



HOMOGENIZATION OF STIFF INCLUSIONS THROUGH NETWORK APPROXIMATION

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ABSTRACT. We investigate the homogenization of inclusions of infinite conductivity, randomly stationary distributed inside a homogeneous conducting medium. A now classical result by Zhikov shows that, under a logarithmic moment bound on the minimal distance between the inclusions, an effective model with finite homogeneous conductivity exists. Relying on ideas from network approximation, we provide a relaxed criterion ensuring homogenization. Several examples not covered by the previous theory are discussed.

1. Introduction. The classical theory of homogenization deals with conductivity matrices $A^\varepsilon = A(\frac{x}{\varepsilon})$ having uniform lower and upper bounds: $\alpha \text{Id} \leq A \leq \beta \text{Id}$, with $0 < \alpha < \beta < +\infty$. Degenerate cases, for which A is allowed to vanish or to take infinite values, still lead to open questions. A problem typical of the first situation is as follows. Given a union $F = \cup_{I \in \mathcal{I}} I$ of closed connected domains I (the *inclusions*), either periodic or random stationary, one considers the following system:

$$\begin{cases} -\Delta u^\varepsilon = f & \text{in } U \setminus F^\varepsilon, \\ u^\varepsilon|_{\partial U} = 0, \\ \partial_\nu u^\varepsilon|_{\partial F^\varepsilon} = 0. \end{cases} \quad (1)$$

Here, U is a smooth bounded domain of \mathbb{R}^3 , and $F^\varepsilon = \cup \varepsilon I$ where the union is restricted to inclusions satisfying $\varepsilon I \subset U$. The Neumann condition corresponds to zero conductivity inside the inclusions, while conductivity is normalized to 1 outside. The problem is to determine a homogenized limit model. Namely, one tries to understand under which conditions the solution u^ε of (1) converges as $\varepsilon \rightarrow 0$ to the solution u^0 of

$$\begin{cases} -\text{div}(A^0 \nabla u^0) = (1 - \lambda)f & \text{in } U, \\ u^0|_{\partial U} = 0, \end{cases} \quad (2)$$

for some non-degenerate effective conductivity matrix A^0 . The constant λ refers to the density of the inclusions, and is the weak limit of 1_{F^ε} . This homogenization problem has become standard, and is discussed extensively in the classical book [27]. In brief, homogenization holds under some mild but uniform regularity requirement

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on the geometry of the inclusions, and in a general random stationary ergodic setting, as long as $\mathbb{R}^3 \setminus F$ is connected. One strategy to show this result is through an adaptation of the classical div-curl approach. With regards to this strategy, a nice feature is that the first equation and the Neumann condition are equivalent to the single equation

$$\operatorname{div}(1_{F^\varepsilon} \nabla u^\varepsilon) = 1_{F^\varepsilon} f \quad \text{in } U$$

meaning that the divergence of the flux $1_{F^\varepsilon} \nabla u^\varepsilon$ is relatively compact in $H^{-1}(U)$. On the contrary, one difficulty is that ∇u^ε is defined only on $U \setminus F^\varepsilon$ and there is no canonical extension as a gradient of an H^1 function over the whole domain U . Still, construction of such a potential field extension inside the inclusions is possible under mild requirements. Let us stress that such extension is a local process, in the sense that extension in one inclusion can be considered independently from the others.

As regards the opposite case of inclusions with infinite conductivity, the theory of homogenization is less complete. The analogue of [1](#) reads:

$$\begin{cases} -\Delta u^\varepsilon = f & \text{in } U \setminus F^\varepsilon, \\ \nabla u^\varepsilon = 0 & \text{in } F^\varepsilon, \\ u^\varepsilon|_{\partial U} = 0, \\ \int_{\partial I^\varepsilon} \partial_\nu u^\varepsilon = 0, \quad \forall I^\varepsilon \in \operatorname{CC}(F^\varepsilon). \end{cases} \quad (3)$$

Here, we denote $I^\varepsilon := \varepsilon I$, that belongs to the set $\operatorname{CC}(F^\varepsilon)$ of connected components of F^ε . On such inclusions, the condition $\nabla u^\varepsilon = 0$ ensures that the potential is constant, which corresponds to infinite conductivity. The integral condition $\int_{\partial I^\varepsilon} \partial_\nu u^\varepsilon = 0$ corresponds to the fact that the total flux through the boundary of the inclusion is zero (a continuous analogue of Kirchoff's nodal rule). Let us note that system [3](#) has an important extension to modelling of suspensions in fluid mechanics, in which the Laplace operator is replaced by the Stokes operator, the electric field being replaced by the Newtonian stress tensor of the fluid. The analogue question of the effective viscosity of passive suspensions has attracted a lot of attention recently, *cf.* [\[25, 2, 30, 26, 20, 21, 22, 19, 16, 15, 17\]](#) among many.

Again, the point is to show convergence to the effective system [2](#). However, contrary to the case of *soft inclusions* [1](#), the case of *stiff inclusions* (in the terminology of [\[27\]](#)) requires more than the connectedness of $\mathbb{R}^3 \setminus F$. Some condition on the distance between particles is needed. The main difference between [1](#) and [3](#), responsible for extra assumptions in the latter case, can be seen when trying to adapt the div-curl approach. The difficulty is reversed. While the potential field ∇u^ε has a natural extension as a potential field over U (just extend u^ε by its constant value inside each inclusion), the divergence of the flux $1_{F^\varepsilon} \nabla u^\varepsilon$ is no longer controlled in $H^{-1}(U)$. The point is then to find a nice extension inside the inclusions of a field (namely ∇u^ε) with given divergence (namely f). This constraint on the divergence makes such extension process non-local, and implies global assumptions on the configuration of the inclusions.

Up to our knowledge, homogenization was so far only established by Zhikov for a random spherical structure $F = \cup_{i \in \mathbb{N}} B_i$, under the assumption that almost surely,

$$\limsup_{N \rightarrow +\infty} \frac{1}{N^3} \sum_{B_i \subset (-N, N)^3} \mu_i < +\infty, \quad \text{where} \quad \mu_i := |\ln(d(B_i, F \setminus B_i))|. \quad (4)$$

See Theorem 2.6 below, or [27, chapter 8] for more. Obviously, in the case where there is a minimal distance $\delta > 0$ between the spheres, such condition is satisfied. As will be discussed later in the paper, the logarithmic term μ_i comes from the fact that given any smooth Dirichlet data φ_i on B_i , there is an H^1 function ϕ with $\phi = \varphi_i$ on B_i and $\phi = 0$ on all other B_j 's, such that its energy $\int_{\mathbb{R}^3} |\nabla \phi|^2$ explodes at most like μ_i . This gives a glimpse on how such assumption may help to build the extension operator alluded to above.

In the opposite very dense setting where for all i , $|\mu_i| \gg 1$, homogenization is not always possible. As a consequence, one can find configurations F and boundary data problems of the form

$$\begin{cases} -\Delta u^\varepsilon = 0 & \text{in } U \setminus F^\varepsilon, \\ \nabla u^\varepsilon = 0 & \text{in } F^\varepsilon, \\ u^\varepsilon|_{\partial U} = \varphi, \\ \int_{\partial I^\varepsilon} \partial_\nu u^\varepsilon = 0, & \forall I^\varepsilon \in \text{CC}(F^\varepsilon) \end{cases} \quad (5)$$

for which

$$\liminf_{\varepsilon \rightarrow 0} \int_U |\nabla u^\varepsilon|^2 = +\infty.$$

Such negative results for dense settings rely notably on the so-called *network approximation method*, as described in the monograph [7]. The idea behind this method is that when the spheres get close to one another, the analysis of systems of type 5 can be simplified: the asymptotic behaviour of the system (notably of its energy), can be deduced from the properties of an underlying weighted graph where:

- nodes of the graph are the spheres of infinite conductivity
- edges are pairs of spheres close to one another, in the sense that they belong to adjacent Voronoi cells
- each edge $e = \{B_i, B_j\}$ has weight $\mu_e := |\ln d(B_i, B_j)|$.

For instance, the energy of the system can be approximated as the parameter $\delta(\varepsilon) := \varepsilon^{-1} \max_{I^\varepsilon \in F^\varepsilon} d(I^\varepsilon, F^\varepsilon \setminus I^\varepsilon)$ goes to zero when ε goes to zero by a reduced discrete energy associated to the graph. Such energy may then be analyzed by tools of graph theory, and its divergence as ε goes to zero established. We refer to [13, 6, 8, 9], and again to the monograph [7] and its bibliography. See also [4, 5, 10] in the context of fluid mechanics and suspensions.

The goal of this paper is to investigate some intermediate situations, in which the condition on the distance between the inclusions 4 is not necessarily satisfied, but homogenization is still possible. To identify situations of this kind, we will rely on two notions:

- the notion of *multigraph of inclusions*, reminiscent of the network approximation just mentioned
- the notion of *short of inclusions*, reminiscent of the study of electrical circuits.

Thanks to these notions, we will formulate two assumptions H1-H2. The weaker one, H1, allows to define the homogenized matrix A^0 , while the stronger one, H2, allows for homogenization. This set of two assumptions is implied by a logarithmic moment bound on the minimal distance, but is much more general. For instance, our homogenization theorems apply to the case of inclusions or clusters of inclusions

of arbitrarily large size, with a mere moment bound on their diameter. Further examples, like inclusions with anisotropic structure, will be discussed.

The outline of the paper is as follows. In Section 2, after a brief reminder on stationary closed sets, we describe the class of inclusions under consideration. For such class, we define multigraphs and shorts of inclusions. We conclude the section by the statement of our homogenization results. In Section 3, we show certain extension properties for potential and solenoidal vector fields, crucial to the proof of homogenization. This proof is then given in Section 4. Eventually, Section 5 provides complements on our main hypothesis **H1-H2**.

2. Statements of the main results.

2.1. Reminder on stationary closed sets. We follow here the definition in the lecture notes [11, chapter 13]. A random closed set is a random variable $F = F(\omega) \subset \mathbb{R}^3$ from a probability space to the set

$$\text{CL} = \{F \subset \mathbb{R}^3, F \text{ closed}\}$$

equipped with the borelian σ -algebra $B(\text{CL})$. We remind that the topology on CL is the topology induced by the sets $\text{CL}_K = \{F, F \cap K \neq \emptyset\}$, K describing the compact subsets of \mathbb{R}^3 . Note that by considering the law P_F of F on CL, we can always assume that the probability space is $(\text{CL}, B(\text{CL}), P_F)$ and that $F(\omega) = \omega$. This is the canonical representation of the random closed set. We now introduce the shift $\tau_x : \text{CL} \rightarrow \text{CL}$, $\tau_x(\omega) = \omega - x$, $x \in \mathbb{R}^3$. We say that the random closed set F is \mathbb{R}^3 -stationary, resp. \mathbb{Z}^3 stationary, if $P_F \circ \tau_x^{-1} = P_F$ for all $x \in \mathbb{R}^3$, resp. $x \in \mathbb{Z}^3$. We say that it is ergodic if under the assumption that $\tau_x(\mathcal{A}) = \mathcal{A}$ for all $x \in \mathbb{R}^3$, resp. $\tau_x(\mathcal{A}) = \mathcal{A}$ for all $x \in \mathbb{Z}^3$, one has $P_F(\mathcal{A}) \in \{0, 1\}$.

Note that there is a trick to turn a \mathbb{Z}^3 -stationary ergodic random closed set into an \mathbb{R}^3 -stationary ergodic random closed set. Namely, given F a \mathbb{Z}^3 -stationary random closed set, one defines

$$\tilde{F} : (\text{CL} \times (0, 1)^3, P_F \otimes \text{Leb}) \rightarrow \text{CL}, \quad (\omega, x) \rightarrow \omega - x.$$

Then, one considers on CL the measure image of $P_F \otimes \text{Leb}$ by \tilde{F} , and denoting $P_{\tilde{F}}$ such a measure, one can check that $P_{\tilde{F}} \circ \tau_x^{-1} = P_{\tilde{F}}$ for all $x \in \mathbb{R}^3$. Indeed, given a borelian \mathcal{A} of CL, by Fubini's theorem,

$$\begin{aligned} P_{\tilde{F}} \circ \tau_x^{-1}(\mathcal{A}) &= P_F \times \text{Leb}(\{(\omega, y), \omega - y \in \mathcal{A} + x\}) \\ &= \int_{(0,1)^3} \int_{\text{CL}} 1_{\mathcal{A}+x}(\omega - y) dP_F(\omega) dy = \int_{(0,1)^3} P_F(\tau_{-x-y}(\mathcal{A})) dy. \end{aligned}$$

By \mathbb{Z}^3 -stationarity of P_F , the integrand is \mathbb{Z}^3 -periodic, and the result follows by the change of variable $y' = y + x$. Moreover, \tilde{F} is ergodic if F is. Indeed, if $\tau_y(\mathcal{A}) = \mathcal{A}$ for all $y \in \mathbb{R}^3$, we find (take $x = 0$ in the previous identity):

$$P_{\tilde{F}}(\mathcal{A}) = \int_{(0,1)^3} P_F(\tau_{-y}(\mathcal{A})) dy = P_F(\mathcal{A}) \in \{0, 1\}.$$

Note that a property that is almost sure with respect to $P_{\tilde{F}}$ will hold for realizations of the form $\omega - x$, that is $F(\omega) - x$, for P_F -almost every ω and almost every x . As the choice $x = 0$ characterizing F made at the beginning is irrelevant, this gives results about the original random closed set F .

More generally, let $(F_i)_{i \in I}$ a family of random closed sets defined on the same probability space, and P its joint law on CL^I . We define the shift $\tau_x : \text{CL}^I \rightarrow \text{CL}^I$,

$\tau_x((\omega_i)_{i \in I}) = (x + \omega_i)_{i \in I}$, $x \in \mathbb{R}^3$. We say that $(F_i)_{i \in I}$ is \mathbb{R}^3 -stationary, resp. \mathbb{Z}^3 -stationary, if $P \circ \tau_x^{-1} = P$ for all $x \in \mathbb{R}^3$, resp. $x \in \mathbb{Z}^3$. We say that it is ergodic if under the assumption that $\tau_x(\mathcal{A}) = \mathcal{A}$ for all $x \in \mathbb{R}^3$, resp. $\tau_x(\mathcal{A}) = \mathcal{A}$ for all $x \in \mathbb{Z}^3$, one has $P(\mathcal{A}) \in \{0, 1\}$. Of course, stationarity of the family is stronger than the stationarity of each of its elements, and is the right notion as soon as one deals with events involving intersections, unions of several random closed sets F_i .

A convenient unified description, that we adopt from now on, is the following. We consider a probability space (Ω, \mathcal{A}, P) , equipped with a family of maps $\tau_x : \Omega \rightarrow \Omega$, $x \in \mathbb{R}^3$, which satisfies:

- i) $(x, \omega) \rightarrow \tau_x(\omega)$ measurable
- ii) $\forall x, y, \quad \tau_{x+y} = \tau_x \circ \tau_y$ (shift)
- iii) $\forall x, \quad P = P \circ \tau_x^{-1}$ (measure preserving)
- iv) $(\forall x, \tau_x(\mathcal{A}) = \mathcal{A}) \Rightarrow P(\mathcal{A}) \in \{0, 1\}$ (ergodicity).

Under this description, an (ergodic) stationary closed set is then a r. v. $F : \Omega \rightarrow \text{CL}$ s.t.

$$F(\tau_x(\omega)) = F(\omega) - x, \quad \forall \omega \in \Omega, \forall x \in \mathbb{R}^3.$$

Introducing the subset \mathcal{F} of Ω defined by

$$\mathcal{F} = \{\omega \in \Omega, 0 \in F(\omega)\} \tag{6}$$

one can notice that

$$F(\omega) = \{x, \tau_x(\omega) \in \mathcal{F}\}.$$

This is the point of view taken in [27].

2.2. Geometry of the inclusions. Our homogenization results apply to a class of ergodic stationary closed sets $F = F(\omega)$ that we now describe. Let F be a closed set and $\text{CC}(F)$ its family of connected components, so that $F = \bigcup_{I \in \text{CC}(F)} I$. Each of this connected component is of course closed. We introduce the following geometric conditions:

- (G1) Regularity of the inclusions : there exists $\mathbf{d} > 0$, such that all $I \in \text{CC}(F)$ is the closure of a C^2 bounded domain satisfying an interior and exterior ball condition with uniform radius \mathbf{d} .
- (G2) Geometry of the *gaps* : there exists $\delta, a > 0$, such that for all $I \in \text{CC}(F)$, the set

$$I \cap \{d(x, F \setminus I) \leq \delta\}$$

has a finite number of connected components I_α , $\alpha = 1, \dots, N_I$, with $\sup_I N_I < +\infty$, and with at most one couple $(J, \beta) \neq (I, \alpha)$ such that $d(I_\alpha, J_\beta) \leq 2\delta$. Moreover, for an appropriate local system of cylindrical coordinates (r, θ, z) :

$$I_\alpha \subset \{z \geq d(I_\alpha, I_\beta)/2 + ar^2\}, \quad I_\beta \subset \{z \leq -d(I_\alpha, I_\beta)/2 - ar^2\}, \tag{7}$$

that is I_α and I_β are separated by paraboloids with uniform curvature.

Remark 1. By 7, there is a unique couple of points $(x_{I,\alpha}, x_{J,\beta}) \in I_\alpha \times J_\beta$ such that

$$|x_{I,\alpha} - x_{J,\beta}| = d(I_\alpha, J_\beta).$$

See Figure 1.

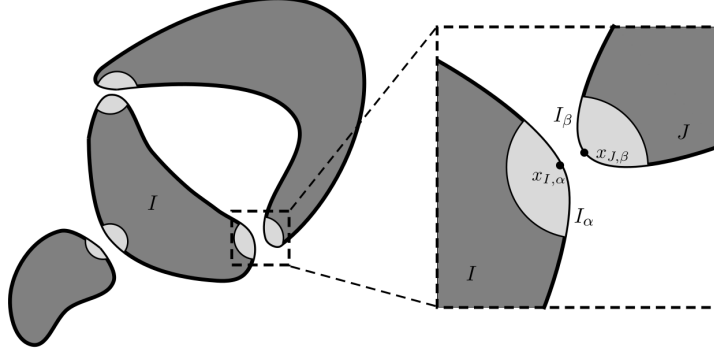


FIGURE 1. Geometry of the inclusions with a close-up on a gap.

Definition 2.1. (Admissible set of inclusions)

We say that an ergodic stationary closed set $F = F(\omega)$ is an admissible set of inclusions if it satisfies (G1)-(G2) almost surely, with \mathbf{d}, δ, a and $\sup_I N_I$ bounded by a deterministic constant.

Remark 2. At this stage, we do not assume anything on the diameter of the inclusions.

