilde{C}_5 \exp(-\bar{C}_1 t)\mathcal{T}.
\end{aligned} \tag{24}$$

Now, we sum up from $(24)_1$ to $(24)_3$ to guarantee that there exists $C \ge 0$, independent of t, N and the initial data, such that
$$\frac{d}{dt}(\mathcal{X} + \mathcal{W} + \mathcal{T}) \le C \exp(-\bar{C}_1 t)(\mathcal{X} + \mathcal{W} + \mathcal{T}).$$
Therefore, we can conclude that
$$\begin{aligned}
&\mathcal{X} + \mathcal{V} + \mathcal{T} \le \mathcal{X} + \mathcal{W} + \mathcal{T} \\
&\qquad \le \exp\left(\frac{C}{\bar{C}_1}\right)(\mathcal{X}(0) + \mathcal{W}(0) + \mathcal{T}(0)) = \exp\left(\frac{C}{\bar{C}_1}\right)(\mathcal{X}(0) + \mathcal{V}(0) + \mathcal{T}(0)),
\end{aligned}$$

yielding the uniform L^2-stability of (21). Moreover, we get the desired results (1), (2) and (3). □

Remark 5. Further, G^∞, estimated in the proof of Theorem 5.3, is independent of the number of particles N, time t, and the given initial data $(X(0), V(0), T(0))$, $(\bar{X}(0), \bar{V}(0), \bar{T}(0))$.

Remark 6. In previous articles [5, 33, 36], the authors verified the uniform stability estimates of the targeted models with regular kernels that are monotonically decreasing, nonnegative, bounded, and Lipschitz continuous. Their verifications were conducted by employing functionals, such as

$$M_{\mathcal{Z}}(t) = \max_{0 \le s \le t} \|Z(s) - \bar{Z}(s)\|$$

with arguments that were too technical and lengthy. However, we used the following substitution

$$u(t) = \|V(t) - \bar{V}(t)\| \exp\left((C_1 - \epsilon)t\right)$$

to obtain a much more improved proof in Theorem 5.3 than those in previous papers.

6. **Conclusion.** In this paper, we provided several sufficient frameworks, independent of the number of particles N, for collision avoidance (that is, global well-posedness) and the emergent dynamics of the CS and TCS models under strongly and weakly singular kernels, respectively. We first derived the dissipative structures with the L^∞-diameters D_X, D_V (and D_T) and then used the Lyapunov functional approach and appropriate bootstrapping arguments with technical estimates to obtain the collision avoidance and asymptotic flocking results. In particular, a collisional phenomenon of the two-particle CSS and TCSS models on $\mathbb{R}^1$ under the weakly singular kernel; therefore we adopted a sufficient framework for the global well-posedness and emergent behavior so that the two models have strictly positive lower bounds on all pairwise distances. Furthermore, to construct an admissible set for the emergent dynamics of the TCSS model, we introduced sufficient frameworks. These were introduced to ensure that the distance between each pair of particles to have a positive lower bound regardless of having weakly or strongly singular kernels, due to the dissipative velocity structure with $\phi(d_X)$ and finite-in-time blow-up when $d_X = 0$, unlike the CSS model. Finally, we described the sufficient frameworks for the L^2-uniform stability results of the TCSS model, which can be used to derive uniform-in-time mean-field limits of the CSS and TCSS models. In summary, this work is meaningful in that it provides sufficient frameworks to enable deriving uniform-in-time mean-field limits from the CS and TCS models with singular kernels to the corresponding kinetic Vlasov equation, respectively. However, several remaining questions require study in future work:

- (Question 1): Can we enlarge the sufficient framework for the emergent dynamics independent of N of the TCSS model with a strongly singular kernel without the strict positivity of each relative distance?
- (Question 2): Can we improve the sufficient frameworks for the uniform stability of the CSS and TCSS models in Section 5 when each nonzero relative distance converges to zero?

- (Question 3): Can we also prove the noncollisional phenomena of the CS and TCS models with singular kernels on complete Riemannian manifolds?

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