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WELL-POSEDNESS AND APPROXIMATE CONTROLLABILITY OF NEUTRAL NETWORK SYSTEMS

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Abstract. In this paper, we study the concept of approximate controllability of retarded network systems of neutral type. On one hand, we reformulate such systems as free-delay boundary control systems on product spaces. On the other hand, we use the rich theory of infinite-dimensional linear systems to derive necessary and sufficient conditions for the approximate controllability. Moreover, we propose a rank condition for which we can easily verify the conditions of controllability. Our approach is mainly based on the feedback theory of regular linear systems in the Salamon-Weiss sense.

1. Introduction.

The main object of this paper is to characterise the approximate controllability of the following retarded network system of neutral type and input delays

$$\begin{cases} \dfrac{\partial}{\partial t}\varrho^j(t,x) = c^j(x)\dfrac{\partial}{\partial x}\varrho^j(t,x) + q^j(x)\varrho^j(t,x) + \displaystyle\sum_{k=1}^{m} L_{jk} z^k(t+\cdot,\cdot),\ x\in(0,1), t\geq 0,\\ \varrho^j(0,x) = g^j(x),\quad x\in(0,1),\\ \mathsf{i}^-_{ij}c^j(1)\varrho^j(t,1) = \mathsf{w}^-_{ij}\displaystyle\sum_{k=1}^{m}\mathsf{i}^+_{ik}c^k(0)\varrho^j(t,0) + \sum_{l=1}^{n_0}\mathrm{k}_{il}v^l(t), \qquad t\geq 0,\\ z^j(\theta,x) = \varphi^j(\theta,x),\ u^j(\theta)=\psi^j(\theta), \qquad \theta\in[-r,0], x\in(0,1),\\ \varrho^j(t,x) = \left[z^j(t,x) - \displaystyle\sum_{k=1}^{m} D_{jk} z^k(t+\cdot,\cdot) - \sum_{i=1}^{n}\mathsf{k}_{ij}u^j(t+\cdot) - \mathrm{b}_{ij}u^j(t)\right] \end{cases} \tag{1}$$

for $i=1,\dots,n$ and $j=1,\dots,m$. Here, $z^j(t,x)$ represents the distribution of the material along an edge e_j of a graph G at the point x and time t, where G is a finite connected graph composed by $n\in\mathbb{N}$ vertices $\alpha_1,\dots,\alpha_n$, and by $m\in\mathbb{N}$ edges $e_1,\dots,e_m$ which are assumed to be normalized on the interval $[0,1]$. Moreover, we assume that the nodes $\alpha_1,\dots,\alpha_n$ exhibit standard Kirchhoff type conditions to be specified later on. Moreover, $z^j(t+\cdot,x):[-r,0]\to\mathbb{C}$ is the history function of z^j,

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and $u^j(t+\cdot):[-r,0]\to\mathbb{C}$ is the history function of the control function u^j. We notice that system (1) arises as a model for linear flows in networks with fading memory.

The study of system (1) is motivated by the several open problems on transport network systems, which is a very active topic for many years [2, 4, 5, 11, 14, 15, 12]. Such research activity is motivated by a broad area of their possible applications, see, e.g. [6], and the interesting mathematical questions that arise from their analysis. For instance, several properties of the transport processes depend on the structure of the network and on the rational relations of the flow velocities, see, e.g. [1, 23] and references therein.

On the other hand, neutral delay systems arise naturally in many practical mathematical models. Typical examples include communication networks, structured population models, chemical processes, tele-operation systems [22, 35]. The qualitative properties (existence, stability, controllability, etc.) for this class of systems have received much attention (see [3], [7], [21], [22], [25], [35] and references therein). For instance, different controllability results for various neutral delay systems have been established recently (see, [24], [10], [29], [27]). In [24], the authors analyze the exact null controllability of neutral systems with distributed state delay by using the moment problem approach. In [10], relative controllability of linear discrete systems with a single constant delay was studied using the so-called discrete delayed matrix exponential. In [29], the authors studied the approximate controllability of linear (continuous-time) systems with state delays via the matrix Lambert W function. The robustness of approximate controllability of linear retarded systems under structured perturbations has been addressed in [27] using the so-called structured distance to non-surjectivity. However, the results established in the aforementioned works become invalid for the transport network system (1), since the operators $D=(D_{jk}), L=(L_{jk}), K_1=(\mathsf{k}_{jk})$ and A_m are supposed to be unbounded. In fact, as we shall see in Section 5, if we take $X=L^p([0,1])^m$ as the state space then $D,L\in\mathcal{L}(W^{1,p}([-r,0],X);X)$ and $K_1\in\mathcal{L}(W^{1,p}([-r,0],\mathbb{C}^n);X)$.

In this paper, we study the concept of approximate controllability of boundary value problems of neutral type with a particular aim to explore new techniques and new questions for control problems of transport network systems. We formulate the problem in the framework of well-posed and regular linear systems and solve it in the operator form. To be precise, we use product spaces and operator matrices to reformulate (11) into a inhomogeneous perturbed Cauchy problem governed by an operator having a perturbed domain. This allows us to use the feedback theory of well-posed and regular linear systems to prove that this operator is a generator. Our approach allows us to easily calculate the spectrum and the resolvent operator of this generator. In this manner, necessary and sufficient conditions of approximate controllability for (11) are formulated and proved by using the feedback theory of regular linear systems and methods of functional analysis. Our main result is that, when the control space is of finite dimension, we prove that the established approximate controllability criteria are reduced to a compact rank condition given in terms of transfer functions of controlled delay systems. As we shall see in Section 4 our approach by transforming the neutral delay system controllability problem into approximate controllability of an abstract perturbed boundary control problem greatly facilitates analysis and offers an alternative approach for the study of controllability in terms of extensive existing knowledge of feedback theory of closed-loop systems. This establishes a framework for investigating the approximate controllability of infinite dimensional neutral delay systems with state and input delays, which may shed some light in solving the approximate controllability of concrete physical problems.

The whole article is organized as follows: we initially present a survey on well-posed and regular linear systems in the Salamon-Weiss sense; Section 2. The results obtained on the well-posedness and spectral theory of boundary value problems of neutral type are discussed in Section 3. Section 4 is devoted to state and prove the main results on approximate controllability of abstract boundary control systems of neutral type. Finally, in Section 5, we show the solvability of transport network systems of neutral type by means of our introduced framework.

2. **Some background on infinite-dimensional linear systems.**

In this section we recall some well-known results and definitions on infinite dimensional linear time-invariant systems. The reader is referred to the papers [25], [31], [32], [34], [33], which was our main reference, if more details or further references are required. For the Hilbert space or Banach space setting, the reader may also refer to [28, 30].

Let X, U, Z be Banach spaces such that $Z \subset X$ with continuous dense embedding, $A_m : Z \longrightarrow X$ be a closed linear (often differential) operator on X (here $D(A_m) = Z$), and boundary linear operators $G, M : Z \subset X \longrightarrow U$.

Consider the following boundary input-output system

$$\begin{cases} \dot{z}(t) = A_m z(t), & t \geq 0,\ z(0) = z_0 \\ Gz(t) = u(t), & t \geq 0, \\ y(t) = Mz(t), & t \geq 0. \end{cases} \tag{2}$$

Notice that the well-posed of the boundary input-output system (2) consists in finding conditions on operators A_m, G and M such as

$$\|z(\tau)\|_X^p + \|y(\cdot)\|^p_{L^p([0,\tau];U)} \leq c(\tau) \left(\|z_0\|_X^p + \|u(\cdot)\|^p_{L^p([0,\tau];U)} \right). \tag{3}$$

for some (hence for every) $\tau > 0$, a constant $c(\tau) > 0$ and $p \geq 1$. To make these statements more clear, some hypothesis are needed.

(A1) the restricted operator $A \subset A_m$ with domain $D(A) = \ker G$ generates a C_0-semigroup $T := (T(t))_{t\geq 0}$ on X;

(A2) the boundary operator G is surjective;

According to assumptions **(A1)** and **(A2)**, for $\mu \in \rho(A)$, the following inverse, called the Dirichlet operator,

$$D_\mu = (G|_{\ker(\mu - A_m)})^{-1} \in \mathcal{L}(U, D(A_m)).$$

exists. Let X_{-1} be the completion of X with respect to the norm $\|z\|_{-1} := \|R(\mu, A)z\|$ for all $z \in X$ and some (hence all) $\mu \in \rho(A)$. This new space is also called the extrapolation space of X with respect to A, which satisfies $D(A) \subset X \subset X_{-1}$ with continuous and dense embedding. We extend the semigroup T to a strongly continuous semigroup $T_{-1} := (T_{-1}(t))_{t\geq 0}$ on X_{-1} (the extrapolation semigroup), whose generator $A_{-1} : X \to X_{-1}$ is the extension of A to X, see e.g. [16]. Let us now define the boundary control operator

$$B = (\mu - A_{-1}) D_\mu \in \mathcal{L}(U, X_{-1}),$$

then $B \in \mathcal{L}(U, X_{-1})$, Range$(B) \cap X = \{0\}$ and

$$(A - A_{-1})_{|Z} = BG, \tag{4}$$

since $\mu D_\mu u = A_m D_\mu u$, $u \in U$. We mention that the operator B is independent of μ due to the resolvent equation. By virtue of formula (4) the boundary input-output system (2) can be reformulated as the following distributed-parameter system

$$\begin{cases} \dot{z}(t) = A_{-1} z(t) + Bu(t), & t \geq 0,\ z(0) = z_0, \\ y(t) = Cz(t), & t \geq 0, \end{cases} \tag{5}$$

where

$$C = M_{|_{D(A)}}.$$

Then the state of the system (5) satisfy the variation of constants formula

$$z(t; z_0, u) = T(t) z_0 + \int_0^t T_{-1}(t-s) Bu(s) ds, \quad t \geq 0, \tag{6}$$

for all $z_0 \in X$ and $u \in L^p([0, +\infty); U)$. Notice that the integral in (6) is taken in the large space X_{-1}. Thus, we need a class of control operators B for which the state of the system (5) takes values in the state space X. This motivated the following definition.