2.3. Multigraph of inclusions. We now associate to a closed set satisfying (G1)-(G2) an unoriented multigraph that we call multigraph of inclusions. Roughly, the nodes of the multigraph are the connected components of the closed set, and we link pieces of these connected components that are δ -close. See Figure 2 for an illustration. In a more formal way:

Definition 2.2. (Multigraph of inclusions)

Let F a closed set satisfying (G1)-(G2). For δ the constant in (G2), the δ -multigraph of inclusions associated to F , called multigraph of inclusions for brevity, is the unoriented multigraph $Gr(F) = (CC(F), Ed(F))$ with set of nodes $CC(F)$ and set of edges $Ed(F)$ made of elements of the form

$$e = [x_{I,\alpha}, x_{J,\beta}], \quad \text{with } I \neq J \in CC(F), \\ 1 \leq \alpha \leq N_I, \quad 1 \leq \beta \leq N_J, \quad d(I_\alpha, J_\beta) \leq \delta.$$

We say that an edge e as above connects I to J , and denote it $I \xleftrightarrow{e} J$. If there exists $e \in Ed(F)$ such that $I \xleftrightarrow{e} J$, we simply note $I \leftrightarrow J$. Note that several edges can connect the same pair of nodes, hence the multigraph structure. This corresponds to multiple gaps for a given pair of inclusions. To each of the edges, we associate a weight μ_e , by the formula

$$\mu_e := |\ln |e|| = |\ln d(I_\alpha, J_\beta)| \quad \text{for } e = [x_{I,\alpha}, x_{J,\beta}], \quad |e| = |x_{J,\beta} - x_{I,\alpha}|. \quad (8)$$

As explained in the introduction, there is a strong analogy between our multigraph of inclusions and the network approximation of [7].

For later use, we further define

Definition 2.3. (Cluster of inclusions)

Let F satisfying (G1)-(G2). A *cluster* of F is a union of all the inclusions that are the nodes of a connected component of the δ -multigraph $Gr(F)$ (not to be

confused with a connected component of F itself, which corresponds to a single inclusion/node).

Remark 3. For F a set of inclusions, and $y \in \mathbb{R}^3$, we shall denote:

- $I_{y,F} \in \text{CC}(F)$ the inclusion containing y , with the convention that $I_{y,F} = \emptyset$ if $y \notin F$.
- $C_{y,F} \in \text{CC}(F)$ the cluster containing y , with the convention that $C_{y,F} = \emptyset$ if $y \notin F$.

In the case $F = F(\omega)$ is stationary, one has clearly

$$I_{y,F(\omega)} = I_{0,F(\tau_y(\omega))}, \quad C_{y,F(\omega)} = C_{0,F(\tau_y(\omega))}.$$

This implies that $(y, \omega) \mapsto \text{diam}(I_{y,F(\omega)})$, $(y, \omega) \mapsto \text{diam}(C_{y,F(\omega)})$ and $(y, \omega) \mapsto \#(C_{y,F(\omega)})$ are stationary random fields. We will apply the ergodic theorem several times to these fields.

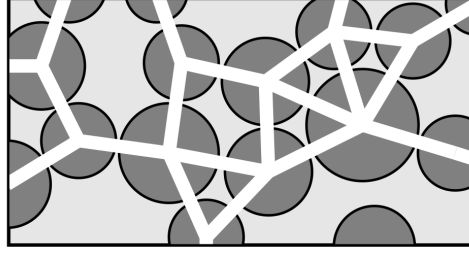


FIGURE 2. Spherical set up. The graph obtained with the whites lines is isomorphic to the multigraph of inclusions.

2.4. Short of inclusions. A last notion we need to explain before stating our main results is the notion of short of inclusions. It is directly inspired from the study of electrical networks and their associated multigraphs:

Definition 2.4. (Short of a multigraph)

Let $G = (V, E)$ a multigraph. Let $I, J \in V$. The short of G at $\{I, J\}$ is the multigraph obtained by identifying nodes I and J (and suppressing all edges joining I and J). More generally, given a set of pairs of nodes S , the short of G at S is the multigraph obtained from G by identifying all pairs in S (and suppressing all edges joining these nodes):

We say that G' is a short of G if there exists a set S of pairs of nodes such that G' is the short of G at S .

For electrical circuits, it is well-known that nodes with the same potential can be shorted, without changing the values of the currents through the remaining edges. This is a useful fact, as one can short the multigraph of an electrical circuit to simplify calculations. In other words, shorting two nodes is the same as imposing the same potential on each of the node, which can be interpreted as having zero resistance between those nodes. We refer to [12, chapter 2] for more.

At the level of the inclusions, an analogue of a short between two nodes consists in bridging the gap between two inclusions, so as to obtain one single connected component out of the two, with a single potential. We introduce the following

Definition 2.5. (Short of inclusions)

Let F a closed set satisfying (G1)-(G2) and $Gr(F)$ its multigraph of inclusions, see Definition 2.2. We say that a closed set $F' \supset F$ is a short of F if

- F' satisfies (G1)-(G2) (with some constants $\mathbf{d}', \delta', a', N'_{I'}$ instead of $\mathbf{d}, \delta, a, N_I$)
- $\text{Ed}(F') \subset \text{Ed}(F)$
- there exists $\eta > 0$, such that for all $e \in \text{Ed}(F) \setminus \text{Ed}(F')$, one has

$$\{d(x, e) < \eta\} \subset F' \subset F \cup \bigcup_{e \in \text{Ed}(F) \setminus \text{Ed}(F')} \{d(x, e) < 2\eta\}.$$

Furthermore, if F is an admissible set of inclusions, see Definition 2.1, F' is an admissible short of F if it is itself an admissible set, and a short of F almost surely for some deterministic η .

As explained above, a short of F is obtained by filling some gaps of F . See Figure 3. If F' is a short of F , $Gr(F')$ is isomorphic to a short of $Gr(F)$.

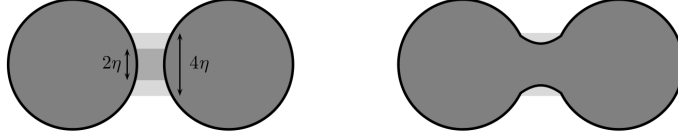


FIGURE 3. Two inclusions configuration, shorted on the right.

2.5. Homogenization results. We now turn to the homogenization problem described roughly in the introduction. For the rest of this section, we assume that $F = F(\omega) = \bigcup_{I \in \text{CC}(F(\omega))} I(\omega)$ is an admissible set of inclusions, see Definition 2.1. Given a parameter $\varepsilon > 0$, $\delta_0 > 0$ and a bounded domain U , we denote

$$I^\varepsilon := \varepsilon I \quad \forall I \in \text{CC}(F), \quad F^\varepsilon := \bigcup_{\substack{I^\varepsilon \subset U, \\ d(I^\varepsilon, \partial U) > \varepsilon \delta_0}} I^\varepsilon. \quad (9)$$

Let $f \in L^{6/5}(U)$, and $u^\varepsilon = u^\varepsilon(\omega) \in H_0^1(U)$ the solution of 3. Existence and uniqueness of u^ε is standard, and comes with the estimate

$$\|\nabla u^\varepsilon\|_{L^2(U)} \leq C \|f\|_{L^{6/5}(U)}.$$

In particular, u^ε has a subsequence that weakly converges in $H_0^1(U)$. The point is to identify the limit u^0 , and to determine under which conditions it satisfies a system like 2, where $\lambda = \mathbb{E} 1_F$ is the average density of the set of inclusions.

As mentioned in the introduction, Zhikov has tackled the homogenization problem, for a collection of random spheres of unit radius, under a logarithmic moment bound on the minimal distance between the spheres. The result extends easily to admissible sets of inclusions, under a uniform bound on their diameter. It can further be expressed in terms of multigraphs of inclusions. Namely, for all $N > 0$, let

$$Q_N := (-N, N)^3, \quad F_N := \bigcup_{\substack{I \in \text{CC}(F), \\ I \subset Q_N}} I. \quad (10)$$

Then the homogenization theorem of Zhikov reads

Theorem 2.6. (Zhikov, [27]) *Let F an admissible set of inclusions satisfying almost surely: $\sup_{I \in \text{CC}(F)} \text{diam}(I) < +\infty$, and such that*

$$\limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \sum_{e \in \text{Ed}(F_N)} \mu_e < +\infty. \quad (11)$$

Then, almost surely, the whole sequence u^ε converges weakly in $H_0^1(U)$ to the solution u^0 of a system of type 2.

Remark 4. As pointed out to us by A. Gloria, it can be seen from the ergodic theorem that: $\sup_{I \in \text{CC}(F)} \text{diam}(I) < +\infty$ a.s. $\Rightarrow \sup_{I \in \text{CC}(F)} \text{diam}(I) \leq D$ a.s for some deterministic D .

As usual in stochastic homogenization, the effective conductivity matrix A^0 can be expressed in terms of a variational problem in probability. Following [27, chapter 8], we first introduce the effective resistance matrix B^0 : it is the symmetric matrix defined by

$$\forall \xi \in \mathbb{R}^3, \quad B^0 \xi \cdot \xi := \inf_z \mathbb{E} \int_{Q_1 \setminus F} |\xi + z|^2,$$

where the infimum is taken over the set of vector fields $z = z(y, \omega) \in L^2(\Omega, L_{loc}^2(\mathbb{R}^3))$ satisfying

- i) z solenoidal, that is $\text{div}_y z = 0$
- ii) z stationary, that is $z(y + y', \omega) = z(y, \tau_{y'}(\omega))$
- iii) $\mathbb{E} z = 0$.

Then, part of the proof of Theorem 2.6 is to show that B^0 is invertible, and that the effective conductivity matrix is given by

$$A^0 = (B^0)^{-1}. \quad (12)$$

Our goal is to relax the assumption in the previous theorem, notably to identify configurations for which no condition on the minimal distance is needed. Our criterion for homogenization will be expressed again through multigraphs of inclusions. A key role will be played by the following discrete energy functional: for F a closed set satisfying (G1)-(G2), and two families $\{u_I\}$, indexed by $I \in \text{CC}(F)$, and $\{b_{IJe}\}$ indexed by triplets with $I, J \in \text{CC}(F)$, and $e \in \text{Ed}(F)$ s.t. $I \xleftrightarrow{e} J$, we introduce the energy functional

$$\begin{aligned} & \mathcal{E}(F, \{u_I\}, \{b_{IJe}\}) \\ &= \sum_{I, J \in \text{CC}(F)} \sum_{\substack{e \in \text{Ed}(F), \\ I \xleftrightarrow{e} J}} \mu_e |b_{IJe} - b_{JIe} + u_I - u_J|^2 + \sum_{I \in \text{CC}(F)} |I| |u_I|^2. \end{aligned} \quad (13)$$

We start with the definition of the homogenized matrix. We remind the notation $I_{0,F}$ from Remark 3. We denote x_S the center of mass of a subset S of \mathbb{R}^3 .

Proposition 1. (Existence of the homogenized matrix)

Let F an admissible set of inclusions. If $\mathbb{E} \text{diam}(I_{0,F})^2 < +\infty$ and if almost surely

$$\limsup_{N \rightarrow +\infty} \inf_{\{u_I\}} \frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{b_{IJe} = \xi \cdot x_I\}) < +\infty \quad (\text{H1})$$

where \mathcal{E} is given in 13, and F_N in 10, then the matrix A^0 introduced in 12 is well-defined.

Compared to the assumptions in Theorem 2.6, those of Proposition 1 are better in two regards.

a) The uniform bound on the diameter of the inclusions (see Remark 4) is replaced by a moment bound. This possibility of considering inclusions of arbitrary large size, interesting in its own, will turn very useful when combined to our homogenization result involving shorts, cf. Theorem 2.7.

b) As mentioned in the introduction, H1 is weaker than 11. Indeed, for inclusions satisfying $\sup_{I \in \text{CC}(F)} \text{diam}(I) < +\infty$, the latter clearly implies

$$\limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \mathcal{E}(F_N, \{0\}, \{\xi \cdot x_I\}) < +\infty.$$

In other words, Zhikov's condition 11 corresponds to the simplest choice $u_I = 0$ for every inclusions I , while H1 corresponds to an optimization over all possible constants u_I . This is not just a technical improvement: as an interesting application, we will prove in Section 5 that H1 is verified by sets F satisfying the moment bound

$$\mathbb{E} \text{diam}(C_{0,F})^2 < +\infty$$

where $C_{0,F}$ is the cluster of F containing 0, cf. Remark 3. Further discussion of H1 will be provided in Section 5, notably its relation to the graph Laplacian or to the subadditive ergodic theorem.

We now turn to the homogenization of 3. It is an interesting open problem to be able to perform homogenization under the mere assumption H1. We must here strengthen it a little.

Theorem 2.7. (Homogenization of stiff inclusions)

Let F an admissible set of inclusions. We assume that there exists an admissible short F' of F and some $s \in (3, 6)$ ($s \in (3, \infty)$ in the case $F' = F$) such that, almost surely:

$$\limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \sup_{\{b_{IJe}\}} \inf_{\{u_I\}} \mathcal{E}(F'_N, \{u_I\}, \{b_{IJe}\}) / \|(b_{IJe})\|_s^2 < +\infty \quad (\text{H2})$$

where

$$\|(b_{IJe})\|_s = \left(\frac{1}{|Q_N|} \sum_{I, J \in \text{CC}(F'_N)} \sum_{\substack{e \in \text{Ed}(F'_N), \\ I \overset{e}{\leftrightarrow} J}} |b_{IJe}|^s \right)^{\frac{1}{s}}.$$

Then, there exists $p = p(s)$ such that under the additional condition:

$$\mathbb{E} \text{diam}(I_{0,F'})^p < +\infty, \quad (14)$$

the solution u^ε of 3 converges a.s. weakly in $H_0^1(U)$ to the solution u^0 of 2, with A^0 defined by 12.

Here are a few remarks, to be complemented in Section 5.

i) Of course, in previous statements, it is enough that all assumptions involving the δ -multigraph of F hold for *some* $\delta > 0$. In practice, one should take δ small, to have a reduced number of edges in the multigraph.

ii) In practice, the following reformulation of H2 will be used:

One can find almost surely an $M = M(\omega) > 0$ satisfying: for all $N > 0$, for any family $\{b_{IJe}\}$ indexed by $I, J, e \in \text{CC}(F_N)^2 \times \text{Ed}(F_N)$ with $I \overset{e}{\leftrightarrow} J$, there exists a

family $\{u_I\}_{I \in CC(F_N)}$ such that

$$\mathcal{E}(F_N, \{u_I\}, \{b_{IJe}\}) \leq M|Q_N| \left(\frac{1}{|Q_N|} \sum_{I, J \in CC(F_N)} \sum_{\substack{e \in \text{Ed}(F_N), \\ I \leftrightarrow J}} |b_{IJe}|^s \right)^{2/s}.$$

iii) Let F an admissible set of inclusions, F' an admissible short of F . If

$$\mathbb{E} \text{diam}(I_{0,F'})^s < +\infty$$

then **H2** implies **H1**. See Lemma 5.5. In particular, by Proposition 1, the matrix A^0 is well-defined.

iv) **H2** is implied by the following logarithmic moment bound, see Lemma 5.4:

$$\limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \sum_{e \in \text{Ed}(F_N)} \mu_e^k < +\infty, \quad k = \left(\frac{s}{2}\right)' = \frac{s}{s-2}. \quad (15)$$

In the case where $F' = F$, s can be taken arbitrarily large in Theorem 2.7, so that this sufficient condition is almost the standard one (one can take any $k > 1$). For a general short F' , we are limited to $s < 6$ for technical reasons, hence to $k > \frac{3}{2}$.

iv) A corollary of Theorem 2.7, to be established in Section 5 and illustrated in Figure 4, is the following:

Corollary 1. *Let F an admissible set of inclusions with a.s. $\sup_{I \in CC(F)} \text{diam}(I) < +\infty$. Assume that $\text{Gr}(F)$ is cycle-free, and that*

$$\mathbb{E} (\sharp C_{0,F})^p < +\infty, \quad \text{for some } p > 2,$$

where $C_{0,F}$ is the cluster of F containing 0, cf. Remark 3. Then, **H2** is satisfied with $F' = F$ and $s = \frac{2p}{p-2}$, so that homogenization holds.

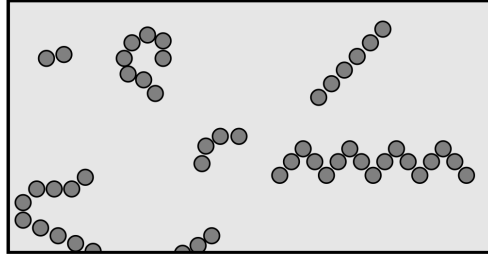


FIGURE 4. Cycle-free configuration set up.

v) Another important corollary of Theorem 2.7 is:

Corollary 2. *Let F an admissible set of inclusions. Let $s \in (3, 6)$ and $p = p(s)$ as in Theorem 2.7. If*

$$\mathbb{E} \text{diam}(C_{0,F})^p < +\infty$$

then homogenization holds.

Indeed, if we simply bridge all pairs of nodes I, J with $I \leftrightarrow J$, we obtain in this way an admissible short F' of F with $\mathbb{E} \text{diam}(I_{0,F'})^p < +\infty$, and with $\text{Ed}(F') = \emptyset$, hence trivially satisfying **H2** and 14. See Figure 5 for an illustration.

Let us point out that the sharper bound

$$\mathbb{E} \exp(\text{diam}(C_{0,F})^\gamma) < +\infty, \quad \text{for one } \gamma > 1$$

has been shown to hold for several examples of admissible sets of inclusions, satisfying strong mixing assumptions, below the critical percolation threshold. We refer to the recent article [18] for details, where, moreover, the homogenization of both Laplace and Stokes equation is obtained in this special case by a different method.

