

## Γ-CONVERGENCE OF QUADRATIC FUNCTIONALS WITH NON UNIFORMLY ELLIPTIC CONDUCTIVITY MATRICES

LORENZA D’ELIA\*

Dipartimento di Matematica, Università di Roma Tor Vergata  
 via della ricerca scientifica 1  
 Roma, 00133, Italy

(Communicated by Andrea Braides)

**ABSTRACT.** We investigate the homogenization through  $\Gamma$ -convergence for the  $L^2(\Omega)$ -weak topology of the conductivity functional with a zero-order term where the matrix-valued conductivity is assumed to be non strongly elliptic. Under proper assumptions, we show that the homogenized matrix  $A^*$  is provided by the classical homogenization formula. We also give algebraic conditions for two and three dimensional 1-periodic rank-one laminates such that the homogenization result holds. For this class of laminates, an explicit expression of  $A^*$  is provided which is a generalization of the classical laminate formula. We construct a two-dimensional counter-example which shows an anomalous asymptotic behaviour of the conductivity functional.

**1. Introduction.** In this paper, for a bounded domain  $\Omega$  of  $\mathbb{R}^d$ , we study the homogenization through  $\Gamma$ -convergence of the conductivity energy with a zero-order term of the type

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_{\Omega} \left\{ A\left(\frac{x}{\varepsilon}\right) \nabla u \cdot \nabla u + |u|^2 \right\} dx, & \text{if } u \in H_0^1(\Omega), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega). \end{cases} \quad (1)$$

The conductivity  $A$  is a  $Y_d$ -periodic, symmetric and non-negative matrix-valued function in  $L^\infty(\mathbb{R}^d)^{d \times d}$ , denoted by  $L_{\text{per}}^\infty(Y_d)^{d \times d}$ , which is not strongly elliptic, *i.e.*

$$\text{ess-inf}_{y \in Y_d} \left( \min \{ A(y) \xi \cdot \xi : \xi \in \mathbb{R}^d, |\xi| = 1 \} \right) \geq 0. \quad (2)$$

This condition holds true when the conductivity energy density has missing derivatives. This occurs, for example, when the quadratic form associated to  $A$  is given by

$$A\xi \cdot \xi := A'\xi' \cdot \xi' \quad \text{for } \xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R},$$

where  $A' \in L_{\text{per}}^\infty(Y_d)^{(d-1) \times (d-1)}$  is symmetric and non-negative matrix. It is known (see *e.g.* [13, Chapters 24 and 25]) that the strongly ellipticity of the matrix  $A$ , *i.e.*

$$\text{ess-inf}_{y \in Y_d} \left( \min \{ A(y) \xi \cdot \xi : \xi \in \mathbb{R}^d, |\xi| = 1 \} \right) > 0, \quad (3)$$

---

2020 *Mathematics Subject Classification.* Primary: 35B27, 35B40, 49J45; Secondary: 74Q05.

*Key words and phrases.* Quadratic functionals, homogenization,  $\Gamma$ -convergence, two-scale convergence, non-local functional.

\* Corresponding author: Lorenza D’Elia.

combined with the boundedness implies a compactness result of the conductivity functional

$$u \in H_0^1(\Omega) \mapsto \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u \cdot \nabla u dx$$

for the  $L^2(\Omega)$ -strong topology. The  $\Gamma$ -limit is given by

$$\int_{\Omega} A^* \nabla u \cdot \nabla u dx,$$

where the matrix-valued function  $A^*$  is defined by the classical homogenization formula

$$A^* \lambda \cdot \lambda := \min \left\{ \int_{Y_d} A(y) (\lambda + \nabla v(y)) \cdot (\lambda + \nabla v(y)) dy : v \in H_{\text{per}}^1(Y_d) \right\}. \quad (4)$$

The  $\Gamma$ -convergence for the  $L^p(\Omega)$ -strong topology, for  $p > 1$ , for the class of integral functionals  $F_{\varepsilon}$  of the form

$$F_{\varepsilon}(u) = \int_{\Omega} f\left(\frac{x}{\varepsilon}, Du\right) dx, \quad \text{for } u \in W^{1,p}(\Omega; \mathbb{R}^m), \quad (5)$$

where  $f : \Omega \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}$  is a Borel function, 1-periodic in the first variable satisfying the standard growth conditions of order  $p$ , namely  $c_1 |M|^p \leq f(x, M) \leq c_2 (|M|^p + 1)$  for any  $x \in \Omega$  and for any real  $(m \times d)$ -matrix  $M$ , has been widely studied and it is a classical subject (see *e.g.* [4, Chapter 12] and [13, Chapter 24]). On the contrary, the  $\Gamma$ -convergence of oscillating functionals for the weak topology on bounded sets of  $L^p(\Omega)$  has been very few analysed. An example of the study of  $\Gamma$ -convergence for the  $L^p(\Omega)$ -weak topology can be found in the paper [6] where, in the context of double-porosity, the authors compare the  $\Gamma$ -limit for non-linear functionals analogous to 5 computed with respect to different topologies and in particular with respect to  $L^p(\Omega)$ -weak topology.

In this paper, we investigate the  $\Gamma$ -convergence for the weak topology on bounded sets (a metrizable topology) of  $L^2(\Omega)$  of the conductivity functional under condition 2. In this case, one has no *a priori*  $L^2(\Omega)$ -bound on the sequence of gradients, which implies a loss of coerciveness of the investigated energy. To overcome this difficulty, we add a quadratic zeroth-order term of the form  $\|u\|_{L^2(\Omega)}^2$ , so that we immediately obtain the coerciveness in the weak topology of  $L^2(\Omega)$  of  $\mathcal{F}_{\varepsilon}$ , namely, for  $u \in H_0^1(\Omega)$ ,

$$\mathcal{F}_{\varepsilon}(u) \geq \int_{\Omega} |u|^2 dx.$$

This estimate guarantees that  $\Gamma$ -limit for the weak topology on bounded sets of  $L^2(\Omega)$  is characterized by conditions (i) and (ii) of the Definition 1.1 below (see [13, Proposition 8.10]), as well as, thanks to a compactness result (see [13, Corollary 8.12]),  $\mathcal{F}_{\varepsilon}$   $\Gamma$ -converges for the weak topology of  $L^2(\Omega)$ , up to subsequences, to some functional. We will show that, under the following assumptions:

- (H1) any two-scale limit  $u_0(x, y)$  of a sequence  $u_{\varepsilon}$  of functions in  $L^2(\Omega)$  with bounded energy  $\mathcal{F}_{\varepsilon}(u_{\varepsilon})$  does not depend on  $y$  (see [1, Theorem 1.2]);
- (H2) the space  $V$  defined by

$$V := \left\{ \int_{Y_d} A^{1/2}(y) \Phi(y) dy : \Phi \in L_{\text{per}}^2(Y_d; \mathbb{R}^d) \right. \\ \left. \text{with } \operatorname{div} \left( A^{1/2}(y) \Phi(y) \right) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d) \right\}$$

agrees with the space  $\mathbb{R}^d$ ,  
the  $\Gamma$ -limit is given by

$$\mathcal{F}_0(u) := \begin{cases} \int_{\Omega} \{A^* \nabla u \cdot \nabla u + |u|^2\} dx, & \text{if } u \in H_0^1(\Omega), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega), \end{cases} \quad (6)$$

where the homogenized matrix  $A^*$  is given through the expected homogenization formula

$$A^* \lambda \cdot \lambda := \inf \left\{ \int_{Y_d} A(y) (\lambda + \nabla v(y)) \cdot (\lambda + \nabla v(y)) dy : v \in H_{\text{per}}^1(Y_d) \right\}. \quad (7)$$

We need to make assumption (H1) since for any sequence  $u_\varepsilon$  with bounded energy, *i.e.*  $\sup_{\varepsilon>0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$ , the sequence  $\nabla u_\varepsilon$  in  $L^2(\Omega; \mathbb{R}^d)$  is not bounded due to the lack of ellipticity of the matrix-valued conductivity  $A(y)$ . Assumption (H2) turns out to be equivalent to the positive definiteness of the homogenized matrix (see Proposition 1).

In the 2D isotropic elasticity setting of [11], the authors make use of similar conditions as (H1) and (H2) in the proof of the main results (see [11, Theorems 3.3 and 3.4]). They investigate the limit in the sense of  $\Gamma$ -convergence for the  $L^2(\Omega)$ -weak topology of the elasticity functional with a zeroth-order term in the case of two-phase isotropic laminate materials where the phase 1 is very strongly elliptic, while the phase 2 is only strongly elliptic. The strong ellipticity of the effective tensor is preserved through a homogenization process expect in the case when the volume fraction of each phase is 1/2, as first evidenced by Gutiérrez [14]. Indeed, Gutiérrez has provided two and three dimensional examples of 1-periodic rank-one laminates such that the homogenized tensor induced by a homogenization process, labelled 1\*-convergence, is not strongly elliptic. These examples have been revisited by means of a homogenization process using  $\Gamma$ -convergence in the two-dimensional case of [10] and in the three-dimensional case of [12].

In the present scalar case, we enlighten assumptions (H1) and (H2) which are the key ingredients to obtain the general  $\Gamma$ -convergence result Theorem 2.1. Using Nguetseng-Allaire [1, 16] two-scale convergence, we prove that for any dimension  $d \geq 2$ , the  $\Gamma$ -limit  $\mathcal{F}_0$  6 for the weak topology of  $L^2(\Omega)$  actually agrees with the one obtained for the  $L^2(\Omega)$ -strong topology under uniform ellipticity 3, replacing the minimum in 4 by the infimum in 7. Assumption (H2) implies the coerciveness of the functional  $\mathcal{F}_0$  showing that its domain is  $H_0^1(\Omega)$  and that the homogenized matrix  $A^*$  is positive definite. More precisely, the positive definiteness of  $A^*$  turns out to be equivalent to assumption (H2) (see Proposition 1). We also provide two and three dimensional 1-periodic rank-one laminates which satisfy assumptions (H1) and (H2) (see Proposition 2 for the two-dimensional case and Proposition 3 for the three-dimensional case). Thanks to Theorem 2.1, the corresponding homogenized matrix  $A^*$  is positive definite. For this class of laminates, an alternative and independent proof of positive definiteness of  $A^*$  is performed using an explicit expression of  $A^*$  (see Proposition 5). This expression generalizes the classical laminate formula for non-degenerate phases (see [17] and also [2, Lemma 1.3.32], [8]) to the case of two-phase rank-one laminates with degenerate and anisotropic phases.

The lack of assumption (H1) may induce a degenerate asymptotic behaviour of the functional  $\mathcal{F}_\varepsilon$  1. We provide a two-dimensional rank-one laminate with two

degenerate phases for which the functional  $\mathcal{F}_\varepsilon$  does  $\Gamma$ -converge for the  $L^2(\Omega)$ -weak topology to a functional  $\mathcal{F}$  which differs from the one given by 6 (see Proposition 4). In this example, any two-scale limit  $u_0(x, y)$  of a sequence with bounded energy  $\mathcal{F}_\varepsilon(u_\varepsilon)$ , depends on the variable  $y$ . Moreover, we give two quite different expressions of the  $\Gamma$ -limit  $\mathcal{F}$  which seem to be original up to the best of our knowledge. The energy density of the first expression is written with Fourier transform of the target function. The second expression appears as a non-local functional due to the presence of a convolution term. However, we do not know if the  $\Gamma$ -limit  $\mathcal{F}$  is a Dirichlet form in the sense of Beurling-Deny [3], since the Markovian property is not stable by the  $L^2(\Omega)$ -weak topology (see Remark 2).

The paper is organized as follows. In Section 2, we prove a general  $\Gamma$ -convergence result (see Theorem 2.1) for the functional  $\mathcal{F}_\varepsilon$  1 with any non-uniformly elliptic matrix-valued function  $A$ , under assumptions (H1) and (H2). In Section 3 we illustrate the general result of Section 2 by periodic two-phase rank-one laminates with two (possibly) degenerate and anisotropic phases in dimension two and three. We provide algebraic conditions so that assumptions (H1) and (H2) are satisfied (see Propositions 2 and 3). In Section 4 we exhibit a two-dimensional counter-example where assumption (H1) fails, which leads us to a degenerate  $\Gamma$ -limit  $\mathcal{F}$  involving a convolution term (see Proposition 4). Finally, in the Appendix we give an explicit formula for the homogenized matrix  $A^*$  for any two-phase rank-one laminates with (possibly) degenerate phases. We also provide an alternative proof of the positive definiteness of  $A^*$  using an explicit expression of  $A^*$  for the class of two-phase rank-one laminates introduced in Section 3 (see Proposition 5).

### Notation.

- For  $i = 1, \dots, d$ ,  $e_i$  denotes the  $i$ -th vector of the canonical basis in  $\mathbb{R}^d$ ;
- $I_d$  denotes the unit matrix of  $\mathbb{R}^{d \times d}$ ;
- $H_{\text{per}}^1(Y_d; \mathbb{R}^n)$  (resp.  $L_{\text{per}}^2(Y_d; \mathbb{R}^n)$ ,  $C_{\text{per}}^\infty(Y_d; \mathbb{R}^n)$ ) is the space of those functions in  $H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^n)$  (resp.  $L_{\text{loc}}^2(\mathbb{R}^d; \mathbb{R}^n)$ ,  $C_{\text{loc}}^\infty(\mathbb{R}^d; \mathbb{R}^n)$ ) that are  $Y_d$ -periodic;
- Throughout, the variable  $x$  will refer to running point in a bounded open domain  $\Omega \subset \mathbb{R}^d$ , while the variable  $y$  will refer to a running point in  $Y_d$  (or  $k + Y_d$ ,  $k \in \mathbb{Z}^d$ );
- We write

$$u_\varepsilon \rightharpoonup u_0$$

with  $u_\varepsilon \in L^2(\Omega)$  and  $u_0 \in L^2(\Omega \times Y_d)$  if  $u_\varepsilon$  two-scale converges to  $u^0$  in the sense of Nguetseng-Allaire (see [1, 16])

- $\mathcal{F}_1$  and  $\mathcal{F}_2$  denote the Fourier transform defined on  $L^1(\mathbb{R})$  and  $L^2(\mathbb{R})$  respectively. For  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , the Fourier transform  $\mathcal{F}_1$  of  $f$  is defined by

$$\mathcal{F}_1(f)(\lambda) := \int_{\mathbb{R}} e^{-2\pi i \lambda x} f(x) dx.$$

**Definition 1.1.** Let  $X$  be a reflexive and separable Banach space endowed with the weak topology  $\sigma(X, X')$ , and let  $\mathcal{F}_\varepsilon : X \rightarrow \mathbb{R}$  be a  $\varepsilon$ -indexed sequence of functionals. The sequence  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges to the functional  $\mathcal{F}_0 : X \rightarrow \mathbb{R}$  for the weak topology of  $X$ , and we write  $\mathcal{F}_\varepsilon \xrightarrow{\Gamma(X)-w} \mathcal{F}_0$ , if for any  $u \in X$ ,

- $\forall u_\varepsilon \rightharpoonup u$ ,  $\mathcal{F}_0(u) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon)$ ,
- $\exists \bar{u}_\varepsilon \rightharpoonup u$  such that  $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\bar{u}_\varepsilon) = \mathcal{F}_0(u)$ .

Such a sequence  $\bar{u}_\varepsilon$  is called a recovery sequence.

Recall that the weak topology of  $L^2(\Omega)$  is metrizable on bounded sets, *i.e.* there exists a metric  $d$  on  $L^2(\Omega)$  such that on every norm bounded subset  $B$  of  $L^2(\Omega)$  the weak topology coincides with the topology induced on  $B$  by the metric  $d$  (see *e.g.* [13, Proposition 8.7]).

