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ON THE LOCAL AND GLOBAL EXISTENCE OF THE HALL EQUATIONS WITH FRACTIONAL LAPLACIAN AND RELATED EQUATIONS

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Abstract. In this paper, we deal with the Hall equations with fractional Laplacian

$$B_t + \operatorname{curl}((\operatorname{curl} B) \times B) + \Lambda B = 0.$$

We begin to prove the existence of unique global in time solutions with sufficiently small initial data in H^k, $k > \frac{5}{2}$. By correcting ΛB logarithmically, we then show the existence of unique local in time solutions. We also deal with the two dimensional systems closely related to the $2\frac{1}{2}$ dimensional version of the above Hall equations. In this case, we show the existence of unique local and global in time solutions depending on whether the damping term is present or not.

1. **Introduction.** The 3D incompressible resistive Hall-Magnetohydrodynamics system (Hall-MHD in short) is the following system of PDEs for (u, p, B):

$$u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla p - \mu \Delta u = 0, \tag{1a}$$

$$B_t + u \cdot \nabla B - B \cdot \nabla u + \operatorname{curl}((\operatorname{curl} B) \times B) - \nu \Delta B = 0, \tag{1b}$$

$$\operatorname{div} u = 0, \quad \operatorname{div} B = 0, \tag{1c}$$

where $u = (u_1, u_2, u_3)$ is the plasma velocity field, p is the pressure, and $B = (B_1, B_2, B_3)$ is the magnetic field. μ and ν are the viscosity and the resistivity constants, respectively. The Hall-MHD is important in describing many physical phenomena [2, 17, 19, 23, 26, 27, 33]. In particular, the Hall MHD explains magnetic reconnection on the Sun which is very important role in acceleration plasma by converting magnetic energy into bulk kinetic energy.

The Hall-MHD recently has been studied intensively. The Hall-MHD can be derived from either two fluids model or kinetic models in a mathematically rigorous way [1]. Global weak solution, local classical solution, global solution for small data, and decay rates are established in [4, 5, 6]. There have been many follow-up results of these papers; see [7, 8, 12, 13, 14, 15, 16, 18, 29, 30, 31, 32, 34, 35] and references therein.

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1.1. **The Hall equation with fractional Laplacian.** We note that the Hall term $\operatorname{curl}((\operatorname{curl} B)\times B)$ is dominant in mathematical analysis of (1) and so we only consider the Hall equations ($(u,p)=0$ in (1)). Also motivated by [7], we consider the Hall equation with fractional Laplacian:

$$B_t + \operatorname{curl}((\operatorname{curl} B)\times B) + \Lambda^{\beta} B = 0, \quad \operatorname{div} B = 0, \tag{2}$$

where we take $\nu = 1$ for simplicity. (2) is locally well-posed [7] when $\beta > 1$. But, we do not know whether (2) is locally well-posed when $\beta = 1$:

$$B_t + \operatorname{curl}((\operatorname{curl} B)\times B) + \Lambda B = 0, \quad \operatorname{div} B = 0. \tag{3}$$

However, we can show the existence of solutions globally in time if initial data is sufficiently small.

Theorem 1.1. *Let $B_0 \in H^k$ with $k > \frac{5}{2}$ and $\operatorname{div} B_0 = 0$. There exists a constant $\epsilon_0 > 0$ such that if $\|B_0\|_{H^k} \le \epsilon_0$, there exists a unique global-in-time solution of (3) satisfying*

$$\|B(t)\|_{H^k}^2 + (1 - C\epsilon_0)\int_0^t \left\|\Lambda^{\frac{1}{2}} B(s)\right\|_{H^k}^2 ds \le \|B_0\|_{H^k}^2 \quad \text{for all } t > 0.$$

Moreover, B decays in time

$$\left\|\Lambda^l B(t)\right\|_{L^2} \le \frac{C_0}{(1+t)^l}, \quad 0 < l \le k, \tag{4}$$

where C_0 depends on $\|B_0\|_{H^k}$ which is expressed in (27) explicitly.

Remark 1. The decay rate (4) is consistent with the decay rates of the linear part of (3).

Remark 2. After this work was completed, the referee pointed out that the same result is proved in [37, Theorem 1.1]. Compared to the proof in [37] where they use the Littlewood-Paley decomposition, we use the standard energy energy estimates and classical commutator estimates.

As one of a minimal modification of (3) to show the existence of unique local in time solutions, we now take a logarithmic correction of (3):

$$B_t + \operatorname{curl}((\operatorname{curl} B)\times B) + \ln(2+\Lambda)\Lambda B = 0, \tag{5}$$

where the Fourier symbol of $\ln(2+\Lambda)\Lambda$ is $\ln(2+|\xi|)|\xi|$.

Theorem 1.2. *Let $B_0 \in H^k$ with $k > \frac{5}{2}$ and $\operatorname{div} B_0 = 0$. There exists $T_* = T_*(\|B_0\|_{H^k}) > 0$ such that there exists a unique local-in-time solution of (5) satisfying*

$$\|B(t)\|_{H^k} \le \ln\left(\frac{1}{e^{-\|B_0\|_{H^k}} - Ct}\right), \quad 0 < t < T^* = \frac{\exp(-\|B_0\|_{H^k})}{C}. \tag{6}$$

1.2. **2D models.** In this paper, we also deal with 2D models closely related to the $2\frac{1}{2}$ dimensional (3). If we take B of the form

$$B(t,x,y) = \left(-\psi_y(t,x,y), \psi_x(t,x,y), Z(t,x,y)\right), \tag{7}$$

we can rewrite (3) as

$$\psi_t + \Lambda\psi = [\psi, Z], \tag{8a}$$

$$Z_t + \Lambda Z = [\Delta\psi, \psi], \tag{8b}$$

where $[f,g] = \nabla f \cdot \nabla^{\perp} g = f_x g_y - f_y g_x$. (7) is used to show a finite-time collapse to a current sheet [3, 20, 21, 24] and is used in [10] to study regularity of stationary weak solutions.

1.2.1. **Case 1.** Although (8) is defined in 2D and has nice cancellation properties (18), the local well-posedness seems unreachable. But, suppose that we redistribute the power of the fractional Laplacians in (8) in such a way that (8b) has the full Laplacian and (8a) is inviscid:

$$\psi_t = [\psi, Z], \quad Z_t - \Delta Z = [\Delta\psi, \psi]. \tag{9}$$

(9) has no direct link to (2), but we may interpret (9) as the $2\frac{1}{2}$ dimensional model of the Hall equations where only B_3 has the full Laplacian in (2). In this case, we can show that (9) is locally well-posed. Let

$$\mathcal{E}(t) = \|\psi(t)\|_{H^4}^2 + \|Z(t)\|_{H^3}^2, \quad \mathcal{E}_0 = \|\psi_0\|_{H^4}^2 + \|Z_0\|_{H^3}^2. \tag{10}$$

Theorem 1.3. *There exists $T_* = T_*(\mathcal{E}_0) > 0$ such that there exists a unique solution of (9) satisfying*

$$\mathcal{E}(t) \le \frac{\mathcal{E}_{01}}{1 - Ct\mathcal{E}_0} \quad \text{for all } 0 < t \le T_* < \frac{1}{C\mathcal{E}_0}.$$

Moreover, we have the following blow-up criterion:

$$\mathcal{E}(t) + \int_0^t \|\nabla Z(s)\|_{H^2}^2 ds < \infty \iff \int_0^t \left(\left\|\nabla^2 Z(s)\right\|_{L^\infty} + \left\|\nabla^2 \psi(s)\right\|_{L^\infty}^2 \right) ds < \infty.$$

Since there is no dissipative effect in the equation of ψ in (9), we only have the local in tim result in Theorem 1.3. Among the possible conditions for the global existence, we find that adding a damping term to the equation of ψ works. More precisely, we deal with the following

$$\psi_t + \psi = [\psi, Z], \quad Z_t - \Delta Z = [\Delta\psi, \psi]. \tag{11}$$

In this case, we can show the existence of global in time solutions with small initial data having regularity one higher than the regularity in Theorem 1.3. Moreover, we can find decay rates of ψ by using the structure of equation of ψ which is a damped transport equation, and this is also the reason why the same method cannot be applied to Z. Let

$$\mathcal{F}(t) = \|\psi(t)\|_{H^5}^2 + \|Z(t)\|_{H^4}^2, \quad \mathcal{F}_0 = \|\psi_0\|_{H^5}^2 + \|Z_0\|_{H^4}^2,$$
$$\mathcal{N}_1(t) = \|\nabla\psi(t)\|_{H^4}^2 + \|\nabla Z(t)\|_{H^4}^2.$$

Theorem 1.4. *There exists a constant $\epsilon_0 > 0$ such that if $\mathcal{F}_0 \le \epsilon_0$, there exists a unique global-in-time solution of (11) satisfying*

$$\mathcal{F}(t) + (1 - C\epsilon_0)\int_0^t \mathcal{N}_1(s) ds \le \mathcal{F}_0 \quad \text{for all } t > 0.$$

Moreover, ψ decays exponentially in time

$$\|\psi(t)\|_{L^2} \le \|\psi_0\|_{L^2} e^{-t}, \quad \left\|\Lambda^k \psi(t)\right\|_{L^2} \le \mathcal{F}_0^{\frac{k-1}{8}} \|\nabla\psi_0\|_{L^2}^{\frac{5-k}{4}} e^{-\frac{(5-k)(1-C\epsilon_0)}{4}t}$$

with $1 \le k < 5$.