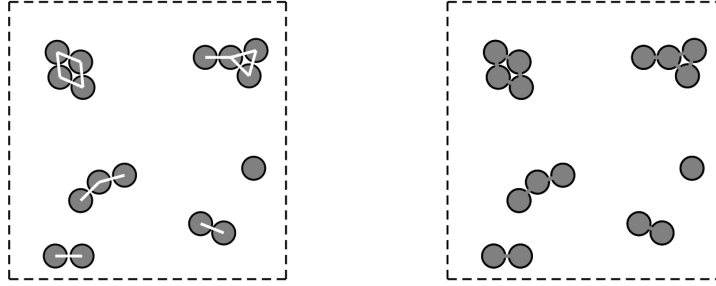


FIGURE 5. On the left, all clusters are far away from the others. On the right, groups of inclusions joined by a grey line form a short F' that verifies $\text{Ed}(F') = \emptyset$.

vi) The local blow-up of the energy between two close inclusions I and J linked by an edge e depends on the geometry of the narrow gap between the two inclusions. The assumption (G2) ensures that the gap is well separated by paraboloids of uniform curvature a , which leads to a blow-up of the energy of order $|\ln |e||$. More precisely, one can show (see Lemma A.1), that the explosion is of order $\frac{|\ln |e||}{a}$, which means that flatter boundaries lead to a greater explosion. One could change the expression of the weight μ_e in 8 and track the dependence on the curvature a . Beyond that, let us remark that the analysis of the local energy explosion as a function of gap geometry was extensively studied in [24] and could be used to relax our geometrical assumption (G2) to include more degenerate curvatures.

vii) In this work, the dimension $n = 3$ case was considered as it is the most pertinent for applications. A similar analysis could be performed in other dimensions. In dimension $n > 3$, a straightforward adaptation of Lemma A.1 shows that there is no blow-up of the energy when inclusions are close. In that case, a simple deterministic bound on the inclusions diameter is enough to perform the whole homogenization process. In dimension $n = 2$, the explosion between two close inclusions in Lemma A.1 is of order $(\sqrt{\nu})^{-1}$. Replacing the weight $\mu_e = |\ln |e||$ by $(\sqrt{|e|})^{-1}$ in 8, the Proposition 1 still holds and the homogenization theorem 2.7 remains correct under the same assumption H2 but with exponent $s > 4$ instead of $s > 3$.

3. Preliminary extension theorems. In this whole section, F is an admissible set of inclusions, U is a bounded domain, and F^ε is defined as in 9. We will show how assumptions H1 or H2 allow to construct suitable extensions of H^1 fields given in the inclusions, that is in F^ε , resp. divergence-free L^2 fields given outside the inclusions, that is in $U \setminus F^\varepsilon$. By suitable, we mean that the extension operator will be bounded uniformly in ε . These extension results will be central to the homogenization process.

3.1. Extension outside the inclusions. The two main results of this paragraph are

Proposition 2. *Let $\xi \in \mathbb{R}^3$. Assume [H1](#). If $\mathbb{E} \text{diam}(I_{0,F})^2 < +\infty$, one can find almost surely $C > 0$ independent of ε and a field $\phi^\varepsilon \in H_0^1(U)$ with*

$$\nabla \phi^\varepsilon = \xi \quad \text{in } F^\varepsilon, \quad \varepsilon^{-1} \|\phi^\varepsilon\|_{L^2(U)} + \|\nabla \phi^\varepsilon\|_{L^2(U)} \leq C|\xi|.$$

Proposition 3. *Assume [H2](#) with $F' = F$ and $s > 3$. For any $\tilde{s} > s$, there exists $p = p(\tilde{s})$ such that if*

$$\mathbb{E} \text{diam}(I_{0,F})^p < +\infty,$$

one can find almost surely $C > 0$ independent of ε satisfying: for all $\varphi^\varepsilon \in W^{1,\tilde{s}}(F^\varepsilon)$, there exists a field $\phi^\varepsilon \in H_0^1(U)$ with

$$\nabla \phi^\varepsilon = \nabla \varphi^\varepsilon \quad \text{in } F^\varepsilon, \quad \varepsilon^{-1} \|\phi^\varepsilon\|_{L^2(U)} + \|\nabla \phi^\varepsilon\|_{L^2(U)} \leq C \|\nabla \varphi^\varepsilon\|_{L^{\tilde{s}}(F^\varepsilon)}.$$

We will focus on the proof of the latter proposition, as the former requires only minor modifications.

Proof. In all the proof, the realization ω is fixed and does not show up in the notations. We will use the following Poincaré-Wirtinger inequality: there exists $r > 0$ such that for all $t \in [1, +\infty)$, one can find $C > 0$ satisfying:

$$\forall I \in \text{CC}(F), \quad \forall u \in W^{1,t}(I), \quad \|u - (u)_I\|_{L^t(I)} \leq C \text{diam}(I)^r \|\nabla u\|_{L^t(I)}, \quad (16)$$

with $(u)_I = \frac{1}{|I|} \int_I u$. A main point here is that the constant is bounded by a power of the diameter. This inequality is known to be true with $r = 1$ for convex domains, and more generally for star-shaped domains, see [[23](#), chapter 12]. We indicate in [Appendix B](#) how to show that this inequality holds with $r = 12$. Obviously, in the special case where $u(x) = \xi \cdot x$, relevant to [Proposition 2](#), the exponent $r = 1$ is enough. Let

$$N := \varepsilon^{-1}, \quad \tilde{F}_N := NF^\varepsilon = \bigcup_{\substack{I \subset F \cap NU, \\ d(I, N\partial U) > \delta_0}} I$$

the set of all the connected components of F included in NU and δ_0 far from the boundary. By a scaling argument, it is enough to prove that there exists a constant $C > 0$ independent of N such that for any $\varphi \in W^{1,\tilde{s}}(\tilde{F}_N)$, one can find $\phi_N \in H_0^1(NU)$ with

$$\nabla \phi_N = \nabla \varphi \quad \text{in } \tilde{F}_N, \quad \|\phi_N\|_{H^1(NU)} \leq CN^{\frac{3\tilde{s}-6}{2\tilde{s}}} \|\nabla \varphi\|_{L^{\tilde{s}}(\tilde{F}_N)}. \quad (17)$$

Let $\varphi \in W^{1,\tilde{s}}(\tilde{F}_N)$ and denote $\varphi_I := \varphi|_I$ for any $I \in \text{CC}(\tilde{F}_N)$. The proof is split into two main steps:

1. Given an arbitrary family $(u_I)_{I \in \text{CC}(\tilde{F}_N)}$, and any $e = [x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(\tilde{F}_N)$, we build a local extension ϕ_e , defined in a neighborhood V_e of e that contains I_α and J_β , such that

$$\phi_e|_I = \varphi_I - \oint_I \varphi_I + u_I, \quad \phi_e|_J = \varphi_J - \oint_J \varphi_J + u_J \quad (18)$$

and

$$\begin{aligned} & \|\phi_e\|_{H^1(V_e)}^2 \\ & \leq C\mu_e|\varphi_I(x_{I,\alpha}) - \oint_I \varphi_I - \varphi_J(x_{J,\beta}) + \oint_J \varphi_J + u_I - u_J|^2 + C(|u_I|^2 + |u_J|^2) \\ & \quad + C(\text{diam}(I)^{2r}\|\nabla\varphi_I\|_{L^2(I)}^2 + \text{diam}(J)^{2r}\|\nabla\varphi_J\|_{L^2(J)}^2) \\ & \quad + C(\text{diam}(I)^{2r}\|\nabla\varphi_I\|_{L^s(I)}^2 + \text{diam}(J)^{2r}\|\nabla\varphi_J\|_{L^s(J)}^2) \end{aligned} \quad (19)$$

2. We show that for a proper choice of the family (u_I) and with the help of the previous local extensions, there exists a global extension ϕ^N satisfying 17.

Step 1. Let $e = [x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(\tilde{F}_N)$. Clearly, to show the existence of ϕ_e satisfying 18-19, one can restrict to the case where

$$\oint_I \varphi_I = 0 \quad \forall I, \quad \text{so that, for all } t \geq 2, \quad \|\varphi_I\|_{W^{1,t}(I)} \leq C \text{diam}(I)^r \|\nabla\varphi_I\|_{L^t(I)}$$

thanks to 16. Let now P_α and P_β two paraboloids that enclose I_α and J_β . By assumption (G2), for appropriate local cylindrical coordinates (r, θ, z) centered at $\frac{x_{I,\alpha} + x_{J,\beta}}{2}$ and of axis along e , we can write

$$P_\alpha = \left\{ z \geq \frac{|e|}{2} + ar^2 \right\}, \quad P_\beta = \left\{ z \leq -\frac{|e|}{2} - ar^2 \right\}.$$

By Lemma A.1, given $d > 0$ there exists a function $w_e \in H^1(P_\alpha \cup P_\beta \cup F_e(d))$ where in the same system of local coordinates:

$$F_e(d) := \left\{ r^2 \leq d^2, \quad ar^2 + \frac{|e|}{2} \geq z \geq -\frac{|e|}{2} - ar^2 \right\}$$

and where

$$\begin{aligned} 0 & \leq w_e \leq 1, \quad w_e|_{P_\alpha} = 1, \quad w_e|_{P_\beta} = 0, \\ \int_{F_e(d)} |\nabla w_e|^2 dx & \leq C\mu_e, \quad \int_{F_e(d)} |\nabla w_e|^2 |x - x_{I,\alpha}|^{2\gamma} dx \leq C_\gamma \quad \forall \gamma > 0. \end{aligned}$$

We take $d > 0$ large enough so that $I_\alpha \cup J_\beta$ lies in the interior of $P_\alpha \cup P_\beta \cup F_e(d)$.

Now, using the Stein extension operator from $W^{1,t}(I)$ to $W^{1,t}(\mathbb{R}^3)$, see [31, Chapter 6], one can find an extension $\tilde{\varphi}_I$ of φ_I on \mathbb{R}^3 such that for all $t \in [2, \tilde{s}]$,

$$\|\tilde{\varphi}_I\|_{W^{1,t}(\mathbb{R}^3)} \leq c \|\varphi_I\|_{W^{1,t}(I)} \leq c' \text{diam}(I)^r \|\nabla\varphi_I\|_{L^t(I)} \quad \forall I \in \text{CC}(F)$$

We remark that the analysis in [31, Chapter 6] provides constants c and c' that are independent of the size of the inclusions. This is however not a crucial point here, as we could handle constants diverging polynomially in the diameter of the inclusions. Eventually, we define

$$\phi_e := w_e(\tilde{\varphi}_I + u_I) + (1 - w_e)(\tilde{\varphi}_J + u_J)$$

that we consider as a function of $W^{1,s}(V_e)$, with

$$V_e := P_\alpha \cup P_\beta \cup F_e(d) \cap \{|z| \leq R\} \quad (20)$$

where R is large enough so that $I_\alpha \cup J_\beta$ lies in the interior of V_e . The intersection with $\{|z| \leq R\}$ is just here to make V_e bounded. Clearly, 18 is satisfied. We then compute

$$\int_{F_e(d)} |\nabla \phi_e|^2 dx \leq \int_{F_e(d)} |\nabla w_e(\tilde{\varphi}_I + u_I - \tilde{\varphi}_J - u_J) + w_e \nabla \tilde{\varphi}_I + (1 - w_e) \nabla \tilde{\varphi}_J|^2 dx.$$

Hence,

$$\begin{aligned}
& \int_{F_e(d)} |\nabla \phi_e|^2 dx \\
& \leq C \left(\int_{F_e(d)} |\nabla w_e|^2 |\tilde{\varphi}_I + u_I - \tilde{\varphi}_J - u_J|^2 dx + \int_{F_e(d)} |\nabla \tilde{\varphi}_I|^2 + |\nabla \tilde{\varphi}_J|^2 dx \right) \\
& \leq C' \left(\int_{F_e(d)} |\nabla w_e|^2 |\varphi_I(x_{I,\alpha}) + u_I - \varphi_J(x_{J,\beta}) - u_J|^2 dx \right. \\
& \quad + \int_{F_e(d)} |\nabla w_e|^2 |\tilde{\varphi}_I - \varphi_I(x_{I,\alpha}) + \varphi_J(x_{J,\beta}) - \tilde{\varphi}_J|^2 dx \\
& \quad \left. + \text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^2(I)}^2 + \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^2(J)}^2 \right).
\end{aligned}$$

Thanks to Morrey's inequality $\dot{W}^{1,s}(\mathbb{R}^3) \hookrightarrow C^{1,\gamma}(\mathbb{R}^3)$, $\gamma = 1 - \frac{3}{s} \in (0, 1)$, one can write

$$|\tilde{\varphi}_I(x) - \varphi_I(x_{I,\alpha})| \leq C|x - x_{I,\alpha}|^\gamma \|\nabla \tilde{\varphi}_I\|_{L^s(\mathbb{R}^3)} \leq C'|x - x_{I,\alpha}|^\gamma \text{diam}(I)^r \|\nabla \varphi_I\|_{L^s(I)}$$

for any x in the gap $F_e(d)$. Thus, we have the following

$$\begin{aligned}
& \int_{F_e(d)} |\nabla w_e|^2 |\tilde{\varphi}_I - \varphi_I(x_{I,\alpha}) + \varphi_J(x_{J,\beta}) - \tilde{\varphi}_J|^2 dx \\
& \leq C \text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^s(I)}^2 \int_{F_e(d)} |\nabla w_e|^2 |x - x_{I,\alpha}|^{2\gamma} dx \\
& \quad + C \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^s(J)}^2 \int_{F_e(d)} |\nabla w_e|^2 |x - x_{J,\beta}|^{2\gamma} dx \\
& \leq C' (\text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^s(I)}^2 + \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^s(J)}^2).
\end{aligned}$$

Finally, combining the previous inequalities entails

$$\begin{aligned}
\int_{F_e(d)} |\nabla \phi_e|^2 dx & \leq C \left(\mu_e |\varphi_I(x_{I,\alpha}) + u_I - \varphi_J(x_{J,\beta}) - u_J|^2 \right. \\
& \quad + \text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^2(I)}^2 + \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^2(J)}^2 \\
& \quad \left. + \text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^s(I)}^2 + \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^s(J)}^2 \right).
\end{aligned}$$

It is even simpler to show that

$$\begin{aligned}
& \int_{V_e \setminus F_e(d)} |\nabla \phi_e|^2 dx + \int_{V_e} |\phi_e|^2 dx \\
& \leq C \left(|u_I|^2 + |u_J|^2 + \text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^2(I)}^2 + \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^2(J)}^2 \right)
\end{aligned}$$

which concludes the derivation of 19, and the first step.

Step 2. We now explain how to construct a global extension $\phi_N \in H_0^1(NU)$ with

$$\phi_N|_I = \varphi_I - \oint_I \varphi_I + u_I, \quad \forall I \in \text{CC}(\tilde{F}_N). \quad (21)$$

For all $e = [x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(\tilde{F}_N)$, we first introduce a function $\chi_e \in C_c^\infty(\mathbb{R}^3)$, with values in $[0, 1]$, satisfying

- $\chi_e = 1$ in a neighborhood of $I_\alpha \cup J_\beta$
- $\chi_e = 0$ in a neighborhood of $\mathbb{R}^3 \setminus V_e$, where V_e was introduced in 20

- $\chi_e = 0$ in a $\delta_0/2$ neighborhood of $N\partial U$
- the supports of χ_e and $\chi_{e'}$ are disjoint for all $e \neq e' \in \text{Ed}(\tilde{F}_N)$
- $|\nabla \chi_e| \leq C$ for some constant C that is uniform in e and N .

Existence of such functions is easily deduced from our geometric assumptions (G1)-(G2). We now set

$$\phi_{N,1} := \sum_{e \in \text{Ed}(\tilde{F}_N)} \chi_e \phi_e.$$

By our choice of functions χ_e and by property 18, one has $\phi_{N,1} \in H_0^1(NU)$, and

$$\phi_{N,1}|_{I_\alpha} = \varphi_I - \int_I \varphi_I + u_I, \quad \forall I \in \text{CC}(\tilde{F}_N), \quad \forall \alpha = 1, \dots, N_I. \quad (22)$$

Moreover, by estimate 19,

$$\begin{aligned} & \|\phi_{N,1}\|_{H^1(NU)}^2 \\ & \leq C \sum_{e=[x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(\tilde{F}_N)} \left(\mu_e \left| \varphi_I(x_{I,\alpha}) - \int_I \varphi_I - \varphi_J(x_{J,\beta}) + \int_J \varphi_J + u_I - u_J \right|^2 \right. \\ & \quad + |u_I|^2 + |u_J|^2 \text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^2(I)}^2 + \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^2(J)}^2 \\ & \quad \left. + \text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^s(I)}^2 + \text{diam}(J)^{2r} \|\nabla \varphi_J\|_{L^s(J)}^2 \right) \end{aligned}$$

It remains to construct some $\phi_{N,2}$ satisfying

$$\phi_{N,2}|_I = \psi_I, \quad \psi_I := \varphi_I - \int_I \varphi_I + u_I - \phi_{N,1}, \quad \forall I \in \text{CC}(\tilde{F}_N), \quad (23)$$

in order for $\phi_N := \phi_{N,1} + \phi_{N,2}$ to satisfy 21. By 22, ψ_I is zero on I_α for all $I \in \text{CC}(\tilde{F}_N)$, for all $1 \leq \alpha \leq N_I$. Thanks to this property and (G1)-(G2), one can find a constant $\nu > 0$ independent of I or N such that for all I , there exists $\tilde{\psi}_I \in H^1(\mathbb{R}^3)$ satisfying

$$\tilde{\psi}_I|_I = \psi_I, \quad \|\tilde{\psi}_I\|_{H^1(\mathbb{R}^3)} \leq C \|\psi_I\|_{H^1(I)}$$

and for all (J, β) with $e = [x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(\tilde{F}_N)$, for the same local coordinates around the edge e as seen before

$$\tilde{\psi}_I = 0 \quad \text{on } J_{\beta,\nu} := J_\beta \cap \{|z| \leq \nu\}.$$

Now, for each $I \in \text{CC}(\tilde{F}_N)$, we introduce $\nu' > 0$ and a smooth function χ_I which is 1 in a ν' -neighborhood of I and 0 outside a $2\nu'$ -neighborhood of I . Thanks to our geometric assumptions, by taking ν' small enough (but independent of I and N), we can ensure that for all $J \neq I$ connected by an edge $e \in \text{Ed}(\tilde{F}_N)$,

$$\text{Supp}(\chi_J) \cap I \subset \cup_{1 \leq \alpha \leq N_I} I_{\alpha,\nu}.$$

We finally set

$$\phi_{N,2} = \sum_{I \in \text{CC}(\tilde{F}_N)} \chi_I \tilde{\psi}_I.$$

The keypoint in the definition of $\phi_{N,2}$ is that for a given I , and for any $J \neq I$, the term $\chi_J \tilde{\psi}_J$ is zero on I : indeed, for all $x \in I$, either $\chi_J(x) = 0$, or $\chi_J(x) \neq 0$

which implies that $x \in \cup_{\alpha} I_{\alpha,\nu}$ so that $\tilde{\psi}_J(x) = 0$. Hence, [23](#) is satisfied, as expected. Moreover, it is easily seen that

$$\|\phi_{N,2}\|_{H^1(NU)}^2 \leq C \left(\|\phi_{N,1}\|_{H^1(NU)}^2 + \sum_{I \in \text{CC}(\tilde{F}_N)} \left(\text{diam}(I)^{2r} \|\nabla \varphi_I\|_{L^2(I)}^{2r} + |I| |u_I|^2 \right) \right)$$

so that eventually

$$\begin{aligned} & \|\phi_N\|_{H^1(NU)}^2 \\ & \leq C \sum_{e=[x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(\tilde{F}_N)} \left(\mu_e |\varphi_I(x_{I,\alpha}) - \int_I \varphi_I - \varphi_J(x_{J,\beta}) + \int_J \varphi_J + u_I - u_J|^2 \right. \\ & \quad \left. + C \sum_{I \in \text{CC}(\tilde{F}_N)} |I| |u_I|^2 + \sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^{2r} (\|\nabla \varphi_I\|_{L^2(I)}^2 + \|\nabla \varphi_I\|_{L^s(I)}^2) \right). \end{aligned}$$

The final step of the proof is to show that for a proper choice of the family $(u_I)_{I \in \text{CC}(\tilde{F}_N)}$, ϕ_N satisfies the bound [17](#). This is done using assumption [H2](#). Namely, we denote $b_{IJe} := \varphi_I(x_{I,\alpha}) - \int_I \varphi_I$, where $e = [x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(\tilde{F}_N)$. Remembering Definition [13](#), one has clearly

$$\begin{aligned} & \|\phi_N\|_{H^1(NU)}^2 \\ & \leq C \mathcal{E}(\tilde{F}_N, \{u_I\}, \{b_{IJe}\}) + C \sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^{2r} (\|\nabla \varphi_I\|_{L^2(I)}^2 + \|\nabla \varphi_I\|_{L^s(I)}^2) \\ & \leq C \mathcal{E}(\tilde{F}_N, \{u_I\}, \{b_{IJe}\}) + C \sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^{2r} (|I|^{\frac{\bar{s}-2}{\bar{s}}} + |I|^{\frac{2(\bar{s}-s)}{s\bar{s}}}) \|\nabla \varphi_I\|_{L^{\bar{s}}(I)}^2 \\ & \leq C \mathcal{E}(\tilde{F}_N, \{u_I\}, \{b_{IJe}\}) + C' \left(\sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^{\frac{2r\bar{s}}{\bar{s}-2}} |I| \right)^{\frac{\bar{s}-2}{\bar{s}}} \|\nabla \varphi\|_{L^{\bar{s}}(\tilde{F}_N)}^2. \end{aligned}$$

Now, taking $N' = 2N \sup_{x \in U} |x|$, one has $NU \subset Q_{N'}$ and $|Q_{N'}| = C|Q_N|$ for a constant C independent of N . Furthermore, we can write $F_{N'} = \tilde{F}_N \cup G$, where the union is disjoint and $Gr(F_{N'})$ is deduced from $Gr(\tilde{F}_N)$ by the addition of nodes and edges. Using the property [\(35\)](#) proved in Section [5](#), we have

$$\mathcal{E}(\tilde{F}_N, \{u_I\}, \{b_{IJe}\}) \leq \mathcal{E}(F_{N'}, \{\bar{u}_I\}, \{\bar{b}_{IJe}\})$$

for any extensions $\{\bar{u}_I\}, \{\bar{b}_{IJe}\}$ of $\{u_I\}, \{b_{IJe}\}$. We make the choice $\bar{b}_{IJe} = 0$ if $e \in \text{Ed}(F_{N'}) \setminus \text{Ed}(\tilde{F}_N)$. Now, using property [H2](#) (in the form mentioned in Remark [i](#)) after Theorem [2.7](#)), there exists, almost surely, a family $\{\bar{u}_I\}_{I \in \text{CC}(F_{N'})}$ such that

$$\begin{aligned} \mathcal{E}(F_{N'}, \{\bar{u}_I\}, \{\bar{b}_{IJe}\}) & \leq M |Q_{N'}| \left(\frac{1}{|Q_{N'}|} \sum_{I, J \in \text{CC}(F_{N'})} \sum_{\substack{e \in \text{Ed}(F_{N'}) \\ I \xleftrightarrow{e} J}} |\bar{b}_{IJe}|^s \right)^{2/s} \\ & \leq M' |Q_N| \left(\frac{1}{|Q_N|} \sum_{I, J \in \text{CC}(\tilde{F}_N)} \sum_{\substack{e \in \text{Ed}(\tilde{F}_N) \\ I \xleftrightarrow{e} J}} |b_{IJe}|^s \right)^{2/s}. \end{aligned}$$

Using one last time the Morrey injection yields

$$\begin{aligned} |b_{IJe}|^s &= |\varphi_I(x_{I,\alpha}) - \varphi_I(x_I)|^s \\ &\leq |x_I - x_{I,\alpha}|^{s-3} \|\nabla \tilde{\varphi}_I\|_{L^s(\mathbb{R}^3)}^s \\ &\leq C \text{diam}(I)^{s-3+rs} \|\nabla \varphi_I\|_{L^s(I)}^s. \end{aligned}$$

Setting $u_I = \bar{u}_I$ for $I \in \text{CC}(\tilde{F}_N)$, we get, back to ϕ_N :

$$\begin{aligned} \|\phi_N\|_{H^1(NU)}^2 &\leq M' |Q_N| \left(\frac{1}{|Q_N|} \sum_{I,J \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^{s-3+rs} \|\nabla \varphi_I\|_{L^s(I)}^s \right)^{\frac{2}{s}} \\ &\quad + C' \left(\sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^{\frac{2r\tilde{s}}{\tilde{s}-2}} |I| \right)^{\frac{\tilde{s}-2}{\tilde{s}}} \|\nabla \varphi\|_{L^{\tilde{s}}(\tilde{F}_N)}^2 \\ &\leq C'' N^{\frac{3\tilde{s}-6}{\tilde{s}}} \left(\frac{1}{|Q_{N'}|} \sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^p |I| \right)^{\frac{\tilde{s}-2}{\tilde{s}}} \|\nabla \varphi\|_{L^{\tilde{s}}(\tilde{F}_N)}^2 \end{aligned}$$

where $p = \max((s-3+rs)\frac{\tilde{s}}{\tilde{s}-s}, \frac{2r\tilde{s}}{\tilde{s}-2})$. As

$$\frac{1}{|Q_{N'}|} \sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^p |I| \leq \frac{1}{|Q_{N'}|} \int_{Q_{N'}} \text{diam}(I_{y,F})^p dy$$

we find by the ergodic theorem that

$$\limsup_{N \rightarrow +\infty} \frac{1}{|Q_{N'}|} \sum_{I \in \text{CC}(\tilde{F}_N)} \text{diam}(I)^p |I| \leq \mathbb{E} \text{diam}(I_{0,F})^p < +\infty$$

which concludes the proof. \square

3.2. Extension inside the inclusions preserving divergence. Our proof of homogenization, based on the div-curl lemma, will require proper extensions of solenoidal vector fields, or of fields with given divergence, inside the inclusions. This is the purpose of

Proposition 4. *Assume that [H2](#) holds with $F' = F$ and $s > 3$. Let $f^\varepsilon \in L^{6/5}(U)$, $p^\varepsilon \in L^2(U \setminus F^\varepsilon)$ such that $\text{div } p^\varepsilon = f^\varepsilon$ in $U \setminus F^\varepsilon$, satisfying the following compatibility conditions:*

$$\int_{\partial I^\varepsilon} p^\varepsilon \cdot \nu = \int_{I^\varepsilon} f^\varepsilon, \quad \forall I^\varepsilon \in \text{CC}(F^\varepsilon).$$

Then, there exists a field $P^\varepsilon \in L^2(U)$ satisfying $P^\varepsilon|_{U \setminus F^\varepsilon} = p^\varepsilon$, $\text{div } P^\varepsilon = f^\varepsilon$ in U . Moreover, given any $t < s' = \frac{s}{s-1}$, there exists $p = p(t)$ such that under the additional hypothesis $\mathbb{E} \text{diam}(I_{0,F})^p < +\infty$, one can choose P^ε satisfying the uniform estimate

$$\|P^\varepsilon\|_{L^t(U)} \leq C_t \left(\|p^\varepsilon\|_{L^2(U \setminus F^\varepsilon)} + \|f^\varepsilon\|_{L^{6/5}(U)} \right).$$

Proof. Let $t < s'$. We introduce \tilde{s} such that $s < \tilde{s} < t'$. We also introduce the solution w of

$$\Delta w = f^\varepsilon \text{ on } U, \quad w|_{\partial U} = 0.$$

It satisfies the estimate

$$\|\nabla w\|_{L^q(U)} \leq C_q \|f^\varepsilon\|_{W^{-1,q}(U)} \leq C'_q \|f^\varepsilon\|_{L^{3q/(3+q)}(U)} \quad \forall q \in (1, 2].$$

Denoting $s^\varepsilon = p^\varepsilon - \nabla w$, it remains to find S^ε satisfying $S^\varepsilon|_{U \setminus F^\varepsilon} = s^\varepsilon$, $\operatorname{div} S^\varepsilon = 0$ in U , and

$$\|S^\varepsilon\|_{L^t(U)} \leq C_t \|s^\varepsilon\|_{L^2(U \setminus F^\varepsilon)}.$$

Then $P^\varepsilon := S^\varepsilon + \nabla w$ will meet all requirements. The idea is to search for S^ε in the form of a gradient in the inclusions. Strictly speaking, for a fixed realization, we introduce a field v^ε (depending on ω) defined on F^ε that verifies in each inclusion I^ε the Neumann problem

$$\begin{cases} \Delta v^\varepsilon = 0 & \text{in } \mathring{I}^\varepsilon, \\ \partial_\nu v^\varepsilon = s^\varepsilon \cdot \nu & \text{in } \partial I^\varepsilon. \end{cases}$$

This Neumann problem is well-posed thanks to the compatibility condition

$$\int_{\partial I^\varepsilon} s^\varepsilon \cdot \nu = \int_{\partial I^\varepsilon} p^\varepsilon \cdot \nu - \int_{\partial I^\varepsilon} \partial_\nu w = \int_{I^\varepsilon} f^\varepsilon - \int_{\mathring{I}^\varepsilon} \Delta w = 0, \quad \forall I^\varepsilon \in \operatorname{CC}(F^\varepsilon).$$

We then define the random field S^ε by $S^\varepsilon = s^\varepsilon$ in $U \setminus F^\varepsilon$ and $S^\varepsilon = \nabla v^\varepsilon$ in F^ε . It is divergence-free on U thanks to the continuity of its normal component through each ∂I^ε . To establish the uniform estimate on S^ε in $L^t(U)$, we proceed by duality. Let $\Phi \in L^{t'}(U)$, with t' the conjugate of t . It admits the following Helmholtz decomposition in each inclusion:

$$\Phi|_{I^\varepsilon} := \mathbb{P}_{\mathring{I}^\varepsilon} \Phi + \nabla \varphi^\varepsilon, \quad \forall I^\varepsilon \in \operatorname{CC}(F^\varepsilon)$$

where for any open set \mathcal{O} , $\mathbb{P}_{\mathcal{O}}$ is the Leray projector, continuous over $L^{\tilde{s}}(\mathcal{O})$. More precisely, we claim that

$$\|\mathbb{P}_{\mathring{I}^\varepsilon} \Phi\|_{L^{\tilde{s}}(I^\varepsilon)} + \|\nabla \varphi^\varepsilon\|_{L^{\tilde{s}}(I^\varepsilon)} \leq C_{\tilde{s}} \operatorname{diam}(I)^R \|\Phi\|_{L^{\tilde{s}}(I^\varepsilon)}, \quad \text{for some } R > 0.$$

Indeed, by scaling, it is enough to show this inequality for $\varepsilon = 1$. To show that the operator norm of $\mathbb{P}_{\mathring{I}}$ (or equivalently $\operatorname{Id} - \mathbb{P}_{\mathring{I}}$) is bounded by a power of $\operatorname{diam}(I)$, one writes $(\operatorname{Id} - \mathbb{P}_{\mathring{I}})f = \nabla u_f$, where

$$\Delta u_f = \operatorname{div} f \quad \text{on } \mathring{I}, \quad \partial_\nu u|_{\partial I} = f \cdot \nu.$$

One must then look carefully at the proof of the inequality $\|\nabla u_f\|_{L^{\tilde{s}}(\mathring{I})} \leq C_I \|f\|_{L^{\tilde{s}}(\mathring{I})}$, and track the dependence of C_I with respect to I . The derivation of this inequality follows the usual scheme: by local charts and straightening of the boundary, one can use that the inequality holds in \mathbb{R}^3 and \mathbb{R}_+^3 . A tedious verification shows that the constant in the inequality involves the constant \mathbf{d} in (G1), the number of charts and the constant in the Poincaré inequality. Under our regularity assumptions, it is controlled by $\operatorname{diam}(I)^R$ for large enough R . We skip the details for brevity.

We now introduce the function $\phi^\varepsilon \in H_0^1(U)$ associated to φ^ε in Proposition 3. In particular, ϕ^ε and φ^ε coincide on each I^ε up to a constant function. We compute

$$\begin{aligned} \int_{F^\varepsilon} \nabla v^\varepsilon \cdot \Phi &= \sum_{I^\varepsilon \in \operatorname{CC}(F^\varepsilon)} \int_{I^\varepsilon} \nabla v^\varepsilon \cdot \nabla \varphi^\varepsilon - \underbrace{\int_{I^\varepsilon} v^\varepsilon \operatorname{div} \mathbb{P}_{\mathring{I}^\varepsilon} \Phi + \int_{\partial I^\varepsilon} v^\varepsilon \mathbb{P}_{\mathring{I}^\varepsilon} \Phi \cdot \nu}_{=0} \\ &= \sum_{I^\varepsilon \in \operatorname{CC}(F^\varepsilon)} \int_{\partial I} s^\varepsilon \cdot \nu \varphi^\varepsilon \text{ using the equation on } v^\varepsilon \\ &= \sum_{I^\varepsilon \in \operatorname{CC}(F^\varepsilon)} \int_{\partial I} s^\varepsilon \cdot \nu \phi^\varepsilon \text{ using that } \int_{\partial I} s^\varepsilon \cdot \nu = 0 \\ &= \int_{U \setminus F^\varepsilon} s^\varepsilon \cdot \nabla \phi^\varepsilon \end{aligned}$$

$$\leq \|s^\varepsilon\|_{L^2(U \setminus F^\varepsilon)} \|\nabla \phi^\varepsilon\|_{L^2(U \setminus F^\varepsilon)} \leq C \|s^\varepsilon\|_{L^2(U \setminus F^\varepsilon)} \|\nabla \varphi^\varepsilon\|_{L^{\tilde{s}}(F^\varepsilon)}$$

where the last inequality comes from Proposition 3. Now,

$$\begin{aligned} \|\nabla \varphi^\varepsilon\|_{L^{\tilde{s}}(F^\varepsilon)} &= \sum_{I^\varepsilon \in \text{CC}(F^\varepsilon)} \|\nabla \varphi^\varepsilon\|_{L^{\tilde{s}}(I^\varepsilon)} \leq C \sum_{I^\varepsilon \in \text{CC}(F^\varepsilon)} \text{diam}(I)^{R\tilde{s}} \|\Phi\|_{L^{\tilde{s}}(I^\varepsilon)} \\ &\leq C \left(\sum_{I^\varepsilon \in \text{CC}(F^\varepsilon)} |I^\varepsilon| \text{diam}(I)^p \right)^{\frac{t' - \tilde{s}}{t'}} \|\Phi\|_{L^{t'}(F^\varepsilon)}, \quad p = \frac{R\tilde{s}t'}{t' - \tilde{s}}. \end{aligned}$$

Again,

$$\sum_{I^\varepsilon \in \text{CC}(F^\varepsilon)} |I^\varepsilon| \text{diam}(I)^p \leq C \int_U \text{diam}(I_{x/\varepsilon, F})^p \xrightarrow[\varepsilon \rightarrow 0]{} C |U| \mathbb{E} \text{diam}(I_{0, F}^p) < +\infty$$

so that we end up with

$$\begin{aligned} \int_U S^\varepsilon \cdot \Phi &= \int_{U \setminus F^\varepsilon} s^\varepsilon \cdot \Phi + \int_{F^\varepsilon} \nabla v^\varepsilon \cdot \Phi \\ &\leq C \|s^\varepsilon\|_{L^2(U \setminus F^\varepsilon)} \left(\|\Phi\|_{L^2(U \setminus F^\varepsilon)} + \|\Phi\|_{L^{t'}(F^\varepsilon)} \right) \\ &\leq C' \|s^\varepsilon\|_{L^2(U \setminus F^\varepsilon)} \|\Phi\|_{L^{t'}(U)}. \end{aligned}$$

This concludes the proof. \square

4. Proofs of the main results. Here, again, F is an admissible set of inclusions.

4.1. Existence of the homogenized matrix - Proposition 1. The goal of this section is to define properly the matrix A^0 , describing the effective viscosity of the conducting medium. We follow here the approach developed in [27, chapter 8]. We first introduce the so-called resistance matrix, that is the symmetric matrix defined by:

$$\forall \xi \in \mathbb{R}^3, \quad B^0 \xi \cdot \xi := \inf_z \mathbb{E} \int_{Q_1 \setminus F} |\xi + z|^2,$$

where the infimum is taken over vector fields $z = z(y, \omega) \in L^2(\Omega, L_{loc}^2(\mathbb{R}^3))$ that are solenoidal, stationary and mean-free. An equivalent formulation of the variational problem is

$$\forall \xi \in \mathbb{R}^3, \quad B^0 \xi \cdot \xi := \inf_{Z \in \mathcal{V}_{sol}^2(\Omega)} \int_{\Omega \setminus \mathcal{F}} |\xi + Z|^2, \quad (24)$$

where:

- \mathcal{F} is the subset of Ω defined in 6, so that $F(\omega) = \{x \in \mathbb{R}^3, \tau_x(\omega) \in \mathcal{F}\}$
- $\mathcal{V}_{sol}^2(\Omega) = \{Z \in L^2(\Omega), \quad \mathbb{E} Z = 0, \quad y \rightarrow Z(\tau_y(\omega)) \text{ solenoidal vector field}\}.$

We remind that introducing the other subspace of vector fields

$$L_{pot}^2(\Omega) = \{U \in L^2(\Omega), \quad y \rightarrow U(\tau_y(\omega)) \text{ potential vector field}\}$$

one has the orthogonal decomposition $L^2(\Omega) = \mathcal{V}_{sol}^2(\Omega) \oplus L_{pot}^2(\Omega)$.

Still following [27, chapter 8], if we now denote

$$X : \text{the closure in } L^2(\Omega \setminus \mathcal{F}) \text{ of the space } \{Z|_{\Omega \setminus \mathcal{F}}, Z \in \mathcal{V}_{sol}^2(\Omega)\} \quad (25)$$

then there exists a unique minimizer $Z \in X$ attaining the infimum, and it satisfies the Euler-Lagrange equation:

$$\int_{\Omega \setminus \mathcal{F}} (\xi + Z) \cdot Z' = 0, \quad \forall Z' \in \mathcal{V}_{sol}^2(\Omega).$$

In particular, $1_{\mathcal{F}}(\xi + Z) \in L^2_{\text{pot}}(\Omega)$, and $B^0\xi = \mathbb{E} 1_{\mathcal{F}}(\xi + Z)$.

The last step of proof of Proposition 1 is showing that the matrix B^0 above is invertible. Therefore, we use Lemma 8.7 of [27], which provides a sufficient condition:

Lemma 4.1. [27, Lemma 8.7] *Assume that for any $\xi \in \mathbb{R}^3$, and for any ω in a subset of positive measure, there exists a sequence of potential vector fields $v^\varepsilon \in L^2(U)$ satisfying*

$$v^\varepsilon|_{F^\varepsilon(\omega)} = 0, \quad v^\varepsilon \rightarrow \xi \text{ weakly in } L^2(U), \quad \limsup_{\varepsilon \rightarrow 0} \|v^\varepsilon\|_{L^2} \leq C|\xi| \text{ for some } C > 0.$$

Then, B^0 is positive definite.

The keypoint is that under H1, the assumptions of the lemma are granted by Proposition 2: one can take $v^\varepsilon = \xi - \nabla\phi^\varepsilon = \nabla(\xi \cdot x - \phi^\varepsilon)$, with ϕ^ε as in Proposition 2. This concludes the proof of Proposition 1.

4.2. Homogenization without short. We prove in this section part of Theorem 2.7. Namely, we focus on the case where H2 is satisfied with $F' = F$, for some $s > 3$. The reason for treating this special case separately is that it is much easier : indeed, the arguments of [27, chapter 8] rely on the existence of proper extensions of solenoidal vector fields, or of fields with given divergence, inside the inclusions. As such extensions are granted by Proposition 4, they adapt straightforwardly. The proof of the general case, given in the next section, will be more involved (and due to technical difficulty limited to $s < 6$).

First, by Remark iii) after Theorem 2.7, H2 implies H1. Hence, we can apply Proposition 1, so that $A^0 = (B^0)^{-1}$ is well-defined. Let $f \in L^{6/5}(U)$, and $u^\varepsilon \in H^1_0(U)$ the solution of 3 (with implicit dependence on ω), where domain F^ε is defined in 9. From a simple energy estimate, u^ε is bounded in $H^1_0(U)$ uniformly in ω and ε . Hence, almost surely, u^ε has a subsequence that converges weakly to some u^0 . The goal is to show that u^0 satisfies 2. By uniqueness of this accumulation point, this will mean that the whole sequence converges to u^0 . From now on, for the sake of brevity, we denote u^ε the converging subsequence.

Let now $\xi \in \mathbb{R}^3$, and Z the minimizer of problem 24. As $Z \in X$, cf. 25, there exists a sequence $Z^\nu \in \mathcal{V}^2_{\text{sol}}(\Omega)$ and $\|Z^\nu - Z\|_{L^2(\Omega \setminus \mathcal{F})} \rightarrow 0$ as $\nu \rightarrow 0$. Let \bar{Z} , resp. \bar{Z}^ν , the extension of Z , resp. of $Z^\nu|_{\Omega \setminus \mathcal{F}}$, by $-\xi$ on \mathcal{F} . We remind that $\xi + \bar{Z} \in L^2_{\text{pot}}(\Omega)$, with $\mathbb{E}(\xi + \bar{Z}) = B^0\xi$. We finally set

$$\bar{z}(y, \omega) = \bar{Z}(\tau_y(\omega)), \quad \bar{z}^\nu(y, \omega) = \bar{Z}^\nu(\tau_y(\omega)), \quad z^\nu(y, \omega) = Z^\nu(\tau_y(\omega)).$$

Let $p^\varepsilon = \nabla u^\varepsilon$. By Proposition 4, for any $t < s'$, assuming 14 for large enough p , one can extend p^ε into a field P^ε such that

$$P^\varepsilon|_{U \setminus F^\varepsilon} = p^\varepsilon, \quad \operatorname{div} P^\varepsilon = f 1_{U \setminus F^\varepsilon} \text{ in } U, \quad \|P^\varepsilon\|_{L^t(U)} \leq C_t.$$

The last bound implies weak convergence of (a subsequence of) P^ε towards some P^0 in $L^t(U)$. By the ergodic theorem, $f 1_{U \setminus F^\varepsilon}$ converges weakly to $f(1 - \lambda)$, with $\lambda = \mathbb{E} 1_F$, in $L^{6/5}(U)$. Hence, $\operatorname{div} P^0 = (1 - \lambda)f$ in U . Let now $\varphi \in C^\infty_c(U)$, and u^0 the weak limit of (a subsequence of) u^ε in $H^1_0(U)$. The point is to show that, as $\varepsilon \rightarrow 0$:

$$\int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + \bar{z}(x/\varepsilon)) dx \rightarrow \int_U \varphi(x) \nabla u^0(x) \cdot \xi dx \quad (26)$$

as well as

$$\int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + \bar{z}(x/\varepsilon)) dx \rightarrow \int_U \varphi(x) P^0 \cdot B^0 \xi dx. \quad (27)$$

Identifying the limits, it follows that $B^0 P^0 = \nabla u^0$, so that $P^0 = A^0 \nabla u^0$ and as $\operatorname{div} P^0 = (1 - \lambda f)$, we recover system 2.

The proof of 26-27 is an adaptation of the one in [27], so that we indicate only the main elements and the changes that are needed. As regards 26, we write

$$\begin{aligned} \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + \bar{z}(x/\varepsilon)) dx &= \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}(x/\varepsilon) - \bar{z}^\nu(x/\varepsilon)) dx \\ &\quad + \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}^\nu(x/\varepsilon) - z^\nu(x/\varepsilon)) dx \\ &\quad + \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + z^\nu(x/\varepsilon)) dx. \end{aligned} \quad (28)$$

The first term at the r.h.s. satisfies

$$\begin{aligned} &\left| \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}(x/\varepsilon) - \bar{z}^\nu(x/\varepsilon)) dx \right| \\ &\leq \|\varphi\|_\infty \|\nabla u^\varepsilon\|_{L^2(U)} \|\bar{z}(x/\varepsilon) - \bar{z}^\nu(x/\varepsilon)\|_{L^2(U)} \end{aligned}$$

so that, by the uniform L^2 bound on ∇u^ε and the ergodic theorem:

$$\begin{aligned} &\left| \limsup_\varepsilon \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}(x/\varepsilon) - \bar{z}^\nu(x/\varepsilon)) dx \right| \\ &\leq C \|\bar{Z} - \bar{Z}^\nu\|_{L^2(\Omega)} = C \|Z - Z^\nu\|_{L^2(\Omega \setminus \mathcal{F})} \end{aligned}$$

and finally

$$\limsup_\nu \limsup_\varepsilon \left| \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}(x/\varepsilon) - z^\nu(x/\varepsilon)) dx \right| = 0.$$

For the second term at the r.h.s. of 28, we notice that $\nabla u^\varepsilon \cdot (\bar{z}^\nu(\cdot/\varepsilon) - z^\nu(\cdot/\varepsilon))$ is zero in F^ε , because ∇u^ε is zero there, and in $U \setminus \varepsilon F$, because $\bar{z}^\nu = z^\nu$ there. However, it does not *a priori* vanish in $(\varepsilon F) \cap U \setminus F^\varepsilon$. This corresponds to inclusions I^ε in εF that intersect $U_{\delta_0 \varepsilon}$, where

$$U_\eta := \{x \in U, d(x, \partial U) \leq \eta\}, \quad \eta > 0.$$

A crucial point is that, under the moment condition $\mathbb{E} \operatorname{diam}(I_{0,F})^3 < +\infty$, by a direct adaptation of the proof of Lemma 4.2 and 30 below, one has almost surely,

$$\sup \{ \operatorname{diam}(I^\varepsilon), I^\varepsilon \in \operatorname{CC}(\varepsilon F), I^\varepsilon \cap U_{\delta_0 \varepsilon} \neq \emptyset \} = o(1) \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, for any $\eta > 0$, for ε small enough, one has $(\varepsilon F \cap U) \setminus F^\varepsilon \subset U_\eta$, so that

$$\begin{aligned} &\left| \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}^\nu(x/\varepsilon) - z^\nu(x/\varepsilon)) dx \right| \\ &= \left| \int_{(\varepsilon F \cap U) \setminus F^\varepsilon} \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + z^\nu(x/\varepsilon)) dx \right| \\ &\leq \|\varphi\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^2(U_\eta)} \|\xi + z^\nu(\cdot/\varepsilon)\|_{L^2(U_\eta)} \leq C \|\xi + z^\nu(\cdot/\varepsilon)\|_{L^2(U_\eta)} \end{aligned}$$

and by the ergodic theorem

$$\limsup_\varepsilon \left| \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}^\nu(x/\varepsilon) - z^\nu(x/\varepsilon)) dx \right| \leq C \|1_{U_\eta}\|_{L^2} \|\xi + Z^\nu\|_{L^2(\Omega)} \leq C' \eta^{1/2}.$$

As η is arbitrary, it follows that

$$\limsup_\varepsilon \left| \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\bar{z}^\nu(x/\varepsilon) - z^\nu(x/\varepsilon)) dx \right| = 0.$$

Finally, as regards the third term at the r.h.s. of 28, by the div-curl lemma and the ergodic theorem, for any given ν ,

$$\begin{aligned} \lim_{\varepsilon} \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + z^\nu(x/\varepsilon)) dx &= \int_U \varphi(x) \nabla u^0(x) \cdot (\xi + \mathbb{E} Z^\nu) dx \\ &= \int_U \varphi(x) \nabla u^0(x) \cdot \xi dx \end{aligned}$$

where the last equality comes from the property $\mathbb{E} Z^\nu = 0$. Combining all previous relations yields 26.

As regards 27, we want again to rely on the div-curl lemma but switching the potential and solenoidal vector fields. Therefore, we write

$$\int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + \bar{z}(x/\varepsilon)) dx = \int_U \varphi(x) P^\varepsilon(x) \cdot \nabla w^\varepsilon dx$$

taking into account that $(\xi + \bar{z}(x/\varepsilon))$ is a potential vector field, hence can be written ∇w^ε . Moreover,

$$\nabla w^\varepsilon \rightarrow \mathbb{E}(\xi + \bar{Z}) = B^0 \xi \quad \text{weakly in } L^2(U), \text{ almost surely.}$$

If $(P^\varepsilon)_{\varepsilon>0}$ was bounded in $L^2(U)$, one could conclude directly by the div-curl lemma. As it is only bounded in $L^t(U)$ for $t < s'$, one must use an approximation of w^ε by the truncation

$$\begin{aligned} w^{\varepsilon, l(x)} &= w^\varepsilon(x) \quad \text{if } |w^\varepsilon(x)| \leq l, \\ w^{\varepsilon, l(x)} &= l \quad \text{if } w^\varepsilon(x) \geq l, \\ w^{\varepsilon, l(x)} &= -l \quad \text{if } w^\varepsilon(x) \leq -l. \end{aligned}$$

We refer to [27, chapter 8, page 286] for implementation of this argument.

4.3. Homogenization with short - Theorem 2.7. We tackle the proof of Theorem 2.7 in the general case where F' is an admissible short of F .

First, we introduce the sequence of admissible shorts $(F^\kappa)_{\kappa \in (0,1]}$, defined by the following properties : for all $\kappa \in (0,1)$, F' is a short of F^κ and F^κ is a short of F , with

$$\text{Ed}(F^\kappa) = \text{Ed}(F') \cup \left\{ e \in \text{Ed}(F) \setminus \text{Ed}(F'), \quad |e| \geq \kappa \right\}.$$

In other words, F^κ is deduced from F' by removing bridges corresponding to gaps of size larger than κ . Obviously, almost surely, for every closed ball B , $F^\kappa \cap B = F \cap B$ for $\kappa \leq \kappa_B$ small enough.

Lemma 4.2. *If F' satisfies H2 and the moment bound 14 for $p = 3$, then F^κ satisfies H2 for all $\kappa > 0$.*

Proof. We will first show that,

$$\text{almost surely, } \forall \kappa \in \mathbb{Q} \cap (0,1], \quad \sup_{I \in CC(F^\kappa), I \cap Q_N \neq \emptyset} \text{diam}(I) = o(N). \quad (29)$$

Indeed, let $\kappa \in \mathbb{Q} \cap (0,1]$. Clearly, $\text{diam}(I_{0,F^\kappa}) \leq \text{diam}(I_{0,F'})$, so $\mathbb{E} \text{diam}(I_{0,F^\kappa})^3 < +\infty$. Let $\eta > 0$, and consider the event

$$A_N = \{\omega, \text{ there exists } I \in CC(F^\kappa(\omega)), I \cap Q_N \neq \emptyset, I \cap Q_{N(1+\eta)}^c \neq \emptyset\}.$$

We recall that all inclusions satisfy an inner sphere condition with uniform deterministic radius. Hence, there exists a (deterministic) set of points x_1, \dots, x_{K_N} of $\partial Q_{N+\frac{1}{2}}$

with $K_N \leq CN^2$ for a deterministic constant C and such that any $I \in CC(F^\kappa)$ with $I \cap Q_N \neq \emptyset$, $I \cap Q_{N(1+\eta)}^c \neq \emptyset$ contains at least an x_i . It follows that

$$\mathbb{P}(A_N) \leq \sum_{i=1}^{K_N} \mathbb{P}(\text{diam}(I_{x_i, F^\kappa}) \geq \eta N) \leq CN^2 \mathbb{P}(\text{diam}(I_{0, F^\kappa}) \geq \eta N).$$

The moment bound implies that $\sum P(A_N) < +\infty$, and it follows from Borel-Cantelli Lemma that $\mathbb{P}(\limsup A_N) = 0$. In other words, for all $\eta > 0$, for ω in a set of full measure, there exists N such that

$$\sup_{I \in CC(F^\kappa), I \cap Q_N \neq \emptyset} \text{diam}(I) \leq \eta N.$$

By taking a countable subset of η (and as κ describes the countable subset $\mathbb{Q} \cap (0, 1]$), one can find a set of full measure independent of κ and η , which proves 29. Let us remark that for κ large enough, namely for $\kappa \geq \delta'$ with δ' the constant in (G1) associated to F' , one has $F^\kappa = F'$, so that 29 implies

$$\text{almost surely, } \sup_{I' \in CC(F'), I' \cap Q_N \neq \emptyset} \text{diam}(I') = o(N). \quad (30)$$

We now turn to the proof of the lemma. Let $N \geq 1$, and $\{b_{IJe}\}$ a family indexed by $I, J \in F_N^\kappa$, $e \in \text{Ed}(F_N^\kappa)$, $I \xleftrightarrow{e} J$. By 30, for N large enough each $I \in CC(F_N^\kappa)$ is included in a connected component of F'_{2N} . We define a family $\{b'_{I'J'e}\}$ indexed by $I', J' \in F'_{2N}$, $e \in \text{Ed}(F'_{2N})$, $I' \xleftrightarrow{e} J'$ in the following way:

- if I' or J' does not contain any element of $CC(F_N^\kappa)$, $b'_{I'J'e} = 0$
- if I' and J' contain elements of $CC(F_N^\kappa)$, but $e \notin \text{Ed}(F_N^\kappa)$, $b'_{I'J'e} = 0$
- if I' and J' contain elements of $CC(F_N^\kappa)$, and $e \in \text{Ed}(F_N^\kappa)$, $b'_{I'J'e} = b_{IJe}$, where I, J are the unique elements in $CC(F_N^\kappa)$ such that $I \xleftrightarrow{e} J$.

We introduce the family $\{u'_{I'}\}$ indexed by $CC(F'_{2N})$ such that

$$\mathcal{E}(F'_{2N}, \{u'_{I'}\}, \{b'_{I'J'e}\}) = \inf_{\{t'_{I'}\}} \mathcal{E}(F'_{2N}, \{t'_{I'}\}, \{b'_{I'J'e}\}).$$

We then define a family $\{u_I\}$ indexed by $I \in CC(F_N^\kappa)$, as follows:

$$u_I = u'_{I'} \quad \text{for } I' \text{ the single c.c. of } F'_{2N} \text{ containing } I.$$

With this choice, we have

$$\sum_{\substack{I', J' \in CC(F'_{2N}), \\ I' \xleftrightarrow{e} J'}} \sum_{e \in \text{Ed}(F'_{2N})} |b'_{I'J'e}|^s \leq \sum_{I, J \in CC(F_N^\kappa)} \sum_{\substack{e \in \text{Ed}(F_N^\kappa), \\ I \xleftrightarrow{e} J}} |b_{IJe}|^s$$

and

$$\sum_{I \in CC(F_N^\kappa)} |I| |u_I|^2 \leq C \sum_{I' \in CC(F'_{2N})} |I'| |u'_{I'}|^2$$

and

$$\begin{aligned} \mathcal{E}(F_N^\kappa, \{u_I\}, \{b_{IJe}\}) &\leq C \mathcal{E}(F'_{2N}, \{u'_{I'}\}, \{b'_{I'J'e}\}) \\ &\quad + \sum_{I' \in CC(F'_{2N})} \sum_{\substack{I, J \in CC(F_N^\kappa), \\ I, J \subset I'}} \sum_{\substack{e \in \text{Ed}(F_N^\kappa), \\ I \xleftrightarrow{e} J}} |b_{IJe} - b_{JJe}|^2 \mu_e. \end{aligned}$$

Now, by definition of F^κ , connected components I, J of F^κ that are included in a single connected component I' of F' are at distance at least κ , so that $\mu_e \leq |\ln \kappa|$. Hence, the last term is bounded by

$$\begin{aligned} & C |\ln \kappa| \sum_{I, J \in \text{CC}(F_N^\kappa)} \sum_{\substack{e \in \text{Ed}(F_N^\kappa), \\ I \overset{\varepsilon}{\leftrightarrow} J}} |b_{IJe}|^2 \\ & \leq C' |\ln \kappa| |Q_N| \left(\frac{1}{|Q_N|} \sum_{I, J \in \text{CC}(F_N^\kappa)} \sum_{\substack{e \in \text{Ed}(F_N^\kappa), \\ I \overset{\varepsilon}{\leftrightarrow} J}} |b_{IJe}|^s \right)^{\frac{2}{s}}. \end{aligned}$$

The result follows easily from assumption **H2** for F' applied with $2N$ and previous inequalities. \square

We have now all ingredients to perform the proof of our main Theorem 2.7, for a general admissible short F' . First, by Remark iii) after Theorem 2.7, $A^0 = (B^0)^{-1}$ is well-defined. As in section 4.2, given $f \in L^{6/5}(U)$, one has $(u^\varepsilon)_\varepsilon$ bounded in $H_0^1(U)$, and the goal is to show that any weak accumulation point u^0 satisfies 2.

Similarly, we introduce $\xi \in \mathbb{R}^3$, Z the minimizer of problem 24, \tilde{Z} its extension by $-\xi$ in \mathcal{F} , and $\bar{z}(y, \omega) = \tilde{Z}(\tau_y(\omega))$. Let $p^\varepsilon = \nabla u^\varepsilon$. Let $F^{\kappa, \varepsilon}$ defined as in 9, replacing F by F^κ and δ_0 by $\delta_0/2$:

$$I^{\kappa, \varepsilon} := \varepsilon I^\kappa \quad \forall I^\kappa \in \text{CC}(F^\kappa), \quad F^{\kappa, \varepsilon} = \bigcup_{\substack{I^{\kappa, \varepsilon} \subset U, \\ d(I^{\kappa, \varepsilon}, \partial U) \geq \frac{\delta_0}{2} \varepsilon}} I^{\kappa, \varepsilon}.$$

We would like to extend the function $p^\varepsilon|_{U \setminus F^{\kappa, \varepsilon}}$ into some $P^{\kappa, \varepsilon}$ satisfying

$$\text{div } P^{\kappa, \varepsilon} = f 1_{U \setminus F^\varepsilon} \quad \text{in } U,$$

relying on Proposition 4 and the fact that F^κ satisfies **H2**. Note that for all $I^{\kappa, \varepsilon}$ in $\text{CC}(F^{\kappa, \varepsilon})$, one has the compatibility condition

$$\begin{aligned} \int_{\partial I^{\kappa, \varepsilon}} \partial_\nu u^\varepsilon &= \sum_{I^\varepsilon \in \text{CC}(F^\varepsilon), I^\varepsilon \subset I^{\kappa, \varepsilon}} \int_{\partial I^\varepsilon} \partial_\nu u^\varepsilon + \int_{\partial(I^{\kappa, \varepsilon} \setminus F)} \partial_\nu u^\varepsilon \\ &= \int_{\partial(I^{\kappa, \varepsilon} \setminus F^\varepsilon)} \partial_\nu u^\varepsilon = \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} \Delta u^\varepsilon = \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} f = \int_{I^{\kappa, \varepsilon}} f 1_{U \setminus F^\varepsilon}. \end{aligned}$$

But there is a little technicality here, due to the fact that $U \setminus F^{\kappa, \varepsilon}$ is not necessarily included in $U \setminus F^\varepsilon$ so that *a priori* $\text{div } p^\varepsilon \neq f$ on $U \setminus F^{\kappa, \varepsilon}$. This is due to the connected components I of F contained in connected components I^κ of F^κ , such that $I^\varepsilon \subset F^\varepsilon$ and $I^{\kappa, \varepsilon} \not\subset F^{\kappa, \varepsilon}$. Still, one can easily replace such connected components $I^{\kappa, \varepsilon}$ by smaller connected closed sets $\tilde{I}^{\kappa, \varepsilon}$ with $I^\varepsilon \subset \tilde{I}^{\kappa, \varepsilon} \subset I^{\kappa, \varepsilon}$, $d(\tilde{I}^{\kappa, \varepsilon}, \partial U) \geq \frac{\delta_0}{2} \varepsilon$, and such that Proposition 4 applies to $\tilde{F}^{\kappa, \varepsilon} = \cup \tilde{I}^{\kappa, \varepsilon}$ instead of $F^{\kappa, \varepsilon}$. Roughly speaking, one has just to erase in $I^{\varepsilon, \kappa}$ the “beads” that intersect $\{x, d(x, \partial U) \geq \frac{\delta_0}{2} \varepsilon\}$. For brevity, we leave to the reader to verify that no complication occurs replacing $F^{\kappa, \varepsilon}$ by $\tilde{F}^{\kappa, \varepsilon}$, and keep the former notation.

Eventually, by applying Proposition 4, we obtain a field $P^{\kappa, \varepsilon} \in L^2(U)$ with

$$P^{\kappa, \varepsilon}|_{U \setminus F^{\kappa, \varepsilon}} = p^\varepsilon, \quad \text{div } P^{\kappa, \varepsilon} = f 1_{U \setminus F^\varepsilon}.$$

Moreover, for all $t < s'$, where s in the exponent in **H2**, if the moment condition 14 is satisfied for $p = p(t)$ large enough, one has

$$\|P^{\kappa, \varepsilon}\|_{L^t(U)} \leq C_{\kappa, t}.$$

By diagonal extraction, there exists a subsequence in ε common to all $\kappa \in \mathbb{Q} \cap (0, 1]$, and $P^{\kappa,0}$ in $L^t(U)$ such that ignoring the subsequence in the notation:

$$P^{\kappa,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} P^{\kappa,0} \text{ weakly in } L^t(U), \quad \forall \kappa.$$

Let $\varphi \in C_c^\infty(U)$. Proceeding exactly as in Section 4.2 for the proof of 26, we find

$$\int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + \bar{z}(x/\varepsilon)) dx \xrightarrow{\varepsilon \rightarrow 0} \int_U \varphi(x) \nabla u^0(x) \cdot \xi dx. \quad (31)$$

The novel difficulty lies in the adaptation of the proof of 27. We shall prove that

$$\int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + \bar{z}(x/\varepsilon)) dx \xrightarrow{\varepsilon \rightarrow 0} \int_U \varphi(x) P^{\kappa,0}(x) \cdot B^0 \xi dx + \eta(\kappa), \quad \eta(\kappa) \xrightarrow{\kappa \rightarrow 0} 0. \quad (32)$$

Comparing 31 and 32, we get

$$\int_U \varphi(x) P^{\kappa,0}(x) \cdot B^0 \xi dx \xrightarrow{\kappa \rightarrow 0} \int_U \varphi(x) \nabla u^0(x) \cdot \xi dx$$

which shows that $P^{\kappa,0}$ converges in the sense of distributions to $A^0 \nabla u^0$. But we also have

$$\operatorname{div} P^{\kappa,\varepsilon} = f 1_{U \setminus F^\varepsilon} \quad \text{in } U$$

so that sending ε to zero,

$$\operatorname{div} P^{\kappa,0} = (1 - \lambda) f \quad \text{in } U$$

and finally, sending κ to zero, we get 2.

It remains to show 31. We take into account that $\nabla u^\varepsilon(x) = 0$, $\xi + \bar{z}(x/\varepsilon) = 0$ in F^ε , and write:

$$\begin{aligned} & \int_U \varphi(x) \nabla u^\varepsilon(x) \cdot (\xi + \bar{z}(x/\varepsilon)) dx \\ &= \int_{U \setminus F^{\kappa,\varepsilon}} \varphi(x) P^{\kappa,\varepsilon} \cdot (\xi + \bar{z}(x/\varepsilon)) dx + \int_{F^{\kappa,\varepsilon}} \varphi(x) \nabla u^\varepsilon \cdot (\xi + \bar{z}(x/\varepsilon)) dx \\ &= \int_U \varphi(x) P^{\kappa,\varepsilon} \cdot (\xi + \bar{z}(x/\varepsilon)) dx + \int_{F^{\kappa,\varepsilon} \setminus F^\varepsilon} \varphi(x) \nabla u^\varepsilon \cdot (\xi + \bar{z}(x/\varepsilon)) dx \\ &\quad - \int_{F^{\kappa,\varepsilon} \setminus F^\varepsilon} \varphi(x) P^{\kappa,\varepsilon} \cdot (\xi + \bar{z}(x/\varepsilon)) dx \\ &= I^{\kappa,\varepsilon} + J^{\kappa,\varepsilon} - K^{\kappa,\varepsilon}. \end{aligned}$$

The first integral can be treated as in Section 4.2, resulting in

$$I^{\kappa,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \int_U \varphi(x) P^{\kappa,0}(x) \cdot B^0 \xi dx.$$

The second integral is bounded by

$$|J^{\kappa,\varepsilon}| \leq \|\varphi\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^2(U)} \|\xi + \bar{z}(\cdot/\varepsilon)\|_{L^2(F^{\kappa,\varepsilon} \setminus F^\varepsilon)} \leq C \|\xi + \bar{z}(\cdot/\varepsilon)\|_{L^2(U \cap \varepsilon(F^{\kappa,\varepsilon} \setminus F))}$$

where we have used the uniform bound on ∇u^ε in $H_0^1(U)$. From the ergodic theorem, we infer that

$$\limsup_{\varepsilon \rightarrow 0} |J^{\kappa,\varepsilon}| \leq C \left(\int_\Omega \int_{Q_1} 1_{F^{\kappa}(\omega) \setminus F(\omega)}(y) |\xi + \bar{z}(y, \omega)|^2 dy d\mathbb{P}(\omega) \right)^{1/2}.$$

The integral at the right-hand side converges to zero as $\kappa \rightarrow 0$: it follows from the dominated convergence theorem and the pointwise convergence to zero of $(\omega, y) \rightarrow 1_{Q_1 \cap (F_\kappa(\omega) \setminus F(\omega))}(y)$, because

$$(F^\kappa(\omega) \setminus F(\omega)) \cap Q_1 = \emptyset \quad \text{for } \kappa \text{ large enough.}$$

Hence,

$$\limsup_{\varepsilon \rightarrow 0} |J^{\kappa, \varepsilon}| = o(\kappa). \quad (33)$$

We still have to control

$$\begin{aligned} K^{\kappa, \varepsilon} &= \sum_{I^{\kappa, \varepsilon} \in \text{CC}(F^{\kappa, \varepsilon})} \left(\int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} \varphi(x_{I^{\kappa, \varepsilon}}) P^{\kappa, \varepsilon} \cdot (\xi + \bar{z}(x/\varepsilon)) dx \right. \\ &\quad \left. + \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} (\varphi(x) - \varphi(x_{I^{\kappa, \varepsilon}})) P^{\kappa, \varepsilon} \cdot (\xi + \bar{z}(x/\varepsilon)) dx \right) \\ &:= K_1^{\kappa, \varepsilon} + K_2^{\kappa, \varepsilon} \end{aligned}$$

where $x_{I^{\kappa, \varepsilon}}$ is the center of mass of $I^{\kappa, \varepsilon}$. We recall that $\xi + \bar{z}(x/\varepsilon) = \nabla w^\varepsilon(x)$ is a potential field that converges weakly in $L^2(U)$ to $B^0 \xi$ as $\varepsilon \rightarrow 0$. By a proper choice of the additive constant in w^ε , we can always assume that w^ε converges weakly in $L^6(U)$ to $x \rightarrow (B^0 \xi) \cdot x$. Now, we write

$$\begin{aligned} \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} P^{\kappa, \varepsilon} \cdot (\xi + \bar{z}(x/\varepsilon)) dx &= \int_{I^{\kappa, \varepsilon}} P^{\kappa, \varepsilon} \cdot \nabla w^\varepsilon dx \\ &= \int_{I^{\kappa, \varepsilon} \setminus F} f w^\varepsilon + \int_{\partial I^{\kappa, \varepsilon}} P^{\kappa, \varepsilon} \cdot \nu w^\varepsilon \\ &= \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} f w^\varepsilon + \int_{\partial I^{\kappa, \varepsilon}} \partial_\nu u^\varepsilon w^\varepsilon \\ &= \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} f w^\varepsilon + \int_{\partial(I^{\kappa, \varepsilon} \setminus F^\varepsilon)} \partial_\nu u^\varepsilon w^\varepsilon. \end{aligned}$$

Note that for the last equality, we have used that $\int_{I^\varepsilon} \partial_\nu u^\varepsilon = 0$ for all $I^\varepsilon \in \text{CC}(F^\varepsilon)$, and that w^ε is a constant in each I^ε . Finally,

$$\int_{\partial(I^{\kappa, \varepsilon} \setminus F^\varepsilon)} \partial_\nu u^\varepsilon w^\varepsilon = \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} \nabla u^\varepsilon \cdot \nabla w^\varepsilon = \int_{I^{\kappa, \varepsilon} \setminus F^\varepsilon} \nabla u^\varepsilon \cdot (\xi + \bar{z}(\cdot/\varepsilon))$$

resulting in

$$\begin{aligned} |K_1^{\kappa, \varepsilon}| &\leq \|\varphi\|_{L^\infty} \left(\|w^\varepsilon\|_{L^6(U)} \|f 1_{F^{\kappa, \varepsilon} \setminus F^\varepsilon}\|_{L^{6/5}(U)} \right. \\ &\quad \left. + \|\nabla u^\varepsilon\|_{L^2(F^{\kappa, \varepsilon} \setminus F^\varepsilon)} \|\xi + \bar{z}(\cdot/\varepsilon)\|_{L^2(F^{\kappa, \varepsilon} \setminus F^\varepsilon)} \right). \end{aligned}$$

By using the uniform L^6 bound on w^ε , the uniform L^2 bound on ∇u^ε and the ergodic theorem, we end up with

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} |K_1^{\kappa, \varepsilon}| &\leq C \|f\|_{L^{6/5}(U)} \left(\int_{\Omega} \int_{Q_1} 1_{F^\kappa(\omega) \setminus F(\omega)}(y) dy d\mathbb{P}(\omega) \right)^{5/6} \\ &\quad + \int_{\Omega} \int_{Q_1} 1_{F^\kappa(\omega) \setminus F(\omega)}(y) |\xi + \bar{z}(y, \omega)|^2 dy d\mathbb{P}(\omega) \Big)^{1/2} = o(\kappa) \end{aligned}$$

as seen above. It remains to treat

$$\begin{aligned} |K_2^{\kappa,\varepsilon}| &\leq \|\nabla\varphi\|_{L^\infty} \varepsilon \|P^{\kappa,\varepsilon}\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)} \|\xi + \bar{z}(\cdot/\varepsilon)\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)} \\ &\leq C' \varepsilon \|P^{\kappa,\varepsilon}\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)} \end{aligned}$$

using the ergodic theorem to bound the factor $\|\xi + \bar{z}(\cdot/\varepsilon)\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)}$. The last difficulty is to bound $\|P^{\kappa,\varepsilon}\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)}$, because we only have so far a control in L^t , for any $t < s'$. Still, under a large moment bound on $\text{diam}(I_{0,F})$, cf. [14](#), we will now show that

$$\|P^{\kappa,\varepsilon}\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)} \leq C\varepsilon^{3/2-3/t}. \quad (34)$$

Indeed, following the proof of Proposition [4](#), we see that inside each inclusion $I^{\kappa,\varepsilon}$, one has $P^{\kappa,\varepsilon} = \nabla w + \nabla v^{\kappa,\varepsilon}$, where w solves

$$\Delta w = f 1_{U\setminus F^\varepsilon} \quad \text{in } U, \quad w|_{\partial U} = 0$$

so that in particular

$$\|\nabla w\|_{L^2(U)} \leq C\|f\|_{L^{6/5}(U)},$$

while $v^{\kappa,\varepsilon}$ is the solution, mean-free over $\mathring{I}^{\kappa,\varepsilon}$, of

$$\Delta v^{\kappa,\varepsilon} = 0 \quad \text{in } \mathring{I}^{\kappa,\varepsilon}, \quad \partial_\nu v^{\kappa,\varepsilon}|_{\partial I^{\kappa,\varepsilon}} = \partial_\nu(u^\varepsilon - w).$$

A crucial point is that $I^\kappa \setminus F$ is a union of bridges, that are at some uniform distance $r > 0$ from all other inclusions. Denoting I_j^κ , $j = 1 \dots$ such bridges, and introducing for all j , the $r/2$ neighborhood V_j^κ of I_j^κ , we claim that

$$\begin{aligned} \|\nabla v^{\kappa,\varepsilon}\|_{L^2(\varepsilon I_j^\kappa)}^2 &\leq C \left(\varepsilon^{-2} \|v^\kappa\|_{L^2(\varepsilon(V_j^\kappa \cap I^\kappa))}^2 + \|\partial_\nu(u^\varepsilon - w)\|_{H^{-1/2}(\varepsilon(\partial I^\kappa \cap V_j^\kappa))} \right) \\ &\leq C' \left(\varepsilon^{-2} \|v^\kappa\|_{L^2(\varepsilon(V_j^\kappa \cap I^\kappa))}^2 + \|\nabla(u^\varepsilon - w)\|_{L^2(\varepsilon(V_j^\kappa \setminus I^\kappa))}^2 \right). \end{aligned}$$

Indeed, by a scaling argument, it is enough to consider the case $\varepsilon = 1$. The first inequality follows then from a standard elliptic regularity result, while the second one follows from the usual bound

$$\|U \cdot \nu\|_{L^2(\partial\mathcal{O})} \leq C (\|\text{div } U\|_{L^2(\mathcal{O})} + \|U\|_{L^2(\mathcal{O})})$$

applied in the domain $\mathcal{O} = V_j^\kappa \setminus I^\kappa$. Here, we rely on the fact that such domains are far away from other inclusions, so that the constant can be taken uniform in j .

Summing over all j 's and all inclusions I^κ , we end up with

$$\begin{aligned} \|\nabla v^{\kappa,\varepsilon}\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)}^2 &\leq C \left(\varepsilon^{-2} \sum_{I^{\kappa,\varepsilon}} \|v^\kappa\|_{L^2(I^{\kappa,\varepsilon})}^2 + \|\nabla(u^\varepsilon - w)\|_{L^2(U)}^2 \right) \\ &\leq C' \left(\varepsilon^{-2} \sum_{I^{\kappa,\varepsilon}} \|v^\kappa\|_{L^2(I^{\kappa,\varepsilon})}^2 + 1 \right). \end{aligned}$$

As $s > 6$, one has $s' > \frac{6}{5}$. Hence, for t close enough to s' , $W^{1,t}(I^{\kappa,\varepsilon}) \subset L^2(I^{\kappa,\varepsilon})$, with

$$\|v^\kappa\|_{L^2(I^{\kappa,\varepsilon})} \leq C \text{diam}(I^\kappa)^r \varepsilon^{\frac{5}{2}-\frac{3}{t}} \|\nabla v^\kappa\|_{L^t(I^{\kappa,\varepsilon})}.$$

Again, the power of ε is deduced from a scaling argument, while the factor $\text{diam}(I^\kappa)^r$ comes from the Poincaré inequality [16](#) applied to the mean-free function v^κ . We get

eventually

$$\begin{aligned} \|P^{\kappa,\varepsilon}\|_{L^2(F^{\kappa,\varepsilon}\setminus F^\varepsilon)}^2 &\leq C\left(\varepsilon^{3-\frac{6}{t}}\sum_{I^{\kappa,\varepsilon}}\text{diam}(I^\kappa)^{2r}\|P^{\kappa,\varepsilon}\|_{L^t(I^{\kappa,\varepsilon})}^2+1\right) \\ &\leq C\left(\varepsilon^{3-\frac{6}{t}}\sum_{I^{\kappa,\varepsilon}}\text{diam}(I^\kappa)^{2r}\|P^{\kappa,\varepsilon}\|_{L^t(I^{\kappa,\varepsilon})}^t\|P^{\kappa,\varepsilon}\|_{L^t(U)}^{2-t}+1\right) \\ &\leq C'\left(\varepsilon^{3-\frac{6}{t}}\sum_{I^{\kappa,\varepsilon}}\text{diam}(I^\kappa)^{2r}\|P^{\kappa,\varepsilon}\|_{L^t(I^{\kappa,\varepsilon})}^t+1\right). \end{aligned}$$

Using Hölder inequality, we find for any \tilde{t} such that $t < \tilde{t} < s'$:

$$\sum_{I^{\kappa,\varepsilon}}\text{diam}(I^\kappa)^{2r}\|P^{\kappa,\varepsilon}\|_{L^t(I^{\kappa,\varepsilon})}^t \leq C\left(\int_U |\text{diam}(I_{x/\varepsilon,F^\kappa})|^p\right)^{2/p}\|P^{\kappa,\varepsilon}\|_{L^{\tilde{t}}(U)}^{t/\tilde{t}}$$

where $p = 2r(\tilde{t}/t)'$. The first factor is bounded thanks to the ergodic theorem and [14](#), while the second one is bounded thanks to the uniform bound for $P^{\kappa,\varepsilon}$ in $L^{\tilde{t}}(U)$. Inequality [34](#) follows.

Back to $K_2^{\kappa,\varepsilon}$, we deduce that,

$$|K_2^{\kappa,\varepsilon}| \leq C\varepsilon^{5/2-6/t}$$

which goes to zero taking t close enough to s' , by the condition $s < 6$. This concludes the proof of the theorem. \square

5. Discussion of the assumptions. We start here an extended discussion of the assumptions [H1-H2](#). We remind the definition of \mathcal{E} :

$$\mathcal{E}\left(F, \{u_I\}, \{b_{IJe}\}\right) = \sum_{I,J \in \text{CC}(F)} \sum_{\substack{e \in \text{Ed}(F) \\ I \xleftrightarrow{e} J}} \mu_e |b_{IJe} - b_{JIe} + u_I - u_J|^2 + \sum_{I \in \text{CC}(F)} |I| |u_I|^2.$$

It follows from this definition that for closed sets $\bar{F} = F \cup G$ where the union is disjoint (so that $Gr(\bar{F})$ is deduced from $Gr(F)$ by the addition of nodes and/or edges), one has

$$\mathcal{E}\left(F, \{u_I\}, \{b_{IJe}\}\right) \leq \mathcal{E}\left(\bar{F}, \{\bar{u}_I\}, \{\bar{b}_{I,J,e}\}\right) \quad (35)$$

for any extensions $\{\bar{u}_I\}, \{\bar{b}_{I,J,e}\}$ of $\{u_I\}, \{b_{IJe}\}$, meaning that

$$\bar{u}_I = u_I, \quad \forall I \in \text{CC}(F), \quad \bar{b}_{I,J,e} = b_{IJe}, \quad \forall I, J \in \text{CC}(F), \quad e \in \text{Ed}(F), \quad I \xleftrightarrow{e} J.$$

Indeed, the sum in the right-hand side of [35](#) has more (positive) terms than the one at the left-hand side.

5.1. Discussion around [H1](#).

[H1](#) for a short implies [H1](#). An important property of the discrete energy above concerns closed sets $F \subset F'$ with F' a short of F , cf. Definition [2.5](#). Given $\{u_{I'}\}$, indexed by $I' \in \text{CC}(F')$, a family associated to F' , one can associate a family $\{u_I\}$ indexed by $I \in \text{CC}(F)$ as follows:

$$u_I := \xi \cdot x_I + u_{I'} - \xi \cdot x_{I'}, \quad \forall I \in \text{CC}(F), \quad I' \in \text{CC}(F') \quad \text{such that } I \subset I'. \quad (36)$$

We remind that for any set S , x_S is the center of mass of S . Note that by definition of a short, any connected component of F' contains at least one connected component of F . We then claim that for $\text{CC}(F)$ finite,

$$\mathcal{E}(F, \{u_I\}, \{\xi \cdot x_I\}) \leq C \left(\mathcal{E}(F', \{u'_{I'}\}, \{\xi \cdot x_{I'}\}) + \sum_{I' \in \text{CC}(F')} |I'| \text{diam}(I')^2 \right). \quad (37)$$

Indeed, introducing $v_I := u_I - \xi \cdot x_I$, resp. $v'_{I'} = u_{I'} - \xi \cdot x_{I'}$, we write

$$\begin{aligned} \mathcal{E}(F, \{u_I\}, \{\xi \cdot x_I\}) &= \sum_{I, J \in \text{CC}(F)} \mu_{I, J} |v_I - v_J|^2 + \sum_{I \in \text{CC}(F)} |I| |v_I + \xi \cdot x_I|^2 \\ &=: E_1 + E_2, \end{aligned}$$

where $\mu_{I, J} = \sum_{e \in \text{Ed}(F), I \xleftrightarrow{e} J} \mu_e$. As $v_I = v_J$ when I, J are included in the same inclusion of F' , the first term can be bounded by

$$\begin{aligned} E_1 &\leq \sum_{I' \neq J' \in \text{CC}(F')} \sum_{\substack{I, J \in \text{CC}(F), \\ I \subset I', J \subset J'}} \mu_{I, J} |v_I - v_J|^2 \\ &\leq \sum_{I' \neq J' \in \text{CC}(F')} \mu_{I', J'} \sum_{\substack{I, J \in \text{CC}(F), \\ I \subset I', J \subset J', I \leftrightarrow J}} |v_I - v_J|^2 \\ &\leq C \sum_{I' \neq J' \in \text{CC}(F')} \mu_{I', J'} |v'_{I'} - v'_{J'}|^2 \\ &\leq C \mathcal{E}(F', \{u'_{I'}\}, \{\xi \cdot x_{I'}\}). \end{aligned}$$

For the third inequality, we have used the fact that $v_I = v'_{I'}$, $v_J = v'_{J'}$ by our definition of the family $\{u_I\}$, as well as assumption (G2): the number of gaps between two inclusions I' and J' is finite, so that

$$\#\{I, J \in \text{CC}(F), \quad I \subset I', J \subset J', I \leftrightarrow J\} \leq C, \quad \text{for some uniform constant } C.$$

Moreover, $v_I = v'_{I'}$, $v_J = v'_{J'}$ by our definition of the family $\{u_I\}$. Eventually,

$$\begin{aligned} E_2 &= \sum_{I' \in \text{CC}(F')} \sum_{I \in \text{CC}(F), I \subset I'} |I| |v'_{I'} + \xi \cdot x_I|^2 \\ &\leq 2 \sum_{I' \in \text{CC}(F')} \sum_{I \in \text{CC}(F), I \subset I'} |I| (|v'_{I'} + \xi \cdot x_{I'}|^2 + |\xi \cdot x_I - \xi \cdot x_{I'}|^2) \\ &\leq C \left(\sum_{I' \in \text{CC}(F')} |I'| |v'_{I'} + \xi \cdot x_{I'}|^2 + \sum_{I' \in \text{CC}(F')} |I'| \text{diam}(I')^2 \right) \end{aligned}$$

which yields 37. We are now ready to show

Lemma 5.1. *Let F' a short of F satisfying*

$$\begin{aligned} \limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \sum_{I' \in \text{CC}(F'_N)} |I'| \text{diam}(I')^2 &< +\infty \\ \sup_{I' \in \text{CC}(F'), I' \cap Q_N \neq \emptyset} \text{diam}(I') &= o(N) \end{aligned}$$

as well as H1. Then, F itself satisfies H1.

Remark 5. In the case where $F = F(\omega)$ is an admissible set of inclusions, and F' is an admissible short of F , see Definition 2.5, the first two conditions above are consequences of the relation

$$\mathbb{E} \operatorname{diam}(I_{0,F'})^3 < +\infty,$$

see the proof of 30.

Proof. One must realize once again that for $I \in \operatorname{CC}(F_N)$, and $I' \in \operatorname{CC}(F')$ such that $I \subset I'$, one does not have necessarily $I' \in \operatorname{CC}(F'_N)$. Indeed, it may happen that I is contained in Q_N , while I' crosses ∂Q_N . Still, by the second assumption of the lemma, $I' \in \operatorname{CC}(F'_{2N})$, for N large enough. It follows that F'_{2N} can be seen as the short of a closed set \bar{F}_N with $Q_{2N} \supset \bar{F}_N$, and $\bar{F}_N = F_N \cup G_N$ with a disjoint union.

Let now $\{u'_{I'}\}$, indexed by $I' \in \operatorname{CC}(F'_{2N})$, satisfying

$$\mathcal{E}(F'_{2N}, \{u'_{I'}\}, \{\xi \cdot x_{I'}\}) = \inf_{\{t'_{I'}\}} \mathcal{E}(F'_{2N}, \{t'_{I'}\}, \{\xi \cdot x_{I'}\}).$$

We associate to $\{u'_{I'}\}$ the family $\{u_I\}$, indexed by $I \in \bar{F}_N$, as in 36 (replacing F by \bar{F}_N and F' by F'_{2N}). By 37, we have

$$\mathcal{E}(\bar{F}_N, \{u_I\}, \{\xi \cdot x_I\}) \leq C \left(\mathcal{E}(F'_{2N}, \{u'_{I'}\}, \{\xi \cdot x_{I'}\}) + \sum_{I' \in \operatorname{CC}(F'_{2N})} |I'| \operatorname{diam}(I')^2 \right).$$

Furthermore, by using 35, we have

$$\mathcal{E}(F_N, \{u_I\}, \{\xi \cdot x_I\}) \leq \mathcal{E}(\bar{F}_N, \{u_I\}, \{\xi \cdot x_I\})$$

so that combining everything we find

$$\begin{aligned} & \inf_{\{t_I\}} \mathcal{E}(F_N, \{t_I\}, \{\xi \cdot x_I\}) \\ & \leq \mathcal{E}(F_N, \{u_I\}, \{\xi \cdot x_I\}) \leq \mathcal{E}(\bar{F}_N, \{u_I\}, \{\xi \cdot x_I\}) \\ & \leq C \left(\mathcal{E}(F'_{2N}, \{u'_{I'}\}, \{\xi \cdot x_{I'}\}) + \sum_{I' \in \operatorname{CC}(F'_{2N})} |I'| \operatorname{diam}(I')^2 \right) \\ & \leq C \left(\inf_{\{t'_{I'}\}} \mathcal{E}(F'_{N+2D'}, \{t'_{I'}\}, \{\xi \cdot x_{I'}\}) + \sum_{I' \in \operatorname{CC}(F'_{2N})} |I'| \operatorname{diam}(I')^2 \right). \end{aligned}$$

As F' satisfies the first assumption of the lemma and H1, dividing by $|Q_N|$ and sending N to infinity, we see that F satisfies H1. \square

Clusters with a moment bound on the diameter. We prove here

Lemma 5.2. *Let $F = F(\omega)$ an admissible set of inclusions satisfying the moment bound*

$$\mathbb{E} \operatorname{diam}(C_{0,F})^2 < +\infty$$

where $C_{0,F}$ is the cluster of F containing 0, cf. Remark 3. Then, F satisfies H1.

Proof. We want to consider the discrete energy $\mathcal{E}(F_N, \{u_I\}, \{\xi \cdot x_I\})$ for a suitable choice of u_I 's. For each cluster \mathcal{C} of F , and for each $I \in \operatorname{CC}(F_N)$, $I \subset \mathcal{C}$, we set

$u_I := -\xi \cdot (x_I - x_C)$, where as before x_S is the center of mass of S . Then,

$$\begin{aligned} \frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{\xi \cdot x_I\}) &= \frac{1}{|Q_N|} \sum_{I \in \text{CC}(F_N)} |I| |u_I|^2 \\ &\leq \frac{|\xi|^2}{|Q_N|} \sum_{\substack{C \\ \text{cluster of } F}} \sum_{\substack{I \in \text{CC}(F_N) \\ I \subset C}} |I| |x_I - x_C|^2 \leq \frac{|\xi|^2}{|Q_N|} \sum_{\substack{C \\ \text{cluster of } F}} \sum_{\substack{I \in \text{CC}(F_N) \\ I \subset C}} |I| \text{diam}(C)^2 \\ &\leq \frac{|\xi|^2}{|Q_N|} \sum_{\substack{C \\ \text{cluster of } F}} \sum_{\substack{I \in \text{CC}(F_N) \\ I \subset C}} \int_I \text{diam}(C_{x,F})^2 dx \leq \frac{|\xi|^2}{|Q_N|} \int_{Q_N} \text{diam}(C_{x,F})^2 dx. \end{aligned}$$

Sending N to infinity, we end up with

$$\limsup_N \frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{\xi \cdot x_I\}) \leq |\xi|^2 \mathbb{E} \text{diam}(C_{0,F})^2 < +\infty$$

which shows that F satisfies **H1** and concludes the proof of the lemma. \square

Link with the graph Laplacian. Another interesting result starts with the following observation. Assumption **H1** is verified by F if almost surely, there exists $M = M(\omega)$ such that for any $N > 0$, any $\xi \in \mathbb{R}^3$,

$$\inf_{\{u_I\}} \frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{\xi \cdot x_I\}) \leq M |\xi|^2.$$

Writing the Euler equations of the minimization problem at the left-hand side leads to the following linear system

$$u_I + \sum_{J \in \text{CC}(F_N)} \mu_{I,J} (u_I - u_J) = - \sum_{J \in \text{CC}(F_N)} \mu_{I,J} (\xi \cdot x_I - \xi \cdot x_J) \quad \forall I \in \text{CC}(F_N)$$

where we remind that $\mu_{I,J} := \sum_{e \in \text{Ed}(F), I \overset{e}{\leftrightarrow} J} \mu_e$ (and is therefore zero if I and J are not linked by an edge). This linear system can be written into the matrix form :

$$(\mathbb{L}_{F_N} + \mathbb{I})\mathbb{U} = -\mathbb{L}_{F_N}\mathbb{S}$$

where $\mathbb{U} = ((u_I)_{I \in \text{CC}(F_N)})^t$, $\mathbb{S} = (\xi \cdot x_I)_{I \in \text{CC}(F_N)}^t$ and \mathbb{L}_{F_N} is a symmetric matrix of size $|\text{CC}(F_N)| \times |\text{CC}(F_N)|$ defined by

$$[\mathbb{L}_{F_N}]_{IJ} = \begin{cases} \sum_{K \in \text{CC}(F_N)} \mu_{I,K} & \text{if } I = J \\ -\mu_{I,J} & \text{if } I \neq J \end{cases}.$$

This kind of matrix arises in the graph literature as the *weighted Laplacian matrix* for the pondered unoriented graph $Gr(F_N)$, see the first section of [3]. It can be seen as a discrete version of a continuous problem of the form

$$\begin{cases} -\text{div}(\mu \nabla u_N) + u_N = \text{div}(\mu \xi) \text{ in } Q_N \\ \mu \nabla u_N \cdot \nu = 0 \text{ on } \partial Q_N \end{cases} \iff u_N = \arg \min_{v \in H^1(Q_N)} \int_{Q_N} (\mu \nabla v \cdot \nabla v + |v|^2 - 2\xi \cdot \nabla v) dx.$$

The energy of this problem is a *superadditive quantity over sets* and one can expect our discrete minimization problem to verify a similar property. We state

Lemma 5.3. *Let \mathcal{P} a bounded set of \mathbb{R}^3 and denote*

$$F_{\mathcal{P}} = \bigcup_{\substack{I \in \text{CC}(F) \\ I \subset \mathcal{P}}} I, \quad Gr(F_{\mathcal{P}}) = (\text{CC}(F_{\mathcal{P}}), \text{Ed}(F_{\mathcal{P}})).$$

The quantity $\mathcal{H}(\mathcal{P}) := \inf_{\{u_I\}} \mathcal{E}(F_{\mathcal{P}}, \{u_I\}, \{\xi \cdot x_I\})$ is superadditive over sets, which means that for any decomposition $\mathcal{P} = \cup_{k=1}^M \mathcal{P}_k$ where $\{\mathcal{P}_1, \dots, \mathcal{P}_M\}$ are pairwise disjoint, one has almost surely

$$\mathcal{H}(\mathcal{P}) \geq \sum_{k=1}^M \mathcal{H}(\mathcal{P}_k).$$

Proof. It is enough to prove the results for a simple decomposition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, with a boundary Σ between \mathcal{P}_1 and \mathcal{P}_2 . This leads to the following decomposition of the nodes of the graph

$$\text{CC}(F_{\mathcal{P}}) = \text{CC}(F_{\mathcal{P}_1}) \cup \text{CC}(F_{\mathcal{P}_2}) \cup \text{CC}_{\Sigma}$$

where CC_{Σ} is the set of all connected components of $F_{\mathcal{P}}$ that intersect the boundary Σ without being included in \mathcal{P}_1 or \mathcal{P}_2 . The following figure explains the decomposition. In white is the graph $\text{Gr}(F_{\mathcal{P}_1})$ and in black is the graph $\text{Gr}(F_{\mathcal{P}_2})$. In dotted lines is what remains from the graph of $\text{CC}(F_{\mathcal{P}})$.

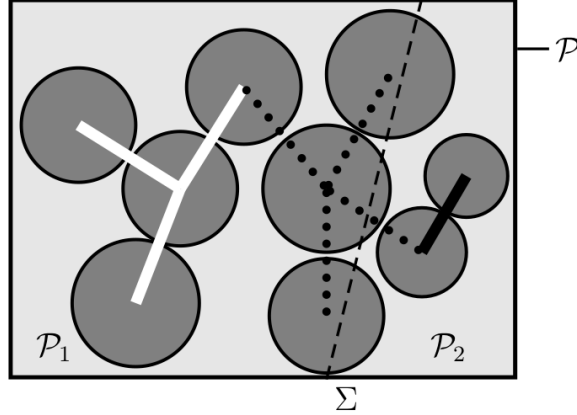


FIGURE 6. Separation of the domain $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ with the boundary Σ .

Let $(u_I)_{I \in \text{CC}(F_{\mathcal{P}})}$ the solution that minimizes $\mathcal{H}(\mathcal{P})$, that we split, up to some permutations, into a vector of the form $(u^1, u^2, u^{\Sigma}) \in \mathbb{R}^{|\text{CC}(F_{\mathcal{P}_1})|} \times \mathbb{R}^{|\text{CC}(F_{\mathcal{P}_2})|} \times \mathbb{R}^{|\text{CC}_{\Sigma}|}$. We compute

$$\begin{aligned} & \mathcal{H}(\mathcal{P}) - \mathcal{H}(\mathcal{P}_1) - \mathcal{H}(\mathcal{P}_2) \\ & \geq \mathcal{H}(\mathcal{P}) - \mathcal{E}(F_{\mathcal{P}_1}, u^1, \{\xi \cdot x_I\}) - \mathcal{E}(F_{\mathcal{P}_2}, u^2, \{\xi \cdot x_I\}) \\ & \geq \sum_{\substack{I \in \text{CC}(F_{\mathcal{P}_1}) \\ J \in \text{CC}_{\Sigma}}} \mu_{I,J} |\xi \cdot (x_I - x_J) + u_I^1 - u_J^{\Sigma}|^2 \\ & + \sum_{\substack{I \in \text{CC}(F_{\mathcal{P}_2}) \\ J \in \text{CC}_{\Sigma}}} \mu_{I,J} |\xi \cdot (x_I - x_J) + u_I^2 - u_J^{\Sigma}|^2 + \sum_{I \in \text{CC}_{\Sigma}} |I| |u_I^{\Sigma}|^2 \geq 0 \end{aligned}$$

which ends the argument. \square

We can then use the superadditive ergodic theorem (cf. [1, 14]), which yields a sufficient condition for **H1** to hold :

Proposition 5. *Assume that*

$$\sup_N \mathbb{E} \frac{1}{|Q_N|} \mathcal{H}(Q_N) < +\infty,$$

*then the admissible set F verifies assumption **H1**.*

5.2. Discussion around **H2**.

Logarithmic moment bound.

Lemma 5.4. *Let F a random closed set satisfying (G1)-(G2). Let $s \in (2, \infty)$ and $k = (\frac{s}{2})'$. If*

$$\limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \sum_{e \in \text{Ed}(F_N)} \mu_e^k < +\infty \quad (38)$$

*then F verifies **H2** with exponent s .*

Proof. Let $N \in \mathbb{N}^*$, and $\{b_{IJe}\}$ a family indexed by I, J in $\text{CC}(F_N)$ and $e \in \text{Ed}(F_N)$ such as $I \xleftrightarrow{e} J$. We want to control the quantity $\mathcal{E}(F_N, \{u_I\}, \{b_{IJe}\})$ by

$$|Q_N| \left(\frac{1}{|Q_N|} \sum_{I, J \in \text{CC}(F_N)} \sum_{e \in \text{Ed}(F_N), I \xleftrightarrow{e} J} |b_{IJe}|^s \right)^{2/s}$$

for a suitable choice of $\{u_I\}_{I \in \text{CC}(F_N)}$. With our logarithmic bound, we may take $u_I = 0$ for all I . We find

$$\mathcal{E}(F_N, \{u_I\}, \{b_{IJe}\}) = \sum_{I, J \in \text{CC}(F_N)} \sum_{e \in \text{Ed}(F_N), I \xleftrightarrow{e} J} |b_{IJe} - b_{JJe}|^2 \mu_e.$$

Using Hölder inequality with $k = \frac{s}{s-2}$ and $k' = \frac{s}{2}$, we have

$$\begin{aligned} & \sum_{I, J \in \text{CC}(F_N)} \sum_{e \in \text{Ed}(F_N), I \xleftrightarrow{e} J} |b_{IJe} - b_{JJe}|^2 \mu_e \\ & \leq \left(\sum_{I, J \in \text{CC}(F_N)} \sum_{e \in \text{Ed}(F_N), I \xleftrightarrow{e} J} \mu_e^k \right)^{\frac{1}{k}} \left(\sum_{I, J \in \text{CC}(F_N)} \sum_{e \in \text{Ed}(F_N), I \xleftrightarrow{e} J} |b_{IJe} - b_{JJe}|^s \right)^{\frac{2}{s}} \\ & \leq C |Q_N| \left(\frac{1}{|Q_N|} \sum_{e \in \text{Ed}(F_N)} \mu_e^k \right)^{\frac{s-2}{s}} \left(\frac{1}{|Q_N|} \sum_{I, J \in \text{CC}(F_N)} \sum_{e \in \text{Ed}(F_N), I \xleftrightarrow{e} J} |b_{IJe}|^s \right)^{\frac{2}{s}} \end{aligned}$$

which together with the bound **38** gives the expected result. \square

H2 implies **H1**.

Lemma 5.5. *Let F an admissible set of inclusions, F' an admissible short of F , $s > 3$. If*

$$\mathbb{E} \text{diam}(I_{0, F'})^s < +\infty$$

*and if **H2** is satisfied by F' , then **H1** is satisfied by F .*

Proof. By Lemma 5.1 and Remark 5, it is enough to show that F' satisfies H1. This can be seen by setting

$$b_{IJe} = \xi \cdot \left(x_I - \frac{x_{I,\alpha} + x_{J,\beta}}{2} \right),$$

$$\forall I, J \in \text{CC}(F'_N), \quad e = [x_{I,\alpha}, x_{J,\beta}] \in \text{Ed}(F'_N), \quad I \xleftrightarrow{e} J.$$

Clearly, from assumptions (G1)-(G2), one has

$$|b_{IJe}| \leq C|\xi|(\text{diam}(I) + \delta) \leq C' \text{diam}(I),$$

and $\sharp \text{Ed}(F'_N) \leq C|Q_N|$. It follows that

$$\begin{aligned} \frac{1}{|Q_N|} \sum_{I, J \in \text{CC}(F'_N)} \sum_{\substack{e \in \text{Ed}(F'_N), \\ I \xleftrightarrow{e} J}} |b_{IJe}|^s &\leq \frac{C}{|Q_N|} |\xi| \sum_{I \in \text{CC}(F_N)} \text{diam}(I)^s \\ &\leq \frac{C'}{|Q_N|} |\xi| \sum_{I \in \text{CC}(F_N)} |I| \text{diam}(I)^s \\ &\leq \frac{C'}{|Q_N|} |\xi| \int_{Q_N} \text{diam}(I_{x,F})^s dx. \end{aligned}$$

Thanks to the ergodic theorem, we end up with

$$\limsup_N \frac{1}{|Q_N|} \sum_{I, J \in \text{CC}(F_N)} \sum_{\substack{e \in \text{Ed}(F_N), \\ I \xleftrightarrow{e} J}} |b_{IJe}|^s \leq C' |\xi| \mathbb{E} I_{0,F}^s.$$

□

Cycle-free graphs. The previous lemma comes from a trivial choice of the family $\{u_I\}$. We will show that if the multigraph of inclusions is cycle-free, there is a better choice, that enables to relax the logarithmic moment bound, and prove Corollary 1.

Proof of Corollary 1. Let $\{b_{IJe}\}$ a family indexed by the triplet $I, J \in \text{CC}(F_N)$, $e \in \text{Ed}(F_N)$, with $I \xleftrightarrow{e} J$. As $Gr(F)$ is cycle-free, there is a single edge e linking the nodes I and J , so that we can note b_{IJ} instead of b_{IJe} for brevity. Given an arbitrary reference inclusion I_{C_N} in each cluster C_N of F_N , we then define a family $\{u_I\}$, $I \in \text{CC}(C_N)$, as follows. For all such inclusion, there is a unique integer $k_I \in \mathbb{N}$ and a unique branch $I_0 = I_{C_N}, \dots, I_{k_I} = I$ connecting I_{C_N} to I (with $k_I = 0$ in the case $I = I_{C_N}$). We define

$$u_I := \sum_{j=0}^{k_I-1} b'_{I,j}, \quad b'_{I,j} := b_{I_j I_{j+1}} - b_{I_{j+1} I_j}.$$

Note that in particular, $u_{I_{C_N}} = 0$. Doing this for all cluster C_N , we get a family $(u_I)_{I \in \text{CC}(F_N)}$. We then compute

$$\begin{aligned} \frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{b_{IJ}\}) &= \frac{1}{|Q_N|} \sum_{I \in \text{CC}(F_N)} |I| |u_I|^2 \\ &= \frac{1}{|Q_N|} \sum_{C_N} \sum_{I \in \text{CC}(C_N)} |I| \left| \sum_{j=0}^{k_I-1} b'_{I,j} \right|^2 \end{aligned}$$

$$\leq \frac{1}{|Q_N|} \sum_{\mathcal{C}_N} \sum_{I \in \text{CC}(\mathcal{C}_N)} |I| k_I \sum_{j=0}^{k_I-1} |b'_{I,j}|^2.$$

Using that $k_I \leq \#\mathcal{C}_N$, and the bound

$$\sum_{j=0}^{k_I-1} |b'_{I,j}|^2 \leq \sum_{[I,J] \in \text{Ed}(\mathcal{C}_N)} |b_{IJ} - b_{JI}|^2$$

we find

$$\begin{aligned} \frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{b_{IJ}\}) &\leq \frac{1}{|Q_N|} \sum_{\mathcal{C}_N} (\#\mathcal{C}_N) \sum_{[I,J] \in \text{Ed}(\mathcal{C})} |b_{IJ} - b_{JI}|^2 \sum_{I \in \text{CC}(\mathcal{C}_N)} |I| \\ &\leq \frac{1}{|Q_N|} \sum_{\mathcal{C}_N} (\#\mathcal{C}_N) |\mathcal{C}_N| \sum_{[I,J] \in \text{Ed}(\mathcal{C})} |b_{IJ} - b_{JI}|^2. \end{aligned}$$

We get, for any $p > 2$, denoting $\tilde{s} := \frac{p}{p-2}$ the conjugate exponent of $\frac{p}{2}$:

$$\begin{aligned} &\frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{b_{IJ}\}) \\ &\leq \left(\frac{1}{|Q_N|} \sum_{\mathcal{C}_N} \sum_{[I,J] \in \text{Ed}(\mathcal{C}_N)} |b_{IJ} - b_{JI}|^{2\tilde{s}} \right)^{\frac{1}{\tilde{s}}} \left(\frac{1}{|Q_N|} \sum_{\mathcal{C}_N} \sum_{[I,J] \in \text{Ed}(\mathcal{C}_N)} (\#\mathcal{C}_N)^{p/2} |\mathcal{C}_N|^{p/2} \right)^{\frac{2}{p}}. \end{aligned}$$

Thanks to the cycle-free hypothesis, we get that $\#\text{Ed}(\mathcal{C}_N) + 1 = \#\mathcal{C}_N$, so much that

$$\begin{aligned} &\frac{1}{|Q_N|} \mathcal{E}(F_N, \{u_I\}, \{b_{IJ}\}) \\ &\leq C \left(\frac{1}{|Q_N|} \sum_{[I,J] \in \text{Ed}(F_N)} |b_{IJ}|^{2\tilde{s}} \right)^{\frac{2}{\tilde{s}}} \left(\frac{1}{|Q_N|} \sum_{\mathcal{C}_N} (\#\mathcal{C}_N)^{p/2+1} |\mathcal{C}_N|^{p/2} \right)^{\frac{2}{p}}. \end{aligned}$$

Finally, we notice that $|\mathcal{C}_N| \leq C \#\mathcal{C}_N$, so that

$$\begin{aligned} \limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \sum_{\mathcal{C}_N} (\#\mathcal{C}_N)^{p/2+1} |\mathcal{C}_N|^{p/2} &\leq C \limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \sum_{\mathcal{C}_N} (\#\mathcal{C}_N)^p |\mathcal{C}_N| \\ &\leq C \limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \int_{Q_N} (\#C_{x,F_N})^p dx \\ &\leq C \limsup_{N \rightarrow +\infty} \frac{1}{|Q_N|} \int_{Q_N} (\#C_{x,F})^p dx \\ &= C \mathbb{E} \#C_{0,F}^p. \end{aligned}$$

This concludes the proof. \square

Appendix A. Keller function. The object of this appendix is the following

Lemma A.1. *Let P_1 and P_2 be two paraboloids be defined in cylindrical coordinates as*

$$P_1 := \{z \leq -ar^2\}, \quad P_2 := \{ar^2 + \nu \leq z\}, \quad \nu > 0 \text{ small.}$$

Furthermore we note $F_{12}(d)$ the gap of width $d > 0$ between the two paraboloids defined by

$$F_{12}(d) := \{r^2 \leq d^2, \quad -ar^2 \leq z \leq \nu + ar^2\}.$$

(see the picture below). Then there exists $w_\nu \in H^1(F_{12}(d) \cup P_1 \cup P_2)$ such that

$$0 \leq w_\nu \leq 1, \quad w_\nu|_{P_1} = 1, \quad w_\nu|_{P_2} = 0,$$

$$\int_{F_{12}(d)} |\nabla w_\nu|^2 dx \leq C \ln \frac{1}{\nu}, \quad \int_{F_{12}(d)} |\nabla w_\nu|^2 |x|^{2\gamma} dx \leq C_\gamma, \quad \forall \gamma > 0,$$

where C, C_γ are constants independent of ν .

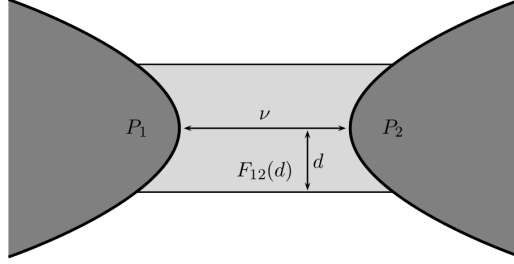


FIGURE 7. Geometry of the gap

Proof. For $x \in F_{12}(d)$, we set

$$w_\nu(x) = \frac{ar^2 + \nu - z}{2ar^2 + \nu}$$

that we extend by 1 on P_1 and 0 on P_2 . Clearly, w_ν is smooth, with values in $[0, 1]$. As regards the L^2 bound on the gradient, it is easily verified that all derivatives are bounded uniformly in ν in L^2 except for $\partial_z w_\nu$, for which one can compute:

$$\begin{aligned} \int_{F_{12}(d)} |\partial_z w_\nu|^2 &= 2\pi \int_0^d \int_{-ar^2}^{\nu+ar^2} \frac{r}{(2ar^2 + \nu)^2} dr dz \\ &= 2\pi \int_0^d \frac{r}{(2ar^2 + \nu)} dr \\ &\leq \frac{C}{4a} \ln \frac{1}{\nu} \end{aligned}$$

by direct computation of the integral. The other bounds can be computed similarly. Note that functions of the type of w_ν are sometimes referred as Keller functions, cf [28]. \square

Appendix B. Poincaré-Wirtinger inequality. We explain here how to obtain inequality (16) with $r = 12$ for inclusions I satisfying (G1). A crucial point, beside the regularity of the boundary, is that the inclusions do not shrink, thanks to the interior ball condition with uniform radius \mathbf{d} .

Our starting point is the following statement: given an open set U Lipschitz-diffeomorphic to the unit ball, with diffeomorphism ϕ satisfying

$$\|\nabla \phi\|_{L^\infty} + \|\nabla \phi^{-1}\|_{L^\infty} \leq M,$$

one has

$$\forall p \in [1, \infty), \quad \int_{U \times U} |f(x) - f(y)|^p dx dy \leq C_M \int_U |\nabla f(x)|^p dx$$

where C_M depends only on M . This can be seen by a direct adaptation of the proof of the Poincaré-Wirtinger inequality for convex domains given in [29, chapter 3, page 5]: just take $z_t = \phi^{-1}((1-t)\phi(x) + t\phi(y))$ in this proof, instead of $z_t = (1-t)x + ty$. It follows easily that for all $A, A' \subset U$, with $|A|, |A'| \geq \eta > 0$,

$$\|f - (f)_A\|_{L^p(U)} \leq C \|\nabla f\|_{L^p(U)}, \quad (39)$$

and then

$$|(f)_A - (f)_{A'}| \leq C \|\nabla f\|_{L^p(U)} \quad (40)$$

where C just depends on η (besides M).

To prove (16), we first introduce a covering $\mathring{I} = \cup_{k=1}^K O_k$, satisfying the following properties:

- i) for each k , O_k is an open set Lipschitz-diffeomorphic to the unit ball, with diffeomorphisms ϕ_k s.t. $\|\nabla \phi_k\|_{L^\infty} + \|\nabla \phi_k^{-1}\|_{L^\infty} \leq M$ for some M independent of k (and I).
- ii) The cardinal K of the covering is bounded by $C \text{diam}(I)^3$.

There are of course many possible choices to satisfy such properties. As regards i), our assumption (G1) ensures that we can cover a vicinity of the boundary by such type of open sets, while the remaining part of I can be covered directly by balls. As regards ii), if the covering satisfies i) and is uniformly locally finite, meaning:

$$\exists M_0 > 0, \quad \forall k, \quad \#\{k', O_k \cap O_{k'} \neq \emptyset\} \leq M_0$$

then the cardinal K is comparable to $|I|$, hence bounded by $C \text{diam}(I)^3$.

Then, given this covering, we write

$$\begin{aligned} \int_I |f - (f)_I|^p &\leq \frac{1}{|I|} \int_{I \times I} |f(x) - f(y)|^p dx dy \\ &= \frac{1}{|I|} \sum_{k, k'} \int_{O_k \times O_{k'}} |f(x) - f(y)|^p dx dy \\ &\leq C \text{diam}(I)^6 \sup_{k, k'} \int_{O_k \times O_{k'}} |f(x) - f(y)|^p dx dy \end{aligned}$$

where we used property ii) and the fact that $\frac{1}{|I|} \geq d^{-3}$, with the constant d in (G1). Now, for any fixed couple k, k' , we take a sequence $U_0 = O_k$, $U_1, \dots, U_J = O_{k'}$ such that

- U_j is a ball for all $j = 1, \dots, J-1$, whose radius is bounded by some $R > 0$ uniform in j and (k, k') .
- $|U_j \cap U_{j+1}| \geq \eta$, $\forall j = 0, \dots, J-1$, for some η uniform in j and k, k' .
- The cardinal J is bounded by $C' \text{diam}(I)^3$ for some constant uniform in k, k' .

We finally write: for all k, k' ,

$$\begin{aligned}
& \int_{O_k \times O_{k'}} |f(x) - f(y)|^p dx dy \\
&= \int_{O_k \times O_{k'}} |f(x) - (f)_{U_0 \cap U_1} + (f)_{U_0 \cap U_1} - (f)_{U_1 \cap U_2} + \cdots + (f)_{U_{J-1} \cap U_J} - f(y)|^p \\
&\leq C J^p \left(\int_{O_k} |f(x) - (f)_{U_0 \cap U_1}|^p dx + \sum_{j=0}^{J-1} |(f)_{U_j \cap U_{j+1}} - (f)_{U_{j+1} \cap U_{j+2}}|^p \right. \\
&\quad \left. + \int_{O_{k'}} |(f)_{U_{J-1} \cap U_J} - f(y)|^p dy \right) \\
&\leq C' J^p \sum_{j=0}^J \|\nabla f\|_{L^p(U_j)}^p \\
&\leq C' J^{p+1} \|\nabla f\|_{L^p(I)}^p \leq C'' \text{diam}(I)^{3p+3} \|\nabla f\|_{L^p(I)}^p,
\end{aligned}$$

where we used (39)-(40) for the second inequality.

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