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ASYMPTOTIC ANALYSIS OF AN ELASTIC MATERIAL REINFORCED WITH THIN FRACTAL STRIPS

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Abstract. We study the asymptotic behavior of a three-dimensional elastic material reinforced with highly contrasted thin vertical strips constructed on horizontal iterated Sierpinski gasket curves. We use Γ-convergence methods in order to study the asymptotic behavior of the composite as the thickness of the strips vanishes, their Lamé constants tend to infinity, and the sequence of the iterated curves converges to the Sierpinski gasket in the Hausdorff metric . We derive the effective energy of the composite. This energy contains new degrees of freedom implying a nonlocal effect associated with thin boundary layer phenomena taking place near the fractal strips and a singular energy term supported on the Sierpinski gasket.

1. **Introduction.** A Reinforced material is a composite building material consisting of two or more materials with different properties. The main objective of studies of reinforced materials is the prediction of their macroscopic behavior from the properties of their individual components as well as from their microstructural characteristics.

The theory of ideal fiber-reinforced composites was initiated by Adkins and Rivlin [4] who studied the deformation of a structure reinforced with thin, flexible and inextensible cords, which lie parallel and close together in smooth surfaces. This theory was further developed by the authors in [44], [1], [2], [3], [45].

The homogenization of elastic materials reinforced with highly contrasted inclusions has been considered by several authors in the two last decades (see for instance [6], [10], [17], [21], [18], and the references therein). The main result is that the materials obtained by the homogenization procedure have new elastic properties.

The homogenization of structures reinforced with fractal inclusions has been considered by various authors, among which [39], [31], [40], [41], [12], [42], [13], and [14]. Lancia, Mosco and Vivaldi studied in [31] the homogenization of transmission problems across highly conductive layers of iterated fractal curves. In [40], Mosco and Vivaldi dealt with the asymptotic behavior of a two-dimensional membrane reinforced with thin polygonal strips of large conductivity surrounding a pre-fractal

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curve obtained after n-iterations of the contractive similarities of the Sierpinski gasket. In [39], they considered an analogous problem with the Koch curve. The same authors considered in [41] a two-dimensional domain reinforced by an increasing number of thin conductive fibers developing a fractal geometry and studied the spectral asymptotic properties of conductive layered-thin-fibers of fractal nature in [42]. The homogenization of insulating fractal surfaces of Koch type approximated by three-dimensional insulating layers has been performed by Capitanelli et al. in [12], [13], and [14]. Due to the physical characteristics of the inclusions, singular energy forms containing fractal energies are obtained in these articles as the limit of non-singular full-dimensional energies. On the other hand, the effective properties of elastic materials fixed on rigid thin self-similar micro-inclusions disposed along two and three dimensional Sierpinski carpet fractals have been recently obtained in [20].

In the present work, we consider the deformation of a three-dimensional elastic material reinforced with highly contrasted thin vertical strips constructed on horizontal iterated Sierpinski gasket curves. Our main purpose is to describe the macroscopic behavior of the composite as the width of the strips tends to zero, their material coefficients tend to infinity, and the sequence of the iterated Sierpinski gasket curves converges to the Sierpinski gasket in the Hausdorff metric.

The asymptotic analysis of problems of this kind was previously studied in [11], [26], [9], and [5], where the authors considered media comprising low dimensional thin inclusions or thin layers of higher conductivity or higher rigidity. The limit problems consist in second order transmission problems. Problems involving thin highly conductive fractal inclusions have been addressed in a series of papers (see for instance [39], [31], [12], [14], and [19]). The obtained mathematical models are elliptic or parabolic boundary value problems involving transmission conditions of order two on the interfaces. The homogenization of three-dimensional elastic materials reinforced by highly rigid fibers with variable cross-section, which may have fractal geometry, has been carried out in [21] . The authors showed that the geometrical changes induced by the oscillations along the fiber-cross-sections can provide jumps of displacement fields or stress fields on interfaces, including fractal ones, due to local concentrations of elastic rigidities. Note that the numerical approximation of second order transmission problems across iterated fractal interfaces has been considered in some few papers among which [32] and [15].

Let us first consider the points $A_1 = (0,0)$, $A_2 = (1,0)$ and $A_3 = \left(1/2, \sqrt{3}/2\right)$ of the xy-plane. Let $\mathcal{V}_0 = \{A_1, A_2, A_3\}$ be the set of vertices of the equilateral triangle $A_1A_2A_3$ of side one. We define inductively

$$\mathcal{V}_{h+1} = \mathcal{V}_h \cup \left(2^{-h}A_2 + \mathcal{V}_h\right) \cup \left(2^{-h}A_3 + \mathcal{V}_h\right). \tag{1}$$

Let us set

$$\mathcal{V}_\infty = \bigcup_{h\in\mathbb{N}} \mathcal{V}_h. \tag{2}$$

The Sierpinski gasket, which is denoted here by Σ, is then defined (see for instance [30]) as the closure of the set $\mathcal{V}_\infty$, that is,

$$\Sigma = \overline{\mathcal{V}_\infty}. \tag{3}$$

We define the graph $\Sigma_h = (\mathcal{V}_h, S_h)$, where S_h is the set of edges $[p,q]$; $p,q \in \mathcal{V}_h$, such that $|p-q| = 2^{-h}$, where $|p-q|$ is the Euclidian distance between p and q. The graph Σ_h is then the standard aproximation of the Sierpinski gasket, which means that the sequence $(\Sigma_h)_h$ converges, as h tends to ∞, to the Sierpinski gasket Σ in the Hausdorff metric.

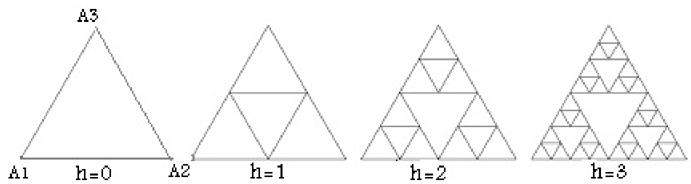

FIGURE 1. The graph Σ_h for $h = 0, 1, 2, 3$.

The edges which belong to S_h can be rearranged as S_h^k; $k = 1, 2, ..., N_h$, where $N_h = 3^{h+1}$.

Let ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\omega$ such that $\Sigma \subset \overline{\omega}$ and

$$\Sigma \cap \partial\omega = \mathcal{V}_0. \tag{4}$$

Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of positive numbers, such that

$$\lim_{h\to\infty} \varepsilon_h 2^h = 0. \tag{5}$$

We define

$$T_h^k = \left(\omega \cap S_h^k\right) \times (-\varepsilon_h, \varepsilon_h) \tag{6}$$

and set

$$T_h = \bigcup_{k\in I_h} T_h^k, \tag{7}$$

where $I_h = \{1, 2, ..., N_h\}$. Denoting $|T_h|$ the 2-dimensional measure of T_h, one can see that

$$|T_h| = \frac{\varepsilon_h 3^{h+1}}{2^h}. \tag{8}$$

Let $\Omega = \omega \times (-1, 1)$. We suppose that $\Omega \backslash T_h$ is the reference configuration of a linear, homogeneous and isotropic elastic material with Lamé coefficients $\mu > 0$ and $\lambda \geq 0$. This means that the deformation tensor $e(u) = (e_{ij}(u))_{i,j=1,2,3}$, with $e_{ij}(u) = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$ for some displacement u, is linked to the stress tensor $\sigma(u) = (\sigma_{ij}(u))_{i,j=1,2,3}$ through Hooke's law

$$\sigma_{ij}(u) = \lambda e_{mm}(u)\,\delta_{ij} + 2\mu e_{ij}(u) \ ; i, j = 1, 2, 3, \tag{9}$$

where the summation convention with respect to repeated indices has been used and will be used in the sequel, and δ_{ij} denotes Kronecker's symbol. We suppose that T_h is the reference configuration of a linear, homogeneous and isotropic elastic material with $\sigma^h(u) = \left(\sigma_{ij}^h(u)\right)_{i,j=1,2,3}$:

$$\sigma_{ij}^h(u) = \lambda_h e_{mm}(u)\,\delta_{ij} + 2\mu_h e_{ij}(u) \ ; i, j = 1, 2, 3,$$

with

$$\lambda_h = \eta_h\lambda_0 \text{ and } \mu_h = \eta_h\mu_0, \tag{10}$$

where λ_0 and μ_0 are positive constants and

$$\eta_h = \frac{1}{\varepsilon_h}\left(\frac{5}{6}\right)^h. \tag{11}$$

The special scaling (10) and (11) of the Lamé-coefficients depend on the structural constants of T_h. The choice of η_h is dictated by the lower bound inequality of assertion 3 of Proposition 6, which will play a crucial role in the asymptotic behavior of the energy F_h.

We suppose that a perfect adhesion occurs between $\Omega\backslash T_h$ and T_h along their common interfaces. We suppose that the material in Ω is submitted to volumic forces with density $f \in L^2\left(\Omega, \mathbb{R}^3\right)$ and is held fixed on $\partial\Omega$. We define the energy functional F_h on $L^2\left(\Omega, \mathbb{R}^3\right)$ through

$$F_h(u) = \begin{cases} \int_{\Omega\backslash T_h} \sigma_{ij}(u)\, e_{ij}(u)\, dx + \int_{T_h} \sigma_{ij}^h(u)\, e_{ij}(u)\, ds dx_3 \\ \qquad\qquad\qquad \text{if } u \in H_0^1\left(\Omega, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right), \\ +\infty \qquad\qquad\quad \text{otherwise,} \end{cases} \tag{12}$$

where ds is the one-dimensional Lebesgue measure on the line segments S_h^k; $k = 1, 2, ..., N_h$. The equilibrium state in Ω is described by the minimization problem

$$\min_{u\in L^2(\Omega,\mathbb{R}^3)\cap L^2(T_h,\mathbb{R}^3)} \left\{ F_h(u) - 2\int_\Omega f.udx \right\}. \tag{13}$$

We use Γ-convergence methods (see for instance [5] and [16]) in order to describe the asymptotic behavior of problem (13) as h goes to ∞. According to the critical term

$$\gamma = \lim_{h\to\infty}\left(-\frac{3^{h+1}}{2^h \ln \varepsilon_h}\right), \tag{14}$$

which is associated with the size of the boundary layers taking place in the neighbourhoods of the fractal strips, we prove that if $\gamma \in (0, +\infty)$ then the effective energy of the composite is given by

$$F_\infty(u,v) = \begin{cases} \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \mu_0 \int_\Sigma d\mathcal{L}_\Sigma(\overline{v}) \\ +\dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma A(s)(u-v).(u-v)\, d\mathcal{H}^d(s) \\ \qquad\qquad \text{if } (u,v) \in H_0^1\left(\Omega, \mathbb{R}^3\right) \times \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma), \\ +\infty \qquad\quad \text{otherwise,} \end{cases} \tag{15}$$

where $\overline{v} = (v_1, v_2)$, $\mathcal{L}_\Sigma(\overline{v})$ is a quadratic measure-valued gradient form supported on Σ (see Proposition 1 in the next Section), $\mathcal{H}^d$ is the d-dimensional Hausdorff measure; d being the fractal dimension of Σ with

$$d = \ln 3/\ln 2, \tag{16}$$

$\mathcal{D}_{\Sigma,\mathcal{E}}$ is the domain of the energy supported on the fractal Σ (see (27) in the next Section), and

$$A(s)=\begin{cases}\operatorname{Diag}\left(1,\frac{2}{(1+\kappa)},\frac{2}{(1+\kappa)}\right) & \text{if } n(s)=\pm(0,1),\\ \begin{pmatrix}\frac{7+\kappa}{4(1+\kappa)} & \frac{\sqrt{3}(\kappa-1)}{4(1+\kappa)} & 0\\ \frac{\sqrt{3}(\kappa-1)}{4(1+\kappa)} & \frac{3\kappa+5}{4(1+\kappa)} & 0\\ 0 & 0 & \frac{2}{(1+\kappa)}\end{pmatrix} & \text{if } n(s)=\pm\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right),\\ \begin{pmatrix}\frac{7+\kappa}{4(1+\kappa)} & \frac{\sqrt{3}(1-\kappa)}{4(1+\kappa)} & 0\\ \frac{\sqrt{3}(1-\kappa)}{4(1+\kappa)} & \frac{3\kappa+5}{4(1+\kappa)} & 0\\ 0 & 0 & \frac{2}{(1+\kappa)}\end{pmatrix} & \text{if } n(s)=\pm\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right),\end{cases}\tag{17}$$

where $\kappa=\dfrac{3\mu+\lambda}{\mu+\lambda}$ and $n(s)$ is the unit normal on $s\in\Sigma$.

The effective energy (15) contains new degrees of freedom implying nonlocal effects associated with thin boundary layer phenomena taking place near the fractal strips and a singular energy term supported on the Sierpinski gasket Σ. The equilibrium of the fractal Σ is asymptotically described by a generalized Laplace equation which is related to the discontinuity of the effective stresses through the following relations (see Corollary 1):

$$\begin{cases}[\sigma_{\alpha 3}|_{x_3=0}]_\Sigma=\dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}A_{\alpha\beta}(s)(U_\beta-V_\beta)\,\mathcal{H}^d \text{ on } \Sigma,\\ \dfrac{\pi\mu\gamma}{(\ln 2)^2}A_{\alpha\beta}(s)(U_\beta-V_\beta)=-\mu_0\Delta_\Sigma V_\alpha \text{ in } \Sigma;\ \alpha,\beta=1,2,\end{cases}\tag{18}$$

where Δ_Σ is the Laplace operator on the Sierpinski gasket, that is, the second order operator in $L^2_{\mathcal{H}^d}(\Sigma,\mathbb{R}^2)$ defined by the form $\mathcal{E}_\Sigma$ in Lemma 2.1 in the next Section under the Dirichlet condition $V_\alpha=0$ on $\mathcal{V}_0=\Sigma\cap\partial\omega$; $\alpha=1,2$, μ_0 is the effective shear modulus of the material occupying the fractal Σ,

$$[\sigma_{\alpha 3}|_{x_3=0}]_\Sigma=\sigma_{\alpha 3}|_{\Sigma\times\{0^+\}}-\sigma_{\alpha 3}|_{\Sigma\times\{0^-\}};\ \alpha=1,2,\tag{19}$$

is the jump of $\sigma_{\alpha 3}|_{x_3=0}$ on $\Sigma\cap\partial\mathcal{T}_m$; $\{\mathcal{T}_m\}_{m\in\mathbb{N}}$ being the network of the interiors of the triangles which are contained in the Sierpinski gasket Σ (see Figure 2).

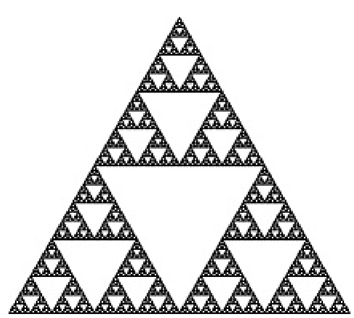

FIGURE 2. The network $\{\mathcal{T}_m\}_{m\in\mathbb{N}}$ where $\sigma_{\alpha 3}|_{\Sigma\times\{0^+\}}$ is the outward normal stress on $\Sigma\cap\partial\mathcal{T}_m$ and $-\sigma_{\alpha 3}|_{\Sigma\times\{0^-\}}$ is the inward normal stress.

If $\gamma = +\infty$ then, for every $(u, v) \in H_0^1(\Omega, \mathbb{R}^3) \times \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)$, $F_\infty(u, v) < \infty \Rightarrow u = v$ on Σ. In this case the energy supported on the structure is given by

$$F_\infty(u) = \begin{cases} \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \mu_0 \int_\Sigma d\mathcal{L}_\Sigma(\overline{u}) \\ \qquad\qquad \text{if } u \in H_0^1(\Omega, \mathbb{R}^3) \cap \left(\mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)\right), \\ +\infty \qquad\qquad \text{otherwise.} \end{cases} \tag{20}$$

If $\gamma = 0$ the displacements u and v are independent. In this case the effective energy of the structure turns out to be

$$F_0(u) = \begin{cases} \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx & \text{if } u \in H_0^1(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{21}$$

The paper is organized as follows: in Section 2 we introduce the energy form and the notion of a measure-valued local energy on the Sierpinski gasket Σ. Section 3 is devoted to compactness results which is useful for the proof of the main result. In Section 4 we formulate the main result of this work. Section 5 is consacred to the proof of the main result. This proof is developed in 3 Subsections: in the first Subsection we study the boundary layers at the interface matrix/strips, in the second Subsection we establish the first condition of the Γ-convergence property, and in the third Subsection we prove the second condition of the Γ-convergence property.

2. **The energy form on the Sierpinski gasket.** In this Section we introduce the energy form and the notion of a measure-valued local energy (or Lagrangian) on the Sierpinski gasket. For the definition and properties of Dirichlet forms and measure energies we refer to [24], [35], and [37].

For any function $w : \mathcal{V}_\infty \longrightarrow \mathbb{R}^2$ we define

$$\mathcal{E}_\Sigma^h(w) = \left(\frac{5}{3}\right)^h \sum_{\substack{p,q \in \mathcal{V}_h \\ |p-q| = 2^{-h}}} |w(p) - w(q)|^2 . \tag{22}$$

Let us define the energy

$$\mathcal{E}_\Sigma(z) = \lim_{h\to\infty} \mathcal{E}_\Sigma^h(z), \tag{23}$$

with domain $\mathcal{D}_\infty = \left\{z : \mathcal{V}_\infty \longrightarrow \mathbb{R}^2 : \mathcal{E}_\Sigma(z) < \infty\right\}$. Every function $z \in \mathcal{D}_\infty$ can be uniquely extended to be an element of $C\left(\Sigma, \mathbb{R}^2\right)$, still denoted z. Let us set

$$\mathcal{D} = \left\{z \in C\left(\Sigma, \mathbb{R}^2\right) : \mathcal{E}_\Sigma(z) < \infty\right\}, \tag{24}$$

where $\mathcal{E}_\Sigma(z) = \mathcal{E}_\Sigma(z\,|_{\mathcal{V}_\infty})$. Then $\mathcal{D} \subset C\left(\Sigma, \mathbb{R}^2\right) \subset L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$. We define the space $\mathcal{D}_\mathcal{E}$ as

$$\mathcal{D}_\mathcal{E} = \overline{\mathcal{D}}^{\|\cdot\|_{\mathcal{D}_\mathcal{E}}}, \tag{25}$$

where $\|.\|_{\mathcal{D}_\mathcal{E}}$ is the intrinsic norm

$$\|z\|_{\mathcal{D}_\mathcal{E}} = \left\{\mathcal{E}_\Sigma(z) + \|z\|^2_{L^2_{\mathcal{H}^d}(\Sigma,\mathbb{R}^2)}\right\}^{1/2}. \tag{26}$$

The space $\mathcal{D}_\mathcal{E}$ is injected in $L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$ and is an Hilbert space with the scalar product associated to the norm (26). Let us now define the space

$$\mathcal{D}_{\Sigma,\mathcal{E}} = \left\{z \in \mathcal{D}_\mathcal{E} : z(A_1) = z(A_2) = z(A_3) = 0\right\}. \tag{27}$$

We denote $\mathcal{E}_\Sigma(.,.)$ the bilinear form defined on $\mathcal{D}_{\Sigma,\mathcal{E}} \times \mathcal{D}_{\Sigma,\mathcal{E}}$ by

$$\mathcal{E}_\Sigma(w,z) = \frac{1}{2}\left(\mathcal{E}_\Sigma(w+z) - \mathcal{E}_\Sigma(w) - \mathcal{E}_\Sigma(z)\right), \forall w,z \in \mathcal{D}_{\Sigma,\mathcal{E}}. \tag{28}$$

One can see that

$$\mathcal{E}_\Sigma(w,z) = \lim_{h\to\infty} \mathcal{E}_\Sigma^h(w,z), \tag{29}$$

where

$$\mathcal{E}_\Sigma^h(w,z) = \left(\frac{5}{3}\right)^h \sum_{\substack{p,q\in\mathcal{V}_h \\ |p-q|=2^{-h}}} (w(p)-w(q)).(z(p)-z(q)). \tag{30}$$

The form $\mathcal{E}_\Sigma(.,.)$ is a closed Dirichlet form in the Hilbert space $L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^2\right)$ and, according to [25, Theorem 4.1], $\mathcal{E}_\Sigma(.,.)$ is a local regular Dirichlet form in $L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^2\right)$, which means that

1. (local property) $w,z \in \mathcal{D}_{\Sigma,\mathcal{E}}$ with supp$[w]$ and supp$[z]$ are disjoint compact sets $\Longrightarrow E_\Sigma(w,z) = 0$,
2. (regularity) $\mathcal{D}_{\Sigma,\mathcal{E}} \cap C_0\left(\Sigma,\mathbb{R}^2\right)$ is dense both in $C_0\left(\Sigma,\mathbb{R}^2\right)$ (the space of functions of $C\left(\Sigma,\mathbb{R}^2\right)$ with compact support) with respect to the uniform norm and in $\mathcal{D}_{\Sigma,\mathcal{E}}$ with respect to the intrinsic norm (26).

The second property implies that $\mathcal{D}_{\Sigma,\mathcal{E}}$ is not trivial (that is $\mathcal{D}_{\Sigma,\mathcal{E}}$ is not made by only the constant functions). Moreover, every function of $\mathcal{D}_{\Sigma,\mathcal{E}}$ possesses a continuous representative. Indeed, according to [36, Theorem 6.3. and example 7_1], the space $\mathcal{D}_{\Sigma,\mathcal{E}}$ is continuously embedded in the space $C^\beta\left(\Sigma,\mathbb{R}^2\right)$ of Hölder continuous functions with $\beta = \ln\frac{5}{3}/\ln 4$.

Now, applying [29, Chap. 6], we have the following result:

Lemma 2.1. *There exists a unique self-adjoint operator $\boldsymbol{\Delta}_\Sigma$ on $L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^2\right)$ with domain*

$$\mathcal{D}_{\Delta_\Sigma} = \left\{ \begin{array}{l} w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^2\right): \\ \boldsymbol{\Delta}_\Sigma w = \begin{pmatrix} \Delta_\Sigma w_1 \\ \Delta_\Sigma w_2 \end{pmatrix} \in L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^2\right) \end{array} \right\} \subset \mathcal{D}_{\Sigma,\mathcal{E}}$$

dense in $L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^2\right)$, such that, for every $w \in \mathcal{D}_{\Delta_\Sigma}$ and $z \in \mathcal{D}_{\Sigma,\mathcal{E}}$,

$$\mathcal{E}_\Sigma(w,z) = -\int_\Sigma (\Delta_\Sigma w).z\frac{d\mathcal{H}^d}{\mathcal{H}^d(\Sigma)}.$$

Let us consider the sequence $(\nu_h)_h$ of measures defined by

$$\nu_h = \frac{1}{Card(\mathcal{V}_h)} \sum_{p\in\mathcal{V}_h} \delta_p, \tag{31}$$

where $Card(\mathcal{V}_h)) = \dfrac{3^{h+1}+3}{2}$ is the number of verticies of $\mathcal{V}_h$ and δ_p is the Dirac measure at the point p. We have the following result:

Lemma 2.2. *The sequence $(\nu_h)_h$ weakly converges in $C(\Sigma)^*$ to the measure*

$$\nu = \mathbf{1}_\Sigma(s)\frac{d\mathcal{H}^d(s)}{\mathcal{H}^d(\Sigma)},$$

where $C(\Sigma)^*$ is the topological dual of the space $C(\Sigma)$.

Proof. Let $\varphi \in C(\Sigma)$. Then, according to the ergodicity result of [22, Theorem 6.1],

$$\begin{aligned}\lim_{h\to\infty}\int_\Sigma \varphi(x)\,d\nu_h &= \lim_{h\to\infty}\sum_{p\in\mathcal{V}_h}\frac{\varphi(p)}{Card(\mathcal{V}_h)}\\ &= \frac{1}{\mathcal{H}^d(\Sigma)}\int_\Sigma \varphi(s,0)\,d\mathcal{H}^d(s).\end{aligned}$$

□

We note that the approximating form $\mathcal{E}_\Sigma^h(.,.)$ can be written as

$$\mathcal{E}_\Sigma^h(w,z)=\int_\Sigma \nabla_h w.\nabla_h z\ d\nu_h, \tag{32}$$

where ν_h is the measure defined in (31) and

$$\nabla_h w.\nabla_h z(p)=\sum_{q:\ |p-q|=2^{-h}}\frac{(w(p)-w(q))}{|p-q|^{\varkappa/2}}.\frac{(z(p)-z(q))}{|p-q|^{\varkappa/2}},$$

where $\varkappa$ is the unique positive number for which the sequence $\left(\mathcal{E}_\Sigma^h(.,.)\right)_h$ has a non trivial limit (see [38] for more details). We note that, according to equality (22), $\varkappa=\ln\dfrac{5}{3}/\ln 2$. We have the following result:

Proposition 1. *For every $w,z\in\mathcal{D}_{\Sigma,\mathcal{E}}$, the sequence of measures $\left(\mathcal{L}_\Sigma^h(w,z)\right)_h$ defined by*

$$\mathcal{L}_\Sigma^h(w,z)(A)=\int_{A\cap\Sigma}\nabla_h w.\nabla_h z\ d\nu_h,\ \forall A\subset\Sigma,$$

weakly converges in the topological dual $C\left(\Sigma,\mathbb{R}^2\right)^*$ of the space $C\left(\Sigma,\mathbb{R}^2\right)$ to a signed finite Radon measure $\mathcal{L}_\Sigma(w,z)$ on Σ, called Lagrangian measure on Σ. Moreover,

$$\mathcal{E}_\Sigma(w,z)=\int_\Sigma d\mathcal{L}_\Sigma(w,z),\ \forall w,z\in\mathcal{D}_{\Sigma,\mathcal{E}}.$$

Proof. The proof follows the lines of the proof of [23, Proposition 2.3] for the von Koch snowflake. Let us set, for every $w\in\mathcal{D}_{\Sigma,\mathcal{E}}$, $\mathcal{L}_\Sigma^h(w)=\mathcal{L}_\Sigma^h(w,w)$. We deduce from (23), (29), and (32) that $\left(\mathcal{L}_\Sigma^h(w)(\Sigma)\right)_h$ is a uniformly bounded sequence. Then, observing that, for every $w\in\mathcal{D}_{\Sigma,\mathcal{E}}$ and every $\varphi e_1\in\mathcal{D}_{\Sigma,\mathcal{E}}\cap C\left(\Sigma,\mathbb{R}^2\right)$, with $e_1=(1,0)$,

$$\int_\Sigma \varphi d\mathcal{L}_\Sigma^h(w)=\mathcal{E}_\Sigma^h(\varphi w,w)-\frac{1}{2}\mathcal{E}_\Sigma^h\left(\varphi e_1,|w|^2 e_1\right), \tag{33}$$

we deduce, taking into account the regularity of the form $\mathcal{E}_\Sigma(.,.)$, that

$$\lim_{h\to\infty}\int_\Sigma \varphi d\mathcal{L}_\Sigma^h(w)=\mathcal{E}_\Sigma(\varphi w,w)-\frac{1}{2}\mathcal{E}_\Sigma\left(\varphi e_1,|w|^2 e_1\right). \tag{34}$$

On the other hand, according to [33, Proposition 1.4.1], the energy form $\mathcal{E}_\Sigma(w)$, which is a Dirichlet form of diffusion type, admits the following integral representation:

$$\mathcal{E}_\Sigma(w)=\int_\Sigma d\mathcal{L}_\Sigma(w), \tag{35}$$

where $\mathcal{L}_\Sigma(w)$ is a positive Radon measure which is uniquely determined by the relation

$$\int_\Sigma \varphi d\mathcal{L}_\Sigma(w)=\mathcal{E}_\Sigma(\varphi w,w)-\frac{1}{2}\mathcal{E}_\Sigma\left(\varphi e_1,|w|^2 e_1\right),\ \forall\varphi\in\mathcal{D}_{\Sigma,\mathcal{E}}\cap C(\Sigma).$$

Thus, combining with (34), the sequence $\left(\mathcal{L}^h_\Sigma(w)\right)_h$ converges in the sense of measures to the measure $\mathcal{L}_\Sigma(w)$. Now, observing that

$$\mathcal{L}^h_\Sigma(w,z) = \frac{1}{2}\left(\mathcal{L}^h_\Sigma(w+z) - \mathcal{L}^h_\Sigma(w) - \mathcal{L}^h_\Sigma(z)\right),$$

we deduce that the sequence $\left(\mathcal{L}^h_\Sigma(w,z)\right)_h$ weakly converges in $C\left(\Sigma,\mathbb{R}^2\right)^*$ to the measure $\mathcal{L}_\Sigma(w,z)$. □

3. **Compactness results.** In this Section we establish the compactness results which is very useful for the proof of the main homogenization result.

Lemma 3.1. *For every sequence $(u_h)_h$; $u_h \in H^1_0\left(\Omega,\mathbb{R}^3\right) \cap H^1\left(T_h,\mathbb{R}^3\right)$, such that $\sup_h F_h(u_h) < +\infty$, we have*

1. $\sup_h \|u_h\|_{H^1_0(\Omega,\mathbb{R}^3)} < +\infty$,
2. $\dfrac{1}{|T_h|}\int_{T_h}|u_h|^2\,dsdx_3 \le C\left\{\|u_h\|^2_{L^2(\Omega,\mathbb{R}^3)} - \dfrac{2^h}{3^{h+1}}\ln\varepsilon_h\right\}$, *where C is a positive constant independent of h.*

Proof. 1. Observing that

$$F_h(u_h) \ge \int_\Omega \sigma_{ij}(u_h)\,e_{ij}(u_h)\,dx,$$

we have, using Korn's inequality (see for instance [43]), that

$$\sup_h \int_\Omega |\nabla u_h|^2\,dx < +\infty. \tag{36}$$

2. Let n^k be the unit normal to S^k_h; $k \in I_h$. Then $n^k = \pm\left(-\sqrt{3}/2, 1/2\right)$, $n^k = \pm\left(\sqrt{3}/2, 1/2\right)$ or $n^k = \pm(0,1)$. Let us denote $s, s^\perp$ the local coordinates defined by

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ if } S^k_h \perp \left(-\sqrt{3}/2, 1/2\right), \tag{37}$$

by

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ if } S^k_h \perp \left(\sqrt{3}/2, 1/2\right), \tag{38}$$

and by

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ if } S^k_h \perp (0,1), \tag{39}$$

where the symbol $\perp$ represents the direction normal to the edge S^k_h. Let $s^k_h = \left(s^k_{1h}, s^k_{2h}\right)$ denotes the center of S^k_h; $k \in I_h$, in the new coordinates. We define, for $\theta \in [0, 2\pi)$ and $r \in (0, \varepsilon_h/2)$,

$$u^k_h(x_1(s), x_2(s), r, \theta) = u_h\left(s, r\sin\theta + s^k_{2h}, r\cos\theta\right). \tag{40}$$

Then, according to (36), we have, for $r_1 \le r_2 < \varepsilon_h/2$ and $\theta \in [0, 2\pi)$,

$$\sum_{k \in I_h}\int_{S^k_h}\int_{r_1}^{r_2}\left|\frac{\partial u^k_h(x_1(s), x_2(s), r, \theta)}{\partial r}\right|^2 r\,dr\,ds \le C. \tag{41}$$

Solving the Euler equation of the following one dimensional minimization problem:

$$\min\left\{\int_{r_1}^{r_2}(\psi')^2\,r\,dr\colon\ \psi(r_1) = 0,\ \psi(r_2) = 1\right\},$$

we deduce that, for every $\theta \in [0, 2\pi)$,

$$\begin{aligned} \ln \frac{r_2}{r_1} \int_{r_1}^{r_2} \left| \frac{\partial u_h^k (x_1(s), x_2(s), r, \theta)}{\partial r} \right|^2 r dr \\ \geq \left| u_h^k (x_1(s), x_2(s), r_2, \theta) - u_h^k (x_1(s), x_2(s), r_1, \theta) \right|^2. \end{aligned} \tag{42}$$

Then, using (41) and (42), we obtain that

$$\begin{aligned} \sum_{k \in I_h} \int_{S_h^k} \int_0^{2\pi} \left| u_h^k (x_1(s), x_2(s), r_2, \theta) - u_h^k (x_1(s), x_2(s), r_1, \theta) \right|^2 d\theta ds \\ \leq C \ln \frac{r_2}{r_1}. \end{aligned} \tag{43}$$

Let us define

$$F(r, \theta) = \sum_{k \in I_h} \int_{S_h^k} \left| u_h^k (x_1(s), x_2(s), r, \theta) \right|^2 ds. \tag{44}$$

We deduce from the inequality (43) that, for $r_1 \leq r_2 < \varepsilon_h/2$,

$$\int_0^{2\pi} F(r_1, \theta) d\theta \leq C \left(\int_0^{2\pi} F(r_2, \theta) d\theta + \ln \frac{r_2}{r_1} \right). \tag{45}$$

Observing that, for $k \in I_h$ and θ_0 fixed in $[0, 2\pi)$,

$$\begin{aligned} \left| u_h^k (x_1(s), x_2(s), r, \theta) - u_h^k (x_1(s), x_2(s), r, \theta_0) \right|^2 \\ = \left| \int_{\theta_0}^{\theta} \frac{\partial u_h^k}{\partial \theta} (x_1(s), x_2(s), r, \phi) d\phi \right|^2 \\ \leq Cr \int_0^{2\pi} \left| \frac{1}{r} \frac{\partial u_h^k}{\partial \theta} (x_1(s), x_2(s), r, \phi) \right|^2 r d\phi, \end{aligned}$$

we deduce that

$$\begin{aligned} \sum_{k \in I_h} \int_{S_h^k} \int_0^{2\pi} \int_0^{\varepsilon_h} \left| u_h^k (x'(s), r, \theta) - u_h^k (x'(s), r, \theta_0) \right|^2 dr d\theta ds \\ \leq \; C\varepsilon_h \sum_{k \in I_h} \int_{C_h^k} \left| \nabla u_h \right|^2 dx_1 dx_2 dx_3 \\ \leq C\varepsilon_h \int_\Omega \left| \nabla u_h \right|^2 dx, \end{aligned} \tag{46}$$

where $x'(s) = (x_1(s), x_2(s))$, C_h^k is the cylinder of radius ε_h around the edge S_h^k. This estimate implies that

$$\int_0^{\varepsilon_h} F(\rho, \theta_0) dr d\theta \quad \leq \quad C \left\{ \int_0^{2\pi} \int_0^{\varepsilon_h} F(r, \theta) dr d\theta + \varepsilon_h \right\}. \tag{47}$$

Now, using (45) and (47), we deduce, by setting $m_h = \dfrac{1}{\varepsilon_h} \dfrac{2^h}{3^{h+1}}$, that, for $\theta_0 = 0$ and π, and for every $r_2 \in [a_h, b_h]$; $a_h = \dfrac{1}{4\sqrt{3}} \left(\dfrac{2}{3} \right)^{h/2}$ and $b_h = 2a_h$,

$$\begin{aligned} m_h \int_{T_h} |u_h|^2 ds dx_3 \quad &= \quad m_h \int_0^{\varepsilon_h} F(r, 0) dr + m_h \int_0^{\varepsilon_h} F(r, \pi) dr \\ &\leq \quad C m_h \left(\int_0^{2\pi} \int_0^{\varepsilon_h} F(r, \theta) dr d\theta + \varepsilon_h \right) \\ &\leq \quad C \left(m_h \int_0^{\varepsilon_h} \left(\int_0^{2\pi} F(r_2, \theta) d\theta + \ln \frac{r_2}{r} \right) dr + \frac{2^h}{3^{h+1}} \right) \\ &\leq \quad C \frac{2^h}{3^{h+1}} \left(\int_0^{2\pi} F(r_2, \theta) d\theta + \ln \frac{r_2}{\varepsilon_h} + 1 \right) \\ &\leq \quad C \left(\left(\frac{2}{3} \right)^{h/2} r_2 \int_0^{2\pi} F(r_2, \theta) d\theta + \frac{2^h}{3^{h+1}} (-\ln \varepsilon_h + 1) \right). \end{aligned}$$

Integrating with respect to r_2 over the interval $[a_h, b_h]$, we obtain that

$$\begin{array}{rcl}\frac{1}{|T_h|}\int_{T_h}|u_h|^2\,dsdx_3 & \leq & C\left(\int_{a_h}^{b_h}\int_0^{2\pi}F(r,\theta)\,rdrd\theta-\frac{2^h}{3^{h+1}}\ln\varepsilon_h\right)\\ & \leq & C\left\{\|u_h\|^2_{L^2(\Omega,\mathbb{R}^3)}-\frac{2^h}{3^{h+1}}\ln\varepsilon_h\right\}.\end{array}$$

□

Let $\mathcal{M}\left(\mathbb{R}^3\right)$ be the space of Radon measures on $\mathbb{R}^3$. We have the following result:

Lemma 3.2. *Let $u_h\in L^2\left(\Omega,\mathbb{R}^3\right)\cap L^2\left(T_h,\mathbb{R}^3\right)$, such that*

$$\sup_h\frac{1}{|T_h|}\int_{T_h}|u_h|^2\,dsdx_3<+\infty.$$

Then, there exists a subsequence of $(u_h)_h$, still denoted $(u_h)_h$, such that

$$u_h\frac{\mathbf{1}_{T_h}(x)}{|T_h|}dsdx_3\underset{h\to\infty}{\overset{*}{\rightharpoonup}}v\mathbf{1}_{\Sigma}(s)\frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(\Sigma)}\ \text{ in }\mathcal{M}\left(\mathbb{R}^3\right),$$

with $v(s,0)\in L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^3\right)$.

Proof. Let us consider the sequence of Radon measures $(\vartheta_h)_h$ on $\mathbb{R}^3$ defined by

$$\vartheta_h=\frac{\mathbf{1}_{T_h}(x)}{|T_h|}dsdx_3.$$

Let $x_h^k=\left(x_{1h}^k,x_{2h}^k\right)$ denotes the center of S_h^k; $k\in I_h$, in Cartesian coordinates. Then, using the ergodicity result of [22, Theorem 6.1], we have, for every $\varphi\in C_0\left(\mathbb{R}^3\right)$,

$$\begin{array}{rcl}\lim\limits_{h\to\infty}\int_{\mathbb{R}^3}\varphi(x)\,d\vartheta_h & = & \lim\limits_{h\to\infty}\sum\limits_{k\in I_h}\frac{2}{3^{h+1}}\varphi\left(x_h^k,0\right)\\ & = & \lim\limits_{h\to\infty}\sum\limits_{k\in I_h}\frac{1}{N_h}\varphi\left(x_h^k,0\right)\\ & = & \frac{1}{\mathcal{H}^d(\Sigma)}\int_\Sigma\varphi(s,0)\,d\mathcal{H}^d(s),\end{array}$$

from which we deduce that $\vartheta_h\underset{h\to\infty}{\overset{*}{\rightharpoonup}}\vartheta$, with

$$\vartheta=\mathbf{1}_\Sigma(s)\frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(\Sigma)}.$$

Let $u_h\in L^2\left(\Omega,\mathbb{R}^3\right)\cap L^2\left(T_h,\mathbb{R}^3\right)$, such that

$$\sup_h\frac{1}{|T_h|}\int_{T_h}|u_h|^2\,dsdx_3<+\infty.$$

As $\vartheta_h\left(\mathbb{R}^3\right)=\frac{1}{|T_h|}\int_{T_h}dsdx_3=1$, we have

$$\begin{array}{rcl}\left|\int_{\mathbb{R}^3}u_h d\vartheta_h\right|^2 & \leq & \int_{\mathbb{R}^3}|u_h|^2\,d\vartheta_h\\ & = & \frac{1}{|T_h|}\int_{T_h}|u_h|^2\,dsdx_3,\end{array}$$

from which we deduce that the sequence $(u_h\vartheta_h)_h$ is uniformly bounded in variation, hence $*$-weakly relatively compact. Possibly passing to a subsequence, we can suppose that the sequence $(u_h\vartheta_h)_h$ converges to some χ. Let $\varphi\in C_0\left(\mathbb{R}^3,\mathbb{R}^3\right)$.

Then, using Fenchel's inequality (also known as the Fenchel-Young inequality, see for instance [7]), we have

$$\begin{aligned}
&\liminf_{h\to\infty}\frac{1}{2}\int_{\mathbb{R}^3}|u_h|^2\,d\vartheta_h\\
&\geq \liminf_{h\to\infty}\left(\int_{\mathbb{R}^3}u_h.\varphi d\vartheta_h-\frac{1}{2}\int_{\mathbb{R}^3}|\varphi|^2\,d\vartheta_h\right)\\
&\geq \langle\chi,\varphi\rangle-\frac{1}{2}\int_{\mathbb{R}^3}|\varphi|^2\,d\vartheta.
\end{aligned}$$

As the left hand side of this inequality is bounded, we deduce that

$$\sup\left\{\langle\chi,\varphi\rangle\,;\,\varphi\in C_0\left(\mathbb{R}^3,\mathbb{R}^3\right),\ \int_\Sigma|\varphi|^2(s,0)\,d\mathcal{H}^d(s)\leq 1\right\}<+\infty,$$

from which we deduce, according to Riesz' representation theorem, that there exists v such that $\chi=v(s,x_3)\,\vartheta$ and $v(s,0)\in L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^3\right)$. □

Proposition 2. *Let $(u_h)_h$; $u_h\in H^1_0\left(\Omega,\mathbb{R}^3\right)\cap H^1\left(T_h,\mathbb{R}^3\right)$, be a sequence, such that $\sup_h F_h(u_h)<+\infty$. There exists a subsequence, still denoted $(u_h)_h$, such that*

1. *$u_h \underset{h\to\infty}{\rightharpoonup} u$ $H^1_0\left(\Omega,\mathbb{R}^3\right)$-weak,*
2. *If $\gamma\in(0,+\infty)$ then*

$$u_h\frac{\mathbf{1}_{T_h}(x)}{|T_h|}dsdx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v(s,0)\,\mathbf{1}_\Sigma(s)\,\frac{d\mathcal{H}^d(s)}{\mathcal{H}^d(\Sigma)},$$

with $v(s,0)\in L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^3\right)$.

3. *If $\gamma\in(0,+\infty)$ then, with η_h given in (11), we have $\overline{v}(s,0)\in\mathcal{D}_{\Sigma,\mathcal{E}}$ and*

$$\liminf_{h\to\infty}\int_{T_h}\sigma^h_{ij}(u_h)\,e_{ij}(u_h)\,dsdx_3\geq\mu_0\mathcal{E}_\Sigma(\overline{v}).$$

Proof. 1. Thanks to Lemma 3.1$_1$, one immediately obtains that, up to some subsequence, $u_h \underset{h\to\infty}{\rightharpoonup} u$ $H^1_0\left(\Omega,\mathbb{R}^3\right)$-weak.

2. If $\gamma\in(0,+\infty)$ then, according to Lemma 3.1$_2$ and Lemma 3.2, one has, up to some subsequence,

$$u_h\frac{\mathbf{1}_{T_h}(x)}{|T_h|}dsdx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v(s,0)\,\mathbf{1}_\Sigma(s)\,\frac{d\mathcal{H}^d(s)}{\mathcal{H}^d(\Sigma)},$$

with $v(s,0)\in L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^3\right)$.

3. One can easily check that

$$\begin{aligned}
&\int_{T_h}\sigma^h_{ij}(u_h)\,e_{ij}(u_h)\,dsdx_3\\
&\quad\geq 2\mu_h\left(\int_{T_h}\left((e_{11}(u_h))^2+2\,(e_{12}(u_h))^2+(e_{22}(u_h))^2\right)dsdx_3\right).
\end{aligned}\tag{48}$$

Computing the strain tensor in the local coordinates (37) and (38), observing that for $S^k_h\perp\left(-\sqrt{3}/2,1/2\right)$ or $S^k_h\perp\left(\sqrt{3}/2,1/2\right)$ the covariant derivative $\dfrac{\partial u^k_{\alpha,h}}{\partial s^\perp}=0$ on

S_h^k; $\alpha=1,2$, we obtain

$$\begin{aligned}
&\int_{S_h^k}\left(\left(e_{11}\left(u_h\right)\right)^2+2\left(e_{12}\left(u_h\right)\right)^2+\left(e_{22}\left(u_h\right)\right)^2\right)ds\\
&=\int_{S_h^k}\left(\frac{1}{4}\left(\frac{\partial u_{1,h}^k}{\partial s}\right)^2+\frac{3}{8}\left(\frac{\partial u_{2,h}^k}{\partial s}\right)^2\right)ds\\
&\geq\frac{1}{4}\int_{S_h^k}\left(\left(\frac{\partial u_{1,h}^k}{\partial s}\right)^2+\left(\frac{\partial u_{2,h}^k}{\partial s}\right)^2\right)ds.
\end{aligned}\tag{49}$$

For $S_h^k\perp(0,1)$, since $\dfrac{\partial u_{\alpha,h}^k}{\partial x_2}=0$ on S_h^k; $\alpha=1,2$, we have

$$\begin{aligned}
&\int_{S_h^k}\left(\left(e_{11}\left(u_h\right)\right)^2+2\left(e_{12}\left(u_h\right)\right)^2+\left(e_{22}\left(u_h\right)\right)^2\right)ds\\
&=\int_{S_h^k}\left(\frac{\partial u_{1,h}^k}{\partial x_1}\right)^2+\frac{1}{2}\left(\frac{\partial u_{2,h}^k}{\partial x_1}\right)^2ds\\
&\geq\frac{1}{4}\int_{S_h^k}\left(\left(\frac{\partial u_{1,h}^k}{\partial x_1}\right)^2+\left(\frac{\partial u_{2,h}^k}{\partial x_1}\right)^2\right)ds.
\end{aligned}\tag{50}$$

According to (48) and (10), we deduce from (49) and (50) that

$$\begin{aligned}
&\int_{T_h}\sigma_{ij}^h\left(u_h\right)e_{ij}\left(u_h\right)dsdx_3\\
&\geq\frac{\mu_h}{2}\int_{T_h}\left(\frac{\partial u_{1,h}^k}{\partial s}\right)^2+\left(\frac{\partial u_{2,h}^k}{\partial s}\right)^2dsdx_3\\
&\quad\geq 2^h\varepsilon_h\mu_h\sum_{k\in I_h}\frac{1}{2\varepsilon_h}\int_{-\varepsilon_h}^{\varepsilon_h}\left(u_{\alpha,h}\left(p^k,x_3\right)-u_{\alpha,h}\left(q^k,x_3\right)\right)^2dx_3\\
&\quad=2^h\varepsilon_h\eta_h\mu_0\sum_{k\in I_h}\frac{1}{2\varepsilon_h}\int_{-\varepsilon_h}^{\varepsilon_h}\left(u_{\alpha,h}\left(p^k,x_3\right)-u_{\alpha,h}\left(q^k,x_3\right)\right)^2dx_3\\
&\quad\geq\mu_0\left(\frac{5}{3}\right)^h\sum_{\substack{p,q\in\mathcal{V}_h\\|p-q|=2^{-h}}}\left(\frac{1}{2\varepsilon_h}\int_{-\varepsilon_h}^{\varepsilon_h}\left(u_{\alpha,h}\left(p,x_3\right)-u_{\alpha,h}\left(q,x_3\right)\right)dx_3\right)^2.
\end{aligned}\tag{51}$$

Let us set $\overline{u}_h=\left(u_{1,h},u_{2,h}\right)$ and $\widetilde{\overline{u}}_h=\dfrac{1}{2\varepsilon_h}\int_{-\varepsilon_h}^{\varepsilon_h}\overline{u}_h\left(.,x_3\right)dx_3$. We introduce the harmonic extension of $\widetilde{\overline{u}}_h\mid_{\mathcal{V}_h}$ obtained by the *decimation* procedure (see for instance [30, Proposition 1] and [8, Corollary1]):

We define the function $H_{h+1}\widetilde{\overline{u}}_h:\mathcal{V}_{h+1}\longrightarrow\mathbb{R}^2$ as the unique minimizer of the problem

$$\min\left\{\mathcal{E}_\Sigma^{h+1}\left(w\right);\,w:\mathcal{V}_{h+1}\longrightarrow\mathbb{R}^2,\,w=\widetilde{\overline{u}}_h\text{ on }\mathcal{V}_h\right\}.\tag{52}$$

Then $\mathcal{E}_\Sigma^{h+1}\left(H_{h+1}\widetilde{\overline{u}}_h\right)=\mathcal{E}_\Sigma^h\left(\widetilde{\overline{u}}_h\right)$. For $m>h$, we define the function $H_m\widetilde{\overline{u}}_h$ from $\mathcal{V}_m$ into $\mathbb{R}^2$ by

$$H_m\widetilde{\overline{u}}_h=H_m\left(H_{m-1}\left(\ldots\left(H_{h+1}\widetilde{\overline{u}}_h\right)\right)\right).$$

For every $m>h$ we have $H_m\widetilde{\overline{u}}_h\mid_{\mathcal{V}_h}=\widetilde{\overline{u}}_h\mid_{\mathcal{V}_h}$ and

$$\mathcal{E}_\Sigma^m\left(H_m\widetilde{\overline{u}}_h\right)=\mathcal{E}_\Sigma^h\left(\widetilde{\overline{u}}_h\right).\tag{53}$$

Now we define, for a fixed $h \in \mathbb{N}$, the function $H\widetilde{\overline{u}}_h$ on $\mathcal{V}_\infty$ as follows. For $p \in \mathcal{V}_\infty$, we choose $m \geq h$ such that $p \in \mathcal{V}_m$ and set

$$H\widetilde{\overline{u}}_h(p) = H_m\widetilde{\overline{u}}_h(p). \tag{54}$$

As $\sup_h \int_{T_h} \sigma_{ij}^h(u_h)\, e_{ij}(u_h)\, dsdx_3 < \infty$, we have, according to (51) and (53),

$$\sup_h \mathcal{E}_\Sigma\left(H\widetilde{\overline{u}}_h\right) = \sup_h \mathcal{E}_\Sigma^h\left(\widetilde{\overline{u}}_h\right) < +\infty, \tag{55}$$

from which we deduce, using Section 2, that $H\widetilde{\overline{u}}_h$ has a unique continuous extension on Σ, still denoted $H\widetilde{\overline{u}}_h$, and that the sequence $\left(H\widetilde{\overline{u}}_h\right)_h$ is bounded in $\mathcal{D}_{\Sigma,\mathcal{E}}$. Therefore, there exists a subsequence, still denoted $\left(H\widetilde{\overline{u}}_h\right)_h$, weakly converging to some $\overline{u}^* \in \mathcal{D}_{\Sigma,\mathcal{E}}$, with

$$\mathcal{E}_\Sigma(\overline{u}^*) \leq \liminf_{h\to\infty} \mathcal{E}_\Sigma\left(H\widetilde{\overline{u}}_h\right) \leq \liminf_{h\to\infty} \mathcal{E}_\Sigma^h\left(\widetilde{\overline{u}}_h\right). \tag{56}$$

On the other hand, using Lemma 3.2, we have, for every $\varphi \in C_0\left(\Sigma, \mathbb{R}^2\right)$,

$$\begin{aligned} \lim_{h\to\infty} \frac{1}{\mathcal{H}^d(\Sigma)} \int_\Sigma H\widetilde{\overline{u}}_h.\varphi d\mathcal{H}^d(s) &= \lim_{h\to\infty} \int_{\mathbb{R}^3} \overline{u}_h.\varphi d\upsilon_h \\ &= \frac{1}{\mathcal{H}^d(\Sigma)} \int_\Sigma \overline{v}(s,0).\varphi d\mathcal{H}^d(s), \end{aligned}$$

which implies that $\overline{u}^*(s) = \overline{v}(s,0)$. Therefore $\overline{v}(s,0) \in \mathcal{D}_{\Sigma,\mathcal{E}}$ and, according to (51) and (56),

$$\liminf_{h\to\infty} \int_{T_h} \sigma_{ij}^h(u_h)\, e_{ij}(u_h)\, dsdx_3 \geq \mu_0 \mathcal{E}_\Sigma(\overline{v}).$$

$\square$

4. **The main result.** In this Section we state the main result of this work. According to Proposition 2 we introduce the following topology τ:

Definition 4.1. We say that a sequence $(u_h)_h$; $u_h \in H_0^1\left(\Omega, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right)$, τ-converges to (u,v) if

$$\begin{cases} u_h \underset{h\to\infty}{\rightharpoonup} u \ \text{ in } \mathbf{H}^1\left(\Omega, \mathbb{R}^3\right)\text{-weak}, \\ u_h \dfrac{\mathbf{1}_{T_h}(x)}{|T_h|} dsdx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v\mathbf{1}_\Sigma(s) \dfrac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(\Sigma)} \ \text{ in } \mathcal{M}\left(\mathbb{R}^3\right), \end{cases}$$

with $v \equiv v(s,0) \in L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^3\right)$.

Our main result in this work reads as follows:

Theorem 4.2. *If $\gamma \in (0, +\infty)$ then*

1. *(*$\limsup$ *inequality) for every $(u,v) \in H_0^1\left(\Omega, \mathbb{R}^3\right) \times \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)$ there exists a sequence $(u_h)_h$; $u_h \in H_0^1\left(\Omega, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right)$, such that $(u_h)_h$ τ-converges to (u,v) and*
$$\limsup_{h\to\infty} F_h(u_h) \leq F_\infty(u,v),$$
where F_∞ is the functional defined in (15),
2. *(*$\liminf$ *inequality) for every sequence $(u_h)_h$; $u_h \in H_0^1\left(\Omega, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right)$, such that $(u_h)_h$ τ-converges to (u,v), we have $\overline{v} \in \mathcal{D}_{\Sigma,\mathcal{E}}$ and*
$$\liminf_{h\to\infty} F_h(u_h) \geq F_\infty(u,v).$$

Before proving this Theorem, let us write the homogenized problem obtained at the limit as $h \longrightarrow \infty$.

Corollary 1. *Problem (13) admits a unique solution U_h which, under the hypothesis of Theorem 4.2, τ-converges to $(U,V) \in H_0^1\left(\Omega,\mathbb{R}^3\right) \times \mathcal{D}_{\Delta_\Sigma} \times L^2_{\mathcal{H}^d}(\Sigma)$ solution of the problem*

$$\left\{\begin{array}{rcll} -\sigma_{ij,j}(U) & = & f_i & \text{in } \Omega, \\ -\mu_0\Delta_{\alpha,\Sigma}(V_\alpha) & = & \dfrac{\pi\mu\gamma}{(\ln 2)^2}A_{\alpha\beta}(s)(U_\beta - V_\beta)\,;\ \alpha,\beta=1,2, & \text{in } \Sigma, \\ \left[\sigma_{\alpha 3}|_{x_3=0}\right]_\Sigma & = & \dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}A_{\alpha\beta}(s)(U_\beta - V_\beta)\,\mathcal{H}^d & \text{on } \Sigma, \\ U_3 & = & V_3 & \text{on } \Sigma, \\ U & = & 0 & \text{on } \partial\Omega, \\ V_\alpha & = & 0;\ \alpha=1,2, & \text{on } \mathcal{V}_0. \end{array}\right. \tag{57}$$

Proof. One can easily check that problem (13) has a unique solution $U_h \in H_0^1\left(\Omega,\mathbb{R}^3\right) \cap H^1\left(T_h,\mathbb{R}^3\right)$. Now, observing that

$$F_h(U_h) - 2\int_\Omega f.U_h dx \le F_h(0) = 0,$$

we deduce, using the fact that $\lim\limits_{h\to\infty}\eta_h = +\infty$, the Korn inequality, and the Poincaré inequality, that

$$\begin{array}{l} \int_\Omega |\nabla U_h|^2 dx \\ \le \int_\Omega \sigma_{ij}(U_h)\, e_{ij}(U_h)\, dx + \int_{T_h} \sigma^h_{ij}(U_h)\, e_{ij}(U_h)\, ds dx_3 \\ \le 2\|f\|_{L^2(\Omega,\mathbb{R}^3)}\|U_h\|_{L^2(\Omega,\mathbb{R}^3)} \le C\|\nabla U_h\|_{L^2(\Omega,\mathbb{R}^9)}, \end{array}$$

from which we deduce that $\sup_h F_h(U_h) < +\infty$. Then, using Proposition 2 and Theorem 4.2, we deduce, according to [16, Theorem 7.8]), that the sequence $(U_h)_h$ τ-converges to the solution (U,V) of the problem

$$\min_{(\xi,\zeta)\in V}\left\{\begin{array}{l} \int_\Omega \sigma_{ij}(\xi)\, e_{ij}(\xi)\, dx + \mu_0\int_\Sigma d\mathcal{L}_\Sigma(\zeta) \\ +\dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\int_\Sigma A(s)(\xi-\zeta).(\xi-\zeta)\, d\mathcal{H}^d(s) \\ -2\int_\Omega f.\xi dx \end{array}\right\}, \tag{58}$$

where $V = H_0^1\left(\Omega,\mathbb{R}^3\right) \times \mathcal{D}_{\Delta_\Sigma} \times L^2_{\mathcal{H}^d}(\Sigma)$. On the other hand, according to [27, Theorem 6], the trace of $\xi \in H^1\left(\Omega,\mathbb{R}^3\right)$ on $\omega\cap\Sigma$ exists for $\mathcal{H}^d$-almost-every $x \in \omega\cap\Sigma$ and belongs to the Besov space $B^2_d\left(\Sigma,\mathbb{R}^3\right)$ of functions $\psi : \Sigma \longrightarrow \mathbb{R}^3$ such that

$$\int_\Sigma |\psi(x)|^2 d\mathcal{H}^d(x) + \int_{\substack{\Sigma \\ |x-y|<1}}\int_\Sigma \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{2d}} d\mathcal{H}^d(x)\, d\mathcal{H}^d(y) < +\infty. \tag{59}$$

Then, according to Lemma 2.1, we obtain from (58), using for example [46, Theorems 3.1 and 3.3], that $\overline{v} \in \mathcal{D}_{\Delta_\Sigma}$ and for every $(\xi,\zeta) \in H_0^1\left(\Omega,\mathbb{R}^3\right) \times \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)$,

$$\begin{array}{l} \int_\Omega \left(-\sigma_{ij,j}(U) - f_i\right)\xi_i dx - \dfrac{\mu_0}{\mathcal{H}^d(\Sigma)}\int_\Sigma \left(\Delta_{\alpha,\Sigma}\overline{V}\right)\zeta_\alpha d\mathcal{H}^d(s) \\ +\dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\int_\Sigma A(s)(U-V).(\xi-\zeta)\, d\mathcal{H}^d(s) \\ \\ -\left\langle \left[\sigma_{i3}|_{x_3=0}\right]_\Sigma, \xi_i\right\rangle_{B^2_{-d}(\Sigma,\mathbb{R}^3),B^2_d(\Sigma,\mathbb{R}^3)} = 0, \end{array} \tag{60}$$

$B^2_{-d}\left(\Sigma,\mathbb{R}^3\right)$ being the dual space of $B^2_d\left(\Sigma,\mathbb{R}^3\right)$ (see [28, p. 291]). Since $(V_1,V_2)\in D_{\Sigma,\mathcal{E}}\subset L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^2\right)$, the trace of U on Σ belongs to $B^2_d\left(\Sigma,\mathbb{R}^3\right)\subset L^2_{\mathcal{H}^d}\left(\Sigma,\mathbb{R}^3\right)$, and, according to Lemma 2.1, $\Delta_{\alpha,\Sigma}$; $\alpha=1,2$, is a second order operator in $L^2_{\mathcal{H}^d}\left(\Sigma\right)$ defined by the form $\mathcal{E}_\Sigma$ under the Dirichlet condition $V_\alpha=0$ on $\mathcal{V}_0$; $\alpha=1,2$, the transmission condition

$$-\mu_0\Delta_{\alpha,\Sigma}\left(V_\alpha\right)=\frac{\pi\mu\gamma}{\left(\ln 2\right)^2}A_{\alpha\beta}\left(s\right)\left(U_\beta-V_\beta\right);\ \alpha,\beta=1,2,\text{ in }\Sigma,$$

in problem (57) is well posed. □

5. Proof of the main result.

The proof of Theorem 4.2 is given in three steps.

5.1. Step 1: Boundary layers.

We consider here a local problem associated with boundary layers in the vicinity of the strips. We denote w^m; $m=1,2$, the solution of the following boundary value problem:

$$\left\{\begin{array}{rcll}\sigma_{ij,j}\left(w^m\right)(y) & = & 0 & \forall y\in\mathbb{R}^{2+};\ i=1,2,\\ w^m\left(y_1,0\right) & = & e_m & \forall y_1\in\left]-1,1\right[,\\ \sigma_{i2}\left(w^m\right)\left(y_1,0\right) & = & 0 & \forall y_1\in\mathbb{R}\setminus\left]-1,1\right[,\\ w^m_m\left(y\right) & = & -\dfrac{\ln|y|}{\ln 2} & \text{as } |y|\to\infty,\ y_2>0,\\ \left|w^m_p\right|(y) & \leq & C & \text{for }\left\{\begin{array}{l}p=2\text{ if }m=1,\\ p=1\text{ if }m=2,\end{array}\right.\end{array}\right.\tag{61}$$

where $\mathbb{R}^{2+}=\left\{y=(y_1,y_2)\in\mathbb{R}^2;\ y_2>0\right\}$ and $e_m=(\delta_{1m},\delta_{2m})$; $m=1,2$. The displacement w^m; $m=1,2$, which belongs to the space $H^1_{loc}\left(\mathbb{R}^{2+},\mathbb{R}^2\right)$, is given (see for instance [34] and [18]) by

$$\begin{array}{rcl}w^1_1\left(y\right) & = & \dfrac{1}{4\pi\mu}\int_{-1}^{1}\xi\left(t\right)\left(\begin{array}{l}-\left(1+\kappa\right)\ln\left(\sqrt{\left(y_1-t\right)^2+\left(y_2\right)^2}\right)\\ +\dfrac{2\left(y_2\right)^2}{\left(y_1-t\right)^2+\left(y_2\right)^2}\end{array}\right)dt,\\ w^1_2\left(y\right) & = & \dfrac{1}{4\pi\mu}\int_{-1}^{1}\xi\left(t\right)\left(\begin{array}{l}-\left(1-\kappa\right)\arctan\left(\dfrac{y_2}{y_1-t}\right)\\ +\dfrac{2y_2\left(y_1-t\right)}{\left(y_1-t\right)^2+\left(y_2\right)^2}\end{array}\right)dt\end{array}\tag{62}$$

and

$$\begin{array}{rcl}w^2_1\left(y\right) & = & \dfrac{1}{4\pi\mu}\int_{-1}^{1}\xi\left(t\right)\left(\begin{array}{l}\left(1-\kappa\right)\arctan\left(\dfrac{y_2}{y_1-t}\right)\\ +\dfrac{2y_2\left(y_1-t\right)}{\left(y_1-t\right)^2+\left(y_2\right)^2}\end{array}\right)dt,\\ w^2_2\left(y\right) & = & \dfrac{1}{4\pi\mu}\int_{-1}^{1}\xi\left(t\right)\left(\begin{array}{l}-\left(1+\kappa\right)\ln\left(\sqrt{\left(y_1-t\right)^2+\left(y_2\right)^2}\right)\\ -\dfrac{2\left(y_2\right)^2}{\left(y_1-t\right)^2+\left(y_2\right)^2}\end{array}\right)dt,\end{array}\tag{63}$$

where

$$\xi\left(t\right)=\left\{\begin{array}{ll}\dfrac{4\mu}{\left(1+\kappa\right)\ln 2}\dfrac{1}{\sqrt{1-t^2}} & \text{if } t\in\left]-1,1\right[,\\ 0 & \text{otherwise.}\end{array}\right.\tag{64}$$

One can check that $w^m(y)$; $m = 1, 2$, is also the solution of problem (61) posed in the half-plane $\mathbb{R}^{2-}$:

$$\mathbb{R}^{2-} = \left\{ y = (y_1, y_2) \in \mathbb{R}^2;\ y_2 < 0 \right\}.$$

We introduce the scalar problem

$$\left\{ \begin{array}{rcll} \Delta w(y) & = & 0 & \forall y \in \mathbb{R}^{2+};\ i = 1, 2, \\ w(y_1, 0) & = & 1 & \forall y_1 \in \left]-1, 1\right[, \\ \dfrac{\partial w}{\partial y_2}(y_1, 0) & = & 0 & \forall y_1 \in \mathbb{R} \setminus \left]-1, 1\right[, \\ w(y) & = & -\dfrac{\ln|y|}{\ln 2} & \text{as } |y| \to \infty,\ y_2 > 0. \end{array} \right. \tag{65}$$

The solution of (65) is given by

$$w(y) = \frac{-1}{\pi \ln 2} \int_{-1}^{1} \frac{\ln\left(\sqrt{(y_1 - t)^2 + (y_2)^2}\right)}{\sqrt{1 - t^2}} dt. \tag{66}$$

Observe that $w(y)$ is also the solution of problem (65) posed in the half-plane $\mathbb{R}^{2-}$. We now state the following preliminary result in this section:

Proposition 3. *([18, Proposition 7]). One has*

1. $\displaystyle\lim_{R\to\infty} \frac{1}{\ln R} \int_{B(0,R)\cap\mathbf{R}^{2\pm}} \sigma_{ij}(w^m)\, e_{ij}(w^l)\, dy = \delta_{ml} \frac{2\mu\pi}{(1+\kappa)(\ln 2)^2}$; $m, l = 1, 2$,
2. $\displaystyle\lim_{R\to\infty} \frac{1}{\ln R} \int_{B(0,R)\cap\mathbf{R}^{2\pm}} |\nabla w|^2\, dy = \frac{\pi}{(\ln 2)^2}$, *where $B(0, R)$ is a disc of radius R centred at the origin.*

Let r_h be a positive parameter, such that

$$\lim_{h\to\infty} 2^h r_h = \lim_{h\to\infty} \frac{\varepsilon_h}{r_h} = 0. \tag{67}$$

We define the rotation $\mathcal{R}\left(x_h^k\right)$; $x_h^k = \left(x_{1h}^k, x_{2h}^k\right)$ being the center of S_h^k in Cartesian coordinates, by

$$\mathcal{R}\left(x_h^k\right) = \left\{ \begin{array}{ll} Id_{\mathbb{R}^3} & \text{if } n^k = \pm(0, 1), \\ \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } n^k = \pm\left(-\sqrt{3}/2, 1/2\right), \\ \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } n^k = \pm\left(\sqrt{3}/2, 1/2\right), \end{array} \right. \tag{68}$$

where n^k is the unit normal on S_h^k and $Id_{\mathbb{R}^3}$ is the 3×3 identity marix. Let φ_h^k; $k \in I_h$, be the truncation function defined on $\mathbb{R}^2$ by

$$\varphi_h^k(x) = \left\{ \begin{array}{ll} \dfrac{4\left(r_h^2 - R_{k,h}^2(x)\right)}{3r_h^2} & \text{if } r_h/2 \le R_h^k(x) \le r_h, \\ 1 & \text{if } R_h^k(x) \le r_h/2, \\ 0 & \text{if } R_h^k(x) \ge r_h, \end{array} \right. \tag{69}$$

where $R_h^k(x) = \sqrt{\left(\left(x' - x_h^k\right).n^k\right)^2 + x_3^2}$ with $x' = (x_1, x_2)$. We define, for $k \in I_h$,

$$D_h^k(r_h) = \left\{\left(\left(x - x_h^k\right).n^k, x_3\right) \in \mathbb{R}^2;\ R_h^k(x) < r_h,\ \forall x \in \mathbb{R}^3\right\} \tag{70}$$

and the cylinder

$$Z_h^k = \mathcal{R}\left(x_h^k\right) S_h^k \times D_h^k(r_h). \tag{71}$$

We then set

$$Z_h = \bigcup_{k \in I_h} Z_h^k. \tag{72}$$

We define, the function $w_h^{mk}(x)$; $k \in I_h$ and $m = 1, 2, 3$, by

$$w_h^{1k}(x) = \varphi_h^k(x)\, \mathcal{R}\left(x_h^k\right)\left(e_1 - \frac{1}{\ln \varepsilon_h}\begin{pmatrix} 1 - w\left(\dfrac{x_3}{\varepsilon_h}, \dfrac{\left(x' - x_h^k\right).n^k}{\varepsilon_h}\right) \\ 0 \\ 0 \end{pmatrix}\right), \tag{73}$$

$$w_h^{2k}(x) = \varphi_h^k(x)\, \mathcal{R}\left(x_h^k\right)\left(e_2 - \frac{1}{\ln \varepsilon_h}\begin{pmatrix} 0 \\ 1 - w_1^1\left(\dfrac{x_3}{\varepsilon_h}, \dfrac{\left(x' - x_h^k\right).n^k}{\varepsilon_h}\right) \\ w_2^1\left(\dfrac{x_3}{\varepsilon_h}, \dfrac{\left(x' - x_h^k\right).n^k}{\varepsilon_h}\right) \end{pmatrix}\right) \tag{74}$$

and

$$w_h^{3k}(x) = \varphi_h^k(x)\, \mathcal{R}\left(x_h^k\right)\left(e_3 - \frac{1}{\ln \varepsilon_h}\begin{pmatrix} 0 \\ w_1^2\left(\dfrac{x_3}{\varepsilon_h}, \dfrac{\left(x' - x_h^k\right).n^k}{\varepsilon_h}\right) \\ 1 - w_2^2\left(\dfrac{x_3}{\varepsilon_h}, \dfrac{\left(x' - x_h^k\right).n^k}{\varepsilon_h}\right) \end{pmatrix}\right), \tag{75}$$

where $e_m = (\delta_{1m}, \delta_{2m}, \delta_{3m})$; $m = 1, 2, 3$. We define the local perturbation w_ε^m; $m = 1, 2, 3$, on Ω by

$$w_h^m(x) = w_h^{mk}(x),\ \forall k \in I_h,\ \forall x \in \Omega. \tag{76}$$

We have the following result:

Lemma 5.1. *If $\gamma \in (0, +\infty)$ then, for every $\Phi \in C^1\left(\overline{\Omega}, \mathbb{R}^3\right)$, we have*

$$\lim_{h \to \infty} \int_{Z_h} \sigma_{ij}\left(w_h^m \Phi_m\right) e_{ij}\left(w_h^l \Phi_l\right) dx = \frac{\pi \mu \gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma A(s)\, \Phi(s) . \Phi(s)\, d\mathcal{H}^d(s),$$

where $A(s)$ is the material matrix defined in (17).

Proof. Let us introduce the change of variables

$$\begin{cases} y_1 &= \dfrac{x_3}{\varepsilon_h}, \\ y_2 &= \dfrac{\left(x' - x_h^k\right).n^k}{\varepsilon_h}, \end{cases}$$

on Z_h^k; $k \in I_h$. Then, using the smoothness of Φ and Proposition 3, we have

$$\begin{aligned}
&\lim_{h\to\infty} \int_{Z_h} \sigma_{ij}\left(w_h^m \Phi_m\right) e_{ij}\left(w_h^l \Phi_l\right) dx \\
&= \lim_{h\to\infty} \sum_{k\in I_h} \int_{Z_h^k} \sigma_{ij}\left(w_h^{mk}\right) e_{ij}\left(w_h^{lk}\right) \Phi_m \Phi_l dx \\
&= \lim_{h\to\infty} \frac{3^{h+1}}{2^h \ln^2 \varepsilon_h} \int_{D\left(0,\frac{r_h}{\varepsilon_h}\right)\setminus D(0,1)} \sigma_{ij}\left(w^m\right) e_{ij}\left(w^l\right) dy_1 dy_2 \\
&\times \left(\sum_{k\in I_h} \frac{1}{N_h} \left(\mathcal{R}\left(x_h^k\right)\Phi\right)_m \left(\mathcal{R}\left(x_h^k\right)\Phi\right)_l \left(x_{1h}^k, x_{2h}^k, 0\right)\right) \\
&= \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)\ln 2} \int_\Sigma \left(B\mathcal{R}(s)\Phi(s)\right)_m \left(\mathcal{R}(s)\Phi(s)\right)_l d\mathcal{H}^d(s) \\
&= \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma \mathcal{R}^t(s) B\mathcal{R}(s)\Phi(s).\Phi(s)\, d\mathcal{H}^d(s),
\end{aligned}$$

where $B = \mathrm{Diag}\left(1, \frac{2}{(1+\kappa)}, \frac{2}{(1+\kappa)}\right)$ and $\mathcal{R}(s)$ is the rotation matrix defined by $\mathcal{R}(s) = Id_{\mathbb{R}^3}$ on the faces of Σ which are perpendicular to the vectors $\pm(0,1)$, by $\mathcal{R}(s) = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the faces of Σ which are perpendicular to the vectors $\pm\left(-\sqrt{3}/2, 1/2\right)$, and $\begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the faces of Σ which are perpendicular to the vectors $\pm\left(\sqrt{3}/2, 1/2\right)$. Then observing that

$$\mathcal{R}^t(s) B\mathcal{R}(s) = \mathcal{R}(s) B\mathcal{R}(s) = A(s),$$

we have the result. □

5.2. **Step 2: Lim-sup inequality.** In this Subsection we prove the lim-sup condition of the Γ-convergence property stated in Theorem 4.2. Let $p_h^k = \left(p_{h,1}^k, p_{h,2}^k\right)$, $q_h^k = \left(q_{h,1}^k, q_{h,2}^k\right)$ be the extremities of the line segment S_h^k. Let $v \in C_c^1\left(\omega, \mathbb{R}^3\right)$. Then, we build the following sequence:

$$\begin{array}{rcl}
v_{1,h}^k(x') &=& v_1\left(x_{1h}^k, x_{2h}^k\right) + 2^h \zeta_h^{1,k}(x') \left|v_1\left(p_h^k\right) - v_1\left(q_h^k\right)\right|, \\
v_{2,h}^k(x') &=& v_2\left(x_{1h}^k, x_{2h}^k\right) + 2^h \zeta_h^{2,k}(x') \left|v_2\left(p_h^k\right) - v_2\left(q_h^k\right)\right|, \\
v_{3,h}^k(x') &=& v_3\left(x_{1h}^k, x_{2h}^k\right),
\end{array} \tag{77}$$

for every $x' \in \omega$, where, using the local coordinates (37) for $S_h^k \perp \left(-\sqrt{3}/2, 1/2\right)$,

$$\left\{\begin{array}{rcl}
\zeta_h^{1,k}(x') &=& 2\sqrt{\mu_h}\dfrac{s + p_{h,1}^k/2 - p_{h,2}^k\sqrt{3}/2}{\sqrt{\lambda_h + 2\mu_h}}, \\
\zeta_h^{2,k}(x') &=& \dfrac{2\left(s + p_{h,1}^k/2 - p_{h,2}^k\sqrt{3}/2\right)}{\sqrt{3}},
\end{array}\right. \tag{78}$$

using the local coordinates (38) for $S_h^k \perp \left(\sqrt{3}/2, 1/2\right)$,

$$\left\{\begin{array}{rcl}
\zeta_h^{1,k}(x') &=& \sqrt{2}\sqrt{\mu_h}\dfrac{s - p_{h,1}^k/2 - p_{h,2}^k\sqrt{3}/2}{\sqrt{\lambda_h + 2\mu_h}}, \\
\zeta_h^{2,k}(x') &=& \dfrac{\sqrt{2}\left(s - p_{h,1}^k/2 - p_{h,2}^k\sqrt{3}/2\right)}{\sqrt{3}},
\end{array}\right.$$

and, using the local coordinates (39) for $S_h^k \perp (0,1)$,

$$\begin{cases} \zeta_h^{1,k}(x') &= \sqrt{\dfrac{\mu_h}{2}} \dfrac{\left(x_1 - p_{h,1}^k\right)}{\sqrt{\lambda_h + 2\mu_h}}, \\ \zeta_h^{2,k}(x') &= \dfrac{\left(x_1 - p_{h,1}^k\right)}{\sqrt{2}}. \end{cases}$$

Let us now introduce the intervals $J_h^{p_h^k}$ and $J_h^{q_h^k}$ centred at the points p_h^k and q_h^k respectively, such that

$$S_h^k \cap J_h^{p_h^k} = \left[p_h^k, p_h^k + \mathbf{s}_h\right), \; S_h^k \cap J_h^{q_h^k} = \left(q_h^k - \mathbf{s}_h, q_h^k\right], \tag{79}$$

where $\mathbf{s}_h = \begin{pmatrix} s_h \\ s_h \end{pmatrix}$, such that $\lim_{h\to\infty} 2^h s_h = 0$. Let ψ_h^k be a $C_c^\infty\left(S_h^k \cup J_h^{p_h^k} \cup J_h^{q_h^k}\right)$ test-function, such that

$$\psi_h^k = \begin{cases} 1 & \text{on } S_h^k \backslash J_h^{p_h^k} \cup J_h^{q_h^k}, \\ 0 & \text{on } J_h^{p_h^k} \cup J_h^{q_h^k} \backslash \left(\left(p_h^k, p_h^k + \mathbf{s}_h\right) \cup \left(q_h^k - \mathbf{s}_h, q_h^k\right)\right). \end{cases} \tag{80}$$

We define the test-function v_h by

$$v_h = \psi_h^k v_h^k, \; \forall k \in I_h. \tag{81}$$

We have the following convergences:

Lemma 5.2. *We have*

1. $v_h \dfrac{\mathbf{1}_{T_h}(x)}{|T_h|} ds dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v \mathbf{1}_\Sigma(s) \dfrac{d\mathcal{H}^d(s)}{\mathcal{H}^d(\Sigma)}$,
2. $\lim_{h\to\infty} \int_{T_h} \sigma_{ij}^h(v_h) e_{ij}(v_h) ds dx_3 = \mu_0 \lim_{h\to\infty} \left(\dfrac{5}{3}\right)^h \sum_{\substack{\alpha=1,2 \\ p,q\in\mathcal{V}_h \\ |p-q|=2^{-h}}} |v_\alpha(p) - v_\alpha(q)|^2$.

Proof. 1. Let $\varphi \in C_0\left(\mathbb{R}^3, \mathbb{R}^3\right)$. We have

$$\begin{aligned} \lim_{h\to\infty} \int_{\mathbb{R}^3} \varphi(x) . v_h(x') \frac{\mathbf{1}_{T_h}(x)}{|T_h|} ds dx_3 = & \lim_{h\to\infty} \sum_{k\in I_h} \frac{2v\left(x_{1h}^k, x_{2h}^k\right)}{3^{h+1}} . \varphi\left(x_{1h}^k, x_{2h}^k, 0\right) \\ & + C \lim_{h\to\infty} \sum_{\substack{\alpha=1,2 \\ i=1,2,3 \\ k\in I_h}} \frac{2}{3^{h+1}} \left|v_\alpha\left(p_h^k\right) - v_\alpha\left(q_h^k\right)\right| \varphi_i\left(x_{1h}^k, x_{2h}^k, 0\right), \end{aligned}$$

where C is a positive constant independent of h. On the one hand we have

$$\begin{aligned} \lim_{h\to\infty} \sum_{k\in I_h} \frac{2}{3^{h+1}} v\left(x_{1h}^k, x_{2h}^k\right) . \varphi\left(x_{1h}^k, x_{2h}^k, 0\right) &= \lim_{h\to\infty} \sum_{k\in I_h} \frac{1}{N_h} v\left(x_h^k\right) . \varphi\left(x_h^k, 0\right) \\ &= \frac{1}{\mathcal{H}^d(\Sigma)} \int_\Sigma v(s) . \varphi(s, 0) d\mathcal{H}^d(s). \end{aligned}$$

On the other hand, since

$$\left|v_\alpha\left(p_h^k\right) - v_\alpha\left(q_h^k\right)\right| \leq C \left|p_h^k - q_h^k\right|$$

and $\left|p_h^k - q_h^k\right| = 2^{-h}$, we have

$$\lim_{h\to\infty} \sum_{\substack{\alpha=1,2 \\ i=1,2,3 \\ k\in I_h}} \frac{2}{3^{h+1}} \left|v_\alpha\left(p_h^k\right) - v_\alpha\left(q_h^k\right)\right| \varphi_i\left(x_{1h}^k, x_{2h}^k, 0\right) = 0.$$

2. Computing tensors in local coordinates (37) and (38), we obtain, for $S_h^k \perp \left(-\sqrt{3}/2, 1/2\right)$ or $S_h^k \perp \left(\sqrt{3}/2, 1/2\right)$,

$$\sigma_{ij}^h\left(v_h^k\right) e_{ij}\left(v_h^k\right) = \frac{(\lambda_h + 2\mu_h)}{4}\left(\frac{\partial v_{1,h}^k}{\partial s}\right)^2 + \frac{3\mu_h}{4}\left(\frac{\partial v_{2,h}^k}{\partial s}\right)^2,$$

and if $S_h^k \perp (0,1)$,

$$\sigma_{ij}^h\left(v_h^k\right) e_{ij}\left(v_h^k\right) = (\lambda_h + 2\mu_h)\left(\frac{\partial v_{1,h}^k}{\partial x_1}\right)^2 + \mu_h\left(\frac{\partial v_{2,h}^k}{\partial x_1}\right)^2.$$

Thus, according to (77)-(78), we obtain on each S_h^k; $k \in I_h$,

$$\sigma_{ij}^h\left(v_h^k\right) e_{ij}\left(v_h^k\right) = \mu_h 2^{2h}\left\{\left|v_1\left(p_h^k\right) - v_1\left(q_h^k\right)\right|^2 + \left|v_2\left(p_h^k\right) - v_2\left(q_h^k\right)\right|^2\right\},$$

which implies that

$$\begin{aligned}
&\lim_{h\to\infty} \int_{T_h} \sigma_{ij}^h\left(v_h\right) e_{ij}\left(v_h\right) ds dx_3 \\
&= \mu_0 \lim_{h\to\infty} \eta_h \sum_{k\in I_h, \alpha=1,2} \varepsilon_h 2^h \left|v_\alpha\left(p_h^k\right) - v_\alpha\left(q_h^k\right)\right|^2 \\
&= \mu_0 \lim_{h\to\infty} \left(\frac{5}{3}\right)^h \sum_{k\in I_h, \alpha=1,2} \left|v_\alpha\left(p_h^k\right) - v_\alpha\left(q_h^k\right)\right|^2 \\
&= \mu_0 \lim_{h\to\infty} \left(\frac{5}{3}\right)^h \sum_{\substack{\alpha=1,2 \\ p,q\in\mathcal{V}_h \\ |p-q|=2^{-h}}} \left|v_\alpha(p) - v_\alpha(q)\right|^2.
\end{aligned}$$

□

We prove here the lim-sup condition of the Γ-convergence property stated in Theorem 4.2_1.

Proposition 4. *If $\gamma \in (0, +\infty)$ then, for every $(u, v) \in H_0^1\left(\Omega, \mathbb{R}^3\right) \times \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)$, there exists a sequence $(u_h)_h$, such that $u_h \in H_0^1\left(\Omega, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right)$, $(u_h)_h$ τ-converges to (u, v), and*

$$\limsup_{h\to\infty} F_h\left(u_h\right) \leq F_\infty(u, v).$$

Proof. Let $(u, v) \in H_0^1\left(\Omega, \mathbb{R}^3\right) \times \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)$. Let $(u_n, v_n)_n$ be a sequence in the space $C_c^1\left(\Omega, \mathbb{R}^3\right) \times \left(C_c^1\left(\Omega, \mathbb{R}^3\right) \cap \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)\right)$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ $H^1\left(\Omega, \mathbb{R}^3\right)$-strong, $\overline{v}_n \underset{n\to\infty}{\longrightarrow} \overline{v}$ strongly with respect to the norm (26), and $v_{3,n} \underset{n\to\infty}{\longrightarrow} v_3$ strongly with respect to the norm of $L^2_{\mathcal{H}^d}(\Sigma)$. We define the sequence $\left(u_{n,h}^0\right)_{h,n}$ by

$$u_{n,h}^0 = u_n - w_h^m\left(\left(u_n\right)_m - \left(v_{n,h}\right)_m\right), \tag{82}$$

where $v_{n,h}$ is the test-function (81) associated with v_n, and w_h^m is the perturbation defined in (76). Then $u_{n,h}^0 \in H_0^1\left(\Omega, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right)$ and, using Lemma 5.1, Lemma 5.2, and the fact that the measure $|Z_h|$ of the set Z_h tends to zero as h tends to ∞, that $\left(u_{n,h}^0\right)_h$ τ-converges to (u_n, v_n) as h tends to ∞.

We have

$$\begin{aligned} F_h\left(u^0_{n,h}\right) &= \int_{\Omega\setminus Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u^0_{n,h}\right) dx \\ &+ \int_{Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u^0_{n,h}\right) dx + \int_{T_h} \sigma^h_{ij}\left(v_{n,h}\right) e_{ij}\left(v_{n,h}\right) ds dx_3. \end{aligned} \tag{83}$$

We immediately obtain

$$\lim_{h\to\infty} \int_{\Omega\setminus Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u^0_{n,h}\right) dx = \int_{\Omega} \sigma_{ij}\left(u_n\right) e_{ij}\left(u_n\right) dx.$$

Using Lemma 5.1, it follows that

$$\begin{aligned} &\lim_{h\to\infty} \int_{Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u^0_{n,h}\right) dx \\ &= \lim_{h\to\infty} \int_{Z_h} \sigma_{ij}\left(w^m_h\left(\left(u_n\right)_m - \left(v_{n,h}\right)_m\right)\right) e_{ij}\left(w^m_h\left(\left(u_n\right)_m - \left(v_{n,h}\right)_m\right)\right) \\ &= \frac{\pi\mu\gamma}{\mathcal{H}^d\left(\Sigma\right)\ln 2} \int_{\Sigma} A\left(s\right)\left(u_n - v_n\right).\left(u_n - v_n\right) d\mathcal{H}^d\left(s\right) \end{aligned} \tag{84}$$

and, using Lemma 5.2 and Proposition 1, we obtain

$$\begin{aligned} &\lim_{h\to\infty} \int_{T_h} \sigma^h_{ij}\left(v_{n,h}\right) e_{ij}\left(v_{n,h}\right) ds dx_3 \\ &= \mu_0 \lim_{h\to\infty} \left(\frac{5}{3}\right)^h \sum_{\substack{\alpha=1,2 \\ p,q\in\mathcal{V}_h \\ |p-q|=2^{-h}}} \left|v_{\alpha,n}\left(p,0\right) - v_{\alpha,n}\left(q,0\right)\right|^2 \\ &= \mu_0 \mathcal{E}_{\Sigma}\left(\overline{v}_n\right) \\ &= \mu_0 \int_{\Sigma} d\mathcal{L}_{\Sigma}\left(\overline{v}_n\right). \end{aligned}$$

This yields

$$\begin{aligned} \lim_{h\to\infty} F_h\left(u^0_{n,h}\right) &= \int_{\Omega} \sigma_{ij}\left(u_n\right) e_{ij}\left(u_n\right) dx + \mu_0 \int_{\Sigma} d\mathcal{L}_{\Sigma}\left(\overline{v}_n\right) \\ &+ \frac{\pi\mu\gamma}{\mathcal{H}^d\left(\Sigma\right)\ln 2} \int_{\Sigma} A\left(s\right)\left(u_n - v_n\right).\left(u_n - v_n\right) d\mathcal{H}^d\left(s\right) \\ &= F_{\infty}\left(u_n, v_n\right). \end{aligned} \tag{85}$$

The continuity of F_∞ implies that $\lim_{n\to\infty} \lim_{h\to\infty} F_h\left(u^0_{n,h}\right) = F_\infty\left(u,v\right)$. Then, using the diagonalization argument of [5, Corollary 1.18], we prove the existence of a sequence $\left(u_h\right)_h = \left(u^0_{n(h),h}\right)_h$: $\lim\limits_{h\to\infty} n\left(h\right) = +\infty$, such that

$$\limsup_{h\to\infty} F_h\left(u_h\right) \leq F_\infty\left(u,v\right).$$

□

5.3. **Step 3: Lim-inf inequality.** In this Subsection we prove the second assertion of Theorem 4.2.

Proposition 5. *If $\gamma \in (0,+\infty)$, then, for every sequence $\left(u_h\right)_h$, such that $u_h \in H^1_0\left(\Omega,\mathbb{R}^3\right) \cap H^1\left(T_h,\mathbb{R}^3\right)$ and $\left(u_h\right)_h$ τ-converges to (u,v), we have $\overline{v} \in \mathcal{D}_{\Sigma,\mathcal{E}}$ and*

$$\liminf_{h\to\infty} F_h\left(u_h\right) \geq F_\infty\left(u,v\right).$$

Proof. Let $(u_h)_h$; $u_h \in H_0^1\left(\Omega,\mathbb{R}^3\right) \cap H^1\left(T_h,\mathbb{R}^3\right)$, such that $(u_h)_h$ τ-converges to (u,v). We suppose that $\sup_h F_h(u_h) < +\infty$, otherwise the lim inf inequality is trivial. Then, owing to Proposition 2 and Proposition 1, we have that $\overline{v} \in \mathcal{D}_{\Sigma,\mathcal{E}}$ and

$$\begin{array}{rcl} \liminf\limits_{h\to\infty} \int_{T_h} \sigma_{ij}^h(u_h)\, e_{ij}(u_h)\, ds dx_3 & \geq & \mu_0 \mathcal{E}_\Sigma(\overline{v}) \\ & = & \mu_0 \int_\Sigma d\mathcal{L}_\Sigma(\overline{v}). \end{array} \tag{86}$$

Let $(u_n,v_n)_n \subset C_c^1\left(\Omega,\mathbb{R}^3\right) \times \left(C_c^1\left(\Omega,\mathbb{R}^3\right) \cap \mathcal{D}_{\Sigma,\mathcal{E}} \times L^2_{\mathcal{H}^d}(\Sigma)\right)$, such that

$$u_n \underset{n\to\infty}{\longrightarrow} u \;\, H^1\left(\Omega,\mathbb{R}^3\right) - \text{strong},$$

$\overline{v}_n \underset{n\to\infty}{\longrightarrow} \overline{v}$ strongly with respect to the norm (26), and $v_{3,n} \underset{n\to\infty}{\longrightarrow} v_3$ strongly with respect to the norm of $L^2_{\mathcal{H}^d}(\Sigma)$. Let $\left(u^0_{n,h}\right)_{h,n}$ be the corresponding sequence defined in (82). We have from the definition of the subdifferentiability of convex functionals

$$\begin{array}{r} \int_{Z_h} \sigma_{ij}(u_h)\, e_{ij}(u_h)\, dx \geq \int_{Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u^0_{n,h}\right) dx \\ + 2\int_{Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u_h - u^0_{n,h}\right) dx. \end{array} \tag{87}$$

Due to the structure of the sequence $\left(u^0_{n,h}\right)_h$, we have

$$\begin{array}{r} \int_{Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u_h - u^0_{n,h}\right) dx = \int_{Z_h} \sigma_{ij}(u_n)\, e_{ij}\left(u_h - u^0_{n,h}\right) dx \\ - \int_{Z_h} \sigma_{ij,j}\left(w_h^m (u_n - v_{n,h})_m\right) \left(u_h - u^0_{n,h}\right)_i dx. \end{array} \tag{88}$$

Since $|Z_h|$ tends to zero as h tends to ∞, it follows that

$$\lim_{h\to\infty} \int_{Z_h} \sigma_{ij}(u_n)\, e_{ij}\left(u_h - u^0_{n,h}\right) dx = 0. \tag{89}$$

Using the definition of the perturbation w_h^m and the expressions (62), (63) and (66), we get

$$\begin{array}{l} \left|\int_{Z_h} \sigma_{ij,j}\left(w_h^m (u_n - v_{n,h})_m\right) \left(u_h - u^0_{n,h}\right)_i dx\right| \\ \quad \leq C_n^m \left(\int_{Z_h} \left|\left(u_h - u^0_{n,h}\right)\right|^2 dx\right)^{1/2} \left(1 + \left(\int_{Z_h} \left|\nabla w_h^m(x)\right|^2 dx\right)^{1/2}\right), \end{array} \tag{90}$$

where C_n^m is a positive constant which may depend of n. Then, using Lemma 5.1, we obtain that

$$\lim_{h\to\infty} \int_{Z_h} \sigma_{ij,j}\left(w_h^m (u_n - v_{n,h})_m\right) \left(u_h - u^0_{n,h}\right)_i dx = 0. \tag{91}$$

We deduce from (84) that

$$\begin{array}{l} \lim\limits_{h\to\infty} \int_{Z_h} \sigma_{ij}\left(u^0_{n,h}\right) e_{ij}\left(u^0_{n,h}\right) dx \\ \quad = \dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)\ln 2} \int_\Sigma A(s)\,(u_n - v_n)\,.\,(u_n - v_n)\, d\mathcal{H}^d(s)\,. \end{array} \tag{92}$$

On the other hand, as $|Z_h|$ tends to zero as h tends to ∞, we have

$$\liminf_{h\to\infty} \int_{\Omega\setminus Z_h} \sigma_{ij}(u_h)\, e_{ij}(u_h)\, dx \geq \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx. \tag{93}$$

We deduce from (86)-(93) that

$$\begin{aligned}\liminf_{h\to\infty} F_h(u_h) \quad &\geq \quad \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \mu_0 \int_\Sigma d\mathcal{L}_\Sigma(\overline{v}) \\ &\quad + \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)\ln 2} \int_\Sigma A(s)(u_n - v_n).(u_n - v_n)\, d\mathcal{H}^d(s).\end{aligned}$$

Letting n tend to ∞ in the right hand side of the above inequality, we deduce that

$$\begin{aligned}\liminf_{h\to\infty} F_h(u_h) \quad &\geq \quad \int_\Omega \sigma_{ij}(u)\, e_{ij}(u)\, dx + \mu_0 \int_\Sigma d\mathcal{L}_\Sigma(\overline{v}) \\ &\quad + \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)\ln 2} \int_\Sigma A(s)(u - v).(u - v)\, d\mathcal{H}^d(s),\end{aligned}$$

which is equivalent to

$$\liminf_{h\to\infty} F_h(u_h) \geq F_\infty(u, v).$$

□

This ends the proof of Theorem 4.2.

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