1.2.2. **Case 2.** As another way to redistribute the derivatives in (8), we also deal with

$$\psi_t - \Delta\psi = [\psi, Z], \quad Z_t = [\Delta\psi, \psi]. \tag{12}$$

Let $\mathcal{E}(t)$ and $\mathcal{E}_0$ be defined as before (10).

Theorem 1.5. *There exists $T_* > 0$, which is depending on $\mathcal{E}_0$, such that there exists a unique solution of (12) satisfying*

$$\mathcal{E}(t) \leq \frac{\mathcal{E}_0}{1 - Ct\mathcal{E}_0} \quad \text{for all } 0 < t \leq T_* < \frac{1}{C\mathcal{E}_0}.$$

Moreover, we have the following blow-up criterion

$$\mathcal{E}(t) + \int_0^t \|\nabla\psi\|_{H^4}^2 \, ds < \infty \iff \int_0^t \left\|\nabla^2\psi\right\|_{L^\infty}^2 ds.$$

We now add a damping term to the equation of Z in (12):

$$\psi_t - \Delta\psi = [\psi, Z], \quad Z_t + Z = [\Delta\psi, \psi]. \tag{13}$$

In this case, we can use the same regularity used in Theorem 1.5 because the dissipative effect in ψ helps to control $\Delta\psi$ in the equation of Z. Let $\mathcal{N}_2(t) = \|\nabla\psi(t)\|_{H^5}^2 + \|Z(t)\|_{H^3}^2$.

Theorem 1.6. *There exists a constant $\epsilon_0 > 0$ such that if $\mathcal{E}_0 \leq \epsilon_0$, there exists a unique global-in-time solution of (13) satisfying*

$$\mathcal{E}(t) + (1 - C\epsilon_0)\int_0^t \mathcal{N}_2(s) ds \leq \mathcal{E}_0 \quad \text{for all } t > 0.$$

Remark 3. Compared to Theorem 1.3, we only need one term in the blow-up criterion in Theorem 1.5 which is due to the dissipative effect in the equation of ψ. Compared to Theorem 1.4, the proof of Theorem 1.6 is simpler, but we are not able to derive decay rates of ψ and Z.

2. Preliminaries.

All constants will be denoted by C and we follow the convention that such constants can vary from expression to expression and even between two occurrences within the same expression. And repeated indices are summed over.

The fractional Laplacian $\Lambda^\beta = (\sqrt{-\Delta})^\beta$ has the Fourier transform representation

$$\widehat{\Lambda^\beta f}(\xi) = |\xi|^\beta \widehat{f}(\xi).$$

For $s > 0$, H^s is a energy space equipped with

$$\|f\|_{H^s} = \|f\|_{L^2} + \|f\|_{\dot{H}^s}, \quad \|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2}.$$

In the energy spaces, we have the following interpolations: for $s_0 < s < s_1$

$$\|f\|_{\dot{H}^s} \leq \|f\|_{\dot{H}^{s_0}}^\theta \|f\|_{\dot{H}^{s_1}}^{1-\theta}, \quad s = \theta s_0 + (1-\theta)s_1. \tag{14}$$

2.1. **Inequalities.** We begin with two inequalities in 3D:

$$\|f\|_{L^\infty} \le C\|f\|_{H^s}, \quad s > \frac{3}{2}, \tag{15a}$$

$$\|f\|_{L^p} \le C\|f\|_{\dot{H}^s}, \quad \frac{1}{p} = \frac{1}{2} - \frac{s}{3}. \tag{15b}$$

We also provide the following inequalities in 2D

$$\|f\|_{L^4} \le C\,\|f\|_{L^2}^{\frac{1}{2}}\,\|\nabla f\|_{L^2}^{\frac{1}{2}}, \quad \|f\|_{L^\infty} \le C\,\|f\|_{L^2}^{\frac{1}{2}}\,\|\Delta f\|_{L^2}^{\frac{1}{2}}$$

which will be used repeatedly in the proof of Theorem 1.3, Theorem 1.4, Theorem 1.5, and Theorem 1.6. We also recall

$$\left\|\nabla^2 f\right\|_{L^2} = \|\Delta f\|_{L^2}$$

which holds in any dimension.

We finally provide the Kato-Ponce commutator cstimate [22]

$$\begin{aligned} \left\|\left[\Lambda^k, f\right] g\right\|_{L^2} &= \left\|\Lambda^k(fg) - f\Lambda^k g\right\|_{L^2} \\ &\le C\,\|\nabla f\|_{L^\infty}\left\|\Lambda^{k-1} g\right\|_{L^2} + C\,\|g\|_{L^\infty}\left\|\Lambda^k f\right\|_{L^2} \end{aligned} \tag{16}$$

and the fractional Leibniz rule [11]: for $1 \le p < \infty$ and $p_i, q_i \ne 1$,

$$\begin{aligned} &\|\Lambda^s(fg)\|_{L^p} \le C\,\|\Lambda^s f\|_{L^{p_1}}\,\|g\|_{L^{q_1}} + C\|f\|_{L^{p_2}}\,\|\Lambda^s g\|_{L^{q_2}}\,, \\ &\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}. \end{aligned} \tag{17}$$

2.2. **Commutator.** We recall the commutator $[f,g] = \nabla f \cdot \nabla^\perp g = f_x g_y - f_y g_x$. Then, the commutator has the following properties:

$$\Delta[f,g] = [\Delta f, g] + [f, \Delta g] + 2[f_x, g_x] + 2[f_y, g_y], \tag{18a}$$

$$\int f[f,g] = 0, \tag{18b}$$

$$\int f[g,h] = \int g[h,f]. \tag{18c}$$

3. Proof of Theorem 1.1 and Theorem 1.2.

3.1. Proof of Theorem 1.1.

We recall (3):

$$B_t + \operatorname{curl}((\operatorname{curl} B) \times B) + \Lambda B = 0. \tag{19}$$

3.1.1. **Approximation.** We first approximate (19) by putting $\epsilon\Delta B$ to the right-hand side of (19):

$$B_t + \operatorname{curl}((\operatorname{curl} B) \times B) + \Lambda B = \epsilon\Delta B. \tag{20}$$

We then mollify (20) as follows

$$\begin{aligned} &\partial_t B^{(\epsilon)} + \operatorname{curl}\left(\mathcal{J}_\epsilon\left(\operatorname{curl}\mathcal{J}_\epsilon B^{(\epsilon)}\right) \times \mathcal{J}_\epsilon B^{(\epsilon)}\right) + \Lambda\mathcal{J}_\epsilon^2 B^{(\epsilon)} = \epsilon\mathcal{J}_\epsilon^2\Delta B^{(\epsilon)}, \\ &B_0^{(\epsilon)} = \mathcal{J}_\epsilon B_0, \end{aligned} \tag{21}$$

where $\mathcal{J}_\epsilon$ is the standard mollifier described in [25, Chapter 3.2]. Then, as proved in [4, Proposition 3.1], there exists a unique global-in-time solution $\left\{B^{(\epsilon)}\right\}$ of (21). Since the bounds in Section 3.1.2 are independent of $\epsilon > 0$, we can pass to the limit in a subsequence and show the existence of smooth solutions globally in time when $B_0 \in H^k$, $k > \frac{5}{2}$, is sufficiently small as in [37, Section 3.2].

3.1.2. **A priori estimates.** We begin with the L^2 bound:

$$\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + \left\|\Lambda^{\frac{1}{2}}B\right\|_{L^2}^2 = 0. \tag{22}$$

We now take Λ^k to (19) and take the inner product of the resulting equation with $\Lambda^k B$. Then,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^k B\|_{L^2}^2 + \left\|\Lambda^{\frac{1}{2}+k}B\right\|_{L^2}^2 = -\int \Lambda^k \operatorname{curl}((\operatorname{curl} B)\times B)\cdot \Lambda^k B$$
$$= \int \left(\left[\Lambda^{\frac{1}{2}+k}, B\right]\times \operatorname{curl} B\right)\cdot \Lambda^{k-\frac{1}{2}}\operatorname{curl} B \le \left\|\left[\Lambda^{\frac{1}{2}+k}, B\right]\times \operatorname{curl} B\right\|_{L^2}\left\|\Lambda^{\frac{1}{2}+k}B\right\|_{L^2}.$$

By (16) and (15a) with $k > \frac{5}{2}$,

$$\begin{aligned}\left\|\left[\Lambda^{\frac{1}{2}+k}, B\right]\times \operatorname{curl} B\right\|_{L^2} &\le C\|\nabla B\|_{L^\infty}\left\|\Lambda^{k-\frac{1}{2}}\operatorname{curl} B\right\|_{L^2}\\ &\le C\|B\|_{H^k}\left\|\Lambda^{\frac{1}{2}+k}B\right\|_{L^2}^2.\end{aligned} \tag{23}$$

So, we obtain

$$\frac{d}{dt}\|\Lambda^k B\|_{L^2}^2 + \left\|\Lambda^{\frac{1}{2}+k}B\right\|_{L^2}^2 \le C\|B\|_{H^k}\left\|\Lambda^{\frac{1}{2}+k}B\right\|_{L^2}^2. \tag{24}$$

By (22) and (24),

$$\frac{d}{dt}\|B\|_{H^k}^2 + \left\|\Lambda^{\frac{1}{2}}B\right\|_{H^k}^2 \le C\|B\|_{H^k}\left\|\Lambda^{\frac{1}{2}+k}B\right\|_{L^2}^2.$$

If $\|B_0\|_{H^k} = \epsilon_0$ is sufficiently small, we can derive a uniform bound

$$\|B(t)\|_{H^k}^2 + (1-C\epsilon_0)\int_0^t \left\|\Lambda^{\frac{1}{2}}B(s)\right\|_{H^k}^2 ds \le \|B_0\|_{H^k}^2 \quad \text{for all } t>0. \tag{25}$$

3.1.3. **Uniqueness.** Let B_1 and B_2 be two solutions of (19). Then, $B = B_1 - B_2$ satisfies

$$B_t + \Lambda B + \operatorname{curl}((\operatorname{curl} B_1)\times B) - \operatorname{curl}((\operatorname{curl} B)\times B_2) = 0 \tag{26}$$

with $B_0 = 0$. We take the inner product of (26) with B. By (17) with $k > \frac{5}{2}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + \left\|\Lambda^{\frac{1}{2}}B\right\|_{L^2}^2 &= -\int (\operatorname{curl}((\operatorname{curl} B_1)\times B))\cdot B\\ &= -\int \Lambda^{\frac{1}{2}}(((\operatorname{curl} B_1)\times B))\cdot \Lambda^{-\frac{1}{2}}\operatorname{curl} B\\ &\le C\|\nabla B_1\|_{L^\infty}\left\|\Lambda^{\frac{1}{2}}B\right\|_{L^2}^2 + C\left\|\nabla\Lambda^{\frac{1}{2}}B_1\right\|_{L^6}\|B\|_{L^3}\left\|\Lambda^{\frac{1}{2}}B\right\|_{L^2}\\ &\le C\|\nabla B_1\|_{L^\infty}\left\|\Lambda^{\frac{1}{2}}B\right\|_{L^2}^2 + C\left\|\Lambda^{\frac{5}{2}}B_1\right\|_{L^2}\left\|\Lambda^{\frac{1}{2}}B\right\|_{L^2}^2\\ &\le C\|B_1\|_{H^k}\left\|\Lambda^{\frac{1}{2}}B\right\|_{L^2}^2,\end{aligned}$$

where we use (15b) to control L^6 and L^3 terms. If $C\epsilon_0 < 1$, (25) implies $B = 0$ in L^2 which gives the uniqueness of a solution.

3.1.4. **Decay rates.** By (14), it is enough to derive the decay rate with $k = l$ to show (4). Since

$$\left\|\Lambda^k B\right\|_{L^2}^{\frac{2k+1}{k}} \leq \|B\|_{L^2}^{\frac{1}{k}} \left\|\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2 \leq \|B_0\|_{L^2}^{\frac{1}{k}} \left\|\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2$$

by (14) and (22), we create the following ODE from (24)

$$\frac{d}{dt}\left\|\Lambda^k B\right\|_{L^2}^2 + \frac{1-C\epsilon_0}{\|B_0\|_{L^2}^{\frac{1}{k}}}\left\|\Lambda^k B\right\|_{L^2}^{\frac{2k+1}{k}} \leq 0.$$

By solving this ODE, we find the following decay rate

$$\left\|\Lambda^k B(t)\right\|_{L^2} \leq \frac{\left((2k)^k \|B_0\|_{L^2} \left\|\Lambda^k B_0\right\|_{L^2}\right)}{\left(2k \|B_0\|_{L^2}^{\frac{1}{k}} + (1-C\epsilon_0)\left\|\Lambda^k B_0\right\|_{L^2}^{\frac{1}{k}} t\right)^k}. \tag{27}$$

3.2. **Proof of Theorem 1.2.** We recall (5):

$$B_t + \operatorname{curl}\left((\operatorname{curl} B) \times B\right) + \ln(2+\Lambda)\Lambda B = 0,$$

The the uniqueness part of Theorem 1.2 is the same as that of Theorem 1.1 and we only derive a priori bounds. Let

$$\left\|\sqrt{\ln(2+\Lambda)}\Lambda^s f\right\|_{L^2}^2 = \int \left(\ln(2+|\xi|)\right)|\xi|^{2s}\left|\widehat{f}(\xi)\right|^2 d\xi.$$

We begin with the L^2 bound:

$$\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + \left\|\sqrt{\ln(2+\Lambda)}\Lambda^{\frac{1}{2}} B\right\|_{L^2}^2 = 0. \tag{28}$$

Following the computations in the proof of Theorem 1.1, we also have

$$\frac{d}{dt}\left\|\Lambda^k B\right\|_{L^2}^2 + \left\|\sqrt{\ln(2+\Lambda)}\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2 \leq C\|B\|_{H^k}\left\|\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2. \tag{29}$$

For each $N \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2 &= \int_{|\xi|\leq 2^N} |\xi|^{2k+1}\left|\widehat{B}(\xi)\right|^2 d\xi + \int_{|\xi|\geq 2^N} |\xi|^{2k+1}\left|\widehat{B}(\xi)\right|^2 d\xi \\
&\leq 2^N \int_{|\xi|\leq 2^N} |\xi|^{2k}\left|\widehat{B}(\xi)\right|^2 d\xi + \frac{1}{\ln(2+2^N)}\int_{|\xi|\geq 2^N} \ln(2+|\xi|)|\xi|^{2k+1}\left|\widehat{B}(\xi)\right|^2 d\xi \\
&\leq 2^N \left\|\Lambda^k B\right\|_{L^2}^2 + \frac{1}{\ln(2+2^N)}\left\|\sqrt{\ln(2+\Lambda)}\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2.
\end{aligned}$$

So, (29) is replaced by

$$\begin{aligned}
&\frac{d}{dt}\left\|\Lambda^k B\right\|_{L^2}^2 + \left\|\sqrt{\ln(2+\Lambda)}\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2 \\
&\leq C2^N \left\|\Lambda^k B\right\|_{L^2}^2 \|B\|_{H^k} + \frac{C\|B\|_{H^k}}{\ln(2+2^N)}\left\|\sqrt{\ln(2+\Lambda)}\Lambda^{\frac{1}{2}+k} B\right\|_{L^2}^2.
\end{aligned}$$

We now choose $N > 0$ such that

$$\frac{1}{2}\ln(2+2^N) < C\|B\|_{H^k} < \ln(2+2^N)$$

and so $N \sim \|B\|_{H^k}$. Then, (29) is reduced to

$$\frac{d}{dt}\left\|\Lambda^k B\right\|_{L^2}^2 \leq C\exp\left(\|B\|_{H^k}\right)\|B\|_{H^k}\left\|\Lambda^k B\right\|_{L^2}. \tag{30}$$

By (28) and (30), we obtain

$$\frac{d}{dt}\|B\|_{H^k}^2 \le C\exp\left(\|B\|_{H^k}\right)\|B\|_{H^k}^2$$

and so we have

$$\frac{d}{dt}\|B\|_{H^k} \le C\exp\left(\|B\|_{H^k}\right)\|B\|_{H^k} \le C\exp\left(\|B\|_{H^k}\right).$$

By solving this ODE, we can derive (6).

4. **Proof of Theorem 1.3 and Theorem 1.4.**

4.1. **Proof of Theorem 1.3.** We recall (9):

$$\psi_t = [\psi, Z], \tag{31a}$$

$$Z_t - \Delta Z = [\Delta\psi, \psi]. \tag{31b}$$

4.1.1. **Approximation.** We first approximate (31a) by putting $\epsilon\Delta\psi$ to the right-hand side and mollify the resulting equations as (21). Then, we have

$$\begin{aligned} &\partial_t\psi^{(\epsilon)} = \mathcal{J}_\epsilon[\mathcal{J}_\epsilon\psi^{(\epsilon)}, \mathcal{J}_\epsilon Z^{(\epsilon)}] + \epsilon\mathcal{J}_\epsilon^2\Delta\psi^{(\epsilon)}, \\ &\partial_t Z^{(\epsilon)} - \Delta\mathcal{J}_\epsilon^2 Z^{(\epsilon)} = \mathcal{J}_\epsilon[\Delta\mathcal{J}_\epsilon\psi^{(\epsilon)}, \mathcal{J}_\epsilon\psi^{(\epsilon)}] \end{aligned} \tag{32}$$

with $\psi_0^{(\epsilon)} = \mathcal{J}_\epsilon\psi_0$ and $Z_0^{(\epsilon)} = \mathcal{J}_\epsilon Z_0$. Since (32) is defined in $\mathbb{R}^2$, the proof of the existence of a unique global-in-time solution of (32) is relatively easier than the one to (21). Moreover, the bounds in Section 4.1.2 are independent of $\epsilon > 0$ and so we can pass to the limit in a subsequence and show the existence of smooth solutions locally in time when $\psi_0 \in H^4$ and $Z_0 \in H^3$.

4.1.2. **A priori estimates.** We first note that

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 = \int \psi[\psi, Z] = 0. \tag{33}$$

We next multiply (31a) by $-\Delta\psi$, (31b) by Z, and integrate over $\mathbb{R}^2$. By (18c),

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2 + \|Z\|_{L^2}^2\right) + \|\nabla Z\|_{L^2}^2 = \int\left(-\Delta\psi[\psi, Z] + Z[\Delta\psi, \psi]\right) = 0. \tag{34}$$

We also multiply (31a) by $\Delta^4\psi$, (31b) by $-\Delta^3 Z$ and integrate over $\mathbb{R}^2$. Then,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2\right) + \left\|\Delta^2 Z\right\|_{L^2}^2 &= \int \Delta^4\psi[\psi, Z] - \int \Delta^3 Z[\Delta\psi, \psi] \\ &= \mathcal{R}. \end{aligned} \tag{35}$$

We now compute the right-hand side of (35). By (18a), (18b), and (18c),

$$\begin{aligned} \mathcal{R} = {} & 2\int \Delta^2\psi[\Delta\psi, \Delta Z] + 4\int \Delta^2\psi[\psi_x, \Delta Z_x] + 4\int \Delta^2\psi[\psi_y, \Delta Z_y] \\ & + 4\int \Delta^2\psi[\Delta\psi_x, Z_x] + 4\int \Delta^2\psi[\Delta\psi_y, Z_y] + 4\int \Delta^2\psi[\psi_{xx}, Z_{xx}] \\ & + 8\int \Delta^2\psi[\psi_{xy}, Z_{xy}] + 4\int \Delta^2\psi[\psi_{yy}, Z_{yy}] - 2\int \Delta^2 Z[\Delta\psi_x, \psi_x] \\ & - 2\int \Delta^2 Z[\Delta\psi_y, \psi_y]. \end{aligned} \tag{36}$$

So, we find that the number of derivatives acting on (ψ, ψ, Z) are $(4,4,2)$, $(3,4,3)$, and $(4,2,4)$ up to multiplicative constants. Hence,

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2\right) + \left\|\Delta^2 Z\right\|_{L^2}^2 \\
&\leq C\int \left|\nabla^4\psi\right|\left|\nabla^4\psi\right|\left|\nabla^2 Z\right| + C\int \left|\nabla^3\psi\right|\left|\nabla^4\psi\right|\left|\nabla^3 Z\right| + C\int \left|\nabla^4\psi\right|\left|\nabla^2\psi\right|\left|\nabla^4 Z\right| \\
&\leq C\left\|\Delta^2\psi\right\|_{L^2}^2\left\|\nabla^2 Z\right\|_{L^\infty} + C\left\|\nabla^3\psi\right\|_{L^4}\left\|\Delta^2\psi\right\|_{L^2}\left\|\nabla^3 Z\right\|_{L^4} \\
&+ C\left\|\Delta^2\psi\right\|_{L^2}\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\Delta^2 Z\right\|_{L^2} \\
&\leq C\left\|\Delta^2\psi\right\|_{L^2}^2\left\|\nabla^2 Z\right\|_{L^\infty} + C\left\|\Delta^2\psi\right\|_{L^2}^{\frac{3}{2}}\|\nabla\Delta\psi\|_{L^2}^{\frac{1}{2}}\left\|\Delta^2 Z\right\|_{L^2} \\
&+ C\left\|\Delta^2\psi\right\|_{L^2}\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\Delta^2 Z\right\|_{L^2} \\
&\leq C\mathcal{E}^2 + \frac{1}{4}\left\|\Delta^2 Z\right\|_{L^2}^2 + \delta\left\|\nabla^2 Z\right\|_{L^\infty}^2 \leq C\mathcal{E}^2 + \frac{1}{2}\left\|\Delta^2 Z\right\|_{L^2}^2 + \frac{1}{4}\|\nabla Z\|_{L^2}^2,
\end{aligned}
$$

where we use

$$
\begin{aligned}
\left\|\nabla^2 Z\right\|_{L^\infty}^2 &\leq C\|\Delta Z\|_{L^2}\left\|\Delta^2 Z\right\|_{L^2} \leq C\|\nabla Z\|_{L^2}^{\frac{2}{3}}\left\|\Delta^2 Z\right\|_{L^2}^{\frac{4}{3}} \\
&\leq C\|\nabla Z\|_{L^2}^2 + C\left\|\Delta^2 Z\right\|_{L^2}^2
\end{aligned}
$$

with δ satisfying $4C\delta = 1$. So, we have

$$
\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2\right) + \left\|\Delta^2 Z\right\|_{L^2}^2 \leq C\mathcal{E}^2 + \frac{1}{2}\|\nabla Z\|_{L^2}^2. \tag{37}
$$

By (33), (34), and (37), we derive $\mathcal{E}' \leq C\mathcal{E}^2$ from which we deduce

$$
\mathcal{E}(t) \leq \frac{\mathcal{E}_0}{1 - Ct\mathcal{E}_0} \quad \text{for all } 0 < t \leq T_* < \frac{1}{C\mathcal{E}_0}. \tag{38}
$$

4.1.3. **Uniqueness.** Let (ψ_1, Z_1) and (ψ_2, Z_2) be two solutions of (31) and let $\psi = \psi_1 - \psi_2$ and $Z = Z_1 - Z_2$. Then, (ψ, Z) satisfies the following equations:

$$
\psi_t = [\psi, Z_1] + [\psi_2, Z], \quad Z_t - \Delta Z = [\Delta\psi, \psi_1] + [\Delta\psi_2, \psi]
$$

with $\psi(0, x) = Z(0, x) = 0$. For these equations, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2 + \|Z\|_{L^2}^2\right) + \|\nabla Z\|_{L^2}^2 \\
&= -\int \Delta\psi[\psi, Z_1] - \int \Delta\psi[\psi_2, Z] + \int Z[\Delta\psi, \psi_1] + \int Z[\Delta\psi_2, \psi] \\
&= \text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)}.
\end{aligned}
$$

The first term is bounded using the definition of $[f, g]$ and $\operatorname{div}\nabla^\perp Z_1 = 0$:

$$
\text{(I)} = \int \left(\nabla^\perp Z_1 \cdot \nabla\psi\right)\Delta\psi = -\int \left(\nabla^\perp\partial_l Z_1 \cdot \nabla\psi\right)\partial_l\psi \leq C\left\|\nabla^2 Z_1\right\|_{L^\infty}\|\nabla\psi\|_{L^2}^2.
$$

We next bound (II)+(III) as

$$
\begin{aligned}
\text{(II)}+\text{(III)} &= -\int Z[\Delta\psi, \psi] \leq C\left\|\nabla^2\psi\right\|_{L^\infty}\|\nabla\psi\|_{L^2}\|\nabla Z\|_{L^2} \\
&\leq C\left(\left\|\nabla^2\psi_1\right\|_{L^\infty}^2 + \left\|\nabla^2\psi_2\right\|_{L^\infty}^2\right)\|\nabla\psi\|_{L^2}^2 + \frac{1}{4}\|\nabla Z\|_{L^2}^2.
\end{aligned}
$$

The last term is bounded as

$$
\text{(IV)} \leq C\left\|\nabla^2\psi_2\right\|_{L^\infty}\|\nabla\psi\|_{L^2}\|\nabla Z\|_{L^2} \leq C\left\|\nabla^2\psi_2\right\|_{L^\infty}^2\|\nabla\psi\|_{L^2}^2 + \frac{1}{4}\|\nabla Z\|_{L^2}^2.
$$

So, we have

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2+\|Z\|_{L^2}^2\right)\\&\le C\left(\left\|\nabla^2 Z_1\right\|_{L^\infty}+\left\|\nabla^2\psi_1\right\|_{L^\infty}^2+\left\|\nabla^2\psi_2\right\|_{L^\infty}^2\right)\left(\|\nabla\psi\|_{L^2}^2+\|Z\|_{L^2}^2\right).\end{aligned}\tag{39}$$

By (38), $\left\|\nabla^2\psi_1\right\|_{L^\infty}^2+\left\|\nabla^2\psi_2\right\|_{L^\infty}^2$ is integrable in time. Integrating (34) and (35) in time, we have

$$\int_0^t\left(\|\nabla Z(s)\|_{L^2}^2+\left\|\Delta^2 Z(s)\right\|_{L^2}^2\right)ds<\infty\quad\text{for } 0<t\le\frac{T_*}{2}$$

which gives the integrability of the first term in the parentheses on the right-hand side of (39). By repeating the same argument one more time, we have the uniqueness up to T_*.

4.1.4. **Blow-up criterion.** Let

$$\mathcal{B}(s)=\left\|\nabla^2 Z(s)\right\|_{L^\infty}+\left\|\nabla^2\psi(s)\right\|_{L^\infty}^2.$$

We first deal with

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Delta\psi\|_{L^2}^2+\|\nabla Z\|_{L^2}^2\right)+\|\Delta Z\|_{L^2}^2=\int\Delta^2\psi[\psi,Z]-\int\Delta Z[\Delta\psi,\psi]\\&=2\int\Delta\psi[\psi_x,Z_x]+2\int\Delta\psi[\psi_y,Z_y]\le C\left\|\nabla^2 Z\right\|_{L^\infty}\|\Delta\psi\|_{L^2}^2\end{aligned}$$

and so we have

$$\frac{d}{dt}\left(\|\Delta\psi\|_{L^2}^2+\|\nabla Z\|_{L^2}^2\right)+\|\Delta Z\|_{L^2}^2\le C\left\|\nabla^2 Z\right\|_{L^\infty}\|\Delta\psi\|_{L^2}^2.$$

This implies

$$\begin{aligned}&\|\Delta\psi(t)\|_{L}^2+\|\nabla Z(t)\|_{L^2}^2+\int_0^t\|\Delta Z(s)\|_{L^2}^2\,ds<\infty\\&\iff\int_0^t\left\|\nabla^2 Z(s)\right\|_{L^\infty}ds<\infty.\end{aligned}\tag{40}$$

We also deal with

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\Delta\psi\|_{L^2}^2+\|\Delta Z\|_{L^2}^2\right)+\|\nabla\Delta Z\|_{L^2}^2=-\int\Delta^3\psi[\psi,Z]+\int\Delta^2 Z[\Delta\psi,\psi]\\&=-\int\Delta^2\psi[\Delta\psi,Z]-2\int\Delta^2\psi\left([\psi_x,Z_x]+[\psi_y,Z_y]\right)\\&-2\int\Delta\psi\left([\psi_x,\Delta Z_x]+[\psi_y,\Delta Z_y]\right)=\text{(I)+(II)+(III)}.\end{aligned}$$

As in Section 4.1.3,

$$\text{(I)}=\int\left(\nabla\nabla^\perp Z\cdot\nabla\Delta\psi\right)\cdot\nabla\Delta\psi\le C\left\|\nabla^2 Z\right\|_{L^\infty}\|\nabla\Delta\psi\|_{L^2}^2.\tag{41}$$

We next estimate (II) + (III):

$$\begin{aligned}(\mathrm{II})+(\mathrm{III}) &= -4\int \Delta\psi\left([\Delta\psi_y, Z_y]+[\psi_y,\Delta Z_y]+[\psi_{xy},Z_{xy}]+[\psi_{yy},Z_{yy}]\right)\\ &\le C\int |\nabla^2 Z|\,|\nabla^3\psi|^2 + C\int |\nabla^2\psi|\,|\nabla^3\psi|\,|\nabla^3 Z| \\ &\le C\left\|\nabla^2 Z\right\|_{L^\infty}\|\nabla\Delta\psi\|_{L^2}^2 + C\left\|\nabla^2\psi\right\|_{L^\infty}^2\|\nabla\Delta\psi\|_{L^2}^2+\frac{1}{2}\|\nabla\Delta Z\|_{L^2}^2 .\end{aligned} \tag{42}$$

By (41) and (42), we have

$$\frac{d}{dt}\left(\|\nabla\Delta\psi\|_{L^2}^2+\|\Delta Z\|_{L^2}^2\right)+\|\nabla\Delta Z\|_{L^2}^2 \le C\left(\left\|\nabla^2 Z\right\|_{L^\infty}+\left\|\nabla^2\psi\right\|_{L^\infty}^2\right)\|\nabla\Delta\psi\|_{L^2}^2$$

which implies

$$\|\nabla\Delta\psi(t)\|_{L^2}^2+\|\Delta Z(t)\|_{L^2}^2+\int_0^t\|\nabla\Delta Z(s)\|_{L^2}^2\,ds<\infty \iff \int_0^t \mathcal{B}(s)ds<\infty. \tag{43}$$

We finally deal with

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2+\|\nabla\Delta Z\|_{L^2}^2\right)+\left\|\Delta^2 Z\right\|_{L^2}^2=\int\Delta^4\psi[\psi,Z]-\int\Delta^3 Z[\Delta\psi,\psi]=\mathcal{R}$$

with the same $\mathcal{R}$ in (36). So, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2+\|\nabla\Delta Z\|_{L^2}^2\right)+\left\|\Delta^2 Z\right\|_{L^2}^2\\ &\le C\left\|\nabla^2 Z\right\|_{L^\infty}\left\|\Delta^2\psi\right\|_{L^2}^2+C\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\Delta^2 Z\right\|_{L^2}\left\|\Delta^2\psi\right\|_{L^2}\\ &+C\|\nabla\Delta Z\|_{L^4}\|\nabla\Delta\psi\|_{L^4}\left\|\Delta^2\psi\right\|_{L^2}\\ &\le C\left(\left\|\nabla^2 Z\right\|_{L^\infty}+\left\|\nabla^2\psi\right\|_{L^\infty}^2+\|\nabla\Delta Z\|_{L^2}^{\frac{3}{2}}\|\nabla\Delta\psi\|_{L^2}^{\frac{3}{2}}\right)\left\|\Delta^2\psi\right\|_{L^2}^2+\frac{1}{2}\left\|\Delta^2 Z\right\|_{L^2}^2\end{aligned}$$

which gives

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2+\|\nabla\Delta Z\|_{L^2}^2\right)+\left\|\Delta^2 Z\right\|_{L^2}^2\\ &\le C\left(\mathcal{B}(s)+\|\nabla\Delta Z\|_{L^2}^{\frac{3}{2}}\|\nabla\Delta\psi\|_{L^2}^{\frac{3}{2}}\right)\left\|\Delta^2\psi\right\|_{L^2}^2 .\end{aligned} \tag{44}$$

By (40) and (43), (44) implies

$$\left\|\Delta^2\psi(t)\right\|_{L^2}^2+\|\nabla\Delta Z(t)\|_{L^2}^2+\int_0^t\left\|\Delta^2 Z(s)\right\|_{L^2}^2 ds<\infty \iff \int_0^t\mathcal{B}(s)ds<\infty.$$

4.2. **Proof of Theorem 1.4.** We recall (11):

$$\psi_t+\psi=[\psi,Z],\quad Z_t-\Delta Z=[\Delta\psi,\psi]$$

Since the uniqueness is already proved in Section 4.1.3 even without the damping term, we only focus on the a priori bounds and the decay rates.

4.2.1. **A priori estimates.** We first have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2+\|\psi\|_{L^2}^2=0,\\ &\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2+\|Z\|_{L^2}^2\right)+\|\nabla\psi\|_{L^2}^2+\|\nabla Z\|_{L^2}^2=0.\end{aligned} \tag{45}$$

We now consider the highest order part:

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla\Delta^2\psi\right\|_{L^2}^2+\left\|\Delta^2 Z\right\|_{L^2}^2\right)+\left\|\nabla\Delta^2\psi\right\|_{L^2}^2+\left\|\nabla\Delta^2 Z\right\|_{L^2}^2$$
$$=-\int\Delta^5\psi[\psi,Z]+\int\Delta^4 Z[\Delta\psi,\psi].$$

We compute the right-hand side of this. By (18a), (18b), and (18c),

$$\begin{aligned}
&-\int\Delta^5\psi[\psi,Z]+\int\Delta^4 Z[\Delta\psi,\psi]\\
&=2\int\Delta^3 Z\left[\Delta\psi_x,\psi_x\right]+2\int\Delta^3 Z\left[\Delta\psi_y,\psi_y\right]+2\int\Delta^2 Z\left[\Delta^2\psi_x,\psi_x\right]\\
&+2\int\Delta^2 Z\left[\Delta^2\psi_y,\psi_y\right]+2\int\Delta Z\left[\Delta^2\psi_x,\Delta\psi_x\right]+2\int\Delta Z\left[\Delta^2\psi_y,\Delta\psi_y\right]\\
&-\int\Delta^3\psi[\Delta\psi,\Delta Z]-2\int\Delta^3\psi[\Delta\psi_x,Z_x]-2\int\Delta^3\psi[\Delta\psi_y,Z_y]-2\int\Delta^3\psi[\psi_x,\Delta Z_x]\\
&-2\int\Delta^3\psi[\psi_y,\Delta Z_y]-2\int\Delta^4\psi[\psi_x,Z_x]-2\int\Delta^4\psi[\psi_y,Z_y]-\int\Delta^3\psi[\Delta^2\psi,Z].
\end{aligned}\tag{46}$$

We now count the number of derivatives hitting on (Z,ψ,ψ) using the integration by parts and (18b) and (18c) up to multiplicative constants. Except for the last integral, we have

$$\begin{aligned}
&(6,2,4)\mapsto(5,2,5),\ (5,3,4)\quad(4,2,6)\mapsto(5,5,2),\ (4,3,5)\\
&(2,2,8)\mapsto(3,2,7)\mapsto(4,2,6),\ (3,3,6)\mapsto(5,5,2),\ (4,3,5)\\
&(2,4,6)\mapsto(2,5,5),\ (3,4,5).
\end{aligned}$$

The last integral is

$$\int\left(\nabla^\perp Z\cdot\nabla\Delta^2\psi\right)\Delta^3\psi=-\int\left(\nabla^\perp\partial_l Z\cdot\nabla\Delta^2\psi\right)\partial_l\Delta^2\psi$$

and so this gives $(2,5,5)$. So, the combinations of the numbers of derivatives taken on (Z,ψ,ψ) are

$$(2,5,5),\ (3,4,5),\ (4,3,5),\ (5,2,5),\ (5,3,4).$$

The first and the fourth cases are bounded by

$$C\left\|\nabla^2 Z\right\|_{L^\infty}\left\|\nabla\Delta^2\psi\right\|_{L^2}^2\le C\left\|\nabla^2 Z\right\|_{L^\infty}^2\left\|\nabla\Delta^2\psi\right\|_{L^2}^2+\frac{1}{6}\left\|\nabla\Delta^2\psi\right\|_{L^2}^2,$$
$$C\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\nabla\Delta^2 Z\right\|_{L^2}^2\le C\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\nabla\Delta^2 Z\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla\Delta^2 Z\right\|_{L^2}^2.$$

The second case is bounded by

$$\begin{aligned}
&C\left\|\nabla^3 Z\right\|_{L^4}\left\|\nabla^4\psi\right\|_{L^4}\left\|\nabla\Delta^2\psi\right\|_{L^2}\le C\left\|\Delta Z\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta^2 Z\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta^2\psi\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta^2\psi\right\|_{L^2}^{\frac{3}{2}}\\
&\le C\left\|\Delta Z\right\|_{L^2}^2\left\|\Delta^2\psi\right\|_{L^2}^2\left\|\nabla\Delta^2 Z\right\|_{L^2}^2+\frac{1}{6}\left\|\nabla\Delta^2\psi\right\|_{L^2}^2.
\end{aligned}$$

The third case is bounded by

$$\begin{aligned}
&C\left\|\nabla^4 Z\right\|_{L^4}\left\|\nabla^3\psi\right\|_{L^4}\left\|\nabla\Delta^2\psi\right\|_{L^2}\le C\left\|\Delta^2 Z\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta^2 Z\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta\psi\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta^2\psi\right\|_{L^2}^{\frac{3}{2}}\\
&\le C\left\|\Delta\psi\right\|_{L^2}^2\left\|\Delta^2 Z\right\|_{L^2}^2\left\|\nabla\Delta^2 Z\right\|_{L^2}^2+\frac{1}{6}\left\|\nabla\Delta^2\psi\right\|_{L^2}^2.
\end{aligned}$$

The last one is bounded by

$$
\begin{aligned}
&C\left\|\nabla^3\psi\right\|_{L^4}\left\|\nabla^4\psi\right\|_{L^4}\left\|\nabla\Delta^2 Z\right\|_{L^2} \leq C\left\|\nabla\Delta\psi\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta^2\psi\right\|_{L^2}\left\|\nabla\Delta^2\psi\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta^2 Z\right\|_{L^2}\\
&\leq C\left\|\nabla\Delta\psi\right\|_{L^2}\left\|\nabla\Delta^2\psi\right\|_{L^2}\left\|\nabla\Delta^2 Z\right\|_{L^2} \leq C\left\|\nabla\Delta\psi\right\|_{L^2}^2\left\|\nabla\Delta^2\psi\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla\Delta^2 Z\right\|_{L^2}^2.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla\Delta^2\psi\right\|_{L^2}^2 + \left\|\Delta^2 Z\right\|_{L^2}^2\right) + \left\|\nabla\Delta^2\psi\right\|_{L^2}^2 + \left\|\nabla\Delta^2 Z\right\|_{L^2}^2\\
&\leq C\left\|\nabla^2 Z\right\|_{L^\infty}^2\left\|\nabla\Delta^2\psi\right\|_{L^2}^2 + C\left\|\nabla^2\psi\right\|_{L^\infty}^2\left\|\nabla\Delta^2 Z\right\|_{L^2}^2 + C\left\|\nabla\Delta\psi\right\|_{L^2}^2\left\|\nabla\Delta^2\psi\right\|_{L^2}^2\\
&+ C\left\|\Delta Z\right\|_{L^2}^2\left\|\Delta^2\psi\right\|_{L^2}^2\left\|\nabla\Delta^2 Z\right\|_{L^2}^2 + C\left\|\Delta\psi\right\|_{L^2}^2\left\|\Delta^2 Z\right\|_{L^2}^2\left\|\nabla\Delta^2 Z\right\|_{L^2}^2
\end{aligned} \tag{47}
$$

By (45) and (47),

$$
\mathcal{F}'(t) + \mathcal{N}_1(t) \leq C\left(\mathcal{F}(t) + \mathcal{F}^2(t)\right)\mathcal{N}_1(t).
$$

So, if $\mathcal{F}_0 = \epsilon_0$ is sufficiently small, we obtain

$$
\mathcal{F}(t) + (1 - C\epsilon_0)\int_0^t \mathcal{N}_1(s)ds \leq \mathcal{F}_0 \quad \text{for all } t > 0.
$$

4.2.2. **Decay rates.** From (45), $\|\psi(t)\|_{L^2} \leq \|\psi_0\|_{L^2} e^{-t}$. Since

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 &= -\int \Delta\psi[\psi, Z] = \int\left(\nabla^\perp Z\cdot\nabla\psi\right)\Delta\psi\\
&= -\int\left(\partial_l\nabla^\perp Z\cdot\nabla\psi\right)\partial_l\psi\\
&\leq \left\|\nabla^2 Z\right\|_{L^\infty}\|\nabla\psi\|_{L^2}^2 \leq C\epsilon_0\|\nabla\psi\|_{L^2}^2,
\end{aligned}
$$

we have

$$
\|\nabla\psi(t)\|_{L^2} \leq \|\nabla\psi_0\|_{L^2} e^{-(1-C\epsilon_0)t}.
$$

By using (14), we also obtain

$$
\left\|\Lambda^k\psi(t)\right\|_{L^2} \leq \mathcal{F}_0^{\frac{k-1}{8}}\|\nabla\psi_0\|_{L^2}^{\frac{5-k}{4}} e^{-\frac{(5-k)(1-C\epsilon_0)}{4}t}, \quad 1 \leq k < 5.
$$

5. Proof of Theorem 1.5 and Theorem 1.6.

5.1. Proof of Theorem 1.5.

We recall (12):

$$
\psi_t - \Delta\psi = [\psi, Z], \quad Z_t = [\Delta\psi, \psi].
$$

By applying the same approximation and mollification methods in Section 4.1.1, we can show the existence of smooth solutions locally in time when $\psi_0 \in H^4$ and $Z_0 \in H^3$.

5.1.1. **A priori estimates.** We first have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2+\|\nabla\psi\|_{L^2}^2=0,\\
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2+\|Z\|_{L^2}^2\right)+\|\Delta\psi\|_{L^2}^2=0.
\end{aligned}\tag{48}$$

We next deal with

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2+\|\nabla\Delta Z\|_{L^2}^2\right)+\left\|\nabla\Delta^2\psi\right\|_{L^2}^2=\int\Delta^4\psi[\psi,Z]-\int\Delta^3 Z[\Delta\psi,\psi]=\mathcal{R}$$

with the same $\mathcal{R}$ in (36). In this case, we choose the the number of derivatives acting on (ψ,ψ,Z) different from Section 4.1.2, which are given by $(3,5,2)$, $(2,5,3)$, and $(4,4,2)$ after several integration by parts. Hence, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2+\|\nabla\Delta Z\|_{L^2}^2\right)+\left\|\nabla\Delta^2\psi\right\|_{L^2}^2\\
&\le C\left\|\nabla^2 Z\right\|_{L^2}\left\|\Delta^2\psi\right\|_{L^4}^2+C\|\nabla\Delta Z\|_{L^2}\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\nabla\Delta^2\psi\right\|_{L^2}+\\
&C\|\Delta Z\|_{L^4}\left\|\nabla^3\psi\right\|_{L^4}\left\|\nabla\Delta^2\psi\right\|_{L^2}\\
&\le C\mathcal{E}_1^2+\frac{1}{2}\left\|\nabla\Delta^2\psi\right\|_{L^2}^2
\end{aligned}$$

and so we have the following bound

$$\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2+\|\nabla\Delta Z\|_{L^2}^2\right)+\left\|\nabla\Delta^2\psi\right\|_{L^2}^2\le C\mathcal{E}^2.\tag{49}$$

By (48) and (49), we derive $\mathcal{E}'\le C\mathcal{E}^2$ from which we deduce

$$\mathcal{E}(t)\le\frac{\mathcal{E}_0}{1-Ct\mathcal{E}_0}\quad\text{for all } 0<t\le T_*<\frac{1}{C\mathcal{E}_0}.\tag{50}$$

5.1.2. **Uniqueness.** Let (ψ_1,Z_1) and (ψ_2,Z_2) be two solutions and let $\psi=\psi_1-\psi_2$ and $Z=Z_1-Z_2$. As in Section 4.1.3

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2+\|Z\|_{L^2}^2\right)+\|\Delta\psi\|_{L^2}^2\\
&=-\int\Delta\psi[\psi,Z_1]-\int\Delta\psi[\psi_2,Z]+\int Z[\Delta\psi,\psi_1]+\int Z[\Delta\psi_2,\psi]\\
&=\text{(I)+(II)+(III)+(IV)}.
\end{aligned}$$

The first term three terms are bounded as

$$\begin{aligned}
\text{(I)}&\le\|\nabla Z_1\|_{L^\infty}\|\nabla\psi\|_{L^2}\|\Delta\psi\|_{L^2}\le C\|\nabla Z_1\|_{L^\infty}^2\|\nabla\psi\|_{L^2}^2+\frac{1}{3}\|\Delta\psi\|_{L^2}^2,\\
\text{(II)+(III)}&=-\int Z[\Delta\psi,\psi]\le C\|\nabla Z\|_{L^\infty}\|\nabla\psi\|_{L^2}\|\Delta\psi\|_{L^2}\\
&\le C\left(\|\nabla Z_1\|_{L^\infty}^2+\|\nabla Z_2\|_{L^\infty}^2\right)\|\nabla\psi\|_{L^2}^2+\frac{1}{3}\|\Delta\psi\|_{L^2}^2
\end{aligned}$$

The last term is bounded as

$$\begin{aligned}(\mathrm{IV}) &\le C\left\|\nabla^3\psi_2\right\|_{L^4}\|\nabla\psi\|_{L^4}\|Z\|_{L^2} \le C\left\|\nabla^3\psi_2\right\|_{L^4}\|\nabla\psi\|_{L^2}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{1}{2}}\|Z\|_{L^2}\\ &\le C\left\|\nabla^3\psi_2\right\|_{L^4}^{\frac{4}{3}}\|\nabla\psi\|_{L^2}^{\frac{2}{3}}\|Z\|_{L^2}^{\frac{4}{3}} + \frac{1}{3}\|\Delta\psi\|_{L^2}^2 \le C\left\|\nabla^3\psi_2\right\|_{L^4}^4\|\nabla\psi\|_{L^2}^2\\ &+ C\|Z\|_{L^2}^2 + \frac{1}{3}\|\Delta\psi\|_{L^2}^2\\ &\le C\|\nabla\Delta\psi_2\|_{L^2}^2\left\|\Delta^2\psi_2\right\|_{L^2}^2\|\nabla\psi\|_{L^2}^2 + C\|Z\|_{L^2}^2 + \frac{1}{3}\|\Delta\psi\|_{L^2}^2.\end{aligned}$$

So, we have

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2 + \|Z\|_{L^2}^2\right)\\ &\le C\left(\|\nabla Z_1\|_{L^\infty}^2 + \|\nabla Z_2\|_{L^\infty}^2 + \|\nabla\Delta\psi_2\|_{L^2}^2\left\|\Delta^2\psi_2\right\|_{L^2}^2\right)\left(\|\nabla\psi\|_{L^2}^2 + \|Z\|_{L^2}^2\right).\end{aligned}$$

By (50), the terms in the parentheses are integrable up to $\frac{T_*}{2}$. By repeating the same argument one more time, we have the uniqueness up to T_*.

5.1.3. **Blow-up criterion.** To derive the blow-up criterion, we first bound

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Delta\psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2\right) + \|\nabla\Delta\psi\|_{L^2}^2 = \int\Delta^2\psi[\psi, Z] - \int\Delta Z[\Delta\psi,\psi]\\ &= 2\int\Delta\psi[\psi_x, Z_x] + 2\int\Delta\psi[\psi_y, Z_y] \le C\left\|\nabla^2\psi\right\|_{L^\infty}\|\nabla Z\|_{L^2}\|\nabla\Delta\psi\|_{L^2}\\ &\le C\left\|\nabla^2\psi\right\|_{L^\infty}^2\|\nabla Z\|_{L^2}^2 + \frac{1}{2}\|\nabla\Delta\psi\|_{L^2}^2\end{aligned}$$

and so we have

$$\frac{d}{dt}\left(\|\Delta\psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2\right) + \|\nabla\Delta\psi\|_{L^2}^2 \le C\left\|\nabla^2\psi\right\|_{L^\infty}^2\|\nabla Z\|_{L^2}^2.$$

This implies

$$\begin{aligned}&\|\Delta\psi(t)\|_L^2 + \|\nabla Z(t)\|_{L^2}^2 + \int_0^t\|\nabla\Delta\psi(s)\|_{L^2}^2\,ds < \infty\\ &\iff \int_0^t\left\|\nabla^2\psi(s)\right\|_{L^\infty}^2 ds < \infty\end{aligned} \tag{51}$$

We also take

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\Delta\psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2\right) + \left\|\Delta^2\psi\right\|_{L^2}^2 = -\int\Delta^3\psi[\psi, Z] + \int\Delta^2 Z[\Delta\psi,\psi]\\ &= -\int\Delta^2\psi[\Delta\psi, Z] - 2\int\Delta^2\psi\left([\psi_x, Z_x] + [\psi_y, Z_y]\right)\\ &\quad - 2\int\Delta\psi\left([\psi_x, \Delta Z_x] + [\psi_y, \Delta Z_y]\right)\\ &= (\mathrm{I})+(\mathrm{II})+(\mathrm{III}).\end{aligned}$$

By using the computation in (41),

$$\begin{aligned}(\mathrm{I}) &= \int\left(\nabla\nabla^\perp Z\cdot\nabla\Delta\psi\right)\cdot\nabla\Delta\psi \le C\left\|\nabla^2 Z\right\|_{L^2}\left\|\nabla^3\psi\right\|_{L^4}^2\\ &\le C\left\|\nabla^2 Z\right\|_{L^2}^2\|\nabla\Delta\psi\|_{L^2}^2 + \frac{1}{6}\left\|\Delta^2\psi\right\|_{L^2}^2.\end{aligned}$$

We next estimate (II) + (III) using (42):

$$\begin{aligned}(\mathrm{II})+(\mathrm{III}) &\le C\int \left|\nabla^2 Z\right|\left|\nabla^3\psi\right|^2 + C\int \left|\nabla^2\psi\right|\left|\nabla^4\psi\right|\left|\nabla^2 Z\right| \\ &\le C\left\|\Delta Z\right\|_{L^2}^2\left\|\nabla\Delta\psi\right\|_{L^2}^2 + C\left\|\nabla^2\psi\right\|_{L^\infty}^2\left\|\Delta Z\right\|_{L^2}^2 + \frac{1}{3}\left\|\Delta^2\psi\right\|_{L^2}^2\end{aligned}$$

So, we have

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\nabla\Delta\psi\right\|_{L^2}^2 + \left\|\Delta Z\right\|_{L^2}^2\right) + \left\|\Delta^2\psi\right\|_{L^2}^2 \\ &\le C\left(\left\|\nabla\Delta\psi\right\|_{L^2}^2 + \left\|\nabla^2\psi\right\|_{L^\infty}^2\right)\left\|\Delta Z\right\|_{L^2}^2 .\end{aligned} \tag{52}$$

By (51), (52) implies

$$\begin{aligned}&\left\|\nabla\Delta\psi(t)\right\|_{L^2}^2 + \left\|\Delta Z(t)\right\|_{L^2}^2 + \int \left\|\Delta^2\psi(s)\right\|_{L^2}^2 ds < \infty \\ &\Longleftrightarrow \int_0^t \left\|\nabla^2\psi(s)\right\|_{L^\infty}^2 ds < \infty.\end{aligned} \tag{53}$$

We finally deal with

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \left\|\nabla\Delta Z\right\|_{L^2}^2\right) + \left\|\nabla\Delta^2\psi\right\|_{L^2}^2 = \int \Delta^4\psi[\psi, Z] - \int \Delta^3 Z[\Delta\psi, \psi]$$

where we count the number of derivatives acting on (ψ, ψ, Z) in (46) as $(3, 5, 2)$, $(2, 5, 3)$, and $(4, 4, 2)$. Then, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \left\|\nabla\Delta Z\right\|_{L^2}^2\right) + \left\|\nabla\Delta^2\psi\right\|_{L^2}^2 \\ &\le C\left\|\Delta Z\right\|_{L^2}^2\left\|\Delta^2\psi\right\|_{L^2}^2 \\ &\qquad + C\left\|\nabla^2\psi\right\|_{L^\infty}^2\left\|\nabla\Delta Z\right\|_{L^2}^2 + C\left\|\Delta Z\right\|_{L^4}^2\left\|\nabla^3\psi\right\|_{L^4}^2 + \frac{1}{2}\left\|\nabla\Delta^2\psi\right\|_{L^2}^2 \\ &\le C\left\|\Delta Z\right\|_{L^2}^2\left\|\Delta^2\psi\right\|_{L^2}^2 + C\left\|\nabla^2\psi\right\|_{L^\infty}^2\left\|\nabla\Delta Z\right\|_{L^2}^2 + C\left\|\Delta Z\right\|_{L^2}^2\left\|\nabla\Delta Z\right\|_{L^2}^2 \\ &+ C\left\|\nabla\Delta\psi\right\|_{L^2}^2\left\|\Delta^2\psi\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla\Delta^2\psi\right\|_{L^2}^2\end{aligned}$$

and so we have

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \left\|\nabla\Delta Z\right\|_{L^2}^2\right) + \left\|\nabla\Delta^2\psi\right\|_{L^2}^2 \\ &\le C\left(\left\|\nabla^2\psi\right\|_{L^\infty}^2 + \left\|\Delta Z\right\|_{L^2}^2\right)\left\|\nabla\Delta Z\right\|_{L^2}^2 + C\left(\left\|\nabla\Delta\psi\right\|_{L^2}^2 + \left\|\Delta Z\right\|_{L^2}^2\right)\left\|\Delta^2\psi\right\|_{L^2}^2\end{aligned} \tag{54}$$

By (51) and (53), (54) implies

$$\begin{aligned}&\left\|\Delta^2\psi(t)\right\|_{L^2}^2 + \left\|\nabla\Delta Z(t)\right\|_{L^2}^2 + \int_0^t \left\|\nabla\Delta^2\psi(s)\right\|_{L^2}^2 ds < \infty \\ &\Longleftrightarrow \int_0^t \left\|\nabla^2\psi(s)\right\|_{L^\infty}^2 ds < \infty.\end{aligned}$$

5.2. **Proof of Theorem 1.6.** We recall (13):

$$\psi_t - \Delta\psi = [\psi, Z], \quad Z_t + Z = [\Delta\psi, \psi].$$

Since the uniqueness is already proved in Section 5.1.2 even without the damping term, we only focus on the a priori bounds.

We first have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 = 0, \\ &\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2 + \|Z\|_{L^2}^2\right) + \|\Delta\psi\|_{L^2}^2 + \|Z\|_{L^2}^2 = 0. \end{aligned} \tag{55}$$

We also have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2\right) + \left\|\nabla\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2 \\ &= \int \Delta^4\psi[\psi, Z] - \int \Delta^3 Z[\Delta\psi, \psi] = \mathcal{R} \end{aligned}$$

with the same $\mathcal{R}$ in (36). In this case, we also choose the the number of derivatives acting on (ψ, ψ, Z) as $(3,5,2)$, $(2,5,3)$, and $(4,4,2)$. Hence,

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2\right) + \left\|\nabla\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2 \\ &\leq C\left\|\nabla^2 Z\right\|_{L^2}\left\|\Delta^2\psi\right\|_{L^4}^2 + C\|\nabla\Delta Z\|_{L^2}\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\nabla\Delta^2\psi\right\|_{L^2} \\ &+ C\|\Delta Z\|_{L^4}\left\|\nabla^3\psi\right\|_{L^4}\left\|\nabla\Delta^2\psi\right\|_{L^2} \\ &\leq C\|\nabla Z\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta Z\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta\psi\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta^2\psi\right\|_{L^2}^{\frac{3}{2}} + C\|\nabla\Delta Z\|_{L^2}\left\|\nabla^2\psi\right\|_{L^\infty}\left\|\nabla\Delta^2\psi\right\|_{L^2} \\ &+ C\|\Delta Z\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta Z\|_{L^2}^{\frac{1}{2}}\|\Delta\psi\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta^2\psi\right\|_{L^2}^{\frac{3}{2}} \\ &\leq C\left(\|\nabla Z\|_{L^2}^2\|\nabla\Delta\psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2\|\Delta\psi\|_{L^2}^2 + \left\|\nabla^2\psi\right\|_{L^\infty}^2\right)\|\nabla\Delta Z\|_{L^2}^2 \\ &\quad + \frac{1}{2}\left\|\nabla\Delta^2\psi\right\|_{L^2}. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2\right) + \left\|\nabla\Delta^2\psi\right\|_{L^2}^2 + \|\nabla\Delta Z\|_{L^2}^2 \\ &\leq C\left(\|\nabla Z\|_{L^2}^2\|\nabla\Delta\psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2\|\Delta\psi\|_{L^2}^2 + \left\|\nabla^2\psi\right\|_{L^\infty}^2\right)\|\nabla\Delta Z\|_{L^2}^2 \end{aligned} \tag{56}$$

By (55) and (56),

$$\mathcal{E}'(t) + \mathcal{N}_2(t) \leq C\left(\mathcal{E}(t) + \mathcal{E}^2(t)\right)\mathcal{N}_2(t).$$

So, if $\mathcal{E}_0 = \epsilon_0$ is sufficiently small, we obtain

$$\mathcal{E}(t) + (1 - C\epsilon_0)\int_0^t \mathcal{N}_2(s)ds \leq \mathcal{E}_0 \quad \text{for all } t > 0.$$

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