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INPUT-OUTPUT L^2-WELL-POSEDNESS, REGULARITY AND LYAPUNOV STABILITY OF STRING EQUATIONS ON NETWORKS

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Abstract. We consider the general networks of elastic strings with Neumann boundary feedbacks and collocated observations in this paper. By selecting an appropriate multiplier, we show that this system is input-output L^2-well-posed. Moreover, we verify its regularity by calculating the input-output transfer function of system. In the end, by choosing an appropriate multiplier, we give a method to construct a Lyapunov functional and prove the exponential decay of tree-shaped networks with one fixed root under velocity feedbacks acted on all leaf vertices.

1. **Introduction and main results.** Generally, the motion of elastic strings on network can be formulated by means of a graph ([12, 15, 24, 25]). In this paper, we always suppose that $G = (V(G), E(G))$ is a connected planar metric graph with the vertex set $V(G) = \{p_1, p_2, \dots, p_m\}$ and the edge set $E(G) = \{e_1, e_2, \dots, e_n\}$. Thus, every edge e_j with the length ℓ_j of G can be parameterized by a continuous function π_j with respective to its arc length. If every edge of G is assigned to a direction that coincides with the arc length increasing, G becomes a digraph. Denote by $\mathcal{I}_E(G) = \{1, 2, \dots, n\}$, $\mathcal{I}_V(G) = \{1, 2, \dots, m\}$ and $\mathcal{I}_E(p_j) = \mathcal{I}_E^+(p_j) \cup \mathcal{I}_E^-(p_j)$, where $\mathcal{I}_E^+(p_j)$ and $\mathcal{I}_E^-(p_j)$

$$\mathcal{I}_E^+(p_j) = \{k \in \mathcal{I}_E(G)\,|p_j \text{ is the starting point (tail) of the edge } e_k, e_k \in E\}$$

and

$$\mathcal{I}_E^-(p_j) = \{k \in \mathcal{I}_E(G)\,|p_j \text{ is the final point (head) of the edge } e_k, e_k \in E\}.$$

Then, the number of elements in sets $\mathcal{I}_E(p_j)$, $\mathcal{I}_E^+(p_j)$ and $\mathcal{I}_E^-(p_j)$ are the degree $(\deg(p_j))$, out-degree $(\deg^+(p_j))$ and in-degree $(\deg^-(p_j))$ of the vertex p_j, respectively. The boundary and the interior of G are defined respectively by

$$\partial G = \{p_j \in V(G)|\deg(p_j) = 1\} \text{ and } Int(G) = \{p_j \in V(G)|\deg(p_j) > 1\}.$$

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Denote the set $(\bigcup_{k=1}^{n} e_k)\bigcup V(G)$ by G for convenience, we thus can use a function $\mathbb{w}(z,t)$, from $G\times[0,+\infty)$ to $\mathbb{R}$, to describe the dynamic behavior of a one-dimensional (1-d) wave equations on network G, where z stands for any point of the set G and t is the time. Especially, $\mathbb{w}(p,t)$ is the value of $\mathbb{w}(z,t)$ at the vertex $p\in V(G)$, which describes the dynamic behavior of the vertex p with the time t. Let the restriction of $\mathbb{w}(z,t)$ to the j-th edge be parametrized by $w_j(x,t)$, that is, $w_j(x,t)=\mathbb{w}(z,t)|_{z\in e_j}=\mathbb{w}(\pi_j(x),t)$. Without loss of generality, we assume that every edge has the unit length. We fix a partition of the vertex set $V(G)=\mathfrak{D}\cup\partial G_N\cup(Int(G)\setminus\mathfrak{D})$, where $\partial G_N=\partial G\setminus\mathfrak{D}$. Thus, the string equations on a continuous type network G can be formulated by (see also [8, 13, 15, 16, 22, 24])

$$\begin{cases} \rho_j(x)w_{j,tt}(x,t)=(T_j(x)w_{j,x})_x(x,t), x\in(0,1), j\in\mathcal{I}_E(G),\\ \forall p\in\mathfrak{D},\ w_i(1,t)=w_k(0,t)=\mathbb{w}(p,t)=0, i\in\mathcal{I}_E^-(p), k\in\mathcal{I}_E^+(p),\\ \forall p\in\partial G_N,\ \text{either } T_k(1)w_{k,x}(1,t)=u(p,t), k\in\mathcal{I}_E^-(p),\\ \qquad\text{or } -T_k(0)w_{k,x}(0,t)=u(p,t), k\in\mathcal{I}_E^+(p),\\ \forall p\in Int(G)\setminus\mathfrak{D},\ w_i(1,t)=\mathbb{w}(p,t)=w_k(0,t), i\in\mathcal{I}_E^-(p), k\in\mathcal{I}_E^+(p),\\ \qquad\text{and } \sum\limits_{i\in\mathcal{I}_E^-(p)}T_i(1)w_{i,x}(1,t)-\sum\limits_{k\in\mathcal{I}_E^+(p)}T_k(0)w_{k,x}(0,t)=u(p,t), \end{cases}\tag{1}$$

where $\rho_j(x)$ and $T_j(x)$ $(j=1,\ldots,n)$ are positive and bounded continuous functions, which are the mass density and the tension of the j-th string, respectively, $u(p,t)$ is the input signal at the vertex p. The input $u(p,t)$ may be zero, then $\mathfrak{F}=\{p\in V(G)|u(p,t)\equiv 0\}$ is called the free vertex set. That is, there no is input signal at the free vertex p. $p\in\mathfrak{D}$ is called a fixed vertex of G, since $\mathbb{w}(p,t)=0$. Thus, the network G is fixed on $\mathfrak{D}$, called the Dirichlet set. In this paper, we assume that $\mathfrak{D}$ is not empty and $V(G)\setminus\mathfrak{D}=\{p_{\jmath_1},p_{\jmath_2},\ldots,p_{\jmath_{m_0}}\}$, where m_0 is the number of vertices in the set $V(G)\setminus\mathfrak{D}$. The output of system (1) is

$$y_k(t)=\mathbb{w}_t(p_{\jmath_k},t)\text{ for }p_{\jmath_k}\notin\mathfrak{F}, k=1,\ldots,m_0,\tag{2}$$

where $\mathbb{w}_t(p_{\jmath_k},t)$ is the derivative of $\mathbb{w}(p_{\jmath_k},t)$ with respective to time t.

One of the objective of this paper is to investigate the input-output well-posedness and regularity of 1-d wave equations on networks (1). Let the state space $\mathbb{X}$, the control space $\mathbb{U}$ and the observation space $\mathbb{Y}$ be Hilbert spaces, and $L^2_{loc}([0,+\infty),\mathbb{U})$ be the space of those functions on $[0,+\infty)$ whose restriction to $[0,\tau]$ is in $L^2([0,\tau],\mathbb{U})$, for every $\tau>0$. An infinite-dimensional linear system is input-output L^2-well-posed, if, for every $t\geq 0$, there exists $M_t>0$, only depending on t, such that

$$\|X(t)\|^2+\int_0^t\|Y(s)\|^2ds\leq M_t\left[\|X_0\|^2+\int_0^t\|U(s)\|^2ds\right],$$

where $X(t)$ is the state of system, X_0 is its initial state, $U(\cdot)\in L^2_{loc}([0,+\infty),\mathbb{U})$ is the input of system and $Y(\cdot)\in L^2_{loc}([0,+\infty),\mathbb{Y})$ is the output of system. Denote by $\mathbf{H}(s)$ the input-output transfer function of the well-posed system. Thus, a well-posed system is called regular if $\lim\limits_{s\to+\infty}\mathbf{H}(s)u=\mathcal{D}u, \forall u\in\mathbb{U}$, where $\mathcal{D}$ is a bounded operator from $\mathbb{U}$ to $\mathbb{Y}$. Refer to [7, 20, 21, 23, 26] and references therein for more details on the theory of input-output well-posedness and regularity. Since 1980s, it has been demonstrated that this class of systems is quite general, including many control systems described by partial differential equations with inputs and outputs on internal sub-domains, or on the (partial) boundary of the spatial region (see

[4, 5, 6, 10, 11, 14, 18, 30] and references therein). But the well-posedness and regularity of 1-d networks has received little attention in the literature. In [1], the well-posedness of a tree-shaped network of strings with feedback acting on the root of the tree was shown, based on the d'Alembert formula. One of main contributions of this paper is to prove the L^2-well-posedness and regularity of the general network system of strings (1) with outputs (2), by constructing suitable multipliers with graph theory and the asymptotical theory of fundamental solution. Here we state this result as Theorem 1.1 and its proof is deferred to Section 3.

Theorem 1.1. *Assume that $\mathfrak{D}$ is not empty, $0 < \rho_L \leq \rho_k(x) \leq \rho_U$, $0 < T_L \leq T_k(x) \leq T_U$, and $\rho_k(\cdot), T_k(\cdot) \in C^1[0,1]$, $\forall k \in \mathcal{I}_E(G)$, and that the input $u(p,\cdot) \in L^2_{loc}([0,+\infty),\mathbb{R})$ for every vertex $p \in V(G) \setminus (\mathfrak{D} \cup \mathfrak{F})$, and the outputs are defined by (2). Then the system (1) is input-output L^2-well-posed and regular.*

It is well-known that only tree-shaped networks with one fixed vertex can be exponential decay under appropriate velocity feedbacks; and when there exist more than two fixed vertices or closed cycles in networks of strings, under velocity feedbacks, the networks is at most polynomial decay or is not stable [13]. Based on the observability estimate, the polynomial decay of a planar tree-shaped network of strings under only one vertex being damped (one-node stabilization) were discussed in [1, 2, 8, 24]. Riesz basis approach is used in [15, 28, 29] to prove that the spectrum-determined-growth (SDG) condition holds for the networks, so the stability of closed-loop systems can be determined by their spectral bound. The decay rate of the chain-shaped and star-shaped networks was estimated by choosing a suitable weighted energy functional in [27]. By means of the frequency domain method, the exponential stability of a tree-shaped network was confirmed in [16]. However, Lyapunov stability is more common and intuitive in engineering. So, in this paper, we use the Lyapunov method to study the stability on networks, e.g., the tree-shaped networks shown as Figure 1b and Figure 2. Thus, the second contribution of this paper is to provide a construction method of Lyapunov functional for general tree-shaped networks of strings with one fixed vertex.

For a connected tree $G = (V(G), E(G))$, a vertex is selected as its root, then, G is said to be a rooted tree. A boundary vertex of the rooted tree is called a leaf vertex, or a leaf for short, if it is not the root. That is, ∂G_N is consisted of all leaves of the rooted tree G. An edge incident with a leaf is called a leaf edge. For convenience, we give a hypothesis as follows (see [15, 16]).

Hypotheses 1.2.

(1) The rooted tree-shaped networks G with the fixed root p_r has no input signal at the internal vertex p_k $(k \neq r)$, i.e., $\mathfrak{D} = \{p_r\}$ and $u(p_k, t) = 0$, for $p_k \in Int(G) \backslash \mathfrak{D}$.

(2) $\deg(p_r) = \deg^+(p_r) = 1$, $\mathcal{I}_E^+(p_r) = \{r\}$.

(3) $\deg^-(p_k) = 1$ for all $k \in \mathcal{I}_V(G)$ with $k \neq r$.

Under Hypothesis 1.2, the motion of the tree-shaped network G is described by

$$\begin{cases} \rho_j(x) w_{j,tt}(x,t) = (T_j(x) w_{j,x})_x(x,t), x \in (0,1), j \in \mathcal{I}_E(G), \\ w_r(0,t) = \mathrm{w}(p_r, t) = 0, \\ \forall p \in Int(G) \setminus \{p_r\}, \mathcal{I}_E^-(p) = \{k\}, \\ \quad w_k(1,t) = \mathrm{w}(p,t) = w_i(0,t) \text{ for } i \in \mathcal{I}_E^+(p), \text{ and} \\ \quad T_k(1) w_{k,x}(1,t) = \sum_{i \in \mathcal{I}_E^+(p)} T_i(0) w_{i,x}(0,t), \\ \forall p \in \partial G_N, \mathcal{I}_E^-(p) = \{k\}, T_k(1) w_{k,x}(1,t) = u(p,t). \end{cases} \tag{3}$$

Thus, we have the following result.

Theorem 1.3. *Assume that Hypothesis 1.2 holds and that* $0 < \rho_L \le \rho_k(x) \le \rho_U$, $0 < T_L \le T_k(x) \le T_U$ *on* $[0,1]$ *and* $\rho_k(\cdot), T_k(\cdot) \in C^1[0,1]$, *for every* $k \in \mathcal{I}_E(G)$, *then the tree-shaped network* (3) *is exponentially stable under the output feedback*

$$u(p_{\jmath_k}, t) = -\beta_k y_k(t) = -\beta_k \mathbb{w}_t(p_{\jmath_k}, t), p_{\jmath_k} \in \partial G_N, \tag{4}$$

where $\beta_k > 0$.

Theorem 1.3 will be proven by constructing a suitable Lyapunov functional in Section 4.

Remark 1. Under Hypothesis 1.2 (3), a tree G with the root p_r is also called a branching with the root p_r [3]. In fact, the root may be an internal vertex (i.e.,(2) in Hypothesis 1.2 dose not hold). Hypothesis 1.2 only is for the sake of the proof of Theorem 1.3 and the construction of Lyapunov functional. If Hypothesis 1.2 (3) is not true, we can take a change of variable $x := 1 - x$, such that it is satisfied. Thus, we can also construct a Lyapunov functional such that the result of Theorem 1.3 also holds for all trees with one fixed root and all leaves controlled by velocity feedbacks, like (4). See the example in Subsection 4.2.2 below.

All in all, main contributions of this paper are:

(1): to prove the input-output L^2-well-posedness and regularity of the general network system of strings, by constructing suitable multipliers and the asymptotical theory of fundamental solution, respectively;

(2): to provide a construction method of Lyapunov functional for general tree-shaped networks of strings with one fixed root.

The choice of multipliers for the L^2-well-posedness and the Lyapunov functional, based on graph notions and theories, is the novelty of this paper, which is also the main difficulty of this paper. The paper is organized as follows. The matrix-vector form of network (1) is provided in light of graph theory in Section 2. Theorem 1.1 and Theorem 1.3 are proven in Section 3 and Section 4, respectively. Finally, the conclusions follow in Section 5.

2. The matrix-vector form of system (1) in $\mathbb{R}^n$.

To study the 1-d wave propagation on general networks, we need some fundamental notations, concepts of the graph theory and a proposition. See [3] and [9] for more details about the graph theory.

Definition 2.1. The matrices $\Upsilon^+ = (\upsilon^+_{i,j})_{m\times n}$ and $\Upsilon^- = (\upsilon^-_{i,j})_{m\times n}$, given by

$$\upsilon^+_{i,j} = \begin{cases} 1, & \text{if } \pi_j(0) = p_i, \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \upsilon^-_{i,j} = \begin{cases} 1, & \text{if } \pi_j(1) = p_i, \\ 0, & \text{otherwise,} \end{cases}$$

are called the outgoing incidence matrix and the incoming incidence matrix, respectively. The incidence matrix is defined by $\Upsilon = \Upsilon^+ - \Upsilon^-$.