Definition 2.1. An operator $B \in \mathcal{L}(U, X_{-1})$ is called an admissible control operator for A, if for some $\tau > 0$,

$$\Phi_\tau u := \int_0^\tau T_{-1}(\tau - s) Bu(s) ds,$$

takes values in X for any $u \in L^p([0, +\infty); U)$.

Note that the admissibility of B implies that the state of the system (5) is a continuous X-valued function of t and satisfy

$$z(t) = T(t) z_0 + \Phi_t u, \tag{7}$$

for all $z_0 \in X$ and $u \in L^p_{loc}([0, \infty); U)$. Now for $\alpha > w_0(T)$, let $u \in L^p_\alpha([0, \infty); U)$, the space of all the functions of the form $u(t) = e^{\alpha t} v(t)$, where $v \in L^p([0, \infty); U)$. Then u and z from (7) have Laplace transforms related by

$$\hat{z}(\mu) = R(\mu, A) z_0 + \widehat{\Phi_\bullet u}(\mu), \quad \text{with} \quad \widehat{\Phi_\bullet u}(\mu) = D_\mu \hat{u}(\mu), \ \forall\, \Re e\, \mu > \alpha, \tag{8}$$

where $\alpha \in \mathbb{R}$ and $\hat{u}$ denote the Laplace transform of u.

For each $\tau > 0$, we define the space

$$u \in W^{1,p}_{0,\tau}(U) := \left\{ u \in W^{1,p}([0, \tau]; U) : u(0) = 0 \right\},$$

which is dense in the Lebesgue space $L^2([0, \tau]; U)$. When B is admissible, one can use an integration by parts technique to prove that $\Phi_\tau u \in Z$ for any $\tau \geq 0$ and $u \in W^{1,p}_{0,\tau}(U)$. It makes sense to define the linear operator

$$(\mathbb{F} u)(t) = M \Phi_t u, \quad t \geq 0, \quad u \in W^{1,p}_{0,t}(U). \tag{9}$$

With these notations, it follows that

$$y = \Psi z_0 + \mathbb{F} u, \quad \text{on} \quad [0, \tau]$$

for and $(z_0, u) \in D(A) \times W^{1,p}_{0,\tau}(U)$ and $\tau > 0$, where

$$\Psi z_0 = CT(\cdot) z_0, \quad z_0 \in D(A).$$

So, according to the inequality (3), we are looking for an output function $y(\cdot)$ in the space $L^p_{loc}([0,\infty);U)$ for any $(z_0,u) \in X \times L^p_{loc}([0,\infty);U)$. As a matter of fact, this may not hold for any unbounded operator C. In order to overcome this obstacle, we first introduce the following class of operators C.

Definition 2.2. An operator $C \in \mathcal{L}(D(A),U)$ is called an admissible observation operator for A if for some (hence all) $\tau > 0$, there exists a constant $\gamma := \gamma(\tau) > 0$ such that

$$\int_0^\tau \|CT(s)z\|^p ds \leq \gamma^p \|z\|^p, \tag{10}$$

for all $z \in D(A)$.

If C is an admissible observation operator for A, then for any $\tau > 0$, the map Ψ is bounded from $D(A)$ to $L^p([0,\tau];U)$. Moreover, by density, the operator Ψ extends to $\Psi \in \mathcal{L}(X, L^p([0,\tau];U))$.

As in Weiss [31], we consider the Λ-extension of C for A defined by

$$C_\Lambda z := \lim_{\mu \longrightarrow \infty} C\mu R(\mu,A)z,$$

whose domain $D(C_\Lambda)$ consists of all $z_0 \in X$ for which the limit exists. According to [31], the admissibility of C for A implies that the orbits of the semigroup $T(\cdot)$ satisfies $T(t)z \in D(C_\Lambda)$ for a.e $t > 0$ and all $z \in X$. Moreover, we have

$$\Psi z := C_\Lambda T(\cdot)z, \qquad z \in X, \quad a.e.$$

Definition 2.3. Let $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(D(A),U)$ be admissible control and observation operator for A, respectively. We call the triplet (A,B,C) (or equivalently the system (5)) a well-posed state-space operators on U,X,U, if for every $\tau > 0$ there exits $\kappa := \kappa(\tau)$ such as

$$\|\mathbb{F}u\|_{L^p([0,\tau];U)} \leq \kappa \|u\|_{L^p([0,\tau];U)}, \qquad u \in W^{1,p}_{0,\tau}(U).$$

If the triple (A,B,C) is well-posed, by density, the operator $\mathbb{F}$ is extended to $\mathbb{F} \in \mathcal{L}(L^p([0,\tau];U))$. We introduce

$$\mathbb{F}_\tau u = (\mathbb{F}u)_{|_{[0,\tau]}}, \qquad \tau \geq 0.$$

This operators are called the input-output maps of the system (A,B,C).

The output function y of the well-posed system (A,B,C) is then extended to $y \in L^p_{loc}([0,+\infty);U)$ and satisfies $y = \Psi z_0 + \mathbb{F}u$ for any $z_0 \in X$ and $u \in L^p_{loc}([0,+\infty);U)$. In particular, the feedback law $u = y$ has a sense if only if $(I - \mathbb{F})u = \Psi z_0$ has a unique solution $u \in L^p([0,\tau],U)$ for some $\tau > 0$. This is true if $I - \mathbb{F}$ is invertible in $L^p([0,\tau],U)$. In this case, the identity $I : U \longrightarrow U$ is called an admissible feedback for (A,B,C).

A more appropriate subclass of well-posed state-space operators is defined by:

Definition 2.4. Let (A,B,C) a well-posed state-space operators on U,X,U. Then, the triplet (A,B,C) is called regular state-space operators (with feedthrough zero) if for any $v \in U$, we have

$$\lim_{\tau \longmapsto 0} \frac{1}{\tau} \int_0^\tau (\mathbb{F}(\mathbb{1}_{\mathbb{R}_+} \cdot v))(\sigma) d\sigma = 0.$$

We recall that, if (A,B,C) is called regular state-space operators and has the identity operator $I : U \to U$ as an admissible feedback, the operator

$$A^I = A_{-1} + BC_\Lambda, \;\; D(A^I) = \{z \in D(C_\Lambda) : (A_{-1} + BC_\Lambda)z \in X\}$$

generates a strongly continuous semigroup $T^I := (T^I(t))_{t\geq 0}$ on X such that $T^I(t)z \in D(C_\Lambda)$ for all $z \in X$ and a.e $t > 0$. Moreover, we have

$$T^I(t)z = T(t)z + \int_0^t T_{-1}(t-s)BC_\Lambda T^I(s)zds$$

for all $z \in X$ and $t \geq 0$, see e.g. [28, Chap.7] and [34].

3. **Well-posedness of boundary value problems of neutral type.** In this section, we investigate the well-posedness of the abstract boundary control systems of neutral type described as

$$\begin{cases} \dfrac{d}{dt}(z(t) - Dz_t - K_0u(t) - K_1u_t) \\ = A_m(z(t) - Dz_t - K_0u(t) - K_1u_t) + Lz_t + B_0u(t) + B_1u_t, & t \geq 0, \\ \lim\limits_{t\longrightarrow 0}(z(t) - Dz_t - K_0u(t) - K_1u_t) = \varrho_0, \\ G(z(t) - Dz_t - K_0u(t) - K_1u_t) \\ = M(z(t) - Dz_t - K_0u(t) - K_1u_t) + Kv(t), & t \geq 0, \\ z_0 = \varphi, \quad u_0 = \psi, \end{cases} \tag{11}$$

where the state variable $z(\cdot)$ takes values in a Banach space X and the control functions $u(\cdot), v(\cdot)$ are given in the Banach space $L^p_{loc}([0,\infty); U)$, where U is also a Banach space. K_0, B_0 are bounded linear operator from U to X, whereas K is a boundary control operator from U to the Banach space ∂X. $A_m : D(A_m) \subset X \longrightarrow X$ is a closed, linear differential operator and $G, M : D(A_m) \longrightarrow \partial X$ are unbounded trace operators. The delay operators $D, L : W^{1,p}([-r, 0]; X) \longrightarrow X$ and $K_1, B_1 : W^{1,p}([-r, 0]; U) \longrightarrow X$ are defined by

$$D\varphi = \int_{-r}^0 d\eta(\theta)\varphi(\theta), \qquad L\varphi = \int_{-r}^0 d\gamma(\theta)\varphi(\theta),$$
$$B_1\psi = \int_{-r}^0 d\nu(\theta)\psi(\theta), \qquad K_1\psi = \int_{-r}^0 d\vartheta(\theta)\psi(\theta),$$

for $\varphi \in W^{1,p}([-r, 0], X)$ and $\psi \in W^{1,p}([-r, 0], U)$, where $\eta, \gamma : [-r, 0] \longrightarrow \mathcal{L}(X)$ and $\nu, \vartheta : [-r, 0] \longrightarrow \mathcal{L}(U, X)$ are functions of bounded variations with total variations $|\eta|([-\varepsilon, 0])$, $|\gamma|([-\varepsilon, 0])$, $|\nu|([-\varepsilon, 0])$, and $|\vartheta|([-\varepsilon, 0])$ approach 0 as $\varepsilon \longrightarrow 0$. Also $x_t : [-r, 0] \longrightarrow X$ and $u_t : [-r, 0] \longrightarrow U$ are the histories functions of $z(\cdot)$ and $u(\cdot)$, respectively, and defined by $x_t(\theta) = x(t + \theta)$ and $u_t(\theta) = u(t + \theta)$ for $t \geq 0$ and $\theta \in [-r, 0]$, where $r > 0$ is a real number. The functions $z_0 = \varphi, u_0 = \psi$ are the initial history functions of $z(\cdot)$ and $u(\cdot)$, respectively. Some new notation is needed. Let E be a Banach sapce and $r > 0$ be a real number. Define the operator

$$Q^E_m\xi = \frac{\partial}{\partial\theta}\xi, \quad D(Q^E_m) = W^{1,p}([-r, 0]; E),$$

and

$$Q^E\xi = \frac{\partial}{\partial\theta}\xi, \quad D(Q^E) = \{\xi \in W^{1,p}([-r, 0]; E) : \xi(0) = 0\}.$$