**2. A preliminary general  $\Gamma$ -result.** In this section, we will prove the main result of this paper. As previously announced, up to a subsequence, the sequence of functionals  $\mathcal{F}_\varepsilon$ , given by 1 with non-uniformly elliptic matrix-valued conductivity  $A(y)$ ,  $\Gamma$ -converges for the weak topology on bounded sets of  $L^2(\Omega)$  to some functional. Our aim is to show that  $\Gamma$ -limit is exactly  $\mathcal{F}_0$  when  $u \in H_0^1(\Omega)$ .

**Theorem 2.1.** *Let  $\mathcal{F}_\varepsilon$  be functionals given by 1 with  $A(y)$  a  $Y_d$ -periodic, symmetric, non-negative matrix-valued function in  $L^\infty(\mathbb{R}^d)^{d \times d}$  satisfying 2. Assume the following assumptions*

- (H1) *any two-scale limit  $u_0(x, y)$  of a sequence  $u_\varepsilon$  of functions in  $L^2(\Omega)$  with bounded energy  $\mathcal{F}_\varepsilon(u_\varepsilon)$  does not depend on  $y$ ;*
- (H2) *the space  $V$  defined by*

$$V := \left\{ \int_{Y_d} A^{1/2}(y) \Phi(y) dy : \Phi \in L_{per}^2(Y_d; \mathbb{R}^d) \right. \\ \left. \text{with } \operatorname{div} \left( A^{1/2}(y) \Phi(y) \right) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d) \right\} \quad (8)$$

*agrees with the space  $\mathbb{R}^d$ .*

*Then,  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges for the weak topology of  $L^2(\Omega)$  to  $\mathcal{F}_0$ , *i.e.**

$$\mathcal{F}_\varepsilon \xrightarrow{\Gamma(L^2)-w} \mathcal{F}_0,$$

*where  $\mathcal{F}_0$  is defined by 6 and  $A^*$  is given by 7.*

*Proof.* We split the proof into two steps which are an adaptation of [11, Theorem 3.3] using the sole assumptions (H1) and (H2) in the general setting of conductivity.

**Step 1 -  $\Gamma$ -lim inf inequality.**

Consider a sequence  $\{u_\varepsilon\}_\varepsilon$  converging weakly in  $L^2(\Omega)$  to  $u \in L^2(\Omega)$ . We want to prove that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}_0(u). \quad (9)$$

If the lower limit is  $\infty$  then 9 is trivial. Up to a subsequence, still indexed by  $\varepsilon$ , we may assume that  $\liminf \mathcal{F}_\varepsilon(u_\varepsilon)$  is a limit and we can also assume henceforth that, for some  $0 < C < \infty$ ,

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq C. \quad (10)$$

As  $u_\varepsilon$  is bounded in  $L^2(\Omega)$ , there exists a subsequence, still indexed by  $\varepsilon$ , which two-scale converges to a function  $u_0(x, y) \in L^2(\Omega \times Y_d)$  (see *e.g.* [1, Theorem 1.2]). In other words,

$$u_\varepsilon \rightharpoonup u_0. \quad (11)$$

Assumption (H1) ensures that

$$u_0(x, y) = u(x) \quad \text{is independent of } y, \quad (12)$$

where, according to the link between two-scale and weak  $L^2(\Omega)$ -convergences (see [1, Proposition 1.6]),  $u$  is the weak limit of  $u_\varepsilon$ , *i.e.*

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega).$$

Since all the components of the matrix  $A(y)$  are bounded and  $A(y)$  is non-negative as a quadratic form, in view of [10](#), for another subsequence (not relabeled), we have

$$A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \rightharpoonup \sigma_0(x, y) \quad \text{with } \sigma_0 \in L^2(\Omega \times Y_d; \mathbb{R}^d),$$

and also

$$A^{1/2}\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \rightharpoonup \Theta_0(x, y) \quad \text{with } \Theta_0 \in L^2(\Omega \times Y_d; \mathbb{R}^d). \quad (13)$$

In particular

$$\varepsilon A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \rightharpoonup 0. \quad (14)$$

Consider  $\Phi \in L^2_{\text{per}}(Y_d; \mathbb{R}^d)$  such that

$$\operatorname{div}\left(A^{1/2}(y)\Phi(y)\right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \quad (15)$$

or equivalently,

$$\int_{Y_d} A^{1/2}(y)\Phi(y) \cdot \nabla \psi(y) dy = 0 \quad \forall \psi \in H^1_{\text{per}}(Y_d).$$

Take also  $\varphi \in C^\infty(\overline{\Omega})$ . Since  $u_\varepsilon \in H^1_0(\Omega)$  and in view of [15](#), an integration by parts yields

$$\int_{\Omega} A^{1/2}\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \Phi\left(\frac{x}{\varepsilon}\right) \varphi(x) dx = - \int_{\Omega} u_\varepsilon A^{1/2}\left(\frac{x}{\varepsilon}\right) \Phi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \varphi(x) dx.$$

By using [\[1, Lemma 5.7\]](#),  $A^{1/2}(y)\Phi(y) \cdot \nabla \varphi(x)$  is an admissible test function for the two-scale convergence. Then, we can pass to the two-scale limit in the previous expression with the help of the convergences [11](#) and [13](#) along with [12](#), and we obtain

$$\int_{\Omega \times Y_d} \Theta_0(x, y) \cdot \Phi(y) \varphi(x) dx dy = - \int_{\Omega \times Y_d} u(x) A^{1/2}(y) \Phi(y) \cdot \nabla \varphi(x) dx dy. \quad (16)$$

We prove that the target function  $u$  is in  $H^1(\Omega)$ . Setting

$$N := \int_{Y_d} A^{1/2}(y) \Phi(y) dy, \quad (17)$$

and varying  $\varphi$  in  $C_c^\infty(\Omega)$ , the equality [16](#) reads as

$$\int_{\Omega \times Y_d} \Theta_0(x, y) \cdot \Phi(y) \varphi(x) dx dy = - \int_{\Omega} u(x) N \cdot \nabla \varphi(x) dx$$

Since the integral in the left-hand side is bounded by a constant times  $\|\varphi\|_{L^2(\Omega)}$ , the right-hand side is a linear and continuous map in  $\varphi \in L^2(\Omega)$ . By the Riesz representation theorem, there exists  $g \in L^2(\Omega)$  such that, for any  $\varphi \in C_c^\infty(\Omega)$ ,

$$\int_{\Omega} u(x) N \cdot \nabla \varphi(x) dx = \int_{\Omega} g(x) \varphi(x) dx,$$

which implies that

$$N \cdot \nabla u \in L^2(\Omega). \quad (18)$$

In view of assumption (H2),  $N$  is an arbitrary vector in  $\mathbb{R}^d$  so that we infer from [18](#) that

$$u \in H^1(\Omega). \quad (19)$$

This combined with equality [16](#) leads us to

$$\int_{\Omega \times Y_d} \Theta_0(x, y) \cdot \Phi(y) \varphi(x) dx dy = \int_{\Omega \times Y_d} A^{1/2}(y) \nabla u(x) \cdot \Phi(y) \varphi(x) dx dy. \quad (20)$$

By density, the last equality holds if the test functions  $\Phi(y)\varphi(x)$  are replaced by the set of  $\psi(x, y) \in L^2(\Omega; L^2_{\text{per}}(Y_d; \mathbb{R}^d))$  such that

$$\operatorname{div}_y \left( A^{1/2}(y) \psi(x, y) \right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

or equivalently,

$$\int_{\Omega \times Y_d} \psi(x, y) \cdot A^{1/2}(y) \nabla_y v(x, y) dx dy = 0 \quad \forall v \in L^2(\Omega; H^1_{\text{per}}(Y_d)).$$

The  $L^2(\Omega; L^2_{\text{per}}(Y_d; \mathbb{R}^d))$ -orthogonal to that set is the  $L^2$ -closure of

$$\mathcal{K} := \left\{ A^{1/2}(y) \nabla_y v(x, y) : v \in L^2(\Omega; H^1_{\text{per}}(Y_d)) \right\}.$$

Thus, the equality 20 yields

$$\Theta_0(x, y) = A^{1/2}(y) \nabla u(x) + S(x, y)$$

for some  $S$  in the closure of  $\mathcal{K}$ , i.e. there exists a sequence  $v_n \in L^2(\Omega; H^1_{\text{per}}(Y_d))$  such that

$$A^{1/2}(y) \nabla_y v_n(x, y) \rightarrow S(x, y) \quad \text{strongly in } L^2(\Omega; L^2_{\text{per}}(Y_d; \mathbb{R}^d)).$$

Due to the lower semi-continuity property of two-scale convergence (see [1, Proposition 1.6]), we get

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \|A^{1/2}(x/\varepsilon) \nabla u_\varepsilon\|_{L^2(\Omega; \mathbb{R}^d)}^2 &\geq \|\Theta_0\|_{L^2(\Omega \times Y_d; \mathbb{R}^d)}^2 \\ &= \lim_n \left\| A^{1/2}(y) (\nabla_x u(x) + \nabla_y v_n) \right\|_{L^2(\Omega \times Y_d; \mathbb{R}^d)}^2. \end{aligned}$$

Then, by the weak  $L^2$ -lower semi-continuity of  $\|u_\varepsilon\|_{L^2(\Omega)}$ , we have

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \\ &\geq \lim_n \int_{\Omega \times Y_d} A(y) (\nabla_x u(x) + \nabla_y v_n(x, y)) \cdot (\nabla_x u(x) + \nabla_y v_n(x, y)) dx dy + \int_{\Omega} |u|^2 dx \\ &\geq \int_{\Omega} \inf \left\{ \int_{Y_d} A(y) (\nabla_x u(x) + \nabla_y v(y)) \cdot (\nabla_x u(x) + \nabla_y v(y)) dy : v \in H^1_{\text{per}}(Y_d) \right\} dx \\ &\quad + \int_{\Omega} |u|^2 dx. \end{aligned}$$

Recalling the definition 7, we immediately conclude that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_{\Omega} \{ A^* \nabla u \cdot \nabla u + |u|^2 \} dx,$$

provided that  $u \in H^1_0(\Omega)$ .

It remains to prove that the target function  $u$  is actually in  $H^1_0(\Omega)$ , giving a complete characterization of  $\Gamma$ -limit. To this end, take  $x_0 \in \partial\Omega$  a Lebesgue point for  $u|_{\partial\Omega}$  and for  $\nu(x_0)$ , the exterior normal to  $\Omega$  at point  $x_0$ . Thanks to 19, we know that  $u \in H^1(\Omega)$ , hence, after an integration by parts of the right-hand side of 16, we obtain, for  $\varphi \in C^\infty(\overline{\Omega})$ ,

$$\begin{aligned} \int_{\Omega \times Y_d} \Theta_0(x, y) \cdot \Phi(y) \varphi(x) dx dy &= \int_{\Omega} N \cdot \nabla u(x) \varphi(x) dx \\ &\quad - \int_{\partial\Omega} N \cdot \nu(x) u(x) \varphi(x) d\mathcal{H}, \end{aligned} \quad (21)$$

where  $N$  is given by 17. Varying  $\varphi$  in  $C_c^\infty(\Omega)$ , the first two integrals in 21 are equal and bounded by a constant times  $\|\varphi\|_{L^2(\Omega)}$ . It follows that, for any  $\varphi \in C^\infty(\bar{\Omega})$ ,

$$\int_{\partial\Omega} N \cdot \nu(x) u(x) \varphi(x) d\mathcal{H} = 0,$$

which leads to  $N \cdot \nu(x) u(x) = 0$   $\mathcal{H}$ -a.e. on  $\partial\Omega$ . Since  $x_0$  is a Lebesgue point, we have

$$N \cdot \nu(x_0) u(x_0) = 0. \quad (22)$$

In view of assumption (H2) and the arbitrariness of  $N$ , we can choose  $N$  such that  $N = \nu(x_0)$  so that from 22 we get  $u(x_0) = 0$ . Hence,

$$u \in H_0^1(\Omega).$$

This concludes the proof of the  $\Gamma$ -lim inf inequality.

**Step 2 -  $\Gamma$ -lim sup inequality.**

We use the same arguments of [12, Theorem 2.4] which can easily extend to the conductivity setting. We just give an idea of the proof, which is based on a perturbation argument. For  $\delta > 0$ , let  $A_\delta$  be the perturbed matrix of  $\mathbb{R}^{d \times d}$  defined by

$$A_\delta := A + \delta I_d,$$

where  $I_d$  is the unit matrix of  $\mathbb{R}^{d \times d}$ . Since the matrix  $A$  is non-negative,  $A_\delta$  turns out to be positive definite, hence, the functional  $\mathcal{F}_\varepsilon^\delta$ , defined by 1 with  $A_\delta$  in place of  $A$ ,  $\Gamma$ -converges to the functional  $\mathcal{F}^\delta$  given by

$$\mathcal{F}^\delta(u) := \begin{cases} \int_{\Omega} \{A_\delta^* \nabla u \cdot \nabla u + |u|^2\} dx, & \text{if } u \in H_0^1(\Omega), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega), \end{cases}$$

for the strong topology of  $L^2(\Omega)$  (see *e.g.* [13, Corollary 24.5]). Thanks to the compactness result of  $\Gamma$ -convergence (see *e.g.* [4, Proposition 1.42]), there exists a subsequence  $\varepsilon_j$  such that  $\mathcal{F}_{\varepsilon_j}$   $\Gamma$ -converges for the  $L^2(\Omega)$ -strong topology to some functional  $F^0$ . Let  $u \in H_0^1(\Omega)$  and let  $u_{\varepsilon_j}$  be a recovery sequence for  $\mathcal{F}_{\varepsilon_j}$  which converges to  $u$  for the  $H^1(\Omega)$ -weak topology on bounded sets. Since  $\mathcal{F}_{\varepsilon_j} \leq \mathcal{F}_{\varepsilon_j}^\delta$  and since  $u_{\varepsilon_j}$  belongs to some bounded set of  $H^1(\Omega)$ , from [13, Propositions 6.7 and 8.10] we deduce that

$$\begin{aligned} F^0(u) &\leq \mathcal{F}^\delta(u) \\ &\leq \liminf_{\varepsilon_j \rightarrow 0} \int_{\Omega} \left\{ A_\delta \left( \frac{x}{\varepsilon_j} \right) \nabla u_{\varepsilon_j} \cdot \nabla u_{\varepsilon_j} + |u_{\varepsilon_j}|^2 \right\} dx \\ &\leq \liminf_{\varepsilon_j \rightarrow 0} \int_{\Omega} \left\{ A \left( \frac{x}{\varepsilon_j} \right) \nabla u_{\varepsilon_j} \cdot \nabla u_{\varepsilon_j} + |u_{\varepsilon_j}|^2 \right\} dx + O(\delta) \\ &= F^0(u) + O(\delta). \end{aligned}$$

It follows that  $\mathcal{F}^\delta$  converges to  $F^0$  as  $\delta \rightarrow 0$ . Then, the  $\Gamma$ -limit  $F^0$  of  $\mathcal{F}_{\varepsilon_j}$  is independent on the subsequence  $\varepsilon_j$ . Repeating the same arguments, any subsequence of  $\mathcal{F}_\varepsilon$  has a further subsequence which  $\Gamma$ -converges for the strong topology of  $L^2(\Omega)$  to  $F^0 = \lim_{\delta \rightarrow 0} \mathcal{F}^\delta$ . Thanks to the Urysohn property (see *e.g.* [4, Proposition 1.44]), the whole sequence  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges to the functional  $F^0$  for the strong topology

of  $L^2(\Omega)$ . On the other hand, in light of the definition 7 of  $A^*$ , we get that  $A_\delta^*$  converges to  $A^*$  as  $\delta \rightarrow 0$ , *i.e.*

$$\lim_{\delta \rightarrow 0} A_\delta^* = A^*. \quad (23)$$

Thanks to the Lebesgue dominated convergence theorem and in view of 23, we get that  $F^0 = \lim_{\delta \rightarrow 0} \mathcal{F}^\delta$  is exactly  $\mathcal{F}_0$  given by 6. Therefore,  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges to  $\mathcal{F}_0$  for the  $L^2(\Omega)$ -strong topology.