From the above definition, it follows that

$$\Upsilon^+ = \left(\epsilon_{\jmath_1^+}, \dots, \epsilon_{\jmath_k^+}, \dots, \epsilon_{\jmath_n^+}\right)_{m\times n} \text{ and } \Upsilon^- = \left(\epsilon_{\jmath_1^-}, \dots, \epsilon_{\jmath_k^-}, \dots, \epsilon_{\jmath_n^-}\right)_{m\times n} \tag{5}$$

where $\jmath_k^+, \jmath_k^- \in \mathcal{I}_V(G)$, $k \in \mathcal{I}_E(G)$, $\epsilon_{\jmath_k^+}$ and $\epsilon_{\jmath_k^-}$ are the $\jmath_k^+$-th and the $\jmath_k^-$-th column vector of I_m, the identity matrix of order m, respectively. $\epsilon_{\jmath_k^+}$ shows that the $\jmath_k^+$-th vertex $p_{\jmath_k^+}$ is the starting point (tail) of the edge e_k, and $\epsilon_{\jmath_k^-}$ shows that the $\jmath_k^-$-th vertex $p_{\jmath_k^-}$ is the final point (head) of the edge e_k. We denote an $m_0 \times m$ matrix by

$$P_{\mathbb{D}} = (\epsilon_{\jmath_1}, \epsilon_{\jmath_2}, \ldots, \epsilon_{\jmath_{m_0}})^\top, \; p_{\jmath_k} \in V(G) \setminus \mathfrak{D}, \tag{6}$$

where the vector $\epsilon_{\jmath_k}$ is the $\jmath_k$-th column of the identity matrix I_m. Then $P_{\mathbb{D}}$ is an orthogonal projection from $\mathbb{R}^m$ to $\mathbb{R}^{m_0}$. Thus, $P_{\mathbb{D}}W$ retains the $\jmath_1$-th row, the $\jmath_2$-th row, …, the $\jmath_{m_0}$-th row of an m-row matrix W, and removes the other rows of W. Moreover,

$$P_{\mathbb{D}}P_{\mathbb{D}}^\top = I_{m_0} \in \mathbb{R}^{m_0 \times m_0} \text{ and } P_{\mathbb{D}}^\top P_{\mathbb{D}} = I_{\mathfrak{D}} \in \mathbb{R}^{m \times m}, \tag{7}$$

where I_{m_0} is the identity matrix of order m_0, $I_{\mathfrak{D}}$ is a diagonal matrix whose entries from the $\jmath_1$-th to the $\jmath_{m_0}$-th are one, others are zero.

Let

$$w(x,t) = \begin{pmatrix} w_1(x,t) \\ \vdots \\ w_n(x,t) \end{pmatrix} \text{ and } \boldsymbol{w}(\boldsymbol{p},t) = \begin{pmatrix} \mathbb{w}(p_1,t) \\ \vdots \\ \mathbb{w}(p_m,t) \end{pmatrix},$$

and call them the vectorization of $\mathbb{w}(z,t)$, where $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)^\top$. Thus, the system (1) can be rewritten as

$$\begin{cases} M(x)w_{tt}(x,t) = (T(x)w_x)_x(x,t), \; x \in (0,\;1), \; t > 0, \\ w(0,t) = (\Upsilon^+)^\top \boldsymbol{w}(\boldsymbol{p},t), w(1,t) = (\Upsilon^-)^\top \boldsymbol{w}(\boldsymbol{p},t), \\ P_{\mathbb{D}} \left[\Upsilon^- T(1)w_x(1,t) - \Upsilon^+ T(0)w_x(0,t)\right] = \boldsymbol{u}(t), \end{cases} \tag{8}$$

where $M(x) = \operatorname{diag}(\rho_1(x), \ldots, \rho_n(x))$, $T(x) = \operatorname{diag}(T_1(x), \ldots, T_n(x))$,

$$\boldsymbol{u}(t) = \begin{pmatrix} u(p_{\jmath_1},t) \\ \vdots \\ u(p_{\jmath_{m_0}},t) \end{pmatrix} \text{ with } u(p_{\jmath_k},t) = 0 \text{ as } p_{\jmath_k} \in \mathfrak{F} \text{ and } \boldsymbol{u}(\cdot) \in L^2_{loc}([0,+\infty); \mathbb{R}^{m_0}).$$

The output of system (2) can be read as

$$Y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_{m_0}(t) \end{pmatrix} = P_{\mathbb{D}}\boldsymbol{w}_t(\boldsymbol{p},t) = \begin{pmatrix} \mathbb{w}_t(p_{\jmath_1},t) \\ \vdots \\ \mathbb{w}_t(p_{\jmath_{m_0}},t) \end{pmatrix}, \tag{9}$$

where $y_k(t)$ or $\mathbb{w}_t(p_{\jmath_k},t)$ is 0 when $p_{\jmath_k} \in \mathfrak{F}$, which means that the system (1) has no output signal at free vertex $p_{\jmath_k}$.

Remark 2. Similar to the matrix $P_{\mathbb{D}}$, we introduce matrices

$$P_u = (\epsilon_{\jmath_{k_1}}, \epsilon_{\jmath_{k_2}}, \ldots, \epsilon_{\jmath_{k_{m_u}}})^\top, \text{ for } \; p_{\jmath_{k_i}} \in V(G) \setminus (\mathfrak{D} \cup \mathfrak{F}) \tag{10}$$

and

$$P_u^\perp = (\epsilon_{\jmath_{\hat{k}_1}}, \epsilon_{\jmath_{\hat{k}_2}}, \ldots, \epsilon_{\jmath_{\hat{k}_{m_0-m_u}}})^\top, \text{ for } p_{\jmath_{\hat{k}_i}} \in \mathfrak{F},$$

where the vector $\epsilon_{\jmath_{k_i}}$ and $\epsilon_{\jmath_{\hat{k}_i}}$ are the $\jmath_{k_i}$-th and $\jmath_{\hat{k}_i}$-th column of the identity matrix I_m, respectively. Thus, the last boundary condition in (8) can be rewritten as

$$\begin{pmatrix} P_u^\perp \\ P_u \end{pmatrix} \left[\Upsilon^- T(1)w_x(1,t) - \Upsilon^+ T(0)w_x(0,t)\right] = \begin{pmatrix} 0 \\ P_u P_{\mathbb{D}}^\top \boldsymbol{u}(t) \end{pmatrix},$$

where $P_u P_{\mathbb{D}}^{\top}\boldsymbol{u}(t)$ is the true (nonzero) input signal. According to (2), (7) and (9), the true (nonzero) output signal can be written as

$$Y_u(t) = P_u P_{\mathbb{D}}^{\top} Y(t) = \begin{pmatrix} y_{k_1}(t) \\ \vdots \\ y_{k_{m_u}}(t) \end{pmatrix} = P_u \boldsymbol{w}_t(\boldsymbol{p}, t) = \begin{pmatrix} \mathbb{w}_t(p_{\jmath_{k_1}}, t) \\ \vdots \\ \mathbb{w}_t(p_{\jmath_{k_{m_u}}}, t) \end{pmatrix}.$$

The energy function of system (1) is defined as follows:

$$\mathscr{E}(t) = \frac{1}{2}\int_0^1 [\langle M(x)w_t(x,t),\, w_t(x,t)\rangle + \langle T(x)w_x(x,t),\, w_x(x,t)\rangle]\, dx, \tag{11}$$

where $\langle \cdot, \cdot \rangle$ represents the Euclidean inner product in $\mathbb{R}^n$. Thus it can be derived from integration by parts and (8) that

$$\begin{aligned} \frac{d\mathscr{E}(t)}{dt} &= \langle (\Upsilon^-)T(1)w_x(1,\, t) - (\Upsilon^+)T(0)w_x(0,\, t),\, \boldsymbol{w}_t(\boldsymbol{p}, t)\ \rangle \\ &= \langle \boldsymbol{u}(t),\, P_{\mathbb{D}}\boldsymbol{w}_t(\boldsymbol{p}, t)\ \rangle_{\mathbb{R}^{m_0}} = \langle \boldsymbol{u}(t),\, Y(t)\rangle_{\mathbb{R}^{m_0}}, \end{aligned} \tag{12}$$

which means that the output of system (1), i.e., $Y(t) = P_{\mathbb{D}}\boldsymbol{w}_t(\boldsymbol{p}, t)$, is collocated.

In the end of this section, we introduce the following definition of edge adjacency matrix and its proposition which discloses relationship between this definition and Definition 2.1.

Definition 2.2. An edge in G is said to be a loop, if its tail and head are the same. Let $G = (V(G), E(G))$ be a loopless digraph. The $n \times n$ matrix $B_G^+ = (b_{i,j}^+)_{n\times n}$, defined by

$$b_{i,j}^+ = \begin{cases} 1, & \text{if two different edges } e_i \text{ and } e_j \text{ join at a common tail,} \\ 0, & \text{otherwise,} \end{cases}$$

is called the outgoing edge adjacency matrix of G. The $n\times n$ matrix $B_G^- = (b_{i,j}^-)_{n\times n}$, defined by

$$b_{i,j}^- = \begin{cases} 1, & \text{if two different edges } e_i \text{ and } e_j \text{ join at a common head,} \\ 0, & \text{otherwise,} \end{cases}$$

is called the incoming edge adjacency matrix of G. The $n \times n$ matrix $B_G^{t,h} = (b_{i,j}^{t,h})_{n\times n}$, defined by

$$b_{i,j}^{t,h} = \begin{cases} 1, & \text{if } p \text{ is the tail of } e_i \text{ and the head of } e_j, \text{ for some vertex } p \in V(G), \\ 0, & \text{otherwise,} \end{cases}$$

is called the outgoing-incoming edge adjacency matrix G. The $n \times n$ matrix $B_G^{h,t} = (b_{i,j}^{h,t})_{n\times n}$, where

$$b_{i,j}^{h,t} = \begin{cases} 1, & \text{if } p \text{ is the head of } e_i, \text{ and the tail of } e_j, \text{ for some vertex } p \in V(G), \\ 0, & \text{otherwise,} \end{cases}$$

is called the incoming-outgoing edge adjacency matrix of G. Thus, the matrix $B_G = B_G^+ + B_G^- + B_G^{t,h} + B_G^{h,t}$, is called the edge adjacency matrix of G.

Note that, in Definition 2.2, the diagonal entries of these matrices, B_G, B_G^+, B_G^-, $B_G^{t,h}$ and $B_G^{h,t}$, are all zeros, B_G^+ and B_G^- are symmetrical and $B_G^{t,h} = (B_G^{h,t})^{\top}$. In addition, the following proposition can be derived from Definition 2.1 and 2.2.

Proposition 1. *Assume that G is a loopless digraph, Υ^+ and Υ^- are its outgoing incidence matrix and incoming incidence matrix, respectively. Let*

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m),\ \Lambda^\ominus = \mathrm{diag}\left(\lambda_{\jmath_1^-}, \lambda_{\jmath_2^-}, \ldots, \lambda_{\jmath_n^-}\right)$$

and $\Lambda^\oplus = \mathrm{diag}\left(\lambda_{\jmath_1^+}, \lambda_{\jmath_2^+}, \ldots, \lambda_{\jmath_n^+}\right)$, *where $\jmath_k^+$ and $\jmath_k^-$ defined by (5). Then*

(1): $(\Upsilon^-)\Lambda(\Upsilon^-)^\top = \mathrm{diag}\left(\sum_{k\in\mathcal{I}_E^-(p_1)} \lambda_k, \ldots, \sum_{k\in\mathcal{I}_E^-(p_i)} \lambda_k, \ldots, \sum_{k\in\mathcal{I}_E^-(p_m)} \lambda_k\right)$;

(2): $(\Upsilon^+)\Lambda(\Upsilon^+)^\top = \mathrm{diag}\left(\sum_{k\in\mathcal{I}_E^+(p_1)} \lambda_k, \ldots, \sum_{k\in\mathcal{I}_E^+(p_i)} \lambda_k, \ldots, \sum_{k\in\mathcal{I}_E^+(p_m)} \lambda_k\right)$;

(3): $(\Upsilon^-)^\top\Lambda\Upsilon^+ = [\Lambda^\ominus]B_G^{h,t}$ *and* $(\Upsilon^+)^\top\Lambda\Upsilon^- = [\Lambda^\oplus]B_G^{t,h} = B_G^{t,h}[\Lambda^\ominus]$;

(4): $(\Upsilon^-)^\top\Lambda\Upsilon^- = [\Lambda^\ominus](I + B_G^-)$ *and* $(\Upsilon^+)^\top\Lambda\Upsilon^+ = [\Lambda^\oplus](I + B_G^+)$;

where we agree that $\sum_{s\in\emptyset} \lambda_k = 0$. *Especially,*

$$(\Upsilon^-)^\top\Upsilon^+ = B_G^{h,t},\ (\Upsilon^+)^\top\Upsilon^- = B_G^{t,h},\ (\Upsilon^-)^\top\Upsilon^- = I + B_G^- \ \text{ and } (\Upsilon^+)^\top\Upsilon^+ = I + B_G^+.$$

Remark 3. Assume that $m = n + 1$, and $\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_0, \widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_n)$. Denote by $\lambda_k = \widetilde{\lambda}_{k-1}$, for $k = 1, \ldots, n+1$, then $\widetilde{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$,

$$\widetilde{\Lambda}^\ominus = \mathrm{diag}\left(\lambda_{\jmath_1^-}, \lambda_{\jmath_2^-}, \ldots, \lambda_{\jmath_n^-}\right) = \mathrm{diag}\left(\widetilde{\lambda}_{\jmath_1^- -1}, \widetilde{\lambda}_{\jmath_2^- -1}, \ldots, \widetilde{\lambda}_{\jmath_n^- -1}\right)$$

and

$$\widetilde{\Lambda}^\oplus = \mathrm{diag}\left(\lambda_{\jmath_1^+}, \lambda_{\jmath_2^+}, \ldots, \lambda_{\jmath_n^+}\right) = \mathrm{diag}\left(\widetilde{\lambda}_{\jmath_1^+ -1}, \widetilde{\lambda}_{\jmath_2^+ -1}, \ldots, \widetilde{\lambda}_{\jmath_n^+ -1}\right).$$

3. Proof of Theorem 1.1 and examples.

The proof is divided into two parts: the input-output L^2-well-posedness and the regularity. We first prove the input-output L^2-well-posedness.

3.1. Proof of input-output well-posedness.

Proof. We choose bounded continuous and differentiable functions on $[0,1]$: $\xi_k(x)$, $k \in \mathcal{I}_E(G)$, such that

$$\xi_k(1)/T_k(1) > 2n, -\xi_k(0)/T_k(0) > 2n, k \in \mathcal{I}_E(G), \tag{13a}$$

and

$$\begin{cases} \max_{x\in[0,1]} \left\{\|M^{1/2}(x)\Xi(x)T^{1/2}(x)\|_2\right\} \le c_E, \\ \max_{x\in[0,1]} \left\{\|[\Xi(x)T^{-1}(x)]'T(x)\|_2\right\} \le c_E, \\ \max_{x\in[0,1]} \left\{\|[\Xi(x)M(x)]'M^{-1}(x)\|_2\right\} \le c_E, \end{cases} \tag{13b}$$

where $\Xi(x) = \mathrm{diag}\,(\xi_1(x), \ldots, \xi_n(x))$ and $c_E > 0$.