It is well known that $(Q^E, D(Q^E))$ generate the left shift semigroup

$$(S^E(t)\xi)(\theta) = \begin{cases} 0, & t + \theta \geq 0, \\ \xi(t + \theta), & t + \theta \leq 0, \end{cases}$$

for $t \geq 0$ and $\theta \in [-r, 0]$ and $\xi \in L^p([-r, 0]; E)$; see [16]. For $\mu \in \mathbf{C}$, we define the operator e_μ as

$$e_\mu^E : E \longrightarrow L^p([-r, 0]; E), \quad (e_\mu^E z)(\theta) = e^{\mu\theta} z, \quad z \in E, \ \theta \in [-r, 0].$$

Denote by Q_{-1}^E the extension of Q^E in the extrapolation sense and define the operator

$$\beta^E := -Q_{-1}^E e_0.$$

Then the function $z_t(\cdot)$ is the solution of the following boundary equation

$$\begin{cases} \dot{v}(t, \theta) = Q_m^E v(t, \theta), & t > 0, \ \theta \in [-r, 0], \\ v(t, 0) = z(t), \quad v(0, .) = \xi, & t \geq 0. \end{cases} \tag{12}$$

Moreover, for any $\xi \in L^p([-r, 0]; E)$ and $z \in L^p([-r, \infty); E)$ with $z_0 = \xi$, z_t is given by

$$z_t = S^E(t)\xi + \Phi_t^E z, \quad t \geq 0,$$

where $\Phi_t : L^p([0, \infty); E) \longrightarrow L^p([-r, 0]; E)$ are the linear operators defined by

$$(\Phi^E(t)z)(\theta) = \begin{cases} z(t + \theta), & t + \theta \geq 0, \\ 0, & t + \theta \leq 0, \end{cases}$$

for $t \geq 0$, $z \in L^p([0, \infty); E)$ and $\theta \in [-r, 0]$; see [19].

Next we study the well-posedness of the neutral delay system (11) in the case of $B_0 \equiv 0, K_0 \equiv 0$ and $K \equiv 0$. First, we introduce the Banach spaces

$$\begin{aligned} \mathcal{X} &:= X \times L^p([-r, 0]; X) \times L^p([-r, 0]; U), \\ \mathcal{Z} &:= D(A_m) \times W^{1,p}([-r, 0]; X) \times W^{1,p}([-r, 0]; U), \\ \mathcal{U} &:= \partial X \times X \times U, \end{aligned}$$

equipped with their usual norms. Second, we set

$$\varrho(t) = z(t) - Dz_t - K_1 u_t.$$

By using (12) together with the function

$$t \longmapsto \zeta(t) = \begin{pmatrix} \varrho(t) \\ z_t \\ u_t \end{pmatrix},$$

one can see that the neutral delay system (11) can be rewrite as the following perturbed Cauchy problem

$$\begin{cases} \dot{\zeta}(t) = [\mathcal{A}_{G,M} + \mathcal{P}]\zeta(t), & t \geq 0, \\ \zeta(0) = (\varrho_0, \varphi, \psi)^\top, \end{cases} \tag{13}$$

where $\mathcal{A}_{G,M}$ and $\mathcal{P}$ are linear operators on $\mathcal{X}$ defined by

$$\mathcal{A}_{G,M} := \begin{pmatrix} A_m & L & B_1 \\ 0 & Q_m^X & 0 \\ 0 & 0 & Q_m^U \end{pmatrix}, \qquad D(\mathcal{A}_{G,M}) := \left\{ \begin{pmatrix} \varrho_0 \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{Z} : \begin{matrix} G\varrho_0 = M\varrho_0 \\ \varphi(0) = \varrho_0 + D\varphi + K_1\psi \end{matrix} \right\}$$

$$\mathcal{P} := \begin{pmatrix} 0 & L & B_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D(\mathcal{P}) := \mathcal{Z}.$$

Next we are concerned with the well-posedness of the perturbed Cauchy problem (13). To this end, we first prove that the operator $\mathcal{A}_{G,M}$ is a generator on $\mathcal{X}$. Second, we show that $\mathcal{P}$ is a Miyadera-Voigt perturbation for $\mathcal{A}_{G,M}$. Let assume that the boundary operator G satisfies the assumptions **(A1)** and **(A2)** (see Section 2). On the other hand, we define

$$C := M_{|_{D(A)}}.$$

We also assume that

(A3) the triple operator (A, B, C) is a regular state-space operators on $\partial X, X, \partial X$ with the identity operator $I_{\partial X}$ as an admissible feedback.

We have the following result.

Proposition 1. *Under the assumptions **(A1)-(A3)**, the operator* $(\mathcal{A}_{G,M}, D(\mathcal{A}_{G,M}))$ *generates a strongly continuous semigroup* $(\mathcal{T}_{G,M}(t))_{t\geq 0}$ *on* $\mathcal{X}$.

Proof. To prove our claim we shall use [20, Theorem 4.1]. To this end, we define the operators $\mathcal{G}, \mathcal{M} : \mathcal{Z} \longrightarrow \mathcal{U}$ by

$$\mathcal{G} = \begin{pmatrix} G & 0 & 0 \\ 0 & \delta_0 & 0 \\ 0 & 0 & \delta_0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M & 0 & 0 \\ I & D & K_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we can rewrite the domain of $\mathcal{A}_{G,M}$ as

$$D(\mathcal{A}_{G,M}) = \left\{ \begin{pmatrix} \varrho_0 \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{Z} : \mathcal{G}\begin{pmatrix} \varrho_0 \\ \varphi \\ \psi \end{pmatrix} = \mathcal{M}\begin{pmatrix} \varrho_0 \\ \varphi \\ \psi \end{pmatrix} \right\}.$$

Clearly, by virtue of the assumptions **(A1)** and **(A2)**, the operator $\mathcal{G}$ is surjective and the operator $\mathcal{A} := \mathcal{A}_m$ with $D(\mathcal{A}) := \ker \mathcal{G}$ generates a diagonal C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$. On the other hand, for $\mu \in \rho(A)$,

$$\mathcal{D}_\mu = \begin{pmatrix} D_\mu & 0 & 0 \\ 0 & e_\mu^X & 0 \\ 0 & 0 & e_\mu^U \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} B & 0 & 0 \\ 0 & \beta^X & 0 \\ 0 & 0 & \beta^U \end{pmatrix}. \tag{14}$$

We know from [19, Sect. 3] that β^X, β^U are admissible control operators for Q^X, Q^U, respectively, and by assumption **(A3)** B is admissible for A. Thus the operator $\mathbb{B}$ is an admissible control operator for $\mathcal{A}$. Define the operator

$$\mathcal{C} := \mathcal{M}_{|_{D(\mathcal{A})}}.$$

We know from [19, Theorem 3] that D, K_1 are admissible observation operators for Q^X, Q^U, respectively, and by assumption **(A3)** C is admissible for A. It follows that the operator $\mathcal{C}$ is an admissible observation operator for $\mathcal{A}$. We now prove that the triple $(\mathcal{A}, \mathbb{B}, \mathcal{C})$ is regular with identity operator $I_{\mathcal{U}}$ as an admissible feedback. According to [28, Theorem 4.2.1], the control maps associated to $\mathbb{B}$ are given by

$$\Phi_t \begin{pmatrix} v \\ u \\ w \end{pmatrix} = \begin{pmatrix} \int_0^t T_{-1}(t-s)Bv(s)ds \\ \int_0^t S_{-1}^X(t-s)\beta^X u(s)ds \\ \int_0^t S_{-1}^U(t-s)\beta^U w(s)ds \end{pmatrix}, \quad t \geq 0, \quad \begin{pmatrix} v \\ u \\ w \end{pmatrix} \in L^p([0, +\infty); \mathcal{U}).$$

Let denote by $D_\Lambda, K_{1,\Lambda}, C_\Lambda$ the Yosida extensions of D, K_1, C with respect to Q^X, Q^U and A, respectively. Then, the Yosida extension of $\mathcal{C}$ with respect $\mathcal{A}$ is given by

$$\mathcal{C}_\Lambda = \begin{pmatrix} C_\Lambda & 0 & 0 \\ I & D_\Lambda & K_{1,\Lambda} \\ 0 & 0 & 0 \end{pmatrix}, \quad D(\mathcal{C}_\Lambda) = D(\mathcal{C}_\Lambda) \times D(D_\Lambda) \times D(K_{1,\Lambda}).$$