Now, let us show that  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges to  $\mathcal{F}_0$  for the weak topology of  $L^2(\Omega)$ . Recall that the  $L^2(\Omega)$ -weak topology is metrizable on the closed ball of  $L^2(\Omega)$ . Fix  $n \in \mathbb{N}$  and let  $d_{B_n}$  be any metric inducing the  $L^2(\Omega)$ -weak topology on the ball  $B_n$  centered on 0 and of radius  $n$ . Let  $u \in H_0^1(\Omega)$  and let  $\bar{u}_\varepsilon$  be a recovery sequence for  $\mathcal{F}_\varepsilon$  for the  $L^2(\Omega)$ -strong topology. Since the topology induced by the metric  $d_{B_n}$  on  $B_n$  is weaker than the  $L^2(\Omega)$ -strong topology,  $\bar{u}_\varepsilon$  is also a recovery sequence for  $\mathcal{F}_\varepsilon$  for the  $L^2(\Omega)$ -weak topology on  $B_n$ . Hence,

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\bar{u}_\varepsilon) = \mathcal{F}_0(u),$$

which proves the  $\Gamma$ -lim sup inequality in  $B_n$ . Finally, since any sequence converging weakly in  $L^2(\Omega)$  belongs to some ball  $B_n \subset L^2(\Omega)$ , as well as its limit, it follows that the  $\Gamma$ -lim sup inequality holds true for  $\mathcal{F}_\varepsilon$  for  $L^2(\Omega)$ -weak topology, which concludes the proof.  $\square$

The next proposition provides a characterization of Assumption (H2) in terms of homogenized matrix  $A^*$ .

**Proposition 1.** *Assumption (H2) is equivalent to the positive definiteness of  $A^*$ , or equivalently,*

$$\text{Ker}(A^*) = V^\perp.$$

*Proof.* Consider  $\lambda \in \text{Ker}(A^*)$ . Define

$$H_\lambda^1(Y_d) := \left\{ u \in H_{\text{loc}}^1(\mathbb{R}^d) : \nabla u \text{ is } Y_d\text{-periodic and } \int_{Y_d} \nabla u(y) dy = \lambda \right\}.$$

Recall that  $u \in H_\lambda^1(Y_d)$  if and only if there exists  $v \in H_{\text{per}}^1(Y_d)$  such that  $u(y) = v(y) + \lambda \cdot y$  (see *e.g.* [13, Lemma 25.2]). Since  $A^*$  is non-negative and symmetric, from 7 it follows that

$$0 = A^* \lambda \cdot \lambda = \inf \left\{ \int_{Y_d} A(y) \nabla u(y) \cdot \nabla u(y) dy : u \in H_\lambda^1(Y_d) \right\}.$$

Then, there exists a sequence  $u_n$  of functions in  $H_\lambda^1(Y_d)$  such that

$$\lim_{n \rightarrow \infty} \int_{Y_d} A(y) \nabla u_n(y) \cdot \nabla u_n(y) dy = 0,$$

which implies that

$$A^{1/2} \nabla u_n \rightarrow 0 \quad \text{strongly in } L^2(Y_d; \mathbb{R}^d). \quad (24)$$

Now, take  $\Phi \in L_{\text{per}}^2(Y_d; \mathbb{R}^d)$  such that  $A^{1/2} \Phi$  is a divergence free field in  $\mathbb{R}^d$ . Recall that, since  $u_n \in H_\lambda^1(Y_d)$ , we have that  $\nabla u_n(y) = \nabla v_n(y) + \lambda$ , for some  $v_n \in$

$H_{\text{per}}^1(Y_d)$ . This implies that

$$\begin{aligned} \int_{Y_d} A^{1/2}(y) \nabla u_n(y) \cdot \Phi(y) dy &= \int_{Y_d} \nabla u_n(y) \cdot A^{1/2}(y) \Phi(y) dy \\ &= \lambda \cdot \int_{Y_d} A^{1/2}(y) \Phi(y) dy + \int_{Y_d} \nabla v_n(y) \cdot A^{1/2}(y) \Phi(y) dy \\ &= \lambda \cdot \int_{Y_d} A^{1/2}(y) \Phi(y) dy, \end{aligned} \quad (25)$$

where the last equality is obtained by integrating by parts the second integral combined with the fact that  $A^{1/2}\Phi$  is a divergence free field in  $\mathbb{R}^d$ . In view of convergence [24](#), the integral on the left-hand side of [25](#) converges to 0. Hence, passing to the limit as  $n \rightarrow \infty$  in [25](#) yields

$$0 = \lambda \cdot \left( \int_{Y_d} A^{1/2}(y) \Phi(y) dy \right),$$

for any  $\Phi \in L_{\text{per}}^2(Y_d; \mathbb{R}^d)$  such that  $A^{1/2}\Phi$  is a divergence free field in  $\mathbb{R}^d$ . Therefore  $\lambda \in V^\perp$  which implies that

$$\text{Ker}(A^*) \subseteq V^\perp.$$

Conversely, by [23](#) we already know that

$$\lim_{\delta \rightarrow 0} A_\delta^* = A^*,$$

where  $A_\delta^*$  is the homogenized matrix associated with  $A_\delta = A + \delta I_d$ . Since  $A_\delta$  is strongly elliptic, the homogenized matrix  $A_\delta^*$  is given by

$$A_\delta^* \lambda \cdot \lambda = \min \left\{ \int_{Y_d} A_\delta(y) \nabla u_\delta(y) \cdot \nabla u_\delta(y) dy : u_\delta \in H_\lambda^1(Y_d) \right\}. \quad (26)$$

Let  $\bar{u}_\delta$  be the minimizer of problem [26](#). Therefore, there exists a constant  $C > 0$  such that

$$A_\delta^* \lambda \cdot \lambda = \int_{Y_d} A_\delta(y) \nabla \bar{u}_\delta(y) \cdot \nabla \bar{u}_\delta(y) dy = \int_{Y_d} |A_\delta^{1/2}(y) \nabla \bar{u}_\delta(y)|^2 dy \leq C,$$

which implies that the sequence  $\Phi_\delta(y) := A_\delta^{1/2}(y) \nabla \bar{u}_\delta(y)$  is bounded in  $L_{\text{per}}^2(Y_d; \mathbb{R}^d)$ . Then, up to extract a subsequence, we can assume that  $\Phi_\delta$  converges weakly to some  $\Phi$  in  $L_{\text{per}}^2(Y_d; \mathbb{R}^d)$ .

Now, we show that  $A_\delta^{1/2}$  converges strongly to  $A^{1/2}$  in  $L_{\text{per}}^\infty(Y_d)^{d \times d}$ . Since  $A(y)$  is a symmetric matrix, there exists an orthogonal matrix-valued function  $R$  in  $L_{\text{per}}^\infty(Y_d)^{d \times d}$  such that

$$A(y) = R(y)D(y)R^T(y) \quad \text{for a.e. } y \in Y_d,$$

where  $D$  is a diagonal non-negative matrix-valued function in  $L_{\text{per}}^\infty(Y_d)^{d \times d}$  and  $R^T$  denotes the transpose of  $R$ . It follows that  $A_\delta(y) = A(y) + \delta I_d = R(y)(D(y) + \delta I_d)R^T(y)$ , for a.e.  $y \in Y_d$ . Hence,

$$A_\delta^{1/2}(y) = R(y)(D(y) + \delta I_d)^{1/2}R^T(y) \quad \text{for a.e. } y \in Y_d,$$

which implies that  $A_\delta^{1/2}$  converges strongly to  $A^{1/2} = RD^{1/2}R^T$  in  $L_{\text{per}}^\infty(Y_d)^{d \times d}$ .

Now, passing to the limit as  $\delta \rightarrow 0$  in

$$\text{div}(A_\delta^{1/2}\Phi_\delta) = \text{div}(A_\delta \nabla \bar{u}_\delta) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

we have

$$\operatorname{div}(A^{1/2}\Phi) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

This along with  $\Phi \in L^2_{\text{per}}(Y_d; \mathbb{R}^d)$  implies that  $\Phi$  is a test function for the set  $V$  given by 8. From 26 it follows that

$$A^*_\delta \lambda = \int_{Y_d} A_\delta(y) \nabla \bar{u}_\delta(y) dy = \int_{Y_d} A^{1/2}_\delta(y) \Phi_\delta(y) dy.$$

Hence, taking into account the strong convergence of  $A^{1/2}_\delta$  in  $L^\infty_{\text{per}}(Y_d)^{d \times d}$  and the weak convergence of  $\Phi_\delta$  in  $L^2_{\text{per}}(Y_d; \mathbb{R}^d)$ , we have

$$A^* \lambda = \lim_{\delta \rightarrow 0} A^*_\delta \lambda = \lim_{\delta \rightarrow 0} \int_{Y_d} A^{1/2}_\delta(y) \Phi_\delta(y) dy = \int_{Y_d} A^{1/2}(y) \Phi(y) dy,$$

which implies that  $A^* \lambda \in V$  since  $\Phi$  is a suitable test function for the set  $V$ . Therefore, for  $\lambda \in V^\perp$ ,

$$A^* \lambda \cdot \lambda = 0,$$

so that, since  $A^*$  is a non-negative matrix, we deduce that  $\lambda \in \operatorname{Ker}(A^*)$ . In other words,

$$V^\perp \subseteq \operatorname{Ker}(A^*),$$

which concludes the proof.  $\square$

**3. Two-dimensional and three-dimensional examples.** In this section we provide a geometric setting for which assumptions (H1) and (H2) are fulfilled. We focus on a 1-periodic rank-one laminates of direction  $e_1$  with two phases in  $\mathbb{R}^d$ ,  $d = 2, 3$ . Specifically, we assume the existence of two anisotropic phases  $Z_1$  and  $Z_2$  of  $Y_d$  given by

$$Z_1 = (0, \theta) \times (0, 1)^{d-1} \quad \text{and} \quad Z_2 = (\theta, 1) \times (0, 1)^{d-1},$$

where  $\theta$  denotes the volume fraction of the phase  $Z_1$ . Let  $Z_1^\#$  and  $Z_2^\#$  be the associated subsets of  $\mathbb{R}^d$ , *i.e.* the open periodic sets

$$Z_i^\# := \operatorname{Int} \left( \bigcup_{k \in \mathbb{Z}^d} (\overline{Z_i} + k) \right) \quad \text{for } i = 1, 2.$$

Let  $X_1$  and  $X_2$  be unbounded connected components of  $Z_1^\#$  and  $Z_2^\#$  in  $\mathbb{R}^d$  given by

$$X_1 := (0, \theta) \times \mathbb{R}^{d-1} \quad \text{and} \quad X_2 := (\theta, 1) \times \mathbb{R}^{d-1},$$

and we denote by  $\partial Z$  the interface  $\{y_1 = 0\}$ .

The anisotropic phases are described by two constant, symmetric and non-negative matrices  $A_1$  and  $A_2$  of  $\mathbb{R}^{d \times d}$  which are possibly not positive definite. Hence, the conductivity matrix-valued function  $A \in L^\infty_{\text{per}}(Y_d)^{d \times d}$ , given by

$$A(y_1) := \chi(y_1) A_1 + (1 - \chi(y_1)) A_2 \quad \text{for } y_1 \in \mathbb{R}, \quad (27)$$

where  $\chi$  is the 1-periodic characteristic function of the phase  $Z_1$ , is not strongly elliptic, *i.e.* 2 is satisfied.

**3.1. The two-dimensional case with one degenerate phase.** We are interested in two-phase mixtures in  $\mathbb{R}^2$  with one degenerate phase. We specialize to the case where the non-negative and symmetric matrices  $A_1$  and  $A_2$  of  $\mathbb{R}^{2 \times 2}$  are such that

$$A_1 = \xi \otimes \xi \quad \text{and} \quad A_2 \text{ is positive definite}, \quad (28)$$

for some  $\xi \in \mathbb{R}^2$ . The next proposition establishes the algebraic conditions which provide assumptions (H1) and (H2) of Theorem 2.1.

**Proposition 2.** *Let  $A_1$  and  $A_2$  be the matrices defined by 28. Assume that  $\xi \cdot e_1 \neq 0$  and the vectors  $\xi$  and  $A_2 e_1$  are linearly independent in  $\mathbb{R}^2$ . Then, assumptions (H1) and (H2) are satisfied. In particular, the homogenized matrix  $A^*$ , given by 7, associated to the matrix  $A$  defined by 27 and 28 is positive definite.*

From Theorem 2.1, we easily deduce that the energy  $\mathcal{F}_\varepsilon$  defined by 1 with  $A$  given by 27 and 28  $\Gamma$ -converges to the functional  $\mathcal{F}_0$  given by 6 with conductivity matrix  $A^*$  defined by 7. In the present case, the homogenized matrix  $A^*$  has an explicit expression given in Proposition 5 in the Appendix.

*Proof.* Firstly, let us prove assumption (H1). We adapt the proof of Step 1 of [11, Theorem 3.3] to two-dimensional laminates. In our context, the algebra involved is different due to the scalar setting.