The first equation in (8) multiplied by $\Xi(x)w_x(x,t)$ on both sides, then integrated on $[0,1]$ with respect to x and on $[0,t]$ with respect to t leads to

$$\int_0^t \int_0^1 \langle \Xi(x)w_x(x,t), M(x)w_{tt}(x,t)\rangle dxdt = \int_0^t \int_0^1 \langle \Xi(x)w_x(x,t), (T(x)w_x)_x(x,t)\rangle dxdt. \tag{14}$$

Applying integration by parts, (9), the following equality

$$\int_0^t\int_0^1 \langle \Xi(x)w_{xt}(x,t), M(x)w_t(x,t)\rangle dxdt = \frac{1}{2}\int_0^t \langle w_t(1,t), \Xi(1)M(1)w_t(1,t)\rangle dt$$
$$-\frac{1}{2}\int_0^t \langle w_t(0,t), \Xi(0)M(0)w_t(0,t)\rangle dt - \frac{1}{2}\int_0^t\int_0^1 \langle w_t(x,t), (\Xi(x)M(x))' w_t(x,t)\rangle dxdt,$$

and boundary conditions in (8): $w(0,t) = (\Upsilon^+)^\top \boldsymbol{w}(\boldsymbol{p},t)$, $w(1,t) = (\Upsilon^-)^\top \boldsymbol{w}(\boldsymbol{p},t)$, to the left-hand side of (14) yields

$$\begin{aligned} LHS &= \int_0^1 [\langle \Xi(x)w_x(x,t), M(x)w_t(x,t)\rangle - \langle \Xi(x)w_x(x,0), M(x)w_t(x,0)\rangle]dx \\ &\quad - \int_0^t \int_0^1 \langle \Xi(x)w_{xt}(x,t), M(x)w_t(x,t)\rangle dxdt \\ &= \int_0^1 \langle \Xi(x)w_x(x,t), M(x)w_t(x,t)\rangle dx - \int_0^1 \langle \Xi(x)w_x(x,0), M(x)w_t(x,0)\rangle dx \\ &\quad -\frac{1}{2}\int_0^t \langle Y(t), P_{\mathbb{D}}[\Upsilon^-\Xi(1)M(1)(\Upsilon^-)^\top - \Upsilon^+\Xi(0)M(0)(\Upsilon^+)^\top]P_{\mathbb{D}}^\top Y(t)\rangle dt \\ &\quad +\frac{1}{2}\int_0^t \int_0^1 \langle w_t(x,t), (\Xi(x)M(x))' w_t(x,t)\rangle dxdt. \end{aligned}$$

Similarly, applying integration by parts to the right-hand side of (14) yields

$$\begin{aligned} RHS &= \frac{1}{2}\int_0^t \langle \Xi(1)w_x(1,t), T(1)w_x(1,t)\rangle dt - \frac{1}{2}\int_0^t \langle \Xi(0)w_x(0,t), T(0)w_x(0,t)\rangle dt \\ &\quad -\frac{1}{2}\int_0^t \int_0^1 \langle [\Xi(x)T^{-1}(x)]' T(x)w_x(x,t), T(x)w_x(x,t)\rangle dxdt. \end{aligned}$$

Thus, it can be derived from (14) that

$$\begin{aligned} &\int_0^1 \langle \Xi(x)w_x(x,t), M(x)w_t(x,t)\rangle dx - \int_0^1 \langle \Xi(x)w_x(x,0), M(x)w_t(x,0)\rangle dx \\ &+\frac{1}{2}\int_0^t \int_0^1 \langle w_t(x,t), [\Xi(x)M(x)]' w_t(x,t)\rangle dxdt \\ &+\frac{1}{2}\int_0^t \int_0^1 \langle [\Xi(x)T^{-1}(x)]' T(x)w_x(x,t), T(x)w_x(x,t)\rangle dxdt \\ = &\ \frac{1}{2}\int_0^t \langle \Xi(1)w_x(1,t), T(1)w_x(1,t)\rangle dt - \frac{1}{2}\int_0^t \langle \Xi(0)w_x(0,t), T(0)w_x(0,t)\rangle dt \\ &+\frac{1}{2}\int_0^t \langle Y(t), P_{\mathbb{D}}[\Upsilon^-\Xi(1)M(1)(\Upsilon^-)^\top - \Upsilon^+\Xi(0)M(0)(\Upsilon^+)^\top]P_{\mathbb{D}}^\top Y(t)\rangle dt. \end{aligned} \quad (15)$$

From the boundary condition $P_{\mathbb{D}}\left[\Upsilon^- T(1)w_x(1,t) - \Upsilon^+ T(0)w_x(0,t)\right] = \boldsymbol{u}(t)$ in (8), it is obtained that

$$\begin{aligned} \|\boldsymbol{u}(t)\|^2 &= \langle \boldsymbol{u}(t), \boldsymbol{u}(t)\rangle_{\mathbb{R}^{m_0}} \le 2\langle (P_{\mathbb{D}}\Upsilon^-)^\top P_{\mathbb{D}}\Upsilon^- T(1)w_x(1,t), T(1)w_x(1,t)\rangle \\ &\quad +2\langle \left[P_{\mathbb{D}}\Upsilon^+\right]^\top P_{\mathbb{D}}\Upsilon^+ T(0)w_x(0,t), T(0)w_x(0,t)\rangle. \end{aligned}$$

It can be deduced from Definition 2.2 that

$$0 \le \min_{1\le i\le n}\left\{\sum_{j=1}^{n} b_{i,j}^{-}\right\} \le \max_{1\le i\le n}\left\{\sum_{j=1}^{n} b_{i,j}^{-}\right\} \le n-1$$

and

$$0 \le \min_{1\le i\le n}\left\{\sum_{j=1}^{n} b_{i,j}^{+}\right\} \le \max_{1\le i\le n}\left\{\sum_{j=1}^{n} b_{i,j}^{+}\right\} \le n-1,$$

thus, it follows from (7), the equalities **(4)** in Proposition 1 and (13a) that

$$\Xi(1)T(1)^{-1} - 2(\Upsilon^-)^\top P_{\mathbb{D}}^\top P_{\mathbb{D}}\Upsilon^- = \Xi(1)T(1)^{-1} - 2[I_{\mathfrak{D}}^{\ominus}](I + B_G^-) > 0$$

and

$$-\Xi(0)T(0)^{-1} - 2(\Upsilon^+)^\top P_{\mathbb{D}}^\top P_{\mathbb{D}}\Upsilon^+ = -\Xi(0)T(0)^{-1} - 2[I_{\mathfrak{D}}^{\oplus}](I + B_G^+) > 0,$$

i.e., $\Xi(1)T(1)^{-1} - 2(P_{\mathbb{D}}\Upsilon^-)^\top P_{\mathbb{D}}\Upsilon^-$ and $-\Xi(0)T(0)^{-1} - 2(P_{\mathbb{D}}\Upsilon^+)^\top P_{\mathbb{D}}\Upsilon^+$ are symmetric positive definite matrices. Therefore,

$$\begin{aligned}
&\langle \Xi(1)w_x(1,t), T(1)w_x(1,t)\rangle - \langle \Xi(0)w_x(0,t), T(0)w_x(0,t)\rangle - \|\boldsymbol{u}(t)\|^2\\
\ge\;& \langle \Xi(1)w_x(1,t), T(1)w_x(1,t)\rangle - \langle \Xi(0)w_x(0,t), T(0)w_x(0,t)\rangle\\
&-2\langle (P_{\mathbb{D}}\Upsilon^-)^\top P_{\mathbb{D}}\Upsilon^- T(1)w_x(1,t), T(1)w_x(1,t)\rangle\\
&-2\langle \left[P_{\mathbb{D}}\Upsilon^+\right]^\top P_{\mathbb{D}}\Upsilon^+ T(0)w_x(0,t), T(0)w_x(0,t)\rangle \ge 0. \qquad (16)
\end{aligned}$$

Obviously, the equalities **(1)** and **(2)** in Proposition 1 and (13a) imply that

$$P_{\mathbb{D}}[(\Upsilon^-)\Xi(1)M(1)(\Upsilon^-)^\top - (\Upsilon^+)\Xi(0)M(0)(\Upsilon^+)^\top]P_{\mathbb{D}}^\top = \operatorname{diag}\left(c_{p_{\jmath_1}}, \ldots, c_{p_{\jmath_{m_0}}}\right)$$

with

$$c_{p_{\jmath_k}} = \left[\sum_{i\in\mathcal{I}_E^-(p_{\jmath_k})} \xi_i(1)\rho_i(1) - \sum_{i\in\mathcal{I}_E^+(p_{\jmath_k})} \xi_i(0)\rho_i(0)\right] > 0.$$

So, it yields that

$$\langle Y(t),\, P_{\mathbb{D}}[\Upsilon^-\Xi(1)M(1)(\Upsilon^-)^\top - \Upsilon^+\Xi(0)M(0)(\Upsilon^+)^\top]P_{\mathbb{D}}^\top Y(t)\rangle \ge c_p\|Y(t)\|^2, \quad (17)$$

where $c_p = \min_{k=1,\ldots,m_0}\{c_{p_{\jmath_k}}\}$. From the inequalities in (13b), it can be derived that

$$\left|\int_0^1 \langle \Xi(x)w_x(x,t), M(x)w_t(x,t)\rangle dx\right| \le c_E\mathscr{E}(t) \qquad (18)$$

and

$$\frac{1}{2}\int_0^1 \langle w_t(x,t), (\Xi(x)M(x))' w_t(x,t)\rangle dx + \frac{1}{2}\int_0^1 \langle [\Xi(x)T^{-1}(x)]' T(x)w_x(x,t), T(x)w_x(x,t)\rangle dx \le c_E\mathscr{E}(t). \quad (19)$$

Thus, it follows from (15), (16), (17), (18) and (19) that

$$c_0\left[\mathscr{E}(t) + \mathscr{E}(0) + \int_0^t \mathscr{E}(t)dt\right] \;\ge\; \int_0^t \|\boldsymbol{u}(t)\|^2dt + \int_0^t \|Y(t)\|^2dt \qquad (20)$$

with $c_0^{-1} = \frac{1}{2c_E}\min\{1, c_p\}$. From (12) and (20), it can be deduced that

$$\mathscr{E}(t) \leq \mathscr{E}(0) + c_0\gamma\left[\mathscr{E}(t) + \mathscr{E}(0) + \int_0^t \mathscr{E}(t)dt\right] + \left(\frac{1}{4\gamma} - \gamma\right)\int_0^t \|\boldsymbol{u}(t)\|^2 dt$$

with $0 < \gamma < \min\{1/2, 1/c_0\}$. So, for $t \leq \tau$,

$$\mathscr{E}(t) \leq \frac{1 + c_0\gamma}{1 - c_0\gamma}\mathscr{E}(0) + \frac{1 - 4\gamma^2}{4\gamma(1 - c_0\gamma)}\int_0^\tau \|\boldsymbol{u}(s)\|^2 ds + \frac{c_0\gamma}{1 - c_0\gamma}\int_0^t \mathscr{E}(s)ds. \tag{21}$$

Applying the Gronwall inequality to (21) leads to

$$\mathscr{E}(t) \leq \left[\frac{1 + c_0\gamma}{1 - c_0\gamma}\mathscr{E}(0) + \frac{1 - 4\gamma^2}{4\gamma(1 - c_0\gamma)}\int_0^\tau \|\boldsymbol{u}(s)\|^2 ds\right] e^{\frac{c_0\gamma}{1 - c_0\gamma}t}$$

and

$$\int_0^t \mathscr{E}(s)ds \leq \left[\frac{1 + c_0\gamma}{1 - c_0\gamma}\mathscr{E}(0) + \frac{1 - 4\gamma^2}{4\gamma(1 - c_0\gamma)}\int_0^\tau \|\boldsymbol{u}(s)\|^2 ds\right]\frac{1 - c_0\gamma}{c_0\gamma}\left(e^{\frac{c_0\gamma}{1 - c_0\gamma}t} - 1\right).$$

From (20) and the above two inequalities, it follows that

$$\begin{aligned}
&\int_0^t |Y(t)|^2 dt \leq c_0\left[\mathscr{E}(t) + \mathscr{E}(0) + \int_0^t \mathscr{E}(s)ds\right] \\
\leq\ & \left[c_0\frac{1 + c_0\gamma}{1 - c_0\gamma}e^{\frac{c_0\gamma}{1 - c_0\gamma}t} + c_0 + \frac{1 + c_0\gamma}{\gamma}\left(e^{\frac{c_0\gamma}{1 - c_0\gamma}t} - 1\right)\right]\mathscr{E}(0) \\
&+ \left[\frac{c_0(1 - 4\gamma^2)}{4\gamma(1 - c_0\gamma)}e^{\frac{c_0\gamma}{1 - c_0\gamma}t} + \frac{1 - 4\gamma^2}{4\gamma^2}\left(e^{\frac{c_0\gamma}{1 - c_0\gamma}t} - 1\right)\right]\int_0^\tau \|\boldsymbol{u}(s)\|^2 ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathscr{E}(t) + \int_0^t |Y(s)|^2 ds \leq& \left[\frac{(1 + \gamma)(1 + c_0\gamma)}{\gamma(1 - c_0\gamma)}e^{\frac{c_0\gamma}{1 - c_0\gamma}t} - \frac{1}{\gamma}\right]\mathscr{E}(0) \\
&+ \left[\frac{(1 + \gamma)(1 - 4\gamma^2)}{4\gamma^2(1 - c_0\gamma)}e^{\frac{c_0\gamma}{1 - c_0\gamma}t} - \frac{1 - 4\gamma^2}{4\gamma^2}\right]\int_0^\tau \|\boldsymbol{u}(s)\|^2 ds,
\end{aligned}$$

which shows that $\forall\tau \geq 0$, there exists $M_\tau > 0$ such that

$$\begin{aligned}
&\mathscr{E}(\tau) + \int_0^\tau |Y_u(s)|^2 ds = \mathscr{E}(\tau) + \int_0^\tau |Y(s)|^2 ds \\
&\leq M_\tau\left[\mathscr{E}(0) + \int_0^\tau \|\boldsymbol{u}(s)\|^2 ds\right] = M_\tau\left[\mathscr{E}(0) + \int_0^\tau \|P_u P_{\mathbb{D}}^\top \boldsymbol{u}(s)\|^2 ds\right].
\end{aligned}$$

Therefore, the system (1) is input-output L^2-well-posedness. □

3.2. **Proof of regularity.**

Proof. Applying Laplace transform to the first equation in (1) leads to

$$s^2\rho_j(x)\widehat{w}_{j,ss}(x,s) = (T_j(x)\widehat{w}_{j,x})_x(x,s), x \in (0,1), j \in \mathcal{I}_E(G). \tag{22}$$

We introduce a new independent variable for (22)

$$\theta(x) = \widetilde{a}_j^{-1}\int_0^x \sqrt{\rho_j(x)T_j^{-1}(x)}dx \text{ for } x \in [0,1], \text{ with } \widetilde{a}_j = \int_0^1 \sqrt{\rho_j(x)T_j^{-1}(x)}dx.$$

Obviously, $\theta(x)$ is strictly monotone function on $[0,1]$. Denote by $x(\theta)$ the inverse function of $\theta(x)$, $\theta \in [0,1]$, then

$$\frac{d\theta(x)}{dx} = \widetilde{a}_j^{-1}\sqrt{\frac{\rho_j(x)}{T_j(x)}} > 0 \text{ and } \frac{dx(\theta)}{d\theta} = \widetilde{a}_j\sqrt{\frac{T_j(x(\theta))}{\rho_j(x(\theta))}} > 0.$$

So, the following equalities can be easily calculated

$$\widehat{w}_{j,x}(x(\theta),s) \quad = \quad \widehat{w}_{j,\theta}(x(\theta),s)\frac{d\theta(x)}{dx} = \widetilde{a}_j^{-1}\widehat{w}_{j,\theta}(x(\theta),s)\sqrt{\frac{\rho_j(x(\theta))}{T_j(x(\theta))}}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}[T_j(x(\theta))\widehat{w}_{j,x}(x(\theta),s)] \quad = \quad & \frac{\widehat{w}_{j,\theta}(x(\theta),s)\left[\left(\frac{\rho_j(x(\theta))}{T_j(x(\theta))}\right)^{3/2}\frac{dT_j(x)}{dx} + \frac{\sqrt{T_j(x(\theta))}}{\sqrt{\rho_j(x(\theta))}}\frac{d\rho_j(x)}{dx}\right]}{2\widetilde{a}_j^3} \\ & +\widetilde{a}_j^{-2}\rho_j(x(\theta))\widehat{w}_{j,\theta^2}(x(\theta),s).\end{aligned}$$

Let $\widetilde{w}_j(\theta,s) = \widehat{w}_j(x(\theta),s)$ and

$$\alpha_j(\theta) \quad = \quad \left[\frac{\rho_j(x(\theta))T_j'(x(\theta))}{2\widetilde{a}_j^2\left[T_j(x(\theta))\right]^2} + \frac{\rho_j'(x(\theta))}{2\widetilde{a}_j^2\rho_j(x(\theta))}\right], j \in \mathcal{I}_E(G),$$

where the prime denotes the derivative with respect to θ, then (22) can be reformulated by

$$\widetilde{a}_j^2 s^2 \widetilde{w}_j(\theta,s) = \widetilde{w}_{j,\theta\theta}(\theta,s) + \alpha_j(\theta)\widetilde{w}_{j,\theta}(\theta,s),\ \theta \in (0,\ 1).$$

So, Laplace transform of the system (1) can be written as follows:

$$\begin{cases}\widetilde{A}^2 s^2 \widetilde{w}(\theta,s) = \widetilde{w}_{\theta\theta}(\theta,s)) + \alpha(\theta)\widetilde{w}_\theta(\theta,s),\ \theta \in (0,\ 1),\\ \widetilde{w}(0,s) = \widehat{w}(0,s) = (\Upsilon^+)^\top \widehat{\boldsymbol{w}}(\boldsymbol{p},s), \widetilde{w}(1,s) = \widehat{w}(1,s) = (\Upsilon^-)^\top \widehat{\boldsymbol{w}}(\boldsymbol{p},s),\\ P_{\mathbb{D}}\left[\Upsilon^-\widetilde{T}(1)\widetilde{w}_\theta(1,s) - \Upsilon^+\widetilde{T}(0)\widetilde{w}_\theta(0,s)\right] = \widehat{\boldsymbol{u}}(s),\end{cases} \tag{23}$$

where $\widetilde{T}(0) = \widetilde{A}^{-1}[M(0)T(0)]^{1/2}$, $\widetilde{T}(1) = \widetilde{A}^{-1}[M(1)T(1)]^{1/2}$,

$$\widetilde{A} = \operatorname{diag}\left(\widetilde{a}_1,\ldots,\widetilde{a}_n\right) \text{ and } \alpha(\theta) = \operatorname{diag}\left(\alpha_1(\theta),\ldots,\alpha_n(\theta)\right).$$

Let $\widetilde{\eta}(\theta) = (\widetilde{w}(\theta,s), s^{-1}\widetilde{w}'(\theta,s))^\top$, then (23) shows that $\widetilde{\eta}(\theta)$ satisfies

$$\frac{d\widetilde{\eta}}{d\theta} \quad = \quad s\begin{pmatrix}0 & I\\ \widetilde{A}^2 & 0\end{pmatrix}\widetilde{\eta} + \begin{pmatrix}0 & 0\\ s^{-1}\alpha & 0\end{pmatrix}\widetilde{\eta}. \tag{24}$$

According to the asymptotical theory of fundamental solution ([17, 19]), the fundamental solution matrix of (24) has the form:

$$\widetilde{W}(\theta,s) = (I + s^{-1}Q(\theta,s))\widetilde{W}_0(\theta,s),$$

where $Q(\theta,s) = \begin{pmatrix}Q_{11}(\theta,s) & Q_{12}(\theta,s)\\ Q_{21}(\theta,s) & Q_{22}(\theta,s)\end{pmatrix}$, $\|Q(\theta,s)\|$ is uniformly bounded and

$$\widetilde{W}_0(\theta,s) \quad = \quad \begin{pmatrix}\cosh(s\theta\widetilde{A}) & \widetilde{A}^{-1}\sinh(s\theta\widetilde{A})\\ \widetilde{A}\sinh(s\theta\widetilde{A}) & \cosh(s\theta\widetilde{A})\end{pmatrix}.$$

Thus,

$$\begin{aligned}\widetilde{\eta}(\theta) &= \widetilde{W}(\theta,s)\widetilde{\eta}(0) = (I+s^{-1}Q(\theta,s))\widetilde{W}_0(\theta,s)\widetilde{\eta}(0)\\ &= \begin{pmatrix}\cosh(s\theta\widetilde{A})+s^{-1}W_{11}^Q(s\theta\widetilde{A}) & \widetilde{A}^{-1}\sinh(s\theta\widetilde{A})+s^{-1}W_{12}^Q(s\theta\widetilde{A})\\ \widetilde{A}\sinh(s\theta\widetilde{A})+s^{-1}W_{21}^Q(s\theta\widetilde{A}) & \cosh(s\theta\widetilde{A})+s^{-1}W_{22}^Q(s\theta\widetilde{A})\end{pmatrix}\widetilde{\eta}(0), \quad (25)\end{aligned}$$

where $\widetilde{\eta}(0)=(\widetilde{\eta}_{0,1},\widetilde{\eta}_{0,2})^\top$ and

$$\begin{aligned}W^Q(\theta,s) &= Q(\theta,s)\widetilde{W}_0(\theta,s) = \begin{pmatrix}W_{11}^Q(s\theta\widetilde{A}) & W_{12}^Q(s\theta\widetilde{A})\\ W_{21}^Q(s\theta\widetilde{A}) & W_{22}^Q(s\theta\widetilde{A})\end{pmatrix}\\ &= \begin{pmatrix}Q_{11}\cosh(s\theta\widetilde{A})+Q_{12}\widetilde{A}\sinh(s\theta\widetilde{A}) & Q_{11}\widetilde{A}^{-1}\sinh(s\theta\widetilde{A})+Q_{12}\cosh(s\theta\widetilde{A})\\ Q_{21}\cosh(s\theta\widetilde{A})+Q_{22}\widetilde{A}\sinh(s\theta\widetilde{A}) & Q_{21}\widetilde{A}^{-1}\sinh(s\theta\widetilde{A})+Q_{22}\cosh(s\theta\widetilde{A})\end{pmatrix}.\end{aligned}$$

Let $d(s)=P_{\mathbb{D}}\widehat{\boldsymbol{w}}(\boldsymbol{p},s)$, then $\widehat{\boldsymbol{w}}(\boldsymbol{p},s)=P_{\mathbb{D}}^\top d(s)$, $\widetilde{\eta}(0)=\begin{pmatrix}(P_{\mathbb{D}}\Upsilon^+)^\top & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}d(s)\\ \widetilde{\eta}_{0,2}\end{pmatrix}$ and

$$\widetilde{\eta}(1) = \begin{pmatrix}\widetilde{w}(1,s)\\ s^{-1}\widetilde{w}'(1,s)\end{pmatrix} = \begin{pmatrix}(\Upsilon^-)^\top\widehat{\boldsymbol{w}}(\boldsymbol{p},s)\\ s^{-1}\widetilde{w}'(1,s)\end{pmatrix} = \begin{pmatrix}(\Upsilon^-)^\top P_{\mathbb{D}}^\top d(s)\\ s^{-1}\widetilde{w}'(1,s)\end{pmatrix}.$$

Thus, it follows from (25) and the last boundary condition in (23) that $d(s)$ and $\widetilde{\eta}_{0,2}$ satisfy the linear system of equations:

$$\widetilde{D}(s)\begin{pmatrix}d(s)\\ \widetilde{\eta}_{0,2}\end{pmatrix} = \begin{pmatrix}s^{-1}\widehat{\boldsymbol{u}}(s)\\ 0\end{pmatrix}, \quad (26)$$

where

$$\begin{aligned}\widetilde{D}(s) &= \begin{pmatrix}P_{\mathbb{D}}(\Upsilon^-)\widetilde{T}(1)\widetilde{A}\sinh(s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top & P_{\mathbb{D}}(\Upsilon^-)\widetilde{T}(1)\cosh(s\widetilde{A})-P_{\mathbb{D}}(\Upsilon^+)\widetilde{T}(0)\\ \cosh(s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top-(P_{\mathbb{D}}\Upsilon^-)^\top & \widetilde{A}^{-1}\sinh(s\widetilde{A})\end{pmatrix}\\ &\quad + s^{-1}\begin{pmatrix}P_{\mathbb{D}}(\Upsilon^-)\widetilde{T}(1)W_{21}^Q(s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top & P_{\mathbb{D}}(\Upsilon^-)\widetilde{T}(1)W_{22}^Q(s\widetilde{A})\\ W_{11}^Q(s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top & W_{12}^Q(s\widetilde{A})\end{pmatrix}.\end{aligned}$$

Moreover, it can be deduced that

$$\widetilde{D}(s)\begin{pmatrix}I & 0\\ -\widetilde{A}(P_{\mathbb{D}}\Upsilon^+)^\top & I\end{pmatrix} = \widetilde{D}_{-e}(s)+\widetilde{D}_C\begin{pmatrix}I & 0\\ 0 & \frac{1}{2}\exp(s\widetilde{A})\end{pmatrix}$$

with

$$\widetilde{D}_C = \left[\begin{pmatrix}P_{\mathbb{D}}\Upsilon^+\widetilde{T}(0)\widetilde{A}(P_{\mathbb{D}}\Upsilon^+)^\top & P_{\mathbb{D}}\Upsilon^-\widetilde{T}(1)\\ -(P_{\mathbb{D}}\Upsilon^-)^\top & \widetilde{A}^{-1}\end{pmatrix}+s^{-1}\begin{pmatrix}0 & Q_{21}\widetilde{A}^{-1}+Q_{22}\\ 0 & Q_{11}\widetilde{A}^{-1}+Q_{12}\end{pmatrix}\right]$$

and

$$\begin{aligned}\widetilde{D}_{-e}(s) &= \begin{pmatrix}-P_{\mathbb{D}}\Upsilon^-\widetilde{T}(1)\widetilde{A}\exp(-s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top & P_{\mathbb{D}}\Upsilon^-\widetilde{T}(1)\frac{\exp(-s\widetilde{A})}{2}-P_{\mathbb{D}}\Upsilon^+\widetilde{T}(0)\\ \exp(-s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top & -\widetilde{A}^{-1}\frac{\exp(-s\widetilde{A})}{2}\end{pmatrix}\\ &+s^{-1}\begin{pmatrix}P_{\mathbb{D}}\Upsilon^-\widetilde{T}(1)(Q_{11}-Q_{12}\widetilde{A})\exp(-s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top & \frac{1}{2}\left(-Q_{21}\widetilde{A}^{-1}+Q_{22}\right)\exp(-s\widetilde{A})\\ (Q_{11}-Q_{12}\widetilde{A})\exp(-s\widetilde{A})(P_{\mathbb{D}}\Upsilon^+)^\top & \frac{1}{2}\left(-Q_{11}\widetilde{A}^{-1}+Q_{12}\right)\exp(-s\widetilde{A})\end{pmatrix}.\end{aligned}$$

Thus, the linear system of equations (26) can be reformulated by

$$\left[\widetilde{D}_C+\widetilde{D}_{-e}(s)\begin{pmatrix}I & 0\\ 0 & \frac{1}{2}\exp(-s\widetilde{A})\end{pmatrix}\right]\begin{pmatrix}d(s)\\ \widetilde{\eta}_d(s)\end{pmatrix} = \begin{pmatrix}s^{-1}\widehat{\boldsymbol{u}}(s)\\ 0\end{pmatrix}, \quad (27)$$

where $\widetilde{\eta}_d(s) = \frac{1}{2}\exp(s\widetilde{A})[\widetilde{\eta}_{0,2} + (P_{\mathbb{D}}\Upsilon^+)^\top d(s)]$. Since

$$\begin{pmatrix} I & -P_{\mathbb{D}}\Upsilon^- T(1)\widetilde{A} \\ 0 & I \end{pmatrix} \widetilde{D}_C =$$

$$\begin{pmatrix} \widetilde{D}_\Lambda & 0 \\ -(P_{\mathbb{D}}\Upsilon^-)^\top & \widetilde{A}^{-1} \end{pmatrix} + s^{-1} \begin{pmatrix} 0 & Q_{21}\widetilde{A}^{-1} + Q_{22} - P_{\mathbb{D}}\Upsilon^- T(1)\widetilde{A}\left(Q_{11}\widetilde{A}^{-1} + Q_{12}\right) \\ 0 & Q_{11}\widetilde{A}^{-1} + Q_{12} \end{pmatrix}$$

with

$$\begin{aligned} \widetilde{D}_\Lambda &= P_{\mathbb{D}}\Upsilon^+\widetilde{T}(0)\widetilde{A}(\Upsilon^+)^\top P_{\mathbb{D}}^\top + P_{\mathbb{D}}\Upsilon^-\widetilde{T}(1)\widetilde{A}(\Upsilon^-)^\top P_{\mathbb{D}}^\top \\ &= \operatorname{diag}\left(\sum_{i\in\mathcal{I}_E^+(p_{\jmath_1})} \sqrt{\rho_i(0)T_i(0)}, \ldots, \sum_{i\in\mathcal{I}_E^+(p_{\jmath_{m_0}})} \sqrt{\rho_i(0)T_i(0)} \right) \\ &\quad + \operatorname{diag}\left(\sum_{i\in\mathcal{I}_E^-(p_{\jmath_1})} \sqrt{\rho_i(1)T_i(1)}, \ldots, \sum_{i\in\mathcal{I}_E^-(p_{\jmath_{m_0}})} \sqrt{\rho_i(1)T_i(1)} \right), \end{aligned} \tag{28}$$

$\widetilde{D}_C$ is invertible for $\Re(s) > 0$ large enough, and

$$\begin{aligned} \widetilde{D}_C^{-1} &= \left[\begin{pmatrix} \widetilde{D}_\Lambda & 0 \\ -(P_{\mathbb{D}}\Upsilon^-)^\top & \widetilde{A}^{-1} \end{pmatrix}^{-1} + o(D)\right] \begin{pmatrix} I & -P_{\mathbb{D}}\Upsilon^- T(1)\widetilde{A} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{D}_\Lambda^{-1} & -\widetilde{D}_\Lambda^{-1}(P_{\mathbb{D}}\Upsilon^-)T(1)\widetilde{A} \\ \widetilde{A}(P_{\mathbb{D}}\Upsilon^-)^\top \widetilde{D}_\Lambda^{-1} & \widetilde{A}\left[I - (P_{\mathbb{D}}\Upsilon^-)^\top \widetilde{D}_\Lambda^{-1}(P_{\mathbb{D}}\Upsilon^-)T(1)\widetilde{A}\right] \end{pmatrix} + o(D), \end{aligned} \tag{29}$$

where $o(D)$ stands for a matrix which tends to zero matrix with appropriate rows and columns as $s \to +\infty$. Thus, it follows from

$$\widetilde{D}_{-e}(s) \begin{pmatrix} I & 0 \\ 0 & \frac{1}{2}\exp(-s\widetilde{A}) \end{pmatrix} \to 0 \text{ as } s \to +\infty,$$

(27) and (29) that

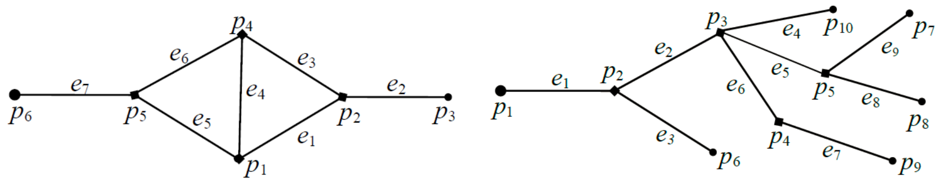
$$\begin{pmatrix} d(s) \\ \widetilde{\eta}_d \end{pmatrix} = [\widetilde{D}_C^{-1} + o(D)] \begin{pmatrix} s^{-1}\widehat{\boldsymbol{u}}(s) \\ 0 \end{pmatrix} \text{ and } d(s) = s^{-1}[\widetilde{D}_\Lambda^{-1} + o(D)]\widehat{\boldsymbol{u}}(s),$$

which implies that $\widehat{Y}(s) = sP_{\mathbb{D}}\widehat{\boldsymbol{w}}(\boldsymbol{p}, s) = sd(s) = [\widetilde{D}_\Lambda^{-1} + o(D)]\widehat{\boldsymbol{u}}(s)$. So, it can be deduced from Remark 2, Proposition 1 and (28) that

$$\begin{aligned} &\lim_{s\to+\infty} \mathbf{H}(s) = P_u P_{\mathbb{D}}^\top \widetilde{D}_\Lambda^{-1} P_{\mathbb{D}} P_u^\top \\ =\ & \operatorname{diag}\left(\sum_{i\in\mathcal{I}_E^+(p_{\jmath_{k_1}})} \sqrt{\rho_i(0)T_i(0)} + \sum_{i\in\mathcal{I}_E^-(p_{\jmath_{k_1}})} \sqrt{\rho_i(1)T_i(1)}, \ldots, \right. \\ & \left. \sum_{i\in\mathcal{I}_E^+(p_{\jmath_{k_{m_u}}})} \sqrt{\rho_i(0)T_i(0)} + \sum_{i\in\mathcal{I}_E^-(p_{\jmath_{k_{m_u}}})} \sqrt{\rho_i(1)T_i(1)} \right)^{-1}, \end{aligned} \tag{30}$$

i.e., the system (1) is regular. $\square$

3.3. **Examples.** Here, we give two networks: one is a network with cycles, another is a tree-shaped network with the fixed root p_1, shown in Figure 1.



(A) A network with cycles

(B) A tree-shaped network with the root p_1

FIGURE 1. Networks consisting of strings with one fixed vertex

3.3.1. *A network with cycles.* See Figure 1a, the motion of strings on the network G is governed by

$$\begin{cases} \rho_j(x)w_{j,tt}(x,t) = (T_j(x)w_{j,x})_x(x,t),\ x\in(0,1), j=1,2,3,4,5,6,7,\\ w_1(0,t)=w_4(1,t)=w_5(1,t), w_3(1,t)=w_2(0,t)=w_1(1,t),\\ w_3(0,t)=w_4(0,t)=w_6(1,t), w_5(0,t)=w_6(0,t)=w_7(1,t), w_7(0,t)=0,\\ T_1(1)w_{1,x}(1,t)+T_3(1)w_{3,x}(1,t)-T_2(0)w_{2,x}(0,t)=0,\\ T_6(1)w_{6,x}(1,t)-[T_4(0)w_{4,x}(0,t)+T_3(0)w_{3,x}(0,t)]=0,\\ T_7(1)w_{7,x}(1,t)-[T_5(0)w_{5,x}(0,t)+T_6(0)w_{6,x}(0,t)]=0,\\ T_4(1)w_{4,x}(1,t)+T_5(1)w_{5,x}(1,t)-T_1(0)w_{1,x}(0,t)=u(p_1,t),\\ T_2(1)w_{2,x}(1,t)=u(p_3,t), \end{cases} \tag{31}$$

where $\mathfrak{D}=\{p_6\}$. The expressions $P_{\mathbb{D}}$ and P_u ((6) and (10)) lead to

$$P_{\mathbb{D}}^{\top} = \begin{pmatrix} \overset{\jmath_1}{\epsilon_1} & \overset{\jmath_2}{\epsilon_2} & \overset{\jmath_3}{\epsilon_3} & \overset{\jmath_4}{\epsilon_4} & \overset{\jmath_5}{\epsilon_5} \end{pmatrix} \text{ and } P_u^{\top} = \begin{pmatrix} \overset{\jmath_{k_1}}{\epsilon_1} & \overset{\jmath_{k_2}}{\epsilon_3} \end{pmatrix},$$

where the vector ϵ_k is the k-th column of the identity matrix I_6. So, it follows from $\jmath_{k_1}=1$, $\jmath_{k_2}=3$, the definitions of $\mathcal{I}_E^+(p_j)$ and $\mathcal{I}_E^-(p_j)$, and (30) that

$$\lim_{s\to+\infty}\mathbf{H}(s) = \operatorname{diag}\left(\left[\sqrt{\rho_1(0)T_1(0)}+\sum_{k\in\{4,5\}}\sqrt{\rho_k(1)T_k(1)}\right]^{-1}, [\sqrt{\rho_2(1)T_2(1)}]^{-1}\right),$$

i.e., the system (31) is regular.

3.3.2. *A tree-shaped network with one fixed root.* See Figure 1b, the motion of strings on the tree-shaped network G is governed by

$$\begin{cases} \rho_j(x)w_{j,tt}(x,t) = (T_j(x)w_{j,x})_x(x,t),\ x \in (0,1), j = 1,\ldots,9, \\ w_1(0,t) = 0, w_1(1,t) = w_2(0,t) = w_3(0,t), w_6(1,t) = w_7(0,t), \\ w_2(1,t) = w_4(0,t) = w_5(0,t) = w_6(0,t), w_5(1,t) = w_8(0,t) = w_9(0,t), \\ T_1(1)w_{1,x}(1,t) = T_2(0)w_{2,x}(0,t) + T_3(0)w_{3,x}(0,t), \\ T_2(1)w_{2,x}(1,t) = T_4(0)w_{4,x}(0,t) + T_5(0)w_{5,x}(0,t) + T_6(0)w_{6,x}(0,t), \\ T_5(1)w_{5,x}(1,t) = T_8(0)w_{8,x}(0,t) + T_9(0)w_{9,x}(0,t), \\ T_6(1)w_{6,x}(1,t) = T_7(0)w_{7,x}(0,t), T_3(1)w_{3,x}(1,t) = u(p_6,t), \\ T_9(1)w_{9,x}(1,t) = u(p_7,t), T_8(1)w_{8,x}(1,t) = u(p_8,t), \\ T_7(1)w_{7,x}(1,t) = u(p_9,t), T_4(1)w_{4,x}(1,t) = u(p_{10},t). \end{cases} \tag{32}$$

where $\mathfrak{D} = \{p_1\}$. The expressions $P_{\mathbb{D}}$ and P_u ((6) and (10)) lead to

$$P_{\mathbb{D}}^\top = \overset{\jmath_1\quad \jmath_2\quad \jmath_3\quad \jmath_4\quad \jmath_5\quad \jmath_6\quad \jmath_7\quad \jmath_8\quad \jmath_9}{(\epsilon_2\ \ \epsilon_3\ \ \epsilon_4\ \ \epsilon_5\ \ \epsilon_6\ \ \epsilon_7\ \ \epsilon_8\ \ \epsilon_9\ \ \epsilon_{10})} \text{ and } P_u^\top = \overset{\jmath_{k_1}\quad \jmath_{k_2}\quad \jmath_{k_3}\quad \jmath_{k_4}\quad \jmath_{k_5}}{(\ \epsilon_6\ \ \ \epsilon_7\ \ \ \epsilon_8\ \ \ \epsilon_9\ \ \ \epsilon_{10})},$$

where the vector ϵ_k is the k-th column of the identity matrix I_{10}. Thus, from (28) and (30), it can be derived that

$$\lim_{s\to+\infty} \mathbf{H}(s)$$
$$= \operatorname{diag}\left(\sqrt{\rho_3(1)T_3(1)}, \sqrt{\rho_9(1)T_9(1)}, \sqrt{\rho_8(1)T_8(1)}, \sqrt{\rho_7(1)T_7(1)}, \sqrt{\rho_4(1)T_4(1)}\right)^{-1},$$

i.e., the system (32) is regular.

4. **Proof of Theorem 1.3 and examples.** For the tree-shaped network $G = (V(G), E(G))$, the number of vertices m is equal to the number of edges plus one, i.e., $m = n + 1$. Without loss of generality, let $\mathfrak{D} = \{p_1\}$, then, the number of vertices in the set $V(G) \setminus \mathfrak{D}$ is $m_0 = n$. It follows from Hypothesis 1.2 and (5) that $P_{\mathbb{D}} = (0, I_n)_{n,n+1}$ and the index set $\{\jmath_1^-, \jmath_2^-, \ldots, \jmath_n^-\} = \{2, 3, \ldots, n+1\}$. Thus, it is derived from the system (8) and Proposition 1 that $(\Upsilon^-)w(1,t) = D_G^- \boldsymbol{w}(\boldsymbol{p},t)$,

$$\boldsymbol{w}(\boldsymbol{p},t) = \begin{pmatrix} 0 \\ \mathbb{w}(p_2,t) \\ \vdots \\ \mathbb{w}(p_{n+1},t) \end{pmatrix} = D_G^-\Upsilon^- w(1,t) \text{ and } w(0,t) = (\Upsilon^+)^\top \boldsymbol{w}(\boldsymbol{p},t) = B_G^{t,h} w(1,t),$$

where

$$D_G^- = \Upsilon^-(\Upsilon^-)^\top = \operatorname{diag}\left(\deg^-(p_1), \deg^-(p_2), \ldots, \deg^-(p_{n+1})\right) = \operatorname{diag}(0, 1, \ldots, 1)$$

and $(D_G^-)^\ominus = I_n$. Since $P_{\mathbb{D}}\Upsilon^- \left[(\Upsilon^-)^\top D_G^- P_{\mathbb{D}}^\top\right] = I_n = \left[(\Upsilon^-)^\top D_G^- P_{\mathbb{D}}^\top\right] P_{\mathbb{D}}\Upsilon^-$, it follows from the last boundary condition in the system (8) and Proposition 1 that

$$\begin{aligned} (\Upsilon^-)^\top D_G^- P_{\mathbb{D}}^\top \boldsymbol{u}(t) &= T(1)w_x(1,t) - (\Upsilon^-)^\top D_G^- \Upsilon^+ T(0) w_x(0,t) \\ &= T(1)w_x(1,t) - B_G^{h,t} T(0) w_x(0,t). \end{aligned}$$

In the feedback control law (4), $\beta_k > 0$ for $p_{\jmath_k} \in \partial G_N$. Now, we supplement $\beta_k = 0$ for $p_{\jmath_k} \in \mathfrak{F}$, and let $\beta = \operatorname{diag}(\beta_1, \ldots, \beta_n)$, then (4) can be formulated by

$$\boldsymbol{u}(t) = -\beta P_{\mathbb{D}} \boldsymbol{w}_t(\boldsymbol{p},t) \text{ with } \beta_k = \begin{cases} 0, & \text{if } p_{\jmath_k} \in \mathfrak{F}, \\ > 0, & \text{if } p_{\jmath_k} \in \partial G_N. \end{cases} \tag{33}$$

In addition, it can be obtained from $B_G^- = 0$ and Proposition 1 that

$$(\Upsilon^-)^\top D_G^- P_{\mathbb{D}}^\top \boldsymbol{u}(t) = -(\Upsilon^-)^\top D_G^- P_{\mathbb{D}}^\top \beta P_{\mathbb{D}} D_G^- \Upsilon^- w_t(1,t).$$

Denote by $\beta^- = (\Upsilon^-)^\top D_G^- P_{\mathbb{D}}^\top \beta P_{\mathbb{D}} D_G^- \Upsilon^- = (\Upsilon^-)^\top \left(\begin{smallmatrix} 0 & 0 \\ 0 & \beta \end{smallmatrix}\right) \Upsilon^-$, then it is derived from (33), Proposition 1 and Remark 3 that

$$\beta^- = \operatorname{diag}\left(\beta_1^-, \dots, \beta_n^-\right), \text{ with } \beta_k^- = \beta_{\jmath_k^- - 1} = \begin{cases} 0, \text{ if } \deg(p_{\jmath_k^-}) > 1, \\ > 0, \text{ if } \deg(p_{\jmath_k^-}) = 1, \end{cases} \tag{34}$$

where $\jmath_k^- - 1 \in \{1, 2, \dots, n\}$, $k \in \mathcal{I}_E(G)$. Therefore, the closed-loop system (3)-(4) can be reformulated by (see also [15, 28])

$$\begin{cases} M(x)w_{tt}(x,t) = (T(x)w_x)_x(x,t), \ x \in (0,\ 1),\ t > 0, \\ w(0,t) = B_G^{t,h} w(1,t), \\ T(1)w_x(1,t) - B_G^{h,t} T(0) w_x(0,t) = -\beta^- w_t(1,t). \end{cases} \tag{35}$$

Remark 4. Since $p_{\jmath_k^-}$ is the final point (head) of the edge e_k for $k \in \mathcal{I}_E(G)$, according to Definition 2.2, all entries in k-th row of $B_G^{h,t}$ are zeros for boundary vertex $p_{\jmath_k^-} \in \partial G_N$, i.e.,$\deg(p_{\jmath_k^-}) = 1$. Moreover, for internal vertex $p_{\jmath_k^-}$, i.e.,$\deg(p_{\jmath_k^-}) > 1$, $\beta_k^- = 0$. Hence, it can be followed from (34) that $\langle B_G^{h,t} z,\ \beta^- v \rangle_{\mathbb{R}^n} \equiv 0$, for all $z, v \in \mathbb{R}^n$.

Let $\mathscr{V}_b(t) = \int_0^1 \langle b(x) w_x(x,t),\ M(x) w_t(x,t) \rangle_{\mathbb{R}^n} dx$ with the diagonal matrix $b(x) = \operatorname{diag}(b_1(x), \dots, b_n(x))$ satisfying the following condition.

Condition 4.1.

(1) For every $k \in \mathcal{I}_E(G)$, $b_k(\cdot) \in C^1[0,1]$ and there exist positive constants $c_{\rho L}$, $c_{\rho U}$, c_{TL} and c_{TU} such that $c_{\rho L} \le [b_k(x)\rho_k(x)]' \le c_{\rho U}$ and $c_{TL} \le [b_k(x) T_k^{-1}(x)]' \le c_{TU}$, for all $x \in [0,1]$.

(2) For every edge e_k, corresponding to the component $b_k(x)$,

$$\begin{cases} \min\left\{ 2\left[\frac{b_k(1)(\beta_k^-)^2}{T_k(1)} + b_k(1)\rho_k(1)\right]^{-1} \beta_k^-,\ \frac{1}{c_b} \right\} > c_V > 0 & \text{as } p_{\jmath_k^-} \in \partial G_N, \\ b_k(1)\rho_k(1) \le \sum_{i \in \mathcal{I}_E^+(p_{\jmath_k^-})} b_i(0)\rho_i(0), & \text{as } p_{\jmath_k^-} \in Int(G), \end{cases}$$

where $c_b = \max_{x\in[0,1]}\{\|M(x)\|_2, \|b^2(x)T^{-1}(x)\|_2\} > 0$ (see (36) below), the final point (head) of the edge e_k is $p_{\jmath_k^-}$ and $\mathcal{I}_E^-(p_{\jmath_k^-}) = \{k\}$.

(3) For every edge e_k, corresponding to the component $b_k(x)$,

$$\begin{cases} \deg^+(p_{\jmath_k^+}) \sum\limits_{i \in \mathcal{I}_E^-(p_{\jmath_k^+})} \frac{b_i(1)}{T_i(1)} < \frac{b_k(0)}{T_k(0)}, & \text{as } p_{\jmath_k^+} \in Int(G), \\ \frac{b_k(0)}{T_k(0)} \ge 0, & \text{as } p_{\jmath_k^+} = p_1 \ (k \in \mathcal{I}_E^+(p_1)), \end{cases}$$

where the starting point (tail) of the edge e_k is the vertex $p_{\jmath_k^+}$.

According to (1) and (2) in Condition 4.1, obviously, the following inequality

$$\begin{aligned} |\mathscr{V}_b(t)| \ &\le\ \frac{1}{2}\int_0^1 \langle b(x)w_x(x,t),\ b(x)w_x(x,t) \rangle_{\mathbb{R}^n} dx \\ &\quad + \frac{1}{2}\int_0^1 \langle M(x)w_t(x,t),\ M(x)w_t(x,t) \rangle_{\mathbb{R}^n} dx \le c_b \mathscr{E}(t) \end{aligned} \tag{36}$$

holds, and there exist $c_L > 0$ and $c_U > 0$ such that

$$c_L\mathscr{E}(t) \le O(\mathscr{E}) = \frac{1}{2}\int_0^1 \langle\, w_t(x,t),\, (b(x)M(x))' w_t(x,t)\,\rangle_{\mathbb{R}^n} dx$$
$$+\frac{1}{2}\int_0^1 \langle\, [b(x)T^{-1}(x)]'T(x)w_x(x,t),\, T(x)w_x(x,t)\,\rangle_{\mathbb{R}^n} dx \le c_U\mathscr{E}(t). \quad (37)$$

Now, we construct a Lyapunov functional

$$\mathscr{V}(t) = \mathscr{E}(t) + c_V\mathscr{V}_b(t), \forall t > 0,$$

where $0 < c_b c_V < 1$, then

$$(1 - c_b c_V)\mathscr{E}(t) \le \mathscr{V}(t) \le (1 + c_b c_V)\mathscr{E}(t), \forall t > 0. \quad (38)$$

In what follows, using this Lyapunov functional, we prove Theorem 1.3.

4.1. **Proof of Theorem 1.3.**

Proof. Using (35) and the following two equalities

$$\int_0^1 \langle b(x)w_{xt}(x,t), M(x)w_t(x,t)\rangle dx = -\frac{1}{2}\int_0^1 \langle w_t(x,t), (b(x)M(x))' w_t(x,t)\rangle dx$$
$$+\frac{1}{2}\langle [b(1)M(1) - B_G^{h,t} b(0)M(0)B_G^{t,h}]w_t(1,t), w_t(1,t)\rangle$$

and

$$\int_0^1 \langle b(x)w_x(x,t), M(x)w_{tt}(x,t)\rangle dx = \frac{1}{2}\langle b(1)w_x(1,t), T(1)w_x(1,t)\rangle$$
$$-\frac{1}{2}\langle b(0)w_x(0,t), T(0)w_x(0,t)\rangle - \frac{1}{2}\int_0^1 \langle [b(x)T^{-1}(x)]'T(x)w_x(x,t)), T(x)w_x(x,t)\rangle dx,$$

we can get that

$$\begin{aligned}\frac{d\mathscr{V}_b(t)}{dt} \;=\; & \frac{1}{2}\langle [b(1)T(1)^{-1}(\beta^-)^2 + b(1)M(1) - B_G^{h,t} b(0)M(0)B_G^{t,h}]w_t(1,t), w_t(1,t)\rangle \\ & + \frac{1}{2}\langle [T(0)B_G^{t,h} b(1)T(1)^{-1}B_G^{h,t}T(0) - T(0)b(0)]w_x(0,t), w_x(0,t)\rangle \\ & - \langle b(1)T(1)^{-1}B_G^{h,t}T(0)w_x(0,t), \beta^- w_t(1,t)\rangle - O(\mathscr{E}). \end{aligned} \quad (39)$$

Moveover, it follows from (11) and (35) that

$$\frac{d\mathscr{E}(t)}{dt} = \langle T(1)w_x(1,t) - B_G^{h,t}T(0)w_x(0,t), w_t(1,t)\rangle = -\langle \beta^- w_t(1,t), w_t(1,t)\rangle. \quad (40)$$

Thus, it can be obtained from Remark 4, (37), (39) and (40) that

$$\frac{d\mathscr{V}(t)}{dt} = \frac{d\mathscr{E}(t)}{dt} + c_V\frac{d\mathscr{V}_b(t)}{dt}$$
$$\le -c_V c_L \mathscr{E}(t) - \langle C_\beta w_t(1,t), w_t(1,t)\rangle - \frac{c_V}{2}\langle C_b T(0)w_x(0,t), T(0)w_x(0,t)\rangle, \quad (41)$$

where

$$C_\beta = \beta^- - \frac{c_V}{2}\left[b(1)T(1)^{-1}(\beta^-)^2 + b(1)M(1) - B_G^{h,t} b(0)M(0)B_G^{t,h}\right]$$

and

$$C_b = b(0)T(0)^{-1} - B_G^{t,h} b(1)T(1)^{-1}B_G^{h,t}.$$

Next, we prove that the symmetric matrices C_β and C_b are positive semi-definite. Denote by $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{n+1})$, where

$$\lambda_j = \sum_{i\in\mathcal{I}_E^+(p_j)} b_i(0)\rho_i(0) = \begin{cases} b_1(0)\rho_1(0), & \text{as } j=1, \\ 0, & \text{as } p_j \in \partial G_N, \\ \sum_{i\in\mathcal{I}_E^+(p_j)} b_i(0)\rho_i(0), & \text{as } p_j \in Int(G)\setminus \mathfrak{D}. \end{cases}$$

From Proposition 1, $B_G^- = 0$ and (34), it can be obtained that

$$\begin{aligned} C_\beta &= \beta^- - \frac{1}{2}c_V\left[b(1)T(1)^{-1}(\beta^-)^2 + b(1)M(1) - (\Upsilon^-)^\top\Upsilon^+ b(0)M(0)(\Upsilon^+)^\top\Upsilon^-\right] \\ &= \beta^- - \frac{c_V}{2}\left[b(1)T(1)^{-1}(\beta^-)^2 + b(1)M(1)\right] + \frac{c_V}{2}\mathrm{diag}\left(\lambda_{\jmath_1^-}, \cdots, \lambda_{\jmath_n^-}\right) \\ &= \mathrm{diag}\left(c_{\beta,1}, \ldots, c_{\beta,n}\right), \end{aligned}$$

where

$$c_{\beta,k} = \beta_k^- - \frac{c_V}{2}\left[\frac{b_k(1)(\beta_k^-)^2}{T_k(1)} + b_k(1)\rho_k(1) - \lambda_{\jmath_k^-}\right], k = 1, \ldots, n.$$

Thus, (2) in Condition 4.1 and (34) lead to

$$\begin{cases} c_{\beta,k} = \beta_k^- - \frac{c_V}{2}\left[\frac{b_k(1)(\beta_k^-)^2}{T_k(1)} + b_k(1)\rho_k(1)\right] \geq 0, & \text{as } p_{\jmath_k^-} \in \partial G_N, \\ c_{\beta,k} = \sum_{i\in\mathcal{I}_E^+(p_{\jmath_k^-})} b_i(0)\rho_i(0) - b_k(1)\rho_k(1) \geq 0, & \text{as } p_{\jmath_k^-} \in Int(G), \end{cases}$$

i.e., C_β is a positive semi-definite matrix. It follows from Proposition 1 that

$$\begin{aligned} C_b &= b(0)T^{-1}(0) - (\Upsilon^+)^\top\Upsilon^- b(1)T(1)^{-1}(\Upsilon^-)^\top\Upsilon^+ \\ &= b(0)T^{-1}(0) - (\Upsilon^+)^\top \mathrm{diag}\left(\sum_{i\in\mathcal{I}_E^-(p_1)}\frac{b_i(1)}{T_i(1)}, \cdots, \sum_{i\in\mathcal{I}_E^-(p_{n+1})}\frac{b_i(1)}{T_i(1)}\right)\Upsilon^+ \\ &= b(0)T^{-1}(0) - \mathrm{diag}\left(\sum_{i\in\mathcal{I}_E^-(p_{\jmath_1^+})}\frac{b_i(1)}{T_i(1)}, \cdots, \sum_{i\in\mathcal{I}_E^-(p_{\jmath_n^+})}\frac{b_i(1)}{T_i(1)}\right)(I + B_G^+). \end{aligned}$$

The k-th row of C_b matches the edge e_k, whose tail and head are the vertices $p_{\jmath_k^+}$ and $p_{\jmath_k^-}$, respectively. In light of Definition 2.2, the number of 1 in the k-th row of $I + B_G^+$ is $\deg^+(p_{\jmath_k^+})$. Thus, when $\frac{b_k(0)}{T_k(0)} > \deg^+(p_{\jmath_k^+})\sum_{i\in\mathcal{I}_E^-(p_{\jmath_k^+})}\frac{b_i(1)}{T_i(1)}$ for the internal vertex $p_{\jmath_k^+}$, the k-th row of C_b is strictly (row) diagonally dominant. When $\frac{b_k(0)}{T_k(0)} > 0$ for the root $p_{\jmath_k^+} = p_1$, the k-th row of C_b has only one non-zero diagonal element $\frac{b_k(0)}{T_k(0)}$, since $\deg^+(p_1) = 1$ and $\deg^-(p_1) = 0$, according to Hypothesis 1.2; when $\frac{b_k(0)}{T_k(0)} = 0$ for the root $p_{\jmath_k^+} = p_1$, the entries in k-th row of C_b are zeros. Hence, the matrix C_b is positive semi-definite under (3) in Condition 4.1.

Thus, it follows from (38) and (41) that

$$\frac{d\mathscr{V}(t)}{dt} \leq -\frac{c_V c_L}{1 + c_b c_V}\mathscr{V}(t),$$

which, together with (38), implies that the system (35), i.e., the closed-loop system (3)-(4), is exponential stable. $\square$

Remark 5. The construction of matrix multiplier $b(x)$ is crucial in the proof, here, we discuss its choice. A path in G is a non-empty subgraph $P = (V_P, E_P)$ of the form $V_P = \{p_{\iota_0}, p_{\iota_1}, \dots, p_{\iota_k}\}$ and $E_P = \{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$, where $p_{\iota_1}, \dots, p_{\iota_k}$ and $e_{i_1}, \dots, e_{i_k}$ are all distinct vertices and edges, respectively, the edge e_{i_j} is joined $p_{\iota_{j-1}}$ and p_{ι_j}, $j = 1, \dots, k$. The number of edges, k, is called the length of path, a path of length k is called a k-path. The vertices p_{ι_0} and p_{ι_k}, linked by P, are called its ends, so the path is also denoted by $P(p_{\iota_0}, p_{\iota_k})$. If p_{ι_0} and p_{ι_k} are the same vertex, then $P(p_{\iota_0}, p_{\iota_k})$ is called a cycle. For a connected tree $G = (V(G), E(G))$ with the root p_r, for every vertex $p \in V(G) \setminus \{p_r\}$, there is a unique path connecting p and p_r, denoted by $P(p_r, p)$. The length of $P(p_r, p)$ is denoted by m_p and let $d_P = \max_{p \in V(G) \setminus \{p_r\}} \{m_p\}$. We define sets: $V_0(G) = \{p_r\}$ and for $k \geq 1$, $V_k(G) = \{p \in V(G) | P(p_r, p) \text{ is a } k\text{-path}\}$ and

$$E_k(G) = \{e \in E(G) \,|\exists p \in V_{k-1}(G), q \in V_k(G) \text{ such that they are joined by } e\},$$

then $V(G) = \bigcup_{k=0}^{d_P} V_k(G)$ and $E(G) = \bigcup_{k=1}^{d_P} E_k(G)$, where $V_k(G)$s and $E_k(G)$s are mutual disjoint, respectively. Next, $b(x)$ on $[0, 1]$ can be chosen via the following three steps.

Step 1: For every $p \in \partial G_N$, denote the path $P(p_r, p)$ by

$$V_P = \{p_{\iota_0}, p_{\iota_1}, \dots, p_{\iota_{m_p-1}}, p_{\iota_{m_p}}\} \text{ and } E_P = \{e_{i_1}, e_{i_2}, \dots, e_{i_{m_p}}\},$$

where $p_{\iota_0} = p_r$ is the root and $p_{\iota_{m_p}} = p \in \partial G_N$. Let $b_{i_{m_p}}(0) = m_p T_{i_{m_p}}(0)$ and choose a value of $b_{i_{m_p}}(1)$ such that

$$b_{i_{m_p}}(1) > m_p e^{\frac{c_\rho}{T_L \rho_L}} \max \left\{ \frac{T_{i_{m_p}}(0) \rho_{i_{m_p}}(0)}{\rho_{i_{m_p}}(1)}, T_{i_{m_p}}(1) \right\}.$$

Step 2: Beginning with the maximum m_p (the longest path $P(p_r, p)$), for $j = m_p - 1, \dots, 2, 1$, corresponding to $e_{i_k} \in E_j(G) \cap E_P$, we calculate

$$b_{i_k}(1) = \min \left\{ \frac{T_{i_k}(1) \min\limits_{j \in \mathcal{I}_E^+(p_{\iota_k})} \left\{ \frac{b_j(0)}{T_j(0)} \right\}}{1 + \deg^+(p_{\iota_k})}, \frac{\sum\limits_{j \in \mathcal{I}_E^+(p_{\iota_k})} b_j(0) \rho_j(0)}{1 + \rho_{i_k}(1)} \right\},$$

and $b_{i_k}(0) = \frac{1}{2} \min \left\{ \frac{\rho_{i_k}(1)}{\rho_{i_k}(0)}, e^{\frac{-c_\rho}{T_L \rho_L}} \frac{T_{i_k}(0)}{T_{i_k}(1)} \right\} b_{i_k}(1)$. Repeat the above procedure, till the least m_p (the shortest path $P(p_r, p)$).

Step 3: For every edge $e_k \in E(G)$, by virtue of $b_k(0)$ and $b_k(1)$ given by above two steps, we choose $b_k(x)$ as follows:

$$b_k(x) = \left[e^{\frac{c_\rho x}{T_L \rho_L}} \left(\frac{b_k(0)}{T_k(0)} + c_b^{(k)} \right) - c_b^{(k)} \right] T_k(x) \text{ with } c_\rho > |[T_k(x) \rho_k(x)]'|$$

and $c_b^{(k)} = \left(e^{\frac{c_\rho}{T_L \rho_L}} - 1 \right)^{-1} \left[\frac{b_k(1)}{T_k(1)} - e^{\frac{c_\rho}{T_L \rho_L}} \frac{b_k(0)}{T_k(0)} \right] > 0.$

At last, after $b(x)$ is determined by above steps, it is easy to verify that

$$\begin{aligned} &\frac{c_\rho}{T_L \rho_L} \min_{k \in \mathcal{I}_E} \left\{ \frac{b_k(1)}{T_k(1)} \right\} \leq \frac{c_\rho}{T_L \rho_L} \frac{b_k(1)}{T_k(1)} \leq \left[\frac{b_k(x)}{T_k(x)} \right]' \\ \leq\ & \frac{c_\rho e^{\frac{c_\rho}{T_L \rho_L}}}{T_L \rho_L \left(e^{\frac{c_\rho}{T_L \rho_L}} - 1 \right)} \frac{b_k(1)}{T_k(1)} \leq \frac{c_\rho e^{\frac{c_\rho}{T_L \rho_L}}}{T_L \rho_L \left(e^{\frac{c_\rho}{T_L \rho_L}} - 1 \right)} \max_{k \in \mathcal{I}_E} \left\{ \frac{b_k(1)}{T_k(1)} \right\} \end{aligned}$$

and

$$
\begin{aligned}
&c_\rho\left(1-\frac{1}{2e^{\frac{c_\rho}{T_L\rho_L}}}\right)\min_{k\in\mathcal{I}_E}\left\{\frac{b_k(1)}{T_k(1)}\right\}\le c_\rho\left(1-\frac{1}{2e^{\frac{c_\rho}{T_L\rho_L}}}\right)\frac{b_k(1)}{T_k(1)}\le [b_k(x)\rho_k(x)]'\\
&\le\left(\frac{e^{\frac{c_\rho}{T_L\rho_L}}}{e^{\frac{c_\rho}{T_L\rho_L}}-1}\frac{T_U\rho_U}{T_L\rho_L}+1\right)\frac{b_k(1)}{T_k(1)}c_\rho\le\left(\frac{e^{\frac{c_\rho}{T_L\rho_L}}}{e^{\frac{c_\rho}{T_L\rho_L}}-1}\frac{T_U\rho_U}{T_L\rho_L}+1\right)c_\rho\max_{k\in\mathcal{I}_E}\left\{\frac{b_k(1)}{T_k(1)}\right\}.
\end{aligned}
$$

Thus, Condition 4.1 is always fulfilled. Note that the construction of $b(x)$ is not unique, the choice based on Step 1,2 and 3 is just one way of constructing $b(x)$.

4.2. **Examples.** To explain further how to choose the diagonal matrix-valued function $b(x)$ and construct the Lyapunov functional via steps in Remark 5, we provide two tree-shaped networks. The first one is shown in Figure 1b, the underlying tree is a branching with a fixed root p_1, i.e., $\mathfrak{D}=\{p_1\}\subset\partial G$ and Hypothesis 1.2 holds. The second one is shown in Figure 2 and $\mathfrak{D}=\{p_4\}\subset Int(G)$. Its underlying rooted tree is not a branching, i.e., (2) and (3) in Hypothesis 1.2 are not satisfied. A concrete Lyapunov functional will be constructed for the second example.

4.2.1. *A nine-string-tree with collocated velocity feedbacks.* We reconsider the tree-shaped network governed by (32), shown in Figure 1b. The outgoing incidence matrix and the incoming incidence matrix of the underlying tree-shaped graph of system (32) are

$$
\Upsilon^+=\begin{array}{c}
\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9\end{array}\\
\left(\begin{array}{ccccccccc}
1&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{array}\right)
\begin{array}{c} p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8\\ p_9\\ p_{10}\end{array}\\
\begin{array}{ccccccccc} \jmath_1^+ & \jmath_2^+ & \jmath_3^+ & \jmath_4^+ & \jmath_5^+ & \jmath_6^+ & \jmath_7^+ & \jmath_8^+ & \jmath_9^+\end{array}
\end{array}
\tag{42}
$$

and

$$
\Upsilon^-=\begin{array}{c}
\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9\end{array}\\
\left(\begin{array}{ccccccccc}
0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0
\end{array}\right)
\begin{array}{c} p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8\\ p_9\\ p_{10}\end{array},\\
\begin{array}{ccccccccc} \jmath_1^- & \jmath_2^- & \jmath_3^- & \jmath_4^- & \jmath_5^- & \jmath_6^- & \jmath_7^- & \jmath_8^- & \jmath_9^-\end{array}
\end{array}
\tag{43}
$$

respectively.

To choose $b(x)$ satisfying Condition 4.1, we first write down all paths from the root p_1 to leaves ($\partial G_N = \{p_6, p_7, p_8, p_9, p_{10}\}$) in the network (see Table 1).

TABLE 1. Paths from the root p_1 to leaves

	$E_1(G)$	$V_1(G)$	$E_2(G)$	$V_2(G)$	$E_3(G)$	$V_3(G)$	$E_4(G)$	$V_4(G)$	m_p
indices	i_1	ι_1	i_2	ι_2	i_3	ι_3	i_4	ι_4	
$P(p_1,p_6)$	e_1	p_2	e_3	p_6	$*$	$*$	$*$	$*$	2
$P(p_1,p_7)$	e_1	p_2	e_2	p_3	e_5	p_5	e_9	p_7	4
$P(p_1,p_8)$	e_1	p_2	e_2	p_3	e_5	p_5	e_8	p_8	4
$P(p_1,p_9)$	e_1	p_2	e_2	p_3	e_6	p_4	e_7	p_9	4
$P(p_1,p_{10})$	e_1	p_2	e_2	p_3	e_4	p_{10}	$*$	$*$	3

Second, according to Remark 5, we choose the function $b_k(x)$ as follows.

Step 1: For the leaf p_6, the corresponding leaf edge is e_3, then $b_3(0) = 2T_3(0)$ and $b_3(1) > 2e^{\frac{c_\rho}{T_L\rho_L}} \max\left\{\frac{T_3(0)\rho_3(0)}{\rho_3(1)}, T_3(1)\right\}$. For the leaf p_7, the corresponding leaf edge is e_9, then $b_9(0) = 4T_9(0)$ and $b_9(1) > 4e^{\frac{c_\rho}{T_L\rho_L}} \max\left\{\frac{T_9(0)\rho_9(0)}{\rho_9(1)}, T_9(1)\right\}$. For the leaf p_8, the corresponding leaf edge is e_8, then $b_8(0) = 4T_8(0)$ and $b_8(1) > 4e^{\frac{c_\rho}{T_L\rho_L}} \max\left\{\frac{T_8(0)\rho_8(0)}{\rho_8(1)}, T_8(1)\right\}$. For the leaf p_9, the corresponding leaf edge is e_7, then $b_7(0) = 4T_7(0)$ and $b_7(1) > 4e^{\frac{c_\rho}{T_L\rho_L}} \max\left\{\frac{T_7(0)\rho_7(0)}{\rho_7(1)}, T_7(1)\right\}$. For the leaf p_{10}, the corresponding leaf edge is e_4, then $b_4(0) = 3T_4(0)$ and $b_4(1) > 3e^{\frac{c_\rho}{T_L\rho_L}} \max\left\{\frac{T_4(0)\rho_4(0)}{\rho_4(1)}, T_4(1)\right\}$.

Step 2: The maximum $m_p = 4$. For $e_5 = e_{i_3} \in E_3(G) \cap E_{P(p_1,p_7)} = E_3(G) \cap E_{P(p_1,p_8)}$ and $p_{\iota_3} = p_5$, we choose $b_5(0) = \frac{1}{2}\min\left\{\frac{\rho_5(1)}{\rho_5(0)}, e^{\frac{-c_\rho}{T_L\rho_L}}\frac{T_5(0)}{T_5(1)}\right\} b_5(1)$ and

$$b_5(1) = \min\left\{\frac{T_5(1)}{3}\min\left\{\frac{b_8(0)}{T_8(0)}, \frac{b_9(0)}{T_9(0)}\right\}, \frac{b_8(0)\rho_8(0) + b_9(0)\rho_9(0)}{1+\rho_5(1)}\right\}.$$

For $e_6 = e_{i_3} \in E_3(G) \cap E_{P(p_1,p_9)}$ and $p_{\iota_3} = p_4$, we choose

$$b_6(1) = \min\left\{\frac{T_6(1)b_7(0)}{2T_7(0)}, \frac{b_7(0)\rho_7(0)}{1+\rho_5(1)}\right\}$$

and

$$b_6(0) = \frac{1}{2}\min\left\{\frac{\rho_6(1)}{\rho_6(0)}, e^{\frac{-c_\rho}{T_L\rho_L}}\frac{T_6(0)}{T_6(1)}\right\} b_6(1).$$

For $e_2 = e_{i_2} \in E_2(G) \cap E_{P(p_1,p_k)}$, $k = 7, 8, 9, 10$, and $p_{i_2} = p_3$, we choose

$$b_2(1) = \min\left\{\frac{1}{4}T_2(1)\min_{j\in\{4,5,6\}}\left\{\frac{b_j(0)}{T_j(0)}\right\}, \frac{1}{1+\rho_2(1)}\sum_{j\in\{4,5,6\}} b_j(0)\rho_j(0)\right\}$$

and $b_2(0) = \frac{1}{2}\min\left\{\frac{\rho_2(1)}{\rho_2(0)}, e^{\frac{-c_\rho}{T_L\rho_L}}\frac{T_2(0)}{T_2(1)}\right\} b_2(1)$. For $e_1 = e_{i_1} \in E_1(G) \cap E_{P(p_1,p_k)}$, $k = 6, \ldots, 10$, and $p_{\iota_1} = p_2$, we choose $b_1(0) = \frac{1}{2}\min\left\{\frac{\rho_1(1)}{\rho_1(0)}, e^{\frac{-c_\rho}{T_L\rho_L}}\frac{T_1(0)}{T_1(1)}\right\} b_1(1)$ and

$$b_1(1) = \min\left\{\frac{1}{3}T_1(1)\min_{j\in\{2,3\}}\left\{\frac{b_j(0)}{T_j(0)}\right\}, \frac{b_2(0)\rho_2(0) + b_3(0)\rho_3(0)}{1+\rho_1(1)}\right\}.$$

Thus, $b(x)$ is constructed by Step 3 in Remark 5, and all assumptions in Theorem 1.3 are fulfilled. Finally, by use of (4), (34), (42), (43) and $\jmath_k = k+1$ for $k = 1, \dots, 9$, it is shown that the network (32), under velocity feedbacks

$$\begin{cases} u(p_6,t) = -\beta_5 \mathrm{w}_t(p_{\jmath_5},t) = -\beta_5 w_{\jmath_3^-,t}(1,t) = -\beta_3^- w_{3,t}(1,t), \\ u(p_7,t) = -\beta_6 \mathrm{w}_t(p_{\jmath_6},t) = -\beta_6 w_{\jmath_9^-,t}(1,t) = -\beta_9^- w_{9,t}(1,t), \\ u(p_8,t) = -\beta_7 \mathrm{w}_t(p_{\jmath_7},t) = -\beta_7 w_{\jmath_8^-,t}(1,t) = -\beta_8^- w_{8,t}(1,t), \\ u(p_9,t) = -\beta_8 \mathrm{w}_t(p_{\jmath_8},t) = -\beta_8 w_{\jmath_7^-,t}(1,t) = -\beta_7^- w_{7,t}(1,t), \\ u(p_{10},t) = -\beta_9 \mathrm{w}_t(p_{\jmath_9},t) = -\beta_9 w_{\jmath_4^-,t}(1,t) = -\beta_4^- w_{4,t}(1,t), \end{cases}$$

is exponentially stable.

4.2.2. *A six-string-tree with collocated velocity feedbacks.* The tree-shaped network consisting of six strings with one fixed root is shown in Figure 2. The motion of the

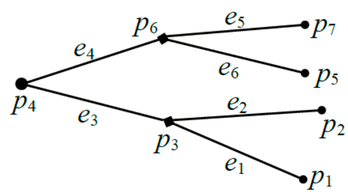


FIGURE 2. The tree-shaped network consisting of six strings with the fixed root p_4

network can be formulated by

$$\begin{cases} \rho_j(x) w_{j,tt}(x,t) = (T_j(x) w_{j,x})_x(x,t),\ x \in (0,1), j = 1, \dots, 6, \\ w_3(0,t) = w_4(0,t) = 0, \\ w_1(0,t) = w_2(0,t) = w_3(1,t), w_6(0,t) = w_5(1,t) = w_4(1,t), \\ T_3(1) w_{3,x}(1,t) - [T_1(0) w_{1,x}(0,t) + T_2(0) w_{2,x}(0,t)] = 0, \\ T_4(1) w_{4,x}(1,t) + T_5(1) w_{5,x}(1,t) - T_6(0) w_{6,x}(0,t)] = 0, \\ T_1(1) w_{1,x}(1,t) = u(p_1,t), T_2(1) w_{2,x}(1,t) = u(p_2,t), \\ -T_5(0) w_{5,x}(0,t) = u(p_7,t), T_6(1) w_{6,x}(1,t) = u(p_5,t), \end{cases} \tag{44}$$

where $\rho_1(x) = \rho_2(x) = \rho_3(x) = 1$, $\rho_4(x) = 1.25 - 0.25x^2$, $\rho_5(x) = 2 - x$, $\rho_6(x) = 1.25 - (2\pi)^{-1}\sin(2\pi x)$, $T_1(x) = 1$, $T_2(x) = 1.5$, $T_3(x) = 2$, $T_4(x) = 1.25 + (2\pi)^{-1}\sin(2\pi x)$, $T_5(x) = 1 + x$ and $T_6(x) = 1 + 0.25x^2$. The outgoing incidence matrix and the incoming incidence matrix are

$$\Upsilon^+ = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{matrix} \\ \begin{matrix} \jmath_1^+ & \jmath_2^+ & \jmath_3^+ & \jmath_4^+ & \jmath_5^+ & \jmath_6^+ \end{matrix} \end{array} \text{ and } \Upsilon^- = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{matrix} \\ \begin{matrix} \jmath_1^- & \jmath_2^- & \jmath_3^- & \jmath_4^- & \jmath_5^- & \jmath_6^- \end{matrix} \end{array}, \tag{45}$$

respectively. The Dirichlet set $\mathfrak{D} = \{p_4\} \subset Int(G)$, and $\deg^-(p_6) = 2$, which means that the rooted tree is not a branching and the system (44) is not the standard form of (3). $V(G) \setminus \mathfrak{D} = \{p_{\jmath_1}, p_{\jmath_2}, p_{\jmath_3}, p_{\jmath_4}, p_{\jmath_5}, p_{\jmath_6}\}$, where $p_{\jmath_k} = p_k$ for $k = 1, 2, 3$, $p_{\jmath_k} = p_{k+1}$ for $k = 4, 5, 6$, and $m_0 = 6$. $\mathfrak{F} = \{p_{\jmath_3}, p_{\jmath_5}\} = \{p_3, p_6\}$. The set of leaves $\partial G_N = \partial G = \{p_1, p_2, p_5, p_7\}$. The system (44) is L^2-well-posedness and regular, according to Theorem 1.1.

Similar to (4), the collocated output feedback is read as

$$u(p_{\jmath_k}, t) = -\beta_k \mathrm{w}_t(p_{\jmath_k}, t), \text{ for } k = 1, 2, 4, 6, \tag{46}$$

where $\beta_k > 0$ and $p_{\jmath_k} \in \partial G_N$, $k = 1, 2, 4, 6$. Hence, the closed-loop system (44) with (46) can be rewritten as

$$\begin{cases} \rho_j(x) w_{j,tt}(x,t) = (T_j(x) w_{j,x})_x(x,t), \ x \in (0,1), j = 1, \ldots, 6, \\ w_3(0,t) = w_4(0,t) = 0, w_1(0,t) = w_2(0,t) = w_3(1,t), \\ w_5(0,t) = w_6(0,t) = w_4(1,t), \\ T_1(1) w_{1,x}(1,t) = -\beta_1 w_{1,t}(1,t), T_2(1) w_{2,x}(1,t) = -\beta_2 w_{2,t}(1,t), \\ T_3(1) w_{3,x}(1,t) - [T_1(0) w_{1,x}(0,t) + T_2(0) w_{2,x}(0,t)] = 0, \\ T_4(1) w_{4,x}(1,t) + T_5(1) w_{5,x}(1,t) - T_6(0) w_{6,x}(0,t) = 0, \\ T_5(0) w_{5,x}(0,t) = \beta_6 w_{5,t}(0,t), T_6(1) w_{6,x}(1,t) = -\beta_4 w_{6,t}(1,t). \end{cases} \tag{47}$$

Notice that the closed-loop system (47) can not be written in the form of (35). Thus, the matrix-vector form of (47) can only be formulated by (8) with Υ^- and Υ^+ being determined by (45), and

$$\boldsymbol{u}(t) = (-\beta_1 w_{1,t}(1,t), -\beta_2 w_{2,t}(1,t), 0, -\beta_4 w_{6,t}(1,t), 0, -\beta_6 w_{5,t}(0,t))^\top. \tag{48}$$

To construct a Lyapunov functional for (47), we do a change of variable $x := 1-x$, for the edge e_5, and let $\widetilde{w}_5(x,t) = w_5(1-x,t)$, $\widetilde{\rho}_5(x) = 1 + x$ and $\widetilde{T}_5(x) = 2 - x$, and for $j \neq 5$, let $\widetilde{w}_j(x,t) = w_j(x,t)$, $\widetilde{\rho}_j(x) = \rho_j(x)$ and $\widetilde{T}_j(x) = T_j(x)$. Thus, (47) is reformulated by

$$\begin{cases} \widetilde{\rho}_j(x) \widetilde{w}_{j,tt}(x,t) = (\widetilde{T}_j(x) \widetilde{w}_{j,x})_x(x,t), \ x \in (0,1), j = 1, \ldots, 6, \\ \widetilde{w}_3(0,t) = \widetilde{w}_4(0,t) = 0, \\ \widetilde{w}_1(0,t) = \widetilde{w}_2(0,t) = \widetilde{w}_3(1,t), \widetilde{w}_5(0,t) = \widetilde{w}_6(0,t) = \widetilde{w}_4(1,t), \\ \widetilde{T}_3(1) \widetilde{w}_{3,x}(1,t) - [\widetilde{T}_1(0) \widetilde{w}_{1,x}(0,t) + \widetilde{T}_2(0) \widetilde{w}_{2,x}(0,t)] = 0, \\ \widetilde{T}_4(1) \widetilde{w}_{4,x}(1,t) - [\widetilde{T}_5(0) \widetilde{w}_{5,x}(0,t) + \widetilde{T}_6(0) \widetilde{w}_{6,x}(0,t)] = 0, \\ \widetilde{T}_1(1) \widetilde{w}_{1,x}(1,t) = -\beta_1 \widetilde{w}_{1,t}(1,t), \widetilde{T}_2(1) \widetilde{w}_{2,x}(1,t) = -\beta_2 \widetilde{w}_{2,t}(1,t), \\ \widetilde{T}_5(1) \widetilde{w}_{5,x}(1,t) = -\beta_6 \widetilde{w}_{5,t}(1,t), \widetilde{T}_6(1) \widetilde{w}_{6,x}(1,t) = -\beta_4 \widetilde{w}_{6,t}(1,t). \end{cases} \tag{49}$$

Thus, (1) and (3) in Hypothesis 1.2 are satisfied and the underlying tree of system (49) is a branching with the root p_4, which is joined with two edges.

In the following, we determine the multiplier $\widetilde{b}(x)$ for (49) due to Remark 5. A simple calculation shows that $0 < 1 = \rho_L \leq \widetilde{\rho}_k(x) \leq \rho_U$, $0 < 1 = T_L \leq \widetilde{T}_k(x) \leq T_U$ and

$$|[\widetilde{T}_k(x) \widetilde{\rho}_k(x)]'| \leq \frac{15}{8} + \frac{1}{4\pi} = c_\rho < 2 = \hat{c}_\rho, \text{ on } [0,1].$$

All paths from the root p_4 to leaves ($\partial G_N = \{p_1, p_2, p_5, p_7\}$) in the tree-shaped network are filled in Table 2. Thus, using Step 1 in Remark 5, we choose $\widetilde{b}_1(0) = 2$ and $\widetilde{b}_1(1) = 2e^{\hat{c}_\rho}$ for the leaf edge e_1; $\widetilde{b}_2(0) = 3$ and $\widetilde{b}_2(1) = 3e^{\hat{c}_\rho}$ for the leaf edge e_2; $\widetilde{b}_6(0) = 2$ and $\widetilde{b}_6(1) = \frac{5}{2}e^{\hat{c}_\rho}$ for the leaf edge e_6; $\widetilde{b}_5(0) = 4$ and $\widetilde{b}_5(1) = 2e^{\hat{c}_\rho}$ for

TABLE 2. Paths from the root p_4 to leaves

	$E_1(G)$	$V_1(G)$	$E_2(G)$	$V_2(G)$	m_p
indices	i_1	ι_1	i_2	ι_2	
$P(p_4, p_1)$	e_3	p_3	e_1	p_1	2
$P(p_4, p_2)$	e_3	p_3	e_2	p_2	2
$P(p_4, p_5)$	e_4	p_6	e_6	p_5	2
$P(p_4, p_7)$	e_4	p_6	e_5	p_7	2

the leaf edge e_5. Using Step 2 in Remark 5, we choose $\widetilde{b}_3(1) = \frac{4}{3}$ and $\widetilde{b}_3(0) = \frac{2}{3}e^{-c_\rho}$ for the edge e_3; $\widetilde{b}_4(1) = \frac{2}{3}$ and $\widetilde{b}_4(0) = \frac{1}{3}e^{-c_\rho}$ for the edge e_4. According to Step 3 in Remark 5, we have

$$\begin{cases} \widetilde{b}_1(x) = 2\left[e^{c_\rho x} + \left(e^{\hat{c}_\rho} - e^{c_\rho}\right)\frac{e^{c_\rho x}-1}{e^{c_\rho}-1}\right], \widetilde{b}_3(x) = \frac{2}{3}\left[e^{c_\rho(x-1)} + \frac{e^{c_\rho x}-1}{e^{c_\rho}-1}\right], \\ \widetilde{b}_2(x) = \frac{3}{2}\widetilde{b}_1(x), \widetilde{b}_4(x) = \frac{\widetilde{T}_4(x)}{2}\widetilde{b}_3(x), \widetilde{b}_5(x) = \widetilde{T}_5(x)\widetilde{b}_1(x), \widetilde{b}_6(x) = \widetilde{T}_6(x)\widetilde{b}_1(x). \end{cases}$$

Hence, we obtain the matrix multiplier $b(x)$ for (47): $b_k(x) = \widetilde{b}_k(x)$ for $k \neq 5$ and $b_5(x) = -\widetilde{b}_5(1-x)$, that is,

$$\begin{cases} b_1(x) = 2\left[e^{c_\rho x} + \frac{e^{\hat{c}_\rho} - e^{c_\rho}}{e^{c_\rho}-1}(e^{c_\rho x} - 1)\right], b_2(x) = \frac{3b_1(x)}{2}, \\ b_3(x) = \frac{2}{3}\left[e^{c_\rho(x-1)} + \frac{e^{c_\rho x}-1}{e^{c_\rho}-1}\right], b_4(x) = \frac{T_4(x)b_3(x)}{2}, \\ b_5(x) = -T_5(x)b_1(1-x), b_6(x) = T_6(x)b_1(x). \end{cases} \tag{50}$$

Thus, the Lyapunov functional for the system (47) is

$$\mathscr{V}(t) = \mathscr{E}(t) + c_V\mathscr{V}_b(t) = \mathscr{E}(t) + c_V\int_0^1 \langle\, b(x)w_x(x,t),\, M(x)w_t(x,t)\,\rangle dx.$$

From (6), (8), (12), (45) and (48), it can be deduced that

$$\frac{d\mathscr{E}(t)}{dt} = -\beta_1 w_{1,t}^2(1,t) - \beta_2 w_{2,t}^2(1,t) - \beta_4 w_{6,t}^2(1,t) - \beta_6 w_{5,t}^2(0,t). \tag{51}$$

Similar to (39), it follows from integration by parts, (6), (8), (44) and (45) that

$$\begin{aligned} \frac{d\mathscr{V}_b(t)}{dt} &= \frac{1}{2}\langle\, C_{\boldsymbol{w}} P_{\mathbb{D}}\boldsymbol{w}_t(\boldsymbol{p},t),\, P_{\mathbb{D}}\boldsymbol{w}_t(\boldsymbol{p},t)\,\rangle_{\mathbb{C}^{m_0}} + \frac{1}{2}\langle\, b(1)w_x(1,t),\, T(1)w_x(1,t)\,\rangle_{\mathbb{C}^n} \\ &\quad -\frac{1}{2}\langle\, b(0)w_x(0,t),\, T(0)w_x(0,t)\,\rangle_{\mathbb{C}^n} - O(\mathscr{E}), \end{aligned} \tag{52}$$

where $O(\mathscr{E})$ satisfies (37),

$$\begin{aligned} C_{\boldsymbol{w}} &= P_{\mathbb{D}}\left[(\Upsilon^-)M(1)b(1)(\Upsilon^-)^\top - (\Upsilon^+)M(0)b(0)(\Upsilon^+)^\top\right]P_{\mathbb{D}}^\top \\ &= \operatorname{diag}\left(2e^{\hat{c}_\rho}, 3e^{\hat{c}_\rho}, -\frac{11}{3}, \frac{25}{8}e^{\hat{c}_\rho}, -\frac{35}{6}, 4e^{\hat{c}_\rho}\right) \end{aligned} \tag{53}$$

and

$$\begin{aligned}
&\frac{1}{2}\langle\, b(1)w_x(1,t),\, T(1)w_x(1,t)\,\rangle - \frac{1}{2}\langle\, b(0)w_x(0,t),\, T(0)w_x(0,t)\,\rangle = \\
&\quad \frac{2}{3}[T_1(0)w_{1,x}(0,t)+T_2(0)w_{2,x}(0,t)]^2 - 2T_1^2(0)w_{1,x}^2(0,t) - 2T_2^2(0)w_{2,x}^2(0,t) \\
&\quad + \frac{2}{3}T_4^2(1)w_{4,x}^2(1,t) - 2T_5^2(1)w_{5,x}^2(1,t) - 2[T_4(1)w_{4,x}(1,t)+T_5(1)w_{5,x}(1,t)]^2 \\
&\qquad + 2e^{\hat{c}_\rho}\left[\beta_1^2 w_{1,t}^2(1,t) + \beta_2^2 w_{2,t}^2(1,t) + \beta_4^2 w_{6,t}^2(1,t) + \beta_6^2 w_{5,t}^2(0,t)\right] \\
&\qquad\quad - \frac{1}{3e^{c_\rho}}T_3^2(0)w_{3,x}^2(0,t) - \frac{1}{6e^{c_\rho}}T_4^2(0)w_{4,x}^2(0,t) \\
&\qquad \le 2e^{\hat{c}_\rho}\left[\beta_1^2 w_{1,t}^2(1,t) + \beta_2^2 w_{2,t}^2(1,t) + \beta_4^2 w_{6,t}^2(1,t) + \beta_6^2 w_{5,t}^2(0,t)\right]. \qquad (54)
\end{aligned}$$

Thus, it follows from (51), (52), (53) and (54) that

$$\begin{aligned}
&\frac{d\mathscr{V}(t)}{dt} = \frac{d\mathscr{E}(t)}{dt} + c_V\frac{d\mathscr{V}_b(t)}{dt} \\
&\quad \le -\left[\beta_1 - c_V e^{\hat{c}_\rho} - c_V e^{\hat{c}_\rho}\beta_1^2\right] w_{1,t}^2(1,t) - \left[\beta_2 - \frac{3}{2}c_V e^{\hat{c}_\rho} - c_V e^{\hat{c}_\rho}\beta_2^2\right] w_{2,t}^2(1,t) \\
&-\left[\beta_4 - \frac{25}{16}c_V e^{\hat{c}_\rho} - c_V e^{\hat{c}_\rho}\beta_4^2\right] w_{6,t}^2(1,t) - \left[\beta_6 - 2c_V e^{\hat{c}_\rho} - c_V e^{\hat{c}_\rho}\beta_6^2\right] w_{5,t}^2(0,t) - c_V O(\mathscr{E}).
\end{aligned}$$

Let $c_V < \min\left\{\frac{\beta_1 e^{-\hat{c}_\rho}}{1+\beta_1^2}, \frac{\beta_2 e^{-\hat{c}_\rho}}{1.5+\beta_2^2}, \frac{\beta_4 e^{-\hat{c}_\rho}}{\frac{25}{16}+\beta_4^2}, \frac{\beta_6 e^{-\hat{c}_\rho}}{2+\beta_6^2}, c_b^{-1}\right\}$, it can be derived that

$$\frac{d\mathscr{V}(t)}{dt} \quad \le \quad -c_V O(\mathscr{E}) \le -c_L c_V \mathscr{E}(t) \le -\frac{c_L c_V}{1+c_V c_b}\mathscr{V}(t).$$

Hence, the system (47) is exponentially stable.

5. **Conclusions.** The multiplier method is applied to discuss the well-posedness and regularity of the open-loop system of strings network and the exponential stability of the closed-loop system in this paper. Especially, a Lyaponuv functional for tree-shaped networks of elastic strings is presented by constructing an appropriate multiplier. This construction method (see Remark 5) may be generalized to other types of networks, e.g., networks of beams etc, even for nonlinear networks, which will be discussed in other papers. In engineering, it is more meaningful work. Additionally, the issues of time-delay and anti-disturbance for networks governed by partial differential equations may also be investigated based on the same methods. These problems are worth exploring in future.

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