Moreover, according to [19, Theorem 3], the triples (Q^X, β^X, D) and (Q^U, β^U, K_1) are regular state-space operators. Now the assumption **(A3)** implies that Rang $\Phi_t \subset D(\mathcal{C}_\Lambda)$ for a.e. $t \geq 0$, cf. [28, Theorem 5.6.5]. We then select

$$\left(\mathbb{F}_\tau \begin{pmatrix} v \\ u \\ w \end{pmatrix}\right)(t) := \mathcal{C}_\Lambda \Phi_t \begin{pmatrix} v \\ u \\ w \end{pmatrix}, \quad t \in [0, \tau], \quad \begin{pmatrix} v \\ u \\ w \end{pmatrix} \in L^p([0, +\infty); \mathcal{U}).$$

Using [19] and the expressions of $\mathcal{C}_\Lambda$ and Φ_t, it is not difficult to see that the triple $(\mathcal{A}, \mathbb{B}, \mathcal{C})$ is well-posed state-space operators. Moreover, as $R(\mu, \mathcal{A}_{-1})\mathbb{B} = \mathcal{D}_\mu$ for $\mu \in \rho(A)$, we have

$$\text{Range}\,(R(\mu, \mathcal{A}_{-1})\mathbb{B}) \subset D(C_\Lambda) \times D(D_\Lambda) \times D(K_{1,\Lambda}) = D(\mathcal{C}_\Lambda),$$

for $\mu \in \rho(\mathcal{A})$, due to **(A3)** and the fact that the triples (Q^X, β^X, D) and (Q^U, β^U, K_1) are regular state-space operators as shown in [19, Theorem 3]. Hence $(\mathcal{A}, \mathbf{B}, \mathcal{C})$ is regular state-space operators on $\mathcal{U}, \mathcal{X}, \mathcal{U}$. We now prove that the identity operator $I_{\mathcal{U}}$ is an admissible feedback for this triple. Clearly, we can write

$$\mathbb{F}_\tau = \begin{pmatrix} \mathbb{F}_\tau^{A,C} & 0 & 0 \\ \Phi_t^A & \mathbb{F}_\tau^{Q^X,D} & \mathbb{F}_\tau^{Q^U,K_1} \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$(\mathbb{F}_\tau^{Q^X,D} u)(t) = D_\Lambda \Phi_t^X u, \quad (\mathbb{F}_\tau^{A,C} v)(t) = C_\Lambda \Phi_t^A v,$$
$$(\mathbb{F}_\tau^{Q^U,K_1} w)(t) = K_{1,\Lambda} \Phi_t^U w,$$

for $t \in [0,\tau]$ and $\begin{pmatrix} v \\ u \\ w \end{pmatrix} \in L^p([0,+\infty);\mathcal{U})$. Thus, $I_{\mathcal{U}} - \mathbb{F}_{\tau_0}$ is invertible in $L^p([0,\tau_0];\mathcal{U})$, due to assumption **(A3)** and the fact that the triple (Q^X, β^X, D) is regular state-space operators with I_X as admissible feedback (see[19]). Hence, by [20, Theorem 4.1] the operator $\mathcal{A}_{G,M}$ generates a strongly continuous semigroup on $\mathcal{X}$. This ends the proof. $\square$

The following result shows the well-posedness of the perturbed Cauchy problem (13).

Theorem 3.1. *The operator $\mathfrak{A} := \mathcal{A}_{G,M} + \mathcal{P}$ generates a strongly continuous semigroup $(\mathfrak{U}(t))_{t\geq 0}$ on $\mathcal{X}$ satisfying $\mathfrak{U}(t)\zeta \in D(\mathcal{C}_\Lambda) \cap D(\mathbb{P}_\Lambda)$ for all $\zeta \in \mathcal{X}$ and almost every $t \geq 0$ and*

$$\mathfrak{U}(t) = \mathcal{T}_{G,M}(t)(t) + \int_0^t \mathcal{T}_{G,M}(t)(t-s)\mathcal{P}\mathfrak{U}(s)ds, \quad \text{on} \quad D(\mathcal{A}_{G,M}),$$
$$\mathcal{T}_{G,M}(t) = \mathcal{T}(t) + \int_0^t \mathcal{T}_{-1}(t-s)\mathbb{B}\mathcal{C}_\Lambda \mathcal{T}_{G,M}(s)ds, \quad \text{on} \quad \mathcal{X}.$$

Proof. We select

$$\mathbb{P} := \mathcal{P}_{|_{D(\mathcal{A})}}.$$

By Proposition 1 and [16, Theorem 3.14], it suffices to check that $\mathbb{P}$ is a Miyadera-Voigt perturbation for $\mathcal{A}_{G,M}$. To this end, we only need to show that $\mathbb{P}$ is an admissible observation operator for $\mathcal{A}_{G,M}$ for a certain $1 < p < \infty$. In fact, by Proposition 1 and [20, Theorem 4.1] the semigroup generated by $\mathcal{A}_{G,M}$ on $\mathcal{X}$ is given by

$$\mathcal{T}_{G,M}(t) = \mathcal{T}(t) + \int_0^t \mathcal{T}_{-1}(t-s)\mathbb{B}\mathcal{C}_\Lambda \mathcal{T}_{G,M}(s)ds, \text{ on } \mathcal{X}, \tag{15}$$

for any $t \geq 0$. Moreover, for any $\tau > 0$ there exists $\gamma := \gamma(\tau) > 0$ such that

$$\int_0^\tau \|\mathcal{C}_\Lambda \mathcal{T}_{G,M}(s)\zeta\|^p ds \leq \gamma^p \|\zeta\|^p \tag{16}$$

for any $\zeta \in \mathcal{X}$. Since D, K_1 are admissible observation operators for Q^X, Q^U, respectively (see [19, Theorem 3]), it follows that $\mathbb{P}$ is admissible observation operator for $\mathcal{A}$. Moreover, The Yosida extension of $\mathbb{P}$ with respect to $\mathcal{A}$ is given by

$$\mathbb{P}_\Lambda = \begin{pmatrix} 0 & L_\Lambda & B_{1,\Lambda} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{P}_\Lambda = X \times D(L_\Lambda) \times D(B_{1,\Lambda}),$$

where $L_\Lambda, B_{1,\Lambda}$ denote the Yosida extension of L, B_1 with respect to Q^X, Q^U, respectively. According to [20, Lemma 3.6], we have

$$(L_\Lambda)_{|W^{1,p}([-r,0];X)} = L \quad \text{and} \quad (B_{1,\Lambda})_{|W^{1,p}([-r,0];U)} = B_1$$

Thus

$$\mathbb{P}_\Lambda = \mathcal{P}, \;\text{ on } X \times W^{1,p}([-r,0];X) \times W^{1,p}([-r,0];U). \tag{17}$$

On the other hand, the same arguments as in the proof of Proposition 1 show that the triple $(\mathcal{A}, \mathbb{B}, \mathbb{P})$ is regular state-space operators. In particular (see e.g. [28, Theorem 2.7]), we have

$$\int_0^\tau \left\| \mathbb{P}_\Lambda \int_0^t \mathcal{T}_{-1}(t-s)\mathbb{B}\xi(s)ds \right\|^p dt \le \delta^p \|\xi\|^p_{L^p[0,\tau];\mathcal{U})},$$

for all $\tau > 0$ and all input $\xi \in L^p[0,\tau];\mathcal{U})$. In particular, by taking $\xi(s) := \mathcal{C}_\Lambda \mathcal{T}_{\mathcal{G},\mathcal{M}}(s)\zeta$ and using (16), we have

$$\int_0^\tau \left\| \mathbb{P}_\Lambda \int_0^t \mathcal{T}_{-1}(t-s)\mathbb{B}\mathcal{C}_\Lambda \mathcal{T}_{G,M}(s)\zeta ds \right\|^p dt \le (\gamma\delta)^p \|\zeta\|^p, \tag{18}$$

for any $\zeta \in D(\mathcal{A}_{G,M})$. Now using (15), (17), (18) and the fact that $\mathcal{P}$ is admissible for $\mathcal{A}$, we obtain that $\mathcal{P}$ is admissible for $\mathcal{A}_{G,M}$. So by [18, Theorem 2.1], the operator $(\mathcal{A}_{G,M} + \mathcal{P}, D(\mathcal{A}_{G,M}))$ generates a strongly continuous semigroup on $\mathcal{X}$. Finally, the rest of the proof follows from [20, Theorem 4.1]. $\square$

In the rest of this section, we will discuss spectral properties of the generator $\mathfrak{A}$ and compute its resolvent operator.