Denote by  $u_0^i$  the restriction of the two-scale limit  $u_0$  in phase  $Z_i$  or  $Z_i^\#$  for  $i = 1, 2$ . In view of 14, for any  $\Phi(x, y) \in C_c^\infty(\Omega \times \mathbb{R}^2; \mathbb{R}^2)$  with compact support in  $\Omega \times Z_1^\#$ , or due to periodicity in  $\Omega \times X_1$ , we deduce that

$$\begin{aligned} 0 &= -\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} A \left( \frac{x}{\varepsilon} \right) \nabla u_\varepsilon \cdot \Phi \left( x, \frac{x}{\varepsilon} \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon \operatorname{div}_y (A_1 \Phi(x, y)) \left( x, \frac{x}{\varepsilon} \right) dx \\ &= \int_{\Omega \times Z_1^\#} u_0^1(x, y) \operatorname{div}_y (A_1 \Phi(x, y)) dx dy \\ &= - \int_{\Omega \times Z_1^\#} A_1 \nabla_y u_0^1(x, y) \cdot \Phi(x, y) dx dy, \end{aligned}$$

so that

$$A_1 \nabla_y u_0^1(x, y) \equiv 0 \quad \text{in } \Omega \times Z_1^\#. \quad (29)$$

Similarly, taking  $\Phi(x, y) \in C_c^\infty(\Omega \times \mathbb{R}^2; \mathbb{R}^2)$  with compact support in  $\Omega \times Z_2^\#$ , or equivalently in  $\Omega \times X_2$ , as test function and repeating the same arguments, we obtain

$$A_2 \nabla_y u_0^2(x, y) \equiv 0 \quad \text{in } \Omega \times Z_2^\#. \quad (30)$$

Due to 29, in phase  $Z_1^\#$  we have

$$\nabla_y u_0^1 \in \operatorname{Ker}(A_1) = \operatorname{Span}(\xi^\perp),$$

where  $\xi^\perp = (-\xi_2, \xi_1) \in \mathbb{R}^2$  is perpendicular to  $\xi = (\xi_1, \xi_2)$ . Hence,  $u_0^1$  reads as

$$u_0^1(x, y) = \theta^1(x, \xi^\perp \cdot y) \quad \text{a.e. } (x, y) \in \Omega \times X_1, \quad (31)$$

for some function  $\theta^1 \in L^2(\Omega \times \mathbb{R})$ . On the other hand, since the matrix  $A_2$  is positive definite, in phase  $Z_2^\#$  the relation 30 implies that

$$u_0^2(x, y) = \theta^2(x) \quad \text{a.e. } (x, y) \in \Omega \times X_2, \quad (32)$$

for some function  $\theta^2 \in L^2(\Omega)$ . Now, consider a constant vector-valued function  $\Phi$  defined on  $Y_2$  such that

$$(A_1 - A_2)\Phi \cdot e_1 = 0 \quad \text{on } \partial Z_1^\#.$$
 (33)

Note that condition 33 is necessary for  $\text{div}_y(A(y)\Phi)$  to be an admissible test function for two-scale convergence. In view of 14 and 32, for any  $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y_2))$ , we obtain

$$\begin{aligned} 0 &= -\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} A(y) \nabla u_\varepsilon \cdot \Phi \varphi \left( x, \frac{x}{\varepsilon} \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon \text{div}_y(A(y)\Phi \varphi(x, y)) \left( x, \frac{x}{\varepsilon} \right) dx \\ &= \int_{\Omega \times Z_1} u_0^1(x, y) \text{div}_y(A_1 \Phi \varphi(x, y)) dx dy \\ &\quad + \int_{\Omega \times Z_2} \theta^2(x) \text{div}_y(A_2 \Phi \varphi(x, y)) dx dy. \end{aligned}$$

Take now  $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^2)$  and use the periodized function

$$\varphi^\#(x, y) := \sum_{k \in \mathbb{Z}^2} \varphi(x, y + k)$$

as new test function. Then, we obtain

$$\begin{aligned} 0 &= \int_{\Omega \times Z_1} u_0^1(x, y) \text{div}_y(A_1 \Phi \varphi^\#(x, y)) dx dy + \int_{\Omega \times Z_2} \theta^2(x) \text{div}_y(A_2 \Phi \varphi^\#(x, y)) dx dy \\ &= \sum_{k \in \mathbb{Z}^2} \int_{\Omega \times (Z_1 + k)} u_0^1(x, y) \text{div}_y(A_1 \Phi \varphi(x, y)) dx dy \\ &\quad + \sum_{k \in \mathbb{Z}^2} \int_{\Omega \times (Z_2 + k)} \theta^2(x) \text{div}_y(A_2 \Phi \varphi(x, y)) dx dy \\ &= \int_{\Omega \times Z_1^\#} u_0^1(x, y) \text{div}_y(A_1 \Phi \varphi(x, y)) dx dy + \int_{\Omega \times Z_2^\#} \theta^2(x) \text{div}_y(A_2 \Phi \varphi(x, y)) dx dy. \end{aligned}$$
 (34)

Recall that  $A_1 = \xi \otimes \xi$ , where  $\xi$  is such that  $\xi \cdot e_1 \neq 0$ . This combined with the linear independence of the vectors  $\xi$  and  $A_2 e_1$  implies that the linear map

$$\Phi \in \mathbb{R}^2 \mapsto (A_1 e_1 \cdot \Phi, A_2 e_1 \cdot \Phi) \in \mathbb{R}^2$$

is one-to-one. Hence, for any  $f \in \mathbb{R}$ , there exists a unique  $\Phi \in \mathbb{R}^2$  such that

$$A_1 \Phi \cdot e_1 = A_2 \Phi \cdot e_1 = f.$$
 (35)

In view of the arbitrariness of  $f$  in 35, we can choose  $\Phi$  such that

$$A_1 e_1 \cdot \Phi = A_2 e_1 \cdot \Phi = 1 \quad \text{on } \partial Z_1^\#.$$
 (36)

Since  $A_1 \nabla_y u_0^1 = 0$  in the distributional sense and  $A_1 = \xi \otimes \xi$ , we deduce that  $u_0^1$  is constant along the direction  $\xi$ . Using Fubini's theorem, we may integrate along straight lines parallel to the vector  $\xi$  where integration by parts is allowed. Therefore, performing an integration by parts in 34 combined with 36, it follows that for any  $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^2)$ ,

$$0 = \int_{\Omega \times \partial Z} v_0(x, y) \varphi(x, y) dx d\mathcal{H}_y,$$

where we have set  $v_0(x, y) := u_0^1(x, y) - \theta^2(x)$ . We conclude that  $v_0(x, \cdot)$  has a trace on  $\partial Z$  for a.e.  $x \in \Omega$  satisfying

$$v_0(x, \cdot) = 0 \quad \text{on } \partial Z. \quad (37)$$

Recall that  $\partial Z = \{y_1 = 0\}$ . Fix  $x \in \Omega$ . Taking into account 31 and 32, the equality 37 reads as

$$\theta^1(x, \xi_1 y_2) = \theta^2(x) \quad \text{on } \partial Z.$$

Since  $\xi \cdot e_1 \neq 0$ , it follows that  $\theta^1$  only depends on  $x$  so that  $u_0^1(x, y)$  agrees with  $\theta^2(x)$ . Finally, we conclude that  $u_0(x, y) := \chi(y_1)u_0^1(x, y) + (1 - \chi(y_1))u_0^2(x, y)$  is independent of  $y$  and hence (H1) is satisfied.

We prove assumption (H2). The proof is a variant of the Step 2 of [11, Theorem 3.4]. For arbitrary  $\alpha, \beta \in \mathbb{R}$ , let  $\Phi$  be a vector-valued function given by

$$A^{1/2}(y)\Phi(y) := \chi(y_1)\alpha\xi + (1 - \chi(y_1))(\alpha\xi + \beta e_2) \quad \text{for a.e. } y \in \mathbb{R}^2. \quad (38)$$

Such a vector field  $\Phi$  does exist, since  $\xi$  is in the range of  $A_1$  and thus the right-hand side of 38 belongs pointwise to the range of  $A$ , or equivalently to the range of  $A^{1/2}$ . Moreover, the difference of two constant phases in 38 is orthogonal to the laminate direction  $e_1$ , so that  $A^{1/2}\Phi$  is a laminate divergence free periodic field in  $\mathbb{R}^2$ . Its average value is given by

$$N := \int_{Y_2} A^{1/2}(y)\Phi(y)dy = \alpha\xi + (1 - \theta)\beta e_2.$$

Hence, due to  $\xi \cdot e_1 \neq 0$  and the arbitrariness of  $\alpha, \beta$ , the set of the vectors  $N$  spans  $\mathbb{R}^2$ , which yields assumption (H2).

From Proposition 1, it immediately follows that the homogenized matrix  $A^*$  is positive definite. For the reader's convenience, the proof of explicit formula of  $A^*$  is postponed to Proposition 5 in the Appendix.  $\square$

**3.2. The three-dimensional case with both degenerate phases.** We are going to deal with three-dimensional laminates where both phases are degenerate. We assume that the symmetric and non-negative matrices  $A_1$  and  $A_2$  of  $\mathbb{R}^{3 \times 3}$  have rank two, hence, there exist  $\eta_1, \eta_2 \in \mathbb{R}^3$  such that

$$\text{Ker}(A_i) = \text{Span}(\eta_i) \quad \text{for } i = 1, 2. \quad (39)$$

The following proposition gives the algebraic conditions so that assumptions required by Theorem 2.1 are satisfied.

**Proposition 3.** *Let  $\eta_1$  and  $\eta_2$  be the vectors in  $\mathbb{R}^3$  defined by 39. Assume that the vectors  $\{e_1, \eta_1, \eta_2\}$  as well as  $\{A_1 e_1, A_2 e_1\}$  are linearly independent in  $\mathbb{R}^3$ . Then, assumptions (H1) and (H2) are satisfied. In particular, the homogenized matrix  $A^*$  given by 7 and associated to the conductivity matrix  $A$  given by 27 and 39 is positive definite.*

Invoking again Theorem 2.1, the energy  $\mathcal{F}_\varepsilon$  defined by 1 with  $A$  given by 27 and 39,  $\Gamma$ -converges for the weak topology of  $L^2(\Omega)$  to  $\mathcal{F}_0$  where the effective conductivity  $A^*$  is given by 7. As in two-dimensional laminate materials,  $A^*$  has an explicit expression (see Proposition 5 in the Appendix).

*Proof.* We first show assumption (H1). The proof is an adaptation of the first step of [11, Theorem 3.3]. Same arguments as in the proof of Proposition 2 show that

$$A_i \nabla_y u_0^i(x, y) \equiv 0 \quad \text{in } \Omega \times Z_i^\# \quad \text{for } i = 1, 2. \quad (40)$$

In view of 39 and 40, in phase  $Z_i^\#$ ,  $u_0^i$  reads as

$$u_0^i(x, y) = \theta^i(x, \eta_i \cdot y) \quad \text{a.e. } (x, y) \in \Omega \times X_i, \quad (41)$$

for some function  $\theta^i \in L^2(\Omega \times \mathbb{R})$  and  $i = 1, 2$ . Now, consider a constant vector-valued function  $\Phi$  on  $Y_3$  such that the transmission condition 33 holds. In view of 14, for any  $\varphi \in C_c^\infty(\Omega, C_{\text{per}}^\infty(Y_3))$ , we obtain

$$\begin{aligned} 0 &= -\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} A(y) \nabla u_\varepsilon \cdot \Phi \varphi \left( x, \frac{x}{\varepsilon} \right) dx \\ &= \int_{\Omega \times Z_1} u_0^1(x, y) \operatorname{div}_y (A_1 \Phi \varphi(x, y)) dx dy \\ &\quad + \int_{\Omega \times Z_2} u_0^2(x, y) \operatorname{div}_y (A_2 \Phi \varphi(x, y)) dx dy. \end{aligned} \quad (42)$$

Take  $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^3)$ . Putting the periodized function

$$\varphi^\#(x, y) := \sum_{k \in \mathbb{Z}^3} \varphi(x, y + k)$$

as test function in 42, we get

$$\int_{\Omega \times Z_1^\#} u_0^1(x, y) \operatorname{div}_y (A_1 \Phi \varphi(x, y)) dx dy + \int_{\Omega \times Z_2^\#} u_0^2(x, y) \operatorname{div}_y (A_2 \Phi \varphi(x, y)) dx dy = 0. \quad (43)$$

Since the vectors  $A_1 e_1$  and  $A_2 e_1$  are independent in  $\mathbb{R}^3$ , the linear map

$$\Phi \in \mathbb{R}^3 \mapsto (A_1 e_1 \cdot \Phi, A_2 e_1 \cdot \Phi) \in \mathbb{R}^2$$

is surjective. In particular, for any  $f \in \mathbb{R}$ , there exists  $\Phi \in \mathbb{R}^3$  such that

$$A_1 \Phi \cdot e_1 = A_2 \Phi \cdot e_1 = f. \quad (44)$$

In view of the arbitrariness of  $f$  in 44, we can choose  $\Phi$  such that 36 is satisfied. Due to 40 and 39, we deduce that  $u_0^i$  is constant along the plane  $\Pi_i$  perpendicular to  $\eta_i$ , for  $i = 1, 2$ . This implies that, thanks to Fubini's theorem, we may integrate along the plane  $\Pi_i$  where an integration by part may be performed. Hence, an integration by parts in 43 combined with 36, yields for any  $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^3)$ ,

$$\int_{\Omega \times \partial Z} [u_0^1(x, y) - u_0^2(x, y)] \varphi(x, y) dx d\mathcal{H}_y = 0,$$

which implies that

$$u_0^1(x, \cdot) = u_0^2(x, \cdot) \quad \text{on } \partial Z. \quad (45)$$

Fix  $x \in \Omega$  and recall that  $\partial Z = \{y_1 = 0\}$ . In view of 41, the relation 45 reads as

$$\theta^1(x, b_1 y_2 + c_1 y_3) = \theta^2(x, b_2 y_2 + c_2 y_3) \quad \text{on } \partial Z, \quad (46)$$

with  $\eta_i = (a_i, b_i, c_i)$  for  $i = 1, 2$ . Due to the independence of  $\{e_1, \eta_1, \eta_2\}$  in  $\mathbb{R}^3$ , the linear map  $(y_2, y_3) \in \mathbb{R}^2 \mapsto (z_1, z_2) \in \mathbb{R}^2$  defined by

$$z_1 := b_1 y_2 + c_1 y_3, \quad z_2 := b_2 y_2 + c_2 y_3,$$

is a change of variables so that 46 becomes

$$\theta^1(x, z_1) = \theta^2(x, z_2) \quad \text{a.e. } z_1, z_2 \in \mathbb{R}.$$

This implies that  $\theta^1$  and  $\theta^2$  depend only on  $x$  and thus  $u_0^1$  and  $u_0^2$  agree with some function  $u \in L^2(\Omega)$ . Finally, we conclude that  $u_0(x, y) = \chi(y_1) u_0^1(x, y) + (1 - \chi(y_1)) u_0^2(x, y)$  is independent of  $y$  and hence (H1) is satisfied.

It remains to prove assumption (H2). To this end, let  $E$  be the subset of  $\mathbb{R}^3 \times \mathbb{R}^3$  defined by

$$E := \{(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : (\xi_1 - \xi_2) \cdot e_1 = 0, \quad \xi_1 \cdot \eta_1 = 0, \quad \xi_2 \cdot \eta_2 = 0\}. \quad (47)$$

For  $(\xi_1, \xi_2) \in E$ , let  $\Phi$  be the vector-valued function defined by

$$A^{1/2}(y)\Phi(y) := \chi(y_1)\xi_1 + (1 - \chi(y_1))\xi_2 \quad \text{a.e. } y \in \mathbb{R}^3. \quad (48)$$

The existence of such a vector field  $\Phi$  is guaranteed by the conditions  $\xi_i \cdot \eta_i = 0$ , for  $i = 1, 2$ , which imply that  $\xi_i$  belongs to the range of  $A_i$  and hence the right-hand side of 48 belongs pointwise to the range of  $A$ , or equivalently to the range of  $A^{1/2}$ . Moreover, since the difference of the phases  $\xi_1$  and  $\xi_2$  is orthogonal to the laminate direction  $e_1$ ,  $A^{1/2}\Phi$  is a laminate divergence free periodic field in  $\mathbb{R}^3$ . Its average value is given by

$$N := \int_{Y_3} A^{1/2}(y)\Phi(y)dy = \theta\xi_1 + (1 - \theta)\xi_2.$$

Note that  $E$  is a linear subspace of  $\mathbb{R}^3 \times \mathbb{R}^3$  whose dimension is three. Indeed, let  $f$  be the linear map defined by

$$(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto ((\xi_1 - \xi_2) \cdot e_1, \xi_1 \cdot \eta_1, \xi_2 \cdot \eta_2) \in \mathbb{R}^3.$$

If we identify the pair  $(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3$  with the vector  $(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6$ , with  $\xi_i = (x_i, y_i, z_i)$ , for  $i = 1, 2$ , the associated matrix  $M_f \in \mathbb{R}^{3 \times 6}$  of  $f$  is given by

$$M_f := \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & b_2 & c_2 \end{pmatrix},$$

with  $\eta_i = (a_i, b_i, c_i)$ ,  $i = 1, 2$ . In view of the linear independence of  $\{e_1, \eta_1, \eta_2\}$ , the rank of  $M_f$  is three, which implies that the dimension of kernel  $\text{Ker}(f)$  is also three. Since the kernel  $\text{Ker}(f)$  agrees with  $E$ , we conclude that the dimension of  $E$  is three.

Now, let  $g$  be the linear map defined by

$$(\xi_1, \xi_2) \in E \mapsto \theta\xi_1 + (1 - \theta)\xi_2 \in \mathbb{R}^3.$$

Let us show that  $g$  is invertible. To this end, consider  $(\xi_1, \xi_2) \in \text{Ker}(g)$ . From the definition of the map  $g$ ,  $\text{Ker}(g)$  consists of all vectors  $(\xi_1, \xi_2) \in E$  of the form

$$\left( \xi_1, \frac{\theta}{\theta - 1}\xi_1 \right). \quad (49)$$

In view of the definition of  $E$  given by 47, the vector 49 satisfies the conditions

$$\left( \xi_1 - \frac{\theta}{\theta - 1}\xi_1 \right) \cdot e_1 = 0, \quad \xi_1 \cdot \eta_1 = 0, \quad \frac{\theta}{\theta - 1}\xi_1 \cdot \eta_2 = 0.$$

This combined with the linear independence of  $\{e_1, \eta_1, \eta_2\}$  implies that

$$\xi_1 \in \{e_1, \eta_1, \eta_2\}^\perp = \{0\}.$$

Hence,  $\text{Ker}(g) = \{(0, 0)\}$  which implies along with the fact that the dimension of  $E$  is three that  $g$  is invertible. This proves that all the vectors of  $\mathbb{R}^3$  can be attained through the map  $g$  so that assumption (H2) is satisfied.

Thanks to Proposition 1, the homogenized matrix  $A^*$  turns out to be positive definite. The proof of the explicit expression of  $A^*$  is given in Proposition 5 in the Appendix.  $\square$

**4. A two-dimensional counter-example.** In this section we are going to construct a counter-example of two-dimensional laminates with two degenerate phases, where the lack of assumption (H1) provides an anomalous asymptotic behaviour of the functional  $\mathcal{F}_\varepsilon$  [1](#).

Let  $\Omega := (0, 1)^2$  and let  $e_2$  be the laminate direction. We assume that the non-negative and symmetric matrices  $A_1$  and  $A_2$  of  $\mathbb{R}^{2 \times 2}$  are given by

$$A_1 = e_1 \otimes e_1 \quad \text{and} \quad A_2 = ce_1 \otimes e_1,$$

for some positive constant  $c > 1$ . The presence of  $c \neq 1$  is essential to have oscillation in the conductivity matrix  $A$ . In the present case, the matrix-valued conductivity  $A$  is given by

$$A(y_2) := \chi(y_2)A_1 + (1 - \chi(y_2))A_2 = a(y_2)e_1 \otimes e_1 \quad \text{for } y_2 \in \mathbb{R}, \quad (50)$$

with

$$a(y_2) := \chi(y_2) + c(1 - \chi(y_2)) \geq 1. \quad (51)$$

Thus, the energy  $\mathcal{F}_\varepsilon$ , defined by [1](#) with  $A(y)$  given by [50](#) and [51](#) becomes

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_\Omega \left[ a\left(\frac{x_2}{\varepsilon}\right) \left(\frac{\partial u}{\partial x_1}\right)^2 + |u|^2 \right] dx, & \text{if } u \in H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2}), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2}). \end{cases} \quad (52)$$

We denote by  $*_1$  the convolution with respect to the variable  $x_1$ , i.e. for  $f \in L^1(\mathbb{R}^2)$  and  $g \in L^2(\mathbb{R}^2)$

$$(f *_1 g)(x_1, x_2) = \int_{\mathbb{R}} f(x_1 - t, x_2)g(t, x_2)dt.$$

Throughout this section,  $c_\theta$  denotes the positive constant given by

$$c_\theta := c\theta + 1 - \theta, \quad (53)$$

where  $\theta \in (0, 1)$  is the volume fraction of the phase  $Z_1$  in  $Y_2$ . The following result proves the  $\Gamma$ -convergence of  $\mathcal{F}_\varepsilon$  for the weak topology of  $L^2(\Omega)$  and provides two alternative expressions of the  $\Gamma$ -limit, one of that seems nonlocal due to presence of convolution term (see Remark [2](#) below).

**Proposition 4.** *Let  $\mathcal{F}_\varepsilon$  be the functional defined by [52](#). Then,  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges for the weak topology of  $L^2(\Omega)$  to the functional defined by*

$$\mathcal{F}(u) := \begin{cases} \int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1, & \text{if } u \in H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2}), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2}), \end{cases}$$

where  $\mathcal{F}_2(u)(\lambda_1, \cdot)$  denotes the Fourier transform on  $L^2(\mathbb{R})$  of parameter  $\lambda_1$  with respect to the variable  $x_1$  of the function  $x_1 \mapsto u(x_1, \cdot)$  extended by zero outside  $(0, 1)$  and

$$\hat{k}_0(\lambda_1) := \int_0^1 \frac{1}{4\pi^2 a(y_2) \lambda_1^2 + 1} dy_2. \quad (54)$$

The  $\Gamma$ -limit  $\mathcal{F}$  can be also expressed as

$$\mathcal{F}(u) := \begin{cases} \int_0^1 dx_2 \int_{\mathbb{R}} \left\{ \frac{c}{c_\theta} \left( \frac{\partial u}{\partial x_1} \right)^2 (x_1, x_2) + [\sqrt{\alpha} u(x_1, x_2) + (h *_{\mathbf{1}} u)(x_1, x_2)]^2 \right\} dx_1, & \text{if } u \in H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2}), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2}), \end{cases} \quad (55)$$

where  $c_\theta$  is given by 53 and  $h$  is a real-valued function in  $L^2(\mathbb{R})$  defined by means of its Fourier transform  $\mathcal{F}_2$  on  $L^2(\mathbb{R})$

$$\mathcal{F}_2(h)(\lambda_1) := \sqrt{\alpha + f(\lambda_1)} - \sqrt{\alpha}, \quad (56)$$

where  $\alpha$  and  $f$  are given by

$$\alpha := \frac{c^2\theta + 1 - \theta}{c_\theta^2} > 0, \quad f(\lambda_1) := \frac{(c-1)^2\theta(\theta-1)}{c_\theta^2} \frac{1}{c_\theta 4\pi^2 \lambda_1^2 + 1}. \quad (57)$$

Moreover, any two-scale limit  $u_0(x, y)$  of a sequence  $u_\varepsilon$  with bounded energy  $\mathcal{F}_\varepsilon$  depends on the variable  $y_2 \in Y_1$ .

**Remark 1.** From 57, we can deduce that

$$\alpha + f(\lambda_1) = \frac{1}{c_\theta^2(c_\theta 4\pi^2 \lambda_1^2 + 1)} \{ (c^2\theta + 1 - \theta)c_\theta 4\pi^2 \lambda_1^2 + [(c-1)\theta + 1]^2 \} > 0,$$

for any  $\lambda_1 \in \mathbb{R}$ , so that the Fourier transform of  $h$  is well-defined.

*Proof.* We divide the proof into three steps.

**Step 1 -  $\Gamma$ -lim inf inequality.**

Consider a sequence  $\{u_\varepsilon\}_\varepsilon$  converging weakly in  $L^2(\Omega)$  to  $u \in L^2(\Omega)$ . Our aim is to prove that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u). \quad (58)$$

If the lower limit is  $\infty$  then 58 is trivial. Up to a subsequence, still indexed by  $\varepsilon$ , we may assume that  $\liminf \mathcal{F}_\varepsilon(u_\varepsilon)$  is a limit and we may assume henceforth that, for some  $0 < C < \infty$ ,

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq C. \quad (59)$$

It follows that the sequence  $u_\varepsilon$  is bounded in  $L^2(\Omega)$  and according to [1, Theorem 1.2], a subsequence, still indexed by  $\varepsilon$ , of that sequence two-scale converges to some  $u_0(x, y) \in L^2(\Omega \times Y_2)$ . In other words,

$$u_\varepsilon \rightharpoonup u_0. \quad (60)$$

In view of 51, we know that  $a \geq 1$  so that, thanks to 59, for another subsequence (not relabeled) we have

$$\frac{\partial u_\varepsilon}{\partial x_1} \rightharpoonup \sigma_0(x, y) \quad \text{with } \sigma_0 \in L^2(\Omega \times Y_2). \quad (61)$$

In particular,

$$\varepsilon \frac{\partial u_\varepsilon}{\partial x_1} \rightharpoonup 0. \quad (62)$$

Take  $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y_2))$ . By integration by parts, we obtain

$$\varepsilon \int_{\Omega} \frac{\partial u_\varepsilon}{\partial x_1} \varphi \left( x, \frac{x}{\varepsilon} \right) dx = - \int_{\Omega} u_\varepsilon \left( \varepsilon \frac{\partial \varphi}{\partial x_1} \left( x, \frac{x}{\varepsilon} \right) + \frac{\partial \varphi}{\partial y_1} \left( x, \frac{x}{\varepsilon} \right) \right) dx.$$

Passing to the limit in both terms with the help of 60 and 62 leads to

$$0 = - \int_{\Omega \times Y_2} u_0(x, y) \frac{\partial \varphi}{\partial y_1}(x, y) dx dy,$$

which implies that

$$u_0(x, y) \text{ is independent of } y_1. \quad (63)$$

Due to the link between two-scale and weak  $L^2$ -convergences (see [1, Proposition 1.6]), we have

$$u_\varepsilon \rightharpoonup u(x) = \int_{Y_1} u_0(x, y_2) dy_2 \quad \text{weakly in } L^2(\Omega). \quad (64)$$

Now consider  $\varphi \in C^\infty(\bar{\Omega}; C_{\text{per}}^\infty(Y_2))$  such that

$$\frac{\partial \varphi}{\partial y_1}(x, y) = 0. \quad (65)$$

Since  $u_\varepsilon \in H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2})$ , an integration by parts leads us to

$$\int_{\Omega} \frac{\partial u_\varepsilon}{\partial x_1} \varphi(x, y) dx = - \int_{\Omega} u_\varepsilon \frac{\partial \varphi}{\partial x_1}(x, y) dx.$$

In view of the convergences 60 and 61 together with 63, we can pass to the two-scale limit in the previous expression and we obtain

$$\int_{\Omega \times Y_2} \sigma_0(x, y) \varphi(x, y) dx dy = - \int_{\Omega \times Y_2} u_0(x, y_2) \frac{\partial \varphi}{\partial x_1}(x, y) dx dy. \quad (66)$$

Varying  $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y_2))$ , the left-hand side of 66 is bounded by a constant times  $\|\varphi\|_{L^2(\Omega \times [0, 1])}$  so that the right-hand side is a linear and continuous form in  $\varphi \in L^2(\Omega \times Y_2)$ . By the Riesz representation theorem, there exists  $g \in L^2(\Omega \times Y_2)$  such that, for any  $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y_2))$ ,

$$\int_{\Omega \times Y_2} u_0(x, y_2) \frac{\partial \varphi}{\partial x_1}(x, y) dx dy = \int_{\Omega \times Y_2} g(x, y) \varphi(x, y) dx dy,$$

which yields

$$\frac{\partial u_0}{\partial x_1}(x, y_2) \in L^2(\Omega \times Y_1). \quad (67)$$

Then, an integration by parts with respect to  $x_1$  of the right-hand side of 66 yields, for any  $\varphi \in C^\infty(\bar{\Omega}; C_{\text{per}}^\infty(Y_2))$  satisfying 65,

$$\begin{aligned} \int_{\Omega \times Y_2} \sigma_0(x, y) \varphi(x, y) dx dy &= \int_{\Omega \times Y_2} \frac{\partial u_0}{\partial x_1}(x, y_2) \varphi(x, y) dx dy \\ &\quad - \int_0^1 dx_2 \int_{Y_2} [u_0(1, x_2, y_2) \varphi(1, x_2, y) - u_0(0, x_2, y_2) \varphi(0, x_2, y)] dy. \end{aligned}$$

Since for any  $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y_2))$  the first two integrals are equal and bounded by a constant times  $\|\varphi\|_{L^2(\Omega \times [0, 1])}$ , we conclude that, for any  $\varphi \in C^\infty(\bar{\Omega}; C_{\text{per}}^\infty(Y_2))$  satisfying 65,

$$\int_0^1 dx_2 \int_{Y_2} [u_0(1, x_2, y_2) \varphi(1, x_2, y) - u_0(0, x_2, y_2) \varphi(0, x_2, y)] dy = 0,$$

which implies that

$$u_0(1, x_2, y_2) = u_0(0, x_2, y_2) = 0 \quad \text{a.e. } (x_2, y_2) \in (0, 1) \times Y_1.$$

This combined with 67 yields

$$u_0(x_1, x_2, y_2) \in H_0^1((0, 1)_{x_1}; L^2((0, 1)_{x_2} \times Y_1)).$$

Finally, an integration by parts with respect to  $x_1$  of the right-hand side of 66 implies that, for any  $\varphi \in C^\infty(\bar{\Omega}; C_{\text{per}}^\infty(Y_2))$  satisfying 65,

$$\int_{\Omega \times Y_2} \left( \sigma_0(x, y) - \frac{\partial u_0}{\partial x_1}(x, y_2) \right) \varphi(x, y) dx dy = 0.$$

Since the orthogonal of divergence-free functions is the gradients, from the previous equality we deduce that there exists  $\tilde{u} \in H_{\text{per}}^1(Y_1; L^2(\Omega \times Y_1))$  such that

$$\sigma_0(x, y) = \frac{\partial u_0}{\partial x_1}(x, y_2) + \frac{\partial \tilde{u}}{\partial y_1}(x, y). \quad (68)$$

Now, we show that

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \left( \frac{\partial u_\varepsilon}{\partial x_1} \right)^2 dx \geq \int_{\Omega \times Y_2} a(y_2) \left( \frac{\partial u_0}{\partial x_1}(x, y_2) + \frac{\partial \tilde{u}}{\partial y_1}(x, y) \right)^2 dx dy. \quad (69)$$

To this end, set

$$\sigma_\varepsilon := \frac{\partial u_\varepsilon}{\partial x_1}.$$

Since  $a \in L_{\text{per}}^\infty(Y_1) \subset L_{\text{per}}^2(Y_1)$ , there exists a sequence  $a_k$  of functions in  $C_{\text{per}}^\infty(Y_1)$  such that

$$\|a - a_k\|_{L^2(Y_1)} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad (70)$$

hence, by periodicity, we also have

$$\left\| a\left(\frac{x_2}{\varepsilon}\right) - a_k\left(\frac{x_2}{\varepsilon}\right) \right\|_{L^2(\Omega)} \leq C \|a - a_k\|_{L^2(Y_1)}, \quad (71)$$

for some positive constant  $C > 0$ . On the other hand, since  $\sigma_0$  given by 68 is in  $L^2(\Omega \times Y_2)$ , there exists a sequence  $\psi_n$  of functions in  $C_c^\infty(\Omega; C_{\text{per}}^\infty(Y_2))$  such that

$$\psi_n(x, y) \rightarrow \sigma_0(x, y) \quad \text{strongly in } L^2(\Omega \times Y_2). \quad (72)$$

From the inequality

$$\int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \left( \sigma_\varepsilon - \psi_n\left(x, \frac{x}{\varepsilon}\right) \right)^2 dx \geq 0,$$

we get

$$\begin{aligned} \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \sigma_\varepsilon^2 dx &\geq 2 \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \sigma_\varepsilon \psi_n\left(x, \frac{x}{\varepsilon}\right) dx - \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \psi_n^2\left(x, \frac{x}{\varepsilon}\right) dx \\ &= 2 \int_{\Omega} \left( a\left(\frac{x_2}{\varepsilon}\right) - a_k\left(\frac{x_2}{\varepsilon}\right) \right) \sigma_\varepsilon \psi_n\left(x, \frac{x}{\varepsilon}\right) dx + 2 \int_{\Omega} a_k\left(\frac{x_2}{\varepsilon}\right) \sigma_\varepsilon \psi_n\left(x, \frac{x}{\varepsilon}\right) dx \\ &\quad - \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \psi_n^2\left(x, \frac{x}{\varepsilon}\right) dx. \end{aligned} \quad (73)$$

In view of 71, the first integral on the right-hand side of 73 can be estimated as

$$\begin{aligned} \left| \int_{\Omega} \left( a\left(\frac{x_2}{\varepsilon}\right) - a_k\left(\frac{x_2}{\varepsilon}\right) \right) \sigma_\varepsilon \psi_n\left(x, \frac{x}{\varepsilon}\right) dx \right| &\leq C \|a - a_k\|_{L^2(Y_1)} \|\psi_n\|_{L^\infty(\Omega)} \|\sigma_\varepsilon\|_{L^2(\Omega)} \\ &\leq C \|a - a_k\|_{L^2(Y_1)}. \end{aligned}$$

Hence, passing to the limit as  $\varepsilon \rightarrow 0$  in 73 with the help of 61 leads to

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \sigma_{\varepsilon}^2 dx &\geq -C\|a - a_k\|_{L^2(Y_1)} + 2 \lim_{\varepsilon \rightarrow 0} \int_{\Omega} a_k\left(\frac{x_2}{\varepsilon}\right) \sigma_{\varepsilon} \psi_n\left(x, \frac{x}{\varepsilon}\right) dx \\ &\quad - \lim_{\varepsilon \rightarrow 0} \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \psi_n^2\left(x, \frac{x}{\varepsilon}\right) dx dy \\ &= 2 \int_{\Omega \times Y_2} a_k(y_2) \sigma_0(x, y) \psi_n(x, y) dx dy - C\|a - a_k\|_{L^2(Y_1)} \\ &\quad - \int_{\Omega \times Y_2} a(y_2) \psi_n^2(x, y) dx dy. \end{aligned}$$

Thanks to 70, we take the limit as  $k \rightarrow \infty$  in the previous inequality and we obtain

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \sigma_{\varepsilon}^2 dx &\geq 2 \int_{\Omega \times Y_2} a(y_2) \sigma_0(x, y) \psi_n(x, y) dx dy \\ &\quad - \int_{\Omega \times Y_2} a(y_2) \psi_n^2(x, y) dx dy, \end{aligned}$$

so that in view of 72, passing to the limit as  $n \rightarrow \infty$  leads to

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \sigma_{\varepsilon}^2 dx \geq \int_{\Omega \times Y_2} a(y_2) \sigma_0^2(x, y) dx dy.$$

This combined with 68 proves 69.

By 63, we already know that  $u_0$  does not depend on  $y_1$ . In view of the periodicity of  $\tilde{u}$  with respect to  $y_1$ , an application of Jensen's inequality leads us to

$$\begin{aligned} &\int_{\Omega \times Y_2} a(y_2) \left( \frac{\partial u_0}{\partial x_1}(x, y_2) + \frac{\partial \tilde{u}}{\partial y_1}(x, y) \right)^2 dx dy \\ &= \int_{\Omega} dx \int_{Y_1} a(y_2) dy_2 \int_{Y_1} \left( \frac{\partial u_0}{\partial x_1}(x, y_2) + \frac{\partial \tilde{u}}{\partial y_1}(x, y) \right)^2 dy_1 \\ &\geq \int_{\Omega} dx \int_{Y_1} a(y_2) dy_2 \left( \int_{Y_1} \left[ \frac{\partial u_0}{\partial x_1}(x, y_2) + \frac{\partial \tilde{u}}{\partial y_1}(x, y) \right] dy_1 \right)^2 \\ &= \int_{\Omega} dx \int_{Y_1} a(y_2) \left( \frac{\partial u_0}{\partial x_1} \right)^2(x, y_2) dy_2. \end{aligned}$$

This combined with 69 implies that

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} a\left(\frac{x_2}{\varepsilon}\right) \left( \frac{\partial u_{\varepsilon}}{\partial x_1} \right)^2 dx \geq \int_{\Omega} dx \int_{Y_1} a(y_2) \left( \frac{\partial u_0}{\partial x_1} \right)^2(x, y_2) dy_2. \quad (74)$$

Now, we extend the functions in  $L^2(\Omega)$  by zero with respect to  $x_1$  outside  $(0, 1)$  so that functions in  $H_0^1((0, 1)_{x_1}; L^2(0, 1)_{x_2})$  can be regarded as functions in  $H^1(\mathbb{R}_{x_1}; L^2(0, 1)_{x_2})$ . Due to the weak  $L^2$ -lower semi-continuity of  $\|u_{\varepsilon}\|_{L^2(\Omega)}$  along with 74, we have

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_{\varepsilon}(u_{\varepsilon}) \\ &\geq \int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[ a(y_2) \left( \frac{\partial u_0}{\partial x_1} \right)^2(x_1, x_2, y_2) + |u_0|^2(x_1, x_2, y_2) \right] dx_1. \end{aligned} \quad (75)$$

We minimize the right-hand side with respect to  $u_0(x_1, x_2, y_2) \in H^1(\mathbb{R}_{x_1}; L^2((0, 1)_{x_2} \times Y_1))$  satisfying 64 where the weak limit  $u$  of  $u_{\varepsilon}$  in  $L^2(\Omega)$  is fixed. The minimizer,

still denoted by  $u_0$ , satisfies the Euler equation

$$\int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[ a(y_2) \frac{\partial u_0}{\partial x_1}(x_1, x_2, y_2) \frac{\partial v}{\partial x_1}(x_1, x_2, y_2) + u_0(x_1, x_2, y_2) v(x_1, x_2, y_2) \right] dx_1 = 0$$

for any  $v(x_1, x_2, y_2) \in H^1(\mathbb{R}_{x_1}; L^2((0, 1)_{x_2} \times Y_1))$  such that  $\int_{Y_1} v(x, y_2) dy_2 = 0$ . Then, there exists  $b(x_1, x_2) \in H^{-1}(\mathbb{R}_{x_1}; L^2(\mathbb{R}_{x_2}))$  independent of  $y_2$  such that in distributions sense with respect to the variable  $x_1$ ,

$$-a(y_2) \frac{\partial^2 u_0}{\partial x_1^2}(x_1, x_2, y_2) + u_0(x_1, x_2, y_2) = b(x_1, x_2) \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad (76)$$

for a.e.  $(x_2, y_2) \in (0, 1) \times Y_1$ . Taking the Fourier transform  $\mathcal{F}_2$  on  $L^2(\mathbb{R})$  of parameter  $\lambda_1$  with respect to the variables  $x_1$ , the equation 76 becomes

$$\mathcal{F}_2(u_0)(\lambda_1, x_2, y_2) = \frac{\mathcal{F}_2(b)(\lambda_1, x_2)}{4\pi^2 a(y_2) \lambda_1^2 + 1} \quad \text{a.e. } (\lambda_1, x_2, y_2) \in \mathbb{R} \times (0, 1) \times Y_1. \quad (77)$$

Note that 77 proves in particular that the two-scale limit  $u_0$  does depend on the variable  $y_2$ , since its Fourier transform with respect to the variable  $x_1$  depends on  $y_2$  through the function  $a(y_2)$ .

In light of the definition 54 of  $\hat{k}_0$  and due to 64, integrating 77 with respect to  $y_2 \in Y_1$  yields

$$\mathcal{F}_2(u)(\lambda_1, x_2) = \hat{k}_0(\lambda_1) \mathcal{F}_2(b)(\lambda_1, x_2) \quad \text{a.e. } (\lambda_1, x_2) \in \mathbb{R} \times (0, 1). \quad (78)$$

By using Plancherel's identity with respect to the variable  $x_1$  in the right-hand side of 75 and in view of 77 and 78, we obtain

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) &\geq \int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} (4\pi^2 a(y_2) \lambda_1^2 + 1) |\mathcal{F}_2(u_0)(\lambda_1, x_2, y_2)|^2 d\lambda_1 \\ &= \int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1, \end{aligned}$$

which proves the  $\Gamma$ -lim inf inequality.

**Step 2-**  $\Gamma$ -lim sup inequality.

For the proof of the  $\Gamma$ -lim sup inequality, we need the following lemma whose proof will be given later.

**Lemma 4.1.** *Let  $u \in C_c^\infty(\Omega)$ . For fixed  $x_2 \in (0, 1)$  and  $y_2 \in Y_1$ , let  $b(\cdot, x_2)$  be the distribution (parameterized by  $x_2$ ) defined by*

$$\mathcal{F}_2(b)(\lambda_1, x_2) := \frac{1}{\hat{k}_0(\lambda_1)} \mathcal{F}_2(u)(\lambda_1, x_2), \quad (79)$$

where  $u(\cdot, x_2)$  is extended by zero outside  $(0, 1)$ . Let  $u_0(\cdot, x_2, y_2)$  be the unique solution to problem

$$\begin{cases} -a(y_2) \frac{\partial^2 u_0}{\partial x_1^2}(x_1, x_2, y_2) + u_0(x_1, x_2, y_2) = b(x_1, x_2), & x_1 \in (0, 1), \\ u_0(0, x_2, y_2) = u_0(1, x_2, y_2) = 0, \end{cases} \quad (80)$$

with  $a(y_2)$  given by 51. Then  $b(x_1, x_2)$  is in  $C([0, 1]_{x_2}; L^2(0, 1)_{x_1})$  and  $u_0(x_1, x_2, y_2)$  is in  $C^1([0, 1]^2; L^\infty_{\text{per}}(Y_1))$ .

Let  $u \in C_c^\infty(\Omega)$ . Thanks to Lemma 4.1, there exists a unique solution

$$u_0(x_1, x_2, y_2) \in C^1([0, 1]^2; L_{\text{per}}^\infty(Y_1)) \quad (81)$$

to the problem 80. Taking the Fourier transform  $\mathcal{F}_2$  on  $L^2(\mathbb{R})$  of parameter  $\lambda_1$  with respect to  $x_1$  of the equation in 80 and taking into account 79, we get

$$\mathcal{F}_2(u_0)(\lambda_1, x_2, y_2) = \frac{\mathcal{F}_2(u)(\lambda_1, x_2)}{(4\pi^2 a(y_2)\lambda_1^2 + 1)\hat{k}_0(\lambda_1)} \quad \text{for } (\lambda_1, x_2, y_2) \in \mathbb{R} \times [0, 1] \times Y_1, \quad (82)$$

where  $u_0(\cdot, x_2, y_2)$  and  $u(\cdot, x_2)$  are extended by zero outside  $(0, 1)$ . Integrating 82 over  $y_2 \in Y_1$ , we obtain

$$u(x_1, x_2) = \int_{Y_1} u_0(x_1, x_2, y_2) dy_2 \quad \text{for } (x_1, x_2) \in \mathbb{R} \times (0, 1). \quad (83)$$

Let  $\{u_\varepsilon\}_\varepsilon$  be the sequence in  $L^2(\Omega)$  defined by

$$u_\varepsilon(x_1, x_2) := u_0\left(x_1, x_2, \frac{x_2}{\varepsilon}\right).$$

Recall that rapidly oscillating  $Y_1$ -periodic function  $u_\varepsilon$  weakly converges in  $L^2(\Omega)$  to the mean value of  $u_\varepsilon$  over  $Y_1$ . This combined with 83 implies that  $u_\varepsilon$  weakly converges in  $L^2(\Omega)$  to  $u$ . In other words,

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega).$$

Due to 81, we can apply [1, Lemma 5.5] so that  $u_0(x_1, x_2, y_2)$  and  $\frac{\partial u_0}{\partial x_1}$  are admissible test functions for the two-scale convergence. Hence,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left[ a\left(\frac{x_2}{\varepsilon}\right) \left(\frac{\partial u_0}{\partial x_1}\right)^2 \left(x_1, x_2, \frac{x_2}{\varepsilon}\right) + \left|u_0\left(x_1, x_2, \frac{x_2}{\varepsilon}\right)\right|^2 \right] dx \\ &= \int_{\Omega} dx \int_{Y_1} \left[ a(y_2) \left(\frac{\partial u_0}{\partial x_1}\right)^2 (x_1, x_2, y_2) + |u_0(x_1, x_2, y_2)|^2 \right] dy_2 \\ &= \int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[ a(y_2) \left(\frac{\partial u_0}{\partial x_1}\right)^2 (x_1, x_2, y_2) + |u_0(x_1, x_2, y_2)|^2 \right] dx_1, \quad (84) \end{aligned}$$

where the function  $x_1 \mapsto u_0(x_1, \cdot, \cdot)$  is extended by zero outside  $(0, 1)$ . In view of the definition 54 of  $\hat{k}_0$  and due to 82, the Plancherel identity with respect to the variable  $x_1$  and the Fubini theorem yield

$$\begin{aligned} &\int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[ a(y_2) \left(\frac{\partial u_0}{\partial x_1}\right)^2 (x_1, x_2, y_2) + |u_0(x_1, x_2, y_2)|^2 \right] dx_1 \\ &= \int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} (4\pi^2 a(y_2)\lambda_1^2 + 1) |\mathcal{F}_2(u_0)(\lambda_1, x_2, y_2)|^2 d\lambda_1 \\ &= \int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1. \end{aligned}$$

This together with 84 implies that, for  $u \in C_c^\infty(\Omega)$ ,

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) = \int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1,$$

which proves the  $\Gamma$ -lim sup inequality on  $C_c^\infty(\Omega)$ .

Now, we extend the previous result to any  $u \in H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$ . To this end, we use a density argument (see *e.g.* [5, Remark 2.8]). Recall that the weak topology of  $L^2(\Omega)$  is metrizable on the closed balls of  $L^2(\Omega)$ . Fix  $n \in \mathbb{N}$  and denote  $d_{B_n}$  any metric inducing the  $L^2(\Omega)$ -weak topology on the ball  $B_n$  centered on 0 and of radius  $n$ . Then,  $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$  can be regarded as a subspace of  $L^2(\Omega)$  endowed with the metric  $d_{B_n}$ . On the other hand,  $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$  is a Hilbert space endowed with the norm

$$\|u\|_{H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})} := \left( \left\| \frac{\partial u}{\partial x_1} \right\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

The associated metric  $d_{H_0^1}$  on  $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$  induces a topology which is not weaker than that induced by  $d_{B_n}$ , *i.e.*

$$d_{H_0^1}(u_k, u) \rightarrow 0 \quad \text{implies} \quad d_{B_n}(u_k, u) \rightarrow 0. \quad (85)$$

Recall that  $C_c^\infty(\Omega)$  is a dense subspace of  $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$  for the metric  $d_{H_0^1}$  and that the  $\Gamma$ -lim sup inequality holds on  $C_c^\infty(\Omega)$  for the  $L^2(\Omega)$ -weak topology, *i.e.* for any  $u \in C_c^\infty(\Omega)$ ,

$$\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u) \leq \mathcal{F}(u). \quad (86)$$

A direct computation of  $\hat{k}_0$ , given by 54, shows that

$$\hat{k}_0(\lambda_1) = \frac{c_\theta 4\pi^2 \lambda_1^2 + 1}{(4\pi^2 \lambda_1^2 + 1)(c 4\pi^2 \lambda_1^2 + 1)},$$

which implies that

$$\frac{1}{\hat{k}_0(\lambda_1)} = \frac{c}{c_\theta} 4\pi^2 \lambda_1^2 + f(\lambda_1) + \alpha, \quad (87)$$

where  $f(\lambda_1)$  and  $\alpha$  are given by 57. Hence, there exists a positive constant  $C$  such that

$$\frac{1}{\hat{k}_0(\lambda_1)} \leq C(4\pi^2 \lambda_1^2 + 1). \quad (88)$$

This combined with the Plancherel identity yields

$$\begin{aligned} \mathcal{F}(u) &\leq C \int_0^1 dx_2 \int_{\mathbb{R}} (4\pi^2 \lambda_1^2 + 1) |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1 \\ &= C \int_0^1 dx_2 \int_{\mathbb{R}} \left[ \left( \frac{\partial u}{\partial x_1} \right)^2 (x_1, x_2) + |u(x_1, x_2)|^2 \right] dx_1 \\ &= C \|u\|_{H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})}^2, \end{aligned} \quad (89)$$

where  $u(\cdot, x_2)$  is extended by zero outside  $(0,1)$ . Since  $\mathcal{F}$  is a non-negative quadratic form, from 89 we conclude that  $\mathcal{F}$  is continuous with respect to the metric  $d_{H_0^1}$ .

Now, take  $u \in H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$ . By density, there exists a sequence  $u_k$  in  $C_c^\infty(\Omega)$  such that

$$d_{H_0^1}(u_k, u) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \quad (90)$$

In particular, due to 85, we also have that  $d_{B_n}(u_k, u) \rightarrow 0$  as  $k \rightarrow \infty$ . In view of the weakly lower semi-continuity of  $\Gamma$ -lim sup and the continuity of  $\mathcal{F}$ , we deduce

from 86 that

$$\begin{aligned} \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u) &\leq \liminf_{k \rightarrow \infty} (\Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_k)) \\ &\leq \liminf_{k \rightarrow \infty} \mathcal{F}(u_k) \\ &= \mathcal{F}(u), \end{aligned}$$

which proves the  $\Gamma$ -lim sup inequality in  $B_n$ . Since for any  $u \in H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$  the sequence  $u_k$  of functions in  $C_c^\infty(\Omega)$  satisfying 90 belongs to some ball  $B_n$  of  $L^2(\Omega)$ , as well as its limit, the  $\Gamma$ -lim sup property holds true for the sequence  $\mathcal{F}_\varepsilon$  on  $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$ , which concludes the proof of  $\Gamma$ -lim sup inequality.

**Step 3** - Alternative expression of  $\Gamma$ -limit.

The proof of the equality between the two expressions of the  $\Gamma$ -limit  $\mathcal{F}$  relies on the following lemma whose proof will be given later.

**Lemma 4.2.** *Let  $h \in L^2(\mathbb{R})$  and  $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Then,  $h * u \in L^2(\mathbb{R})$  and*

$$\mathcal{F}_2(h * u) = \mathcal{F}_2(h)\mathcal{F}_2(u) \quad \text{a.e. in } \mathbb{R}. \quad (91)$$

By applying Plancherel's identity with respect to  $x_1$ , for any  $u \in H_0^1(\mathbb{R}_{x_1}; L^2(0,1)_{x_2})$  extended by zero with respect to the variable  $x_1$  outside  $(0,1)$ , we get

$$\begin{aligned} &\int_{\mathbb{R}} |\sqrt{\alpha}u(x_1, x_2) + (h *_1 u)(x_1, x_2)|^2 dx_1 \\ &= \int_{\mathbb{R}} |\sqrt{\alpha}\mathcal{F}_2(u)(\lambda_1, x_2) + \mathcal{F}_2(h *_1 u)(\lambda_1, x_2)|^2 d\lambda_1 \\ &= \int_{\mathbb{R}} \left[ \alpha |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 + 2\sqrt{\alpha} \operatorname{Re} \left( \mathcal{F}_2(u)(\lambda_1, x_2) \overline{\mathcal{F}_2(h *_1 u)(\lambda_1, x_2)} \right) \right. \\ &\quad \left. + |\mathcal{F}_2(h *_1 u)(\lambda_1, x_2)|^2 \right] d\lambda_1. \end{aligned} \quad (92)$$

Recall that the Fourier transform of  $h$ , given by 56, is real. From 92, an application of Lemma 4.2 leads us to

$$\begin{aligned} &\int_{\mathbb{R}} \left[ \alpha |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 + 2\sqrt{\alpha} \operatorname{Re} \left( \mathcal{F}_2(u)(\lambda_1, x_2) \overline{\mathcal{F}_2(h *_1 u)(\lambda_1, x_2)} \right) \right. \\ &\quad \left. + |\mathcal{F}_2(h *_1 u)(\lambda_1, x_2)|^2 \right] d\lambda_1 \\ &= \int_{\mathbb{R}} \left[ \alpha + 2\sqrt{\alpha}\mathcal{F}_2(h)(\lambda_1) + (\mathcal{F}_2(h)(\lambda_1))^2 \right] |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1 \\ &= \int_{\mathbb{R}} [\sqrt{\alpha} + \mathcal{F}_2(h)(\lambda_1)]^2 |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1 \\ &= \int_{\mathbb{R}} [\alpha + f(\lambda_1)] |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1. \end{aligned} \quad (93)$$

On the other hand, by applying Plancherel's identity with respect to  $x_1$ , we obtain

$$\int_{\mathbb{R}} \frac{c}{c_\theta} 4\pi^2 \lambda_1^2 |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1 = \int_{\mathbb{R}} \frac{c}{c_\theta} \left( \frac{\partial u}{\partial x_1} \right)^2 (x_1, x_2) dx_1.$$

In view of the expansion of  $1/\hat{k}_0(\lambda_1)$  given by 87, the previous equality combined with 92 and 93 implies that, for  $u \in H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$  extended by zero with

respect to  $x_1$  outside  $(0, 1)$ ,

$$\begin{aligned} & \int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1 \\ &= \int_0^1 dx_2 \int_{\mathbb{R}} \left\{ \frac{c}{c_\theta} \left( \frac{\partial u}{\partial x_1} \right)^2 (x_1, x_2) + [\sqrt{\alpha} u(x_1, x_2) + (h *_1 u)(x_1, x_2)]^2 \right\} dx_1, \end{aligned}$$

which concludes the proof.  $\square$

*Proof of Lemma 4.1.* In view of 87, the equality 79 becomes

$$\begin{aligned} \mathcal{F}_2(b)(\lambda_1, x_2) &= \left( \frac{c}{c_\theta} 4\pi^2 \lambda_1^2 + \alpha + f(\lambda_1) \right) \mathcal{F}_2(u)(\lambda_1, x_2) \\ &= \mathcal{F}_2 \left( -\frac{c}{c_\theta} \frac{\partial^2 u}{\partial x_1^2} + \alpha u \right) (\lambda_1, x_2) + f(\lambda_1) \mathcal{F}_2(u)(\lambda_1, x_2). \end{aligned} \quad (94)$$

Since

$$f(\lambda_1) = \frac{(c-1)^2 \theta(\theta-1)}{c_\theta^2} \frac{1}{c_\theta 4\pi^2 \lambda_1^2 + 1} = O(\lambda_1^{-2}) \in C_0(\mathbb{R}) \cap L^1(\mathbb{R}),$$

the right-hand side of 94 belongs to  $L^2(\mathbb{R})$  with respect to  $\lambda_1$ , which implies that

$$\mathcal{F}_2(b)(\cdot, x_2) \in L^2(\mathbb{R}).$$

Applying the Plancherel identity, we obtain that  $b(\cdot, x_2) \in L^2(\mathbb{R})$  with respect to  $x_1$ . Since  $u(\cdot, x_2)$  is extended by zero outside  $(0, 1)$ ,  $b(\cdot, x_2)$  is also equal to zero outside  $(0, 1)$  so that

$$b(\cdot, x_2) \in L^2(0, 1). \quad (95)$$

We show that  $b(x_1, \cdot)$  is a continuous function with respect to  $x_2 \in [0, 1]$ . Recall that the continuity of  $x_2 \in [0, 1] \mapsto b(x_1, x_2) \in L^2(0, 1)_{x_1}$  is equivalent to

$$\lim_{t \rightarrow 0} \|b(\cdot, x_2 + t) - b(\cdot, x_2)\|_{L^2(0, 1)_{x_1}} = 0.$$

Thanks to Plancherel's identity, we infer from 79 that

$$\begin{aligned} \|b(\cdot, x_2 + t) - b(\cdot, x_2)\|_{L^2(0, 1)_{x_1}}^2 &= \|\mathcal{F}_2(b)(\cdot, x_2 + t) - \mathcal{F}_2(b)(\cdot, x_2)\|_{L^2(\mathbb{R})_{\lambda_1}}^2 \\ &= \int_{\mathbb{R}} \left| \frac{1}{\hat{k}_0(\lambda_1)} [\mathcal{F}_2(u)(\lambda_1, x_2 + t) - \mathcal{F}_2(u)(\lambda_1, x_2)] \right|^2 d\lambda_1. \end{aligned}$$

In view of 88 and thanks to the Plancherel identity, we obtain

$$\begin{aligned} & \|b(\cdot, x_2 + t) - b(\cdot, x_2)\|_{L^2(0, 1)_{x_1}}^2 \\ & \leq C^2 \int_{\mathbb{R}} |(4\pi^2 \lambda_1^2 + 1)(\mathcal{F}_2(u)(\lambda_1, x_2 + t) - \mathcal{F}_2(u)(\lambda_1, x_2))|^2 d\lambda_1 \\ & \leq C^2 \left\| \mathcal{F}_2 \left( \frac{\partial u}{\partial x_1} \right) (\cdot, x_2 + t) - \mathcal{F}_2 \left( \frac{\partial u}{\partial x_1} \right) (\cdot, x_2) \right\|_{L^2(0, 1)_{\lambda_1}}^2 \\ & \quad + C^2 \|\mathcal{F}_2(u)(\cdot, x_2 + t) - \mathcal{F}_2(u)(\cdot, x_2)\|_{L^2(0, 1)_{\lambda_1}}^2 \\ & = C^2 \left\| \frac{\partial u}{\partial x_1} (\cdot, x_2 + t) - \frac{\partial u}{\partial x_1} (\cdot, x_2) \right\|_{L^2(0, 1)_{x_1}}^2 \\ & \quad + C^2 \|u(\cdot, x_2 + t) - u(\cdot, x_2)\|_{L^2(0, 1)_{x_1}}^2. \end{aligned}$$

By the Lebesgue dominated convergence theorem and since  $u \in C_c^\infty([0, 1]^2)$ , from the previous inequality we conclude that the map  $x_2 \in [0, 1] \mapsto b(x_1, x_2) \in L^2(0, 1)_{x_1}$  is continuous. Hence,

$$b(x_1, x_2) \in C([0, 1]_{x_2}; L^2(0, 1)_{x_1}). \quad (96)$$

To conclude the proof, it remains to show the regularity of  $u_0$ . Note that 80 is a Sturm-Liouville problem with constant coefficient with respect to  $x_1$ , since  $x_2 \in (0, 1)$  and  $y_2 \in Y_1$  play the role of parameters. By 95, we already know that  $b(\cdot, x_2) \in L^2(0, 1)$ , so that thanks to a classical regularity result (see *e.g.* [7] pp. 223-224), the problem 80 admits a unique solution  $u_0(\cdot, x_2, y_2)$  in  $H^2(0, 1)$ . Since  $H^2(0, 1)$  is embedding into  $C^1([0, 1])$ , we have

$$u_0(\cdot, x_2, y_2) \in C^1([0, 1]) \quad \text{a.e. } (x_2, y_2) \in (0, 1) \times Y_1.$$

On the other hand, the solution  $u_0(x_1, x_2, y_2)$  to the Sturm-Liouville problem 80 is explicitly given by

$$u_0(x_1, x_2, y_2) := \int_0^1 G_{y_2}(x_1, s) b(s, x_2) ds, \quad (97)$$

where  $b(x_1, x_2)$  is defined by 79 and 96 and the kernel  $G_{y_2}(x_1, s)$  is given by

$$G_{y_2}(x_1, s) := \frac{1}{\sqrt{a(y_2)} \sinh\left(\frac{1}{\sqrt{a(y_2)}}\right)} \sinh\left(\frac{x_1 \wedge s}{\sqrt{a(y_2)}}\right) \sinh\left(\frac{1 - x_1 \vee s}{\sqrt{a(y_2)}}\right).$$

This combined with 96 and 97 proves that

$$u_0(x_1, x_2, y_2) \in C^1([0, 1]^2, L^\infty_{\text{per}}(Y_1)),$$

which concludes the proof.  $\square$

We prove now the Lemma 4.2 that we used in Step 3 of Proposition 4.

*Proof of Lemma 4.2.* By the convolution property of the Fourier transform on  $L^2(\mathbb{R})$ , we have

$$h * u = \overline{\mathcal{F}_2}(\mathcal{F}_2(h)) * \overline{\mathcal{F}_2}(\mathcal{F}_2(h)) = \overline{\mathcal{F}_1}(\mathcal{F}_2(h)\mathcal{F}_2(u)), \quad (98)$$

where  $\overline{\mathcal{F}_i}$  denotes the conjugate Fourier transform for  $i = 1, 2$ . On the other hand, since  $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and due to Riemann-Lebesgue's lemma, we deduce that  $\mathcal{F}_2(u) = \mathcal{F}_1(u) \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ . This combined with  $\mathcal{F}_2(h) \in L^2(\mathbb{R})$  implies that

$$\mathcal{F}_2(h)\mathcal{F}_2(u) = \mathcal{F}_2(h)\mathcal{F}_1(u) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}).$$

Since  $\overline{\mathcal{F}_1} = \overline{\mathcal{F}_2}$  on  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , from 98 we deduce that

$$h * u = \overline{\mathcal{F}_2}(\mathcal{F}_2(h)\mathcal{F}_2(u)) \in L^2(\mathbb{R}),$$

which yields 91. This concludes the proof.  $\square$

**Remark 2.** Thanks to the Beurling-Deny theory of Dirichlet forms [3], Mosco [15, Theorem 4.1.2] has proved that the  $\Gamma$ -limit  $F$  of a family of Markovian form for the  $L^2(\Omega)$ -strong topology is a Dirichlet form which can be split into a sum of three forms: a strongly local form  $F_d$ , a local form and nonlocal one. More precisely, for  $u \in L^2(\Omega)$  with  $F(u) < \infty$ , we have

$$F(u) = F_d(u) + \int_{\Omega} u^2 k(dx) + \int_{(\Omega \times \Omega) \setminus \text{diag}} (u(x) - u(y))^2 j(dx, dy), \quad (99)$$

where  $F_d$  is called the diffusion part of  $F$ ,  $k$  is a positive Radon measure on  $\Omega$ , called the killing measure, and  $j$  is a positive Radon measure on  $(\Omega \times \Omega) \setminus \text{diag}$ , called the jumping measure. Recall that a Dirichlet form  $F$  is a closed form which satisfies the Markovian property, *i.e.* for any contraction  $T : \mathbb{R} \rightarrow \mathbb{R}$ , such that

$$T(0) = 0, \quad \text{and} \quad \forall x, y \in \mathbb{R}, \quad |T(x) - T(y)| \leq |x - y|,$$

we have  $F \circ T \leq F$ . A  $\Gamma$ -limit form obtained with the  $L^2(\Omega)$ -weak topology does not *a priori* satisfy the Markovian property, since the  $L^2(\Omega)$ -weak convergence does not commute with all contractions  $T$ . An example of a sequence of Markovian forms whose  $\Gamma$ -limit for the  $L^2(\Omega)$ -weak topology does not satisfy the Markovian property is provided in [9, Theorem 3.1]. Hence, the representation formula 99 does not hold in general when the  $L^2(\Omega)$ -strong topology is replaced by the  $L^2(\Omega)$ -weak topology.

In the present context, we do not know if the  $\Gamma$ -limit  $\mathcal{F}$  55 is a Dirichlet form since the presence of the convolution term makes difficult to prove the Markovian property.

**Appendix A. Homogenized formula for a rank-one laminate.** We are going to give an explicit expression of the homogenized matrix  $A^*$  defined by 7, which extends the rank-one laminate formula in the case of a rank-one laminates with degenerate phases. We will recover directly from this expression the positive definiteness of  $A^*$  for the class of rank-one laminates introduced in Section 3. Indeed, by virtue of Theorem 2.1 the positive definiteness of  $A^*$  also follows from assumption (H2) which is established in Proposition 2 and Proposition 3.

Set

$$a := (1 - \theta)A_1 e_1 \cdot e_1 + \theta A_2 e_1 \cdot e_1, \quad (100)$$

with  $\theta \in (0, 1)$  being the volume fraction of phase  $Z_1$ .

**Proposition 5.** *Let  $A_1$  and  $A_2$  be two symmetric and non-negative matrices of  $\mathbb{R}^{d \times d}$ ,  $d \geq 2$ . If  $a$  given by 100 is positive, the homogenized matrix  $A^*$  is given by*

$$A^* = \theta A_1 + (1 - \theta)A_2 - \frac{\theta(1 - \theta)}{a}(A_2 - A_1)e_1 \otimes (A_2 - A_1)e_1. \quad (101)$$

*If  $a = 0$ , the homogenized matrix  $A^*$  is the arithmetic average of the matrices  $A_1$  and  $A_2$ , *i.e.**

$$A^* = \theta A_1 + (1 - \theta)A_2. \quad (102)$$

*Furthermore, if one of the following conditions is satisfied:*

- i) in two dimensions,  $a > 0$  and the matrices  $A_1$  and  $A_2$  are given by 28 with  $\xi \cdot e_1 \neq 0$ ,*
- ii) in three dimensions,  $a > 0$ , the matrices  $A_1$  and  $A_2$  are given by 39 and the vectors  $\{e_1, \eta_1, \eta_2\}$  are independent in  $\mathbb{R}^3$ ,*

*then  $A^*$  is positive definite.*

**Remark 3.** The condition  $a > 0$  agrees with the  $\Gamma$ -convergence results of Propositions 2 and 3. In the two-dimensional framework, the degenerate case  $a = 0$  does not agree with Propositions 2. Indeed,  $a = 0$  implies that  $A_1 e_1 \cdot e_1 = A_2 e_1 \cdot e_1 = 0$  in contradiction to positive definiteness of  $A_2$ . Similar in the three-dimensional setting, where the independence of  $\{e_1, \eta_1, \eta_2\}$  is not compatible with  $a = 0$ . Indeed,  $a = 0$  implies that  $A_i e_1 = A_i \eta_i = 0$ , for  $i = 1, 2$ , which contradicts the fact that  $A_1$  and  $A_2$  have rank two.

*Proof.* Assume that  $a > 0$ . In view of the convergence 23, we already know that

$$\lim_{\delta \rightarrow 0} A_\delta^* = A^*, \quad (103)$$

where, for  $\delta > 0$ ,  $A_\delta^*$  is the homogenized matrix associated to conductivity matrix  $A_\delta$  given by

$$A_\delta(y_1) = \chi(y_1)A_1^\delta + (1 - \chi(y_1))A_2^\delta \quad \text{for } y_1 \in \mathbb{R},$$

with  $A_i^\delta = A_i + \delta I_d$ . Since  $A_1$  and  $A_2$  are non-negative matrices,  $A_\delta$  is positive definite and thus the homogenized matrix  $A_\delta^*$  is given by the lamination formula (see [17] and also [2, Lemma 1.3.32])

$$A_\delta^* = \theta A_1^\delta + (1 - \theta)A_2^\delta - \frac{\theta(1 - \theta)}{(1 - \theta)A_1^\delta e_1 \cdot e_1 + \theta A_2^\delta e_1 \cdot e_1} (A_2^\delta - A_1^\delta) e_1 \otimes (A_2^\delta - A_1^\delta) e_1. \quad (104)$$

If  $a > 0$ , we easily infer from the convergence 103 combined with the lamination formula 104 the expression 101 for  $A^*$ .

We prove that  $A^* x \cdot x \geq 0$  for any  $x \in \mathbb{R}^d$ . From the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} |(A_2 - A_1)e_1 \cdot x| &\leq |A_2 e_1 \cdot x| + |A_1 e_1 \cdot x| \\ &\leq (A_2 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2} + (A_1 e_1 \cdot e_1)^{1/2} (A_1 x \cdot x)^{1/2}. \end{aligned} \quad (105)$$

This combined with the definition 101 of  $A^*$  implies that, for any  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} A^* x \cdot x &= \theta(A_1 x \cdot x) + (1 - \theta)(A_2 x \cdot x) - \theta(1 - \theta)a^{-1} |(A_2 - A_1)e_1 \cdot x|^2 \\ &\geq \theta(A_1 x \cdot x) + (1 - \theta)(A_2 x \cdot x) \\ &\quad - \theta(1 - \theta)a^{-1} [(A_2 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2} + (A_1 e_1 \cdot e_1)^{1/2} (A_1 x \cdot x)^{1/2}]^2 \\ &= a^{-1} [a\theta(A_1 x \cdot x) + a(1 - \theta)(A_2 x \cdot x) - \theta(1 - \theta)(A_2 e_1 \cdot e_1)(A_2 x \cdot x) \\ &\quad - \theta(1 - \theta)(A_1 e_1 \cdot e_1)(A_1 x \cdot x) \\ &\quad - 2\theta(1 - \theta)(A_2 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2} (A_1 e_1 \cdot e_1)^{1/2} (A_1 x \cdot x)^{1/2}]. \end{aligned} \quad (106)$$

In view of definition 100 of  $a$ , we have that

$$\begin{aligned} &a\theta(A_1 x \cdot x) + a(1 - \theta)(A_2 x \cdot x) \\ &= \theta(1 - \theta)(A_1 e_1 \cdot e_1)(A_1 x \cdot x) + \theta^2(A_2 e_1 \cdot e_1)(A_1 x \cdot x) \\ &\quad + (1 - \theta)^2(A_1 e_1 \cdot e_1)(A_2 x \cdot x) \\ &\quad + \theta(1 - \theta)(A_2 e_1 \cdot e_1)(A_2 x \cdot x). \end{aligned}$$

Plugging this equality in 106, we deduce that

$$\begin{aligned} A^* x \cdot x &\geq a^{-1} [\theta^2(A_2 e_1 \cdot e_1)(A_1 x \cdot x) + (1 - \theta)^2(A_1 e_1 \cdot e_1)(A_2 x \cdot x) \\ &\quad - 2\theta(1 - \theta)(A_2 e_1 \cdot e_1)^{1/2} (A_1 x \cdot x)^{1/2} (A_1 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2}] \\ &= a^{-1} [\theta(A_2 e_1 \cdot e_1)^{1/2} (A_1 x \cdot x)^{1/2} - (1 - \theta)(A_1 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2}]^2 \geq 0, \end{aligned} \quad (107)$$

which proves that  $A^*$  is a non-negative definite matrix.

Now, assume  $a = 0$ . Since  $A_1$  and  $A_2$  are non-negative matrices, the condition  $a = 0$  implies  $A_1 e_1 \cdot e_1 = A_2 e_1 \cdot e_1 = 0$  or equivalently  $A_1 e_1 = A_2 e_1 = 0$ . Hence,

$$(A_2^\delta - A_1^\delta) e_1 = (A_2 - A_1) e_1 = 0,$$

which implies that the lamination formula 104 becomes

$$A_\delta^* = \theta A_1^\delta + (1 - \theta) A_2^\delta.$$

This combined with the convergence 103 yields to the expression 102 for  $A^*$ .

To conclude the proof, it remains to prove the positive definiteness of  $A^*$  under the above conditions i) and ii).

**Case (i).**  $d = 2$ ,  $a > 0$  and  $A_1, A_2$  given by 28.

Assume  $A^* x \cdot x = 0$ . Then, the inequality 107 is an equality, which yields in turn equalities in 105. In particular, we have

$$|A_2 e_1 \cdot x| = (A_2 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2} = \|A_2^{1/2} e_1\| \|A_2^{1/2} x\|. \quad (108)$$

Recall that the Cauchy-Schwarz inequality is an equality if and only if one of vectors is a scalar multiple of the other. This combined with 108 leads to  $A_2^{1/2} x = \alpha A_2^{1/2} e_1$  for some  $\alpha \in \mathbb{R}$ , so that, since  $A_2$  is positive definite or equivalently  $A_2^{1/2}$ , we have

$$x = \alpha e_1 \quad \text{for some } \alpha \in \mathbb{R}. \quad (109)$$

From the definition 101 of  $A^*$  and due to the assumption  $\xi \cdot e_1 \neq 0$ , we get

$$A^* e_1 \cdot e_1 = \frac{1}{a} (A_2 e_1 \cdot e_1) (\xi \cdot e_1)^2 > 0. \quad (110)$$

Recall that  $A^* x \cdot x = 0$ . This combined with 109 and 110 implies that  $x = 0$ , which proves that  $A^*$  is positive definite.

**Case (ii).**  $d = 3$ ,  $a > 0$  and  $A_1, A_2$  given by 39.

Assume that  $A^* x \cdot x = 0$ . As in Case (i), we have equalities in 105. In other words,

$$|A_1 e_1 \cdot x| = (A_1 e_1 \cdot e_1)^{1/2} (A_1 x \cdot x)^{1/2}, \quad (111)$$

$$|A_2 e_1 \cdot x| = (A_2 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2}. \quad (112)$$

Let  $p_i(t)$  be the non-negative polynomials of degree 2 defined by

$$p_i(t) := A_i(x + t e_1) \cdot (x + t e_1) \quad \text{for } i = 1, 2.$$

In view of 111, the discriminant of  $p_1(t)$  is zero, so that there exists  $t_1 \in \mathbb{R}$  such that

$$p_1(t_1) = A_1(x + t_1 e_1) \cdot (x + t_1 e_1) = 0. \quad (113)$$

Recall that  $\text{Ker}(A_1) = \text{Span}(\eta_1)$ . Since  $A_1$  is non-negative matrix, we deduce from 113 that  $x + t_1 e_1$  belongs to  $\text{Ker}(A_1)$ , so that

$$x \in \text{Span}(e_1, \eta_1). \quad (114)$$

Similarly, recalling that  $\text{Ker}(A_2) = \text{Span}(\eta_2)$  and using 112, we have

$$x \in \text{Span}(e_1, \eta_2). \quad (115)$$

Since the vectors  $\{e_1, \eta_1, \eta_2\}$  are independent in  $\mathbb{R}^3$ , 114 and 115 imply that

$$x = \alpha e_1 \quad \text{for some } \alpha \in \mathbb{R}.$$

In light of definition 101 of  $A^*$ , we have

$$A^* e_1 \cdot e_1 = \frac{1}{a} (A_1 e_1 \cdot e_1) (A_2 e_1 \cdot e_1) > 0,$$

which implies that  $x = 0$ , since  $A^* x \cdot x = 0$ . This establishes that  $A^*$  is positive definite and concludes the proof.  $\square$

Note that when  $d = 2$  and  $a > 0$  the assumption  $\xi \cdot e_1 \neq 0$  is essential to obtain that  $A^*$  is positive definite. Otherwise, the homogenized matrix  $A^*$  is just non-negative definite as shown by the following counter-example. Let  $A_1$  and  $A_2$  be symmetric and non-negative matrices of  $\mathbb{R}^{2 \times 2}$  defined by

$$A_1 = e_2 \otimes e_2 \quad \text{and} \quad A_2 = I_2.$$

Then, it is easy to check that  $a = \theta > 0$  and  $A^* e_1 \cdot e_1 = 0$ .

**Acknowledgments.** This problem was pointed out to me by Marc Briane during my stay at Institut National des Sciences Appliquées de Rennes. I thank him for the countless fruitful discussions. My thanks also extend to Valeria Chiadò Piat for useful remarks. The author is also a member of the INdAM-GNAMPA project “Analisi variazionale di modelli non-locali nelle scienze applicate”.

## REFERENCES

- [1] G. Allaire, [Homogenization and two-scale convergence](#), *SIAM J. Math. Anal.*, **23** (1992), 1482–1518.
- [2] G. Allaire, [Shape Optimization by the Homogenization Method](#), Springer-Verlag, New York, 2002.
- [3] A. Beurling and J. Deny, [Espaces de dirichlet](#), *Acta Math.*, **99** (1958), 203–224.
- [4] A. Braides, [Γ-Convergence for Beginners](#), Oxford University Press, Oxford, 2002.
- [5] A. Braides, [A handbook of Γ-convergence](#), *Handbook of Differential Equations: Stationary Partial Differential Equations* Vol. **3**, Elsevier, (2006), 101–213.
- [6] A. Braides, V. C. Piat and A. Piatnitski, A variational approach to double-porosity problems, *Asymptot. Anal.*, **39** (2004), 281–308.
- [7] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext series, Springer, New York, 2010.
- [8] M. Briane, Correctors for the homogenization of a laminate, *Adv. Math. Sci. Appl.*, **4** (1994), 357–379.
- [9] M. Briane, Non-Markovian quadratic forms obtained by homogenization, *Boll. Uni. Mate. Ital. Sez. B Artic. Ric. Mat.*, **6** (2003), 323–337.
- [10] M. Briane and G. A. Francfort, [Loss of ellipticity through homogenization in linear elasticity](#), *Math. Mod. Met. Appl. Sci.*, **25** (2015), 905–928.
- [11] M. Briane and G. A. Francfort, [A two-dimensional labile aether through homogenization](#), *Commun. Math. Phys.*, **367** (2019), 599–628.
- [12] M. Briane and A. J. Pallares Martín, [Homogenization of weakly coercive integral functionals in three-dimensional linear elasticity](#), *J. Éc. Polytech. Math.*, **4** (2017), 483–514.
- [13] G. Dal Maso, [An Introduction to Γ-Convergence](#), Volume **8** of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, Boston, 1993.
- [14] S. Gutiérrez, [Laminations in linearized elasticity: The isotropic non-very strongly elliptic case](#), *Q. J. Mech. Appl. Math.*, **57** (2004), 571–582.
- [15] U. Mosco, [Composite media and asymptotic Dirichlet forms](#), *J. Funct. Anal.*, **123** (1994), 368–421.
- [16] G. Nguetseng, [A general convergence result for a functional related to the theory of homogenization](#), *SIAM J. Math. Anal.*, **20** (1989), 608–623.
- [17] L. Tartar, Estimations fines de coefficients homogénéisés, *Ennio De Giorgi Colloquium*, Ed. P. Krée, *Pitman Research Notes in Mathematics*, **125** (1985), 168–187.

Received June 2021; revised July 2021; early access September 2021.

E-mail address: [lorenza.delia@polito.it](mailto:lorenza.delia@polito.it)