Remark 1. According to [20, Theorem 4.1], the operator

$$\mathcal{Q}_D\varphi = Q_m^X, \quad D(\mathcal{Q}_D) = \left\{ \varphi \in W^{1,p}([-r,0];X) : \quad \varphi(0) = \int_{-r}^0 d\eta(\theta)\varphi(\theta) \right\}.$$

generates a strongly continuous semigroup on $L^p([-r,0];X)$. Moreover, for $\mu \in \rho(\mathcal{Q}_D)$ (or equivalently $1 \in \rho(De_\mu^X)$), we have

$$R(\mu, \mathcal{Q}_D) = \left(I + e_\mu^X (I - De_\mu^X)^{-1} D\right) R(\mu, Q^X).$$

Lemma 3.2. *Let assumptions of Proposition 1 be satisfied. Then, for $\mu \in \rho(A)$, we have*

$$\mu \in \rho(\mathcal{A}_{G,M}) \Leftrightarrow 1 \in \rho(MD_\mu) \cap \rho(De_\mu).$$

In this case,

$$\begin{aligned} R(\mu, \mathcal{A}_{G,M}) &= \left(I + \mathcal{D}_\mu (I - \mathcal{M}\mathcal{D}_\mu)^{-1}\mathcal{M}\right) R(\mu, \mathcal{A}) \\ &= \begin{pmatrix} R(\mu, A_{G,M}) & 0 & 0 \\ e_\mu^X (I - De_\mu^X)^{-1} R(\mu, A_{G,M}) & R(\mu, \mathcal{Q}_D) & e_\mu^X (I - De_\mu^X)^{-1} K_1 R(\mu, Q^U) \\ 0 & 0 & R(\mu, Q^U) \end{pmatrix}. \end{aligned} \tag{19}$$

Proof. Under the assumptions **(A1)-(A3)** and according to [20, Theorem 4.1], the operator

$$A_{G,M} = A_m, \;\; D(A_{G,M}) = \{z \in Z : \, Gz = Mz\}$$

generates a strongly continuous semigroup $(T_{G,M}(t))_{t\ge 0}$ on X. Moreover, for $\mu \in \rho(A)$, we have

$$\mu \in \rho(A_{G,M}) \Leftrightarrow 1 \in \rho(MD_\mu).$$

In this case

$$R(\mu, A_{G,M}) = \big(I + D_\mu (I_{\partial X} - MD_\mu)^{-1} M\big) R(\mu, A).$$

On the other hand, for $\mu \in \rho(A)$,

$$I_{\mathcal{U}} - \mathcal{M}\mathcal{D}_\mu = \begin{pmatrix} I_{\partial X} - MD_\mu & 0 & 0 \\ -D_\mu & I_X - De_\mu^X & -K_1 e_\mu^U \\ 0 & 0 & I_X \end{pmatrix}.$$

Thus $I_{\mathcal{U}} - \mathcal{M}\mathcal{D}_\mu$ is invertible if and only if $1 \in \rho(MD_\mu) \cap \rho(De_\mu)$. Hence $\mu \in \rho(\mathcal{A}_{G,M})$ is equivalent to $1 \in \rho(MD_\mu) \cap \rho(De_\mu)$, due to [20, Theorem 4.1]. Finally, the expression (19) is easily obtained by using the following relation

$$R(\mu, \mathcal{A}_{G,M}) = \big(I + \mathcal{D}_\mu (I - \mathcal{M}\mathcal{D}_\mu)^{-1} \mathcal{M}\big)\, R(\mu, \mathcal{A}).$$

$\square$

Let us now discuss the spectrum of the generator $\mathfrak{A}$.

Proposition 2. *Let assumptions of Proposition 1 be satisfied. For $\mu \in \rho(A)$,*

$$\mu \in \rho(\mathfrak{A}) \Leftrightarrow 1 \in \rho(\Delta(\mu)) \Leftrightarrow 1 \in \rho(MD_\mu) \cap \rho(De_\mu^X),$$

where $\Delta(\mu) := e_\mu^X (I - De_\mu^X)^{-1} R(\mu, A_{G,M}) L$. In addition,

$$R(\mu, \mathfrak{A}) = \begin{pmatrix} R(\mu,A_{G,M})[I+L\Gamma(\mu)R(\mu,A_{G,M})] & R(\mu,A_{G,M})LR(1,\Delta(\mu))R(\mu,\mathcal{Q}_D) & \Lambda(\mu)R(\mu,Q^U) \\ \Gamma(\mu)R(\mu,A_{G,M}) & R(1,\Delta(\mu))R(\mu,\mathcal{Q}_D) & \Omega(\mu)R(\mu,Q^U) \\ 0 & 0 & R(\mu,Q^U) \end{pmatrix}, \tag{20}$$

for $1 \in \rho(MD_\mu) \cap \rho(De_\mu^X)$. Here the operator $\mathcal{Q}_D$ is defined in Remark 1 and

$$\Gamma(\mu) := R(1, \Delta(\mu)) e_\mu^X (I - De_\mu^X)^{-1}, \quad \Omega(\mu) := \Gamma(\mu)\Big(K_1 + R(\mu, A_{G,M}) B_1\Big)$$

$$\Lambda(\mu) := R(\mu, A_{G,M})\Big(L\Omega(\mu) + B_1\Big).$$

Proof. Let $\mu \in \rho(A) \cap \rho(A_{G,M})$ and $(x, f, g)^\top \in \mathcal{X}$, we are seeking for $(z, \varphi, \psi)^\top \in D(\mathcal{A}_{G,M})$ such that

$$(\mu - \mathfrak{A})(z, \varphi, \psi)^\top = (x, f, g)^\top. \tag{21}$$

From the proof of Theorem 3.1 (in particular (17)), we have $\mathfrak{A} = \mathcal{A}_{G,M} + \mathcal{P}$ on $D(\mathcal{A}_{G,M})$. Now according to Lemma 3.2, for $\mu \in \rho(A_{G,M}) \cap \rho(\mathcal{A}_{G,M})$,

$$\begin{aligned} \mu - \mathfrak{A} &= \mu - \mathcal{A}_{G,M} - \mathcal{P} \\ &= (\mu - \mathcal{A}_{G,M})(I - R(\mu, \mathcal{A}_{G,M})\mathcal{P}) \qquad (22) \\ &= (\mu - \mathcal{A}_{\mathcal{G},\mathcal{M}}) \begin{pmatrix} I & -R(\mu,A_{G,M})L & -R(\mu,A_{G,M})B_1 \\ 0 & I-e_\mu^X(I-De_\mu^X)^{-1}R(\mu,A_{G,M})L & -e_\mu^X(I-De_\mu^X)^{-1}R(\mu,A_{G,M})B_1 \\ 0 & 0 & I \end{pmatrix}. \end{aligned}$$

So, the above matrix together with the equation (21) gives

$$\begin{aligned} z - R(\mu, A_{G,M})L\varphi - R(\mu, A_{G,M})B_1\psi &= x \\ (I - e_\mu^X (I - De_\mu^X)^{-1} R(\mu, A_{G,M})L)\varphi - e_\mu^X (I - De_\mu^X)^{-1} R(\mu, A_{G,M}) B_1 \psi &= f \qquad (23) \\ \psi &= g. \end{aligned}$$

Assume that $1 \in \rho(\Delta(\mu))$. Then

$$\varphi = R(1, \Delta(\mu)) f + R(1, \Delta(\mu)) e_\mu^X (I - De_\mu^X)^{-1} R(\mu, A_{G,M}) B_1 g.$$

On the other hand, replacing the value of φ in (23),

$$\begin{aligned} z &= x + R(\mu, A_{G,M})LR(1,\Delta(\mu))f \\ &\quad + R(\mu, A_{G,M})\big(LR(1,\Delta(\mu))e_\mu^X(I - De_\mu^X)^{-1}R(\mu, A_{G,M})B_1 + B_1\big)g. \end{aligned}$$

Thus $\mu \in \rho(\mathfrak{A})$ and (20) holds. □

4. **Frequency domain characterization for approximate controllability of neutral delay systems.** In this section, we provide conditions for approximate controllability of the abstract perturbed boundary control systems of neutral type (11). In fact, by using the feedback theory of regular linear systems and methods of functional analysis, necessary and sufficient conditions for approximate controllability are introduced.

4.1. **Abstract setting.** We first reformulate the neutral system(11) as an abstract perturbed boundary control system with input space. To this end, we select

$$\mathcal{B}\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right) = \left(\begin{smallmatrix} B_0 u \\ 0 \\ 0 \end{smallmatrix}\right), \quad \mathcal{K}\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right) = \left(\begin{smallmatrix} Kv \\ K_0 u \\ 0 \end{smallmatrix}\right), \quad u, v \in U.$$

According to Section 3, the neutral system (11) can be written as

$$\begin{cases} \dot{\zeta}(t) &= [\mathcal{A}_m + \mathcal{P}]\zeta(t) + \mathcal{B}u(t), \quad t \geq 0, \\ \zeta(0) &= (\varrho_0, \varphi, \psi)^\top, \\ \mathcal{G}\zeta(t) &= \mathcal{M}\zeta(t) + \mathcal{K}u(t), \qquad t \geq 0. \end{cases} \tag{24}$$

By combining Theorem 3.1 and [20, Theorem 4.3], it follows that the system (24) (hence the system (11)) has a unique solution. This solution coincides with the solution of the open–loop system

$$\begin{cases} \dot{\zeta}(t) &= \mathfrak{A}_{-1}\zeta(t) + (\mathbb{B}\mathcal{K} + \mathcal{B})\left(\begin{smallmatrix} v(t) \\ u(t) \end{smallmatrix}\right) \quad t > 0, \\ \zeta(0) &= (\varrho_0, \varphi, \psi)^\top. \end{cases} \tag{25}$$

In particular, according to [20, Theorem 4.3] the state trajectory of the system (25), for the initial state $\zeta(0)$, is given by

$$\zeta(t) = \mathfrak{U}(t)\zeta(0) + \int_0^t \mathfrak{U}_{-1}(t-s)(\mathbb{B}\mathcal{K} + \mathcal{B})\left(\begin{smallmatrix} v(s) \\ u(s) \end{smallmatrix}\right) ds, \quad t \geq 0. \tag{26}$$

Remark 2. The function $\zeta(\cdot)$ given by (26) is a strong solution of (25), which is defined for any $\zeta(0) \in \mathcal{X}$ and $u, v \in L^p_{loc}([0,\infty); U)$. To obtain a classical solution, one needs more regularities of control functions.

To state our results on approximate controllability of (24), we first define the concept of approximate controllability for (25).

Definition 4.1. According to the equation (26), we define the operator

$$\Phi^{\mathfrak{A}}(t)u := \int_0^t \mathfrak{U}_{-1}(t-s)(\mathbb{B}\mathcal{K} + \mathcal{B})\left(\begin{smallmatrix} v(s) \\ u(s) \end{smallmatrix}\right) ds,$$

for $t \geq 0$ and $u, v \in L^p([0,t]; U)$. The system (24) is said to be $\mathfrak{X}$-approximately controllable if

$$Cl\left(\bigcup_{t \geq 0} P_{\mathfrak{X}}(\Phi^{\mathfrak{A}}(t))\right) = \mathfrak{X},$$

where $\mathfrak{X}$ can be any of $\mathcal{X}$, X, $X \times L^p([-r,0]; X)$, $L^p([-r,0]; X)$ and $L^p([-r,0]; U)$, and $P_{\mathfrak{X}}$ is the projection operator from $\mathcal{X}$ to $\mathfrak{X}$. In particular, $P_{\mathcal{X}} = I$.

Remark 3. Notice that when the system (24) is X-approximately controllable, it means that the corresponding state z is approximately controllable; when it is $X \times L^p([-r,0];X)$-approximately controllable, it means that the corresponding state z and x_t are approximately controllable; when it is $L^p([-r,0];U)$-approximately controllable, it means that the corresponding state u_t is approximately controllable. Apparently, it is always $L^p([-r,0];U)$-approximately controllable.

In the following we will use the duality of product spaces. Assume that X and U are reflexive Banach spaces (or more generally its satisfies the Radon-Nikodym property) and $1 < p < \infty$. Then the dual space $\mathcal{X}'$ of $\mathcal{X}$ is identified with the product space $X' \times L^q([-r,0],X') \times L^q([-r,0],U')$ with q satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4.2. *Under the framework of Definition 4.1 and with the notation in Proposition 2, the following assertions are equivalent:*

(a) *the system (24) is $X \times L^p([-r,0],X)$-approximately controllable,*
(b) *for $1 \in \rho(MD_\mu) \cap \rho(De_\mu^X)$, $x' \in X'$ and $\varphi \in L^q([-r,0],X')$, the fact that*

$$\Big\langle [I + R(\mu, A_{G,M})L\Gamma(\mu)]R(\mu, A_{G,M})\big(BKv + B_0u\big) + R(\mu, A_{G,M})L\Gamma(\mu)K_0u + \Lambda(\mu)e_\mu^U u, x'\Big\rangle + \Big\langle \Gamma(\mu)R(\mu, A_{G,M})\big(BKv + B_0u\big) + \Gamma(\mu)K_0u + \Omega(\mu)e_\mu^U u, \varphi\Big\rangle = 0,\ \forall u, v \in U,$$

implies that $x' = 0$ and $\varphi = 0$.

In this case, there exists $\tau > 0$ such that the system (24) is approximately controllable.

Proof. Let $\mu \in \rho(A)$ such that $1 \in \rho(MD_\mu) \cap \rho(De_\mu) \cap \rho(\Delta(\mu))$. According to Proposition 2, there is

$$R(\mu, \mathfrak{A}) = \begin{pmatrix} R(\mu,A_{G,M})[I+L\Gamma(\mu)R(\mu,A_{G,M})] & R(\mu,A_{G,M})LR(1,\Delta(\mu))R(\mu,\mathcal{Q}_D) & \Lambda(\mu)R(\mu,Q^U) \\ \Gamma(\mu)R(\mu,A_{G,M}) & R(1,\Delta(\mu))R(\mu,\mathcal{Q}_D) & \Omega(\mu)R(\mu,Q^U) \\ 0 & 0 & R(\mu,Q^U) \end{pmatrix}$$

On the other hand, we have

$$(\mathbb{B}\mathcal{K} + \mathcal{B})\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} BKv \\ \beta^X K_0 u \\ \beta^U u \end{pmatrix} + \begin{pmatrix} B_0 u \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} BKv + B_0 u \\ \beta^X K_0 u \\ \beta^U u \end{pmatrix}.$$

Thus,

$$(\mathbb{B}\mathcal{K} + \mathcal{B})\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} R(\mu,A_{G,M})[I+L\Gamma(\mu)R(\mu,A_{G,M})] & R(\mu,A_{G,M})LR(1,\Delta(\mu))R(\mu,\mathcal{Q}_D) & \Lambda(\mu)R(\mu,Q^U) \\ \Gamma(\mu)R(\mu,A_{G,M}) & R(1,\Delta(\mu))R(\mu,\mathcal{Q}_D) & \Omega(\mu)R(\mu,Q^U) \\ 0 & 0 & R(\mu,Q^U) \end{pmatrix} \begin{pmatrix} BKv+B_0u \\ \beta^X K_0 u \\ \beta^U u \end{pmatrix}$$

$$= \begin{pmatrix} [I+R(\mu,A_{G,M})L\Gamma(\mu)]R(\mu,A_{G,M})\big(BKv+B_0u\big)+R(\mu,A_{G,M})L\Gamma(\mu)K_0u+\Lambda(\mu)e_\mu^U u \\ \Gamma(\mu)R(\mu,A_{G,M})\big(BKv+B_0u\big)+\Gamma(\mu)K_0u+\Omega(\mu)e_\mu^U u \\ e_\mu^U u \end{pmatrix},$$

where we have used the fact that $R(\mu, \mathcal{Q}_D) = (I - e_\mu^X D)^{-1} R(\mu, Q^X)$ and $e_\mu^X (I - De_\mu^X)^{-1} = (I - e_\mu^X D)^{-1} e_\mu^X$. Now using the same strategy as in [12, Proposition 3], it follows that $(a) \Leftrightarrow (b)$.

Therefore, according to Remark 3, it is clear that the system (24) is always $L^p([-r,0],U)$-approximately controllable. Then, the statement (a) yields the existence of a time for which system (24) is approximate controllability. $\square$

Corollary 1. *Let the conditions of Theorem 4.2 be satisfied. The system (24) is X-approximately controllable if and only if, for $1 \in \rho(MD_\mu) \cap \rho(De_\mu^X)$ and $x' \in X'$, the fact that*

$$\Big\langle [I + R(\mu, A_{G,M})L\Gamma(\mu)]R(\mu, A_{G,M})\big(BKv + B_0u\big) + R(\mu, A_{G,M})L\Gamma(\mu)K_0u + \Lambda(\mu)e_\mu^U u, x'\Big\rangle = 0,$$

for all $v, u \in U$ implies that $x' = 0$.

This corollary characterize the condition when z, partial state of system (24), can reach all points in X. In general, this is irrelevant to the approximate controllability of system (11).

Corollary 2. *Under the conditions of Theorem 4.2, the system (24) is $L^p([-r, 0], X)$-approximately controllable if and only if, for $1 \in \rho(MD_\mu) \cap \rho(De_\mu^X)$ and $\varphi \in L^q([-r, 0], X')$, the fact that*

$$\Big\langle \Gamma(\mu)R(\mu, A_{G,M})\big(BKv + B_0u\big) + \Gamma(\mu)K_0u + \Omega(\mu)e_\mu^U u, \varphi\Big\rangle = 0, \ \forall v, u \in U, \tag{27}$$

implies that $\varphi = 0$.

This corollary actually describes the approximate controllability of the system (11).

Remark 4. For $\mu \in \rho(A)$ such that $1 \in \rho(MD_\mu) \cap \rho(De_\mu^X)$, we denote by

$$\Xi(\mu) = (I - De_\mu^X - R(\mu, A_{G,M})Le_\mu^X)^{-1}.$$

In view of Proposition 2, the following holds.

$$\Gamma(\mu) := R(1, \Delta(\mu))e_\mu^X(I - De_\mu^X)^{-1} = e_\mu^X \Xi(\mu).$$

Using the above remark we obtain:

Theorem 4.3. *The abstract boundary control system of neutral type (11) is approximately controllable if and only if, for $1 \in \rho(MD_\mu) \cap \rho(De_\mu^X)$ and $x' \in X'$, the fact that*

$$\Big\langle \Xi(\mu)\Big(R(\mu, A_{G,M})\big(BKv + B_0u + B_1e_\mu^U u\big) + K_0u + K_1e_\mu^U u\Big), x'\Big\rangle = 0$$

for all $v, u \in U$, implies that $x' = 0$.

Proof. The state $x(\cdot)$ of the system (11) is approximately controllable if and only if the system (24) is $L^p([-r, 0], X)$-approximately controllable, i.e., the state x_t is approximately controllable. Under the notation of Remark 4, the condition (27) is equivalent to

$$\Big\langle \Xi(\mu)\Big(R(\mu, A_{G,M})\big(BKv + B_0u + B_1e_\mu^U u\big) + K_0u + K_1e_\mu^U u\Big), (e_\mu^X)^\top \varphi\Big\rangle = 0,$$

for all $u, v \in U$, with $(e_\mu^X)^\top \varphi \in X'$, since $\Omega(\mu) = \Gamma(\mu)[K_1 + R(\mu, A_{G,M})B_1]$; see Proposition 2. Replacing $(e_\mu^X)^\top \varphi$ with x' results in the conclusion. $\square$

4.2. **A rank condition for a special case in control spaces.** In this subsection, we study approximate controllability of neutral delay systems when the control space is finite dimensional. We establish a novel rank condition criteria for approximate controllability of such class of systems.

Before going further and stating the main result of this subsection, we need to introduce some notation. We start with the assumption on the control space $U = \mathbb{C}^n$. Then, K_0 and B_0 are finite-rank operators defined by

$$Ku = \sum_{l=1}^{n} K_l v_l, \qquad K_0 u = \sum_{l=1}^{n} K_{0,l} u_l \qquad \text{and} \qquad B_0 u = \sum_{l=1}^{n} B_{0,l} u_l,$$

where $K_{0,l}, B_{0,l} \in X$. Moreover, denote K_1 and B_1 as

$$K_1 = (K_{1,1}, K_{1,2}, \ldots, K_{1,n}) \qquad \text{and} \qquad B_1 = (B_{1,1}, B_{1,2}, \ldots, B_{1,n}),$$

such as $K_{1,l}, B_{1,l} : W^{1,p}([-r,0],\mathbb{C}) \longrightarrow X$ for $l = 1, \ldots, n$. Therefore, for $u = (u_1, \ldots, u_n)^\top \in \mathbb{C}^n$ and $\mu \in \mathbb{C}$, we have

$$\begin{aligned} K_1 e_\mu^U u &= \sum_{l=1}^{n} K_{1,l} e_\mu u_l = \sum_{l=1}^{n} u_l K_{1,l} e_\mu 1 \\ B_1 e_\mu^U u &= \sum_{l=1}^{n} B_{1,l} e_\mu u_l = \sum_{l=1}^{n} u_l B_{1,l} e_\mu 1 \end{aligned} \tag{28}$$

with $(e_\mu 1)(\theta) = e^{\mu\theta}$ for $\theta \in [-r, 0]$.

Throughout the following, we denote the orthogonal space of a set F in X by

$$F^\perp = \{x' \in X';\ \langle y, x' \rangle = 0,\ \ \forall y \in F\}.$$

Remark 5. In view of Theorem 4.3, the abstract boundary control system of neutral type (11) is approximately controllable if and only if, for $1 \in \rho(MD_\mu) \cap \rho(De_\mu) \cap \rho(\Delta(\mu))$,

$$\overline{\Big(\Xi(\mu) D_\mu (I - MD_\mu)^{-1} K \mathbb{C}^n\Big) + \Big(\Xi(\mu)\big(K_0 + K_1 e_\mu^U + R(\mu, A_{G,M})(B_0 + B_1 e_\mu^U)\big)\mathbb{C}^n\Big)} = X. \tag{29}$$

In fact,

$$\begin{aligned} R(\mu, A_{G,M}) BK &= (I + D_\mu (I - MD_\mu)^{-1} M) R(\mu, A) BK \\ &= (I + D_\mu (I - MD_\mu)^{-1} M) D_\mu K \\ &= D_\mu (I - MD_\mu)^{-1} K. \end{aligned}$$

Moreover, according to [8, Proposition 2.14.], the above fact is equivalent to that

$$\Big(\Xi(\mu) D_\mu (I - MD_\mu)^{-1} K \mathbb{C}^n\Big)^\perp \cap \Big(\Xi(\mu)\big(K_0 + K_1 e_\mu^U + R(\mu, A_{G,M})(B_0 + B_1 e_\mu^U)\big)\mathbb{C}^n\Big)^\perp = \{0\}. \tag{30}$$

Next we provide a useful characterization of approximate controllability for system (11). To this end, we denotes by d_l the dimension of

$$\Upsilon_l(\mu) := \Big(\Xi(\mu) D_\mu (I - MD_\mu)^{-1} K_l\Big)^\perp, \ \ l = 1, \ldots, n,$$

and by $(\varphi_l^1, \varphi_l^2, \ldots, \varphi_l^{d_l})$ the associated basis.

In view of Remark 5, the statement in Theorem 4.3 is equivalent to the following theorem:

Theorem 4.4. *Assume that the control space is finite dimensional. Then the neutral delay system* (11) *is approximately controllable if and only if, for* $1 \in \rho(MD_\mu) \cap \rho(De_\mu^X)$,

$$\operatorname{Rank}\begin{pmatrix} \langle \Xi(\mu)\Pi_1'(\mu)+\Xi(\mu)\Pi_1(\mu)e_\mu 1,\varphi_l^1\rangle & \cdots & \langle \Xi(\mu)\Pi_1'(\mu)+\Xi(\mu)\Pi_1(\mu)e_\mu 1,\varphi_l^{d_l}\rangle \\ \langle \Xi(\mu)\Pi_2'(\mu)+\Xi(\mu)\Pi_2(\mu)e_\mu 1,\varphi_l^1\rangle & \cdots & \langle \Xi(\mu)\Pi_2'(\mu)+\Xi(\mu)\Pi_2(\mu)e_\mu 1,\varphi_l^{d_l}\rangle \\ \vdots & & \vdots \\ \langle \Xi(\mu)\Pi_l'(\mu)+\Xi(\mu)\Pi_l(\mu)e_\mu 1,\varphi_l^1\rangle & \dots & \langle \Xi(\mu)\Pi_l'(\mu)+\Xi(\mu)\Pi_l(\mu)e_\mu 1,\varphi_l^{d_l}\rangle \end{pmatrix} = d_l, \tag{31}$$

for $l = 1, \dots, n$, *where*

$$\Pi_l(\mu) := R(\mu, A_{G,M})B_{1,l} + K_{1,l}$$
$$\Pi_l'(\mu) := R(\mu, A_{G,M})B_{0,l} + K_{0,l}.$$

Proof. First, using the fact that

$$\langle \Xi(\mu)\Pi_l'(\mu)u + \Xi(\mu)\Pi_l(\mu)e_\mu u, x'\rangle = \sum_{l=1}^{n} \bar{u}_l \big\langle \Xi(\mu)\Pi_l'(\mu) + \Xi(\mu)\Pi_l(\mu)e_\mu 1, x'\big\rangle,$$

we promptly obtain the following:

$$\Big(\Xi(\mu)(K_0 + K_1 e_\mu^U + R(\mu, A_{G,M})(B_0 + B_1 e_\mu^U))\mathbb{C}^n\Big)^\perp = \Big(\{\Xi(\mu)\Pi_l'(\mu) + \Xi(\mu)\Pi_l(\mu)e_\mu 1 : l = 1, \cdots, n\}\Big)^\perp. \tag{32}$$

According to Remark 5, to prove the claim of the theorem it suffice to prove that the conditions (30) and (31) are equivalent. To this end, we denote by M_l the matrix appearing in (31) and assume that it is not of rank d_l. Then, there exist $v = (v_1, \cdots, v_{d_l}) \in \mathbb{C}^{d_l} \setminus \{0\}$ such that $\mathsf{M}_l v = 0$, i.e,

$$\begin{pmatrix} \sum_{j=1}^{d_l} \bar{v}_j \big\langle \Xi(\mu)\Pi_1'(\mu)+\Xi(\mu)\Pi_1(\mu)e_\mu 1,\varphi_l^j\big\rangle \\ \sum_{j=1}^{d_l} \bar{v}_j \big\langle \Xi(\mu)\Pi_2'(\mu)+\Xi(\mu)\Pi_2(\mu)e_\mu 1,\varphi_l^j\big\rangle \\ \vdots \\ \sum_{j=1}^{d_l} \bar{v}_j \big\langle \Xi(\mu)\Pi_l'(\mu)+\Xi(\mu)\Pi_l(\mu)e_\mu 1,\varphi_l^j\big\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{33}$$

As $(\varphi_l^1, \varphi_l^2, \dots, \varphi_l^{d_l})$ is a basis of $\Upsilon_l(\mu)$, we obtain

$$\sum_{j=1}^{d_l} \bar{v}_j \varphi_l^j \in \Upsilon_l(\mu).$$

On other hand, because of (32) and (33) one can see that

$$\sum_{j=1}^{d_l} \bar{v}_j \varphi_l^j \in \Big(\Xi(\mu)\Pi_l'(\mu) + \Xi(\mu)\Pi_l(\mu)e_\mu 1\Big)^\perp.$$

This is a contradiction, which show that the condition (30) implies (31). The converse is demonstrated in a similar fashion by using the expression (32) and the basis $(\varphi_l^1, \varphi_l^2, \dots, \varphi_l^{d_l})$. The proof is completed. □

Example 1. Consider the perturbed boundary control time-delay system

$$\begin{aligned} \dot{z}(t) &= A_m z(t) + Pz(t-r) + Nu(t-r) + Bu(t), \qquad t \geq 0, \\ Gz(t) &= Mz(t) + Kv(t), \qquad t \geq 0, \\ z(0) &= z_0, z(\theta) = \varphi, u(\theta) = \psi, \qquad \text{for a.e. } \theta \in [-r, 0]. \end{aligned} \tag{34}$$

Here $A_m : D(A_m) \subset X \longrightarrow X$ is a closed, linear differential operator on a reflexive Banach space X and $G, M : D(A_m) \longrightarrow \partial X$ are unbounded trace operators. $B, N :$

$\mathbb{C}^n \longrightarrow X$ are linear bounded operators and $K : \mathbb{C}^n \longrightarrow \partial X$. This system can be obtained from the system (11) by imposing $D = 0$, $K_0 = 0$, $K_1 = 0$, $L = P\delta_{-r}$, and $B_1 = N\delta_{-r}$, where δ_{-r} is the Dirac operator. Thus, according to Theorem 3.1, the following holds.

Corollary 3. *Assume that the operators A_m, G, M satisfy the assumptions **(A1)**-**(A3)**. Then the perturbed boundary control time-delay system (34) is well-posed.*

In this case, for $\mu \in \rho(A) \cap \rho(A_{G,M} + e^{-r\mu}M)$, we have

$$\Xi(\mu) = (\mu I - A_{G,M} - e^{-r\mu}M)^{-1},$$

and

$$\begin{aligned} Kv &= \sum_{i=1}^{n} K_i v^i,\ K_i \in X \qquad Bu = \sum_{i=1}^{n} B_i u^i,\ B_i \in X \\ Nu_t &= \sum_{i=1}^{n} N_i u_t^i,\ N_i \in X. \end{aligned}$$

Corollary 4. *According to Theorem 4.4, the system (34) is approximately controllable if and only if, for $\mu \in \rho(A) \cap \rho(A_{G,M} + e^{-r\mu}M)$,*

$$\text{Rank} \begin{pmatrix} \langle \Xi(\mu)(N_1+B_1e^{-r\mu}), \psi_i^1 \rangle & \langle \Xi(\mu)(N_1+B_1e^{-r\mu}), \psi_i^2 \rangle & \cdots & \langle \Xi(\mu)(N_1+B_1e^{-r\mu}), \psi_i^{d_i} \rangle \\ \langle \Xi(\mu)(N_2+B_2e^{-r\mu}), \psi_i^1 \rangle & \langle \Xi(\mu)(N_2+B_2e^{-r\mu}), \psi_i^2 \rangle & \cdots & \langle \Xi(\mu)(N_2+B_2e^{-r\mu}), \psi_i^{d_i} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \Xi(\mu)(N_l+B_le^{-r\mu}), \psi_i^1 \rangle & \langle \Xi(\mu)(N_l+B_le^{-r\mu}), \psi_i^2 \rangle & \cdots & \langle \Xi(\mu)(N_l+B_ke^{-r\mu}), \psi_i^{d_i} \rangle \end{pmatrix} = d_i.$$

Where $d_i = \ dim\ \big(\Xi(\mu)D_\mu(I - MD_\mu)^{-1}K_i\big)^\perp$ and $(\psi_i^1, \psi_i^2, \ldots, \psi_i^{d_i})$ is the associated basis, for $i = 1, 2, \ldots, n$.

Note that the equation (34) is a slight generalization of the one proposed in [9, Theorem 4.2.6], where the system operator A_m and delay operators are just constant matrices. Thus, the result from the above corollary is more general.

5. **Application: Transport network system of neutral type and input delays.** Let us consider a finite, connected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ and a flow on it (the latter is described by the differential equation (35) below). The graph G is composed by $n \in \mathbb{N}$ vertices $\alpha_1, \ldots, \alpha_n$, and by $m \in \mathbb{N}$ edges $e_1, \ldots, e_m$ which are assumed to be normalized on the interval $[0, 1]$. We shall denote the vertices at the endpoints of the edge e_j by $e_j(1)$ and $e_j(0)$, respectively, and assume that the particles flows from $e_j(1)$ to $e_j(0)$.

This section characterizes approximate controllability of the following system of transport network system of neutral type and input delays:

$$\begin{cases} \dfrac{\partial}{\partial t}\varrho^j(t,x) = c^j(x)\dfrac{\partial}{\partial x}\varrho^j(t,x) + q^j(x)\varrho^j(t,x) + \displaystyle\sum_{k=1}^{m} L_{jk} z^k(t+\cdot,\cdot),\ x \in (0,1), t \geq 0, \\ \varrho^j(0,x) = g^j(x),\ \ x \in (0,1), \\ \mathsf{i}_{ij}^- c^j(1)\varrho^j(t,1) = \mathsf{w}_{ij}^- \displaystyle\sum_{k=1}^{m} \mathsf{i}_{ik}^+ c^k(0)\varrho^j(t,0) + \sum_{l=1}^{n_0} \mathrm{k}_{il} v^l(t), \qquad t \geq 0, \\ z^j(\theta,x) = \varphi^j(\theta,x),\ u^j(\theta) = \psi^j(\theta), \qquad \theta \in [-r,0],\ x \in (0,1), \\ \varrho^j(t,x) = \left[z^j(t,x) - \displaystyle\sum_{k=1}^{m} D_{jk} z^k(t+\cdot,\cdot) - \sum_{i=1}^{n} \mathsf{k}_{ij} u^j(t+\cdot) - \mathrm{b}_{ij} u^j(t) \right] \end{cases} \tag{35}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$ with $n, m \in \mathbb{N}$. Here, the coefficients i^-_{ij} and i^+_{ij} are the entries of the so-called outgoing and incoming incidence matrices of G (denoted by $\mathcal{I}^+$ and $\mathcal{I}^-$), respectively, defined as

$$\mathsf{i}^-_{ij} := \begin{cases} 1, & \text{if } v_i = e_j(1), \\ 0, & \text{otherwise.} \end{cases} \quad , \mathsf{i}^+_{ij} := \begin{cases} 1, & \text{if } v_i = e_j(0), \\ 0, & \text{otherwise.} \end{cases}$$

The coefficients $0 \leq \mathsf{w}^-_{ij}$ determine the proportion of mass leaving vertex v_i into the edge e_j and define a graph matrix called the weighted outgoing incidence matrix of G, denoted by $\mathcal{I}^-_w$. Additionally, we impose *the Kirchhoff condition*

$$\sum_{j=1}^{m} \mathsf{w}^-_{ij} = 1, \ \forall i = 1, \dots, n. \tag{36}$$

Let $X = L^p([0,1]; \mathbb{C}^m)$, $\partial X = \mathbb{C}^n$ and define the operator A_m as

$$(A_m g)^j(x) := c^j(x) \frac{d}{dx} g^j(x) + q^j(x).g^j(x)$$

with domain

$$g \in D(A_m) := \left\{ g = (g^1, \dots, g^m) \in (W^{1,p}[0,1])^m : g(1) \in \operatorname{Range}(\mathcal{I}^-_w)^\top \right\}.$$

Moreover, we define the boundary operators $G, M : D(A_m) \longrightarrow \partial X$ by

$$Gg := g(1), \qquad Mg := c^{-1}(1)\mathbb{B}c(0)g(0). \tag{37}$$

Clearly, G satisfies the assumptions **(A1)** and **(A2)**. Therefore, according to [13, Theorem 3.6], it follows that:

Lemma 5.1. *Define the operators*

$$A := (A_m)_{|_{\ker G}}, \quad B = (\mu - A_{-1})(G_{|_{\ker(\mu - A_m)}})^{-1} \quad C = M_{|_{D(A)}}, \quad \mu \in \mathbb{C}.$$

*Then the triple (A, B, C) satisfy the assumption **(A3)**.*

So, we have.

Lemma 5.2. *The operator*

$$\mathcal{A} := A_m, \quad D(\mathcal{A}) = \left\{ g \in W^{1,p}([0,1], \mathbb{C}^m) : g(1) = c^{-1}(1)\mathbb{B}c(0)g(0) \right\}. \tag{38}$$

generates a strongly continuous semigroups $(T(t))_{t\geq 0}$ on X, where $\mathbb{B} := (\mathcal{I}^-_w)^\top \mathcal{I}^+$ is the weighted (transposed) adjacency matrix of the line graph (i.e., the graph obtained from G by exchanging the role of the vertices and edges).

Proof. For the proof of this result we refer to [13, Theorem 3.6]. □

Moreover, we obtain.

Corollary 5. *For $\mu \in \rho(\mathcal{A})$, we have*

$$R(\mu, \mathcal{A}) = (I + D_\mu (I_{\mathbb{C}^n} - \mathbb{A}_\mu)^{-1} M) R(\mu, A),$$

where

$$(D_\mu v)(x) = diag\left(e^{\xi^j(x,1) - \mu\tau^j(x,1)} \right) v, \quad Mg := c(1)^{-1}\mathbb{B}c(0)g(0)$$

$$(R(\mu, A)f)^j(x) = \int_x^1 e^{\xi^j(x,y) - \mu\tau^j(x,y)} c^j(y) f^j(y) dy$$

for $g \in D(A_m)$, $v \in \mathbb{C}^n$, $f \in \mathbb{C}^m$, $x \in [0,1]$ *and*

$$(\mathbb{A}_\mu)_{ip} = \begin{cases} \mathsf{w}^-_{pj} e^{\xi^j(0,1)-\mu\tau^j(0,1)}, & \text{if } v_i = e_j(0) \text{ and } v_p = e_j(1),\\ 0, & \text{otherwise.}\end{cases}$$

Proof. A proof of this lemma can be found in [13, Corollary 3.8]. □

On the other hand, in order to apply the results of the previous sections, let us assume that

$$D_{jk}(g^k) = \int_{-r}^0 d\eta_{jk}(\theta)g^k(\theta),\ L_{jk}(g^k) = \int_{-r}^0 d\gamma_{jk}(\theta)g^k(\theta),$$
$$\mathsf{k}_{ij}(f^j) = \int_{-r}^0 d\vartheta_{ij}(\theta)f^j(\theta),$$

for $g \in W^{1,p}([-r,0],X)$ and $f \in W^{1,p}([-r,0],\mathbb{C}^n)$, where $\eta, \gamma : [-r,0] \longrightarrow \mathcal{L}(X)$ and $\vartheta : [-r,0] \longrightarrow \mathcal{L}(\mathbb{C}^n, X)$ are functions of bounded variations continuous at zero with $\eta(0) = \gamma(0) = \vartheta(0) = 0$. Thus the system (35) is rewritten in the form (11) with $D = (D_{jk})_{m\times m}$, $L = (L_{jk})_{m\times m}$, $K_0 = (\mathsf{b}_{ij})_{m\times n}$, $K_1 = (\mathsf{k}_{ij})_{m\times n}$ and $B_1 = B_0 \equiv 0$.

Therefore, according to Theorem 3.1, the transport network system of neutral type (35) is well-posed.

Corollary 6. *The operator* $(\mathfrak{A}, D(\mathfrak{A}))$ *defined by*

$$\mathfrak{A} = \begin{pmatrix} \mathcal{A} & L & 0\\ 0 & Q^X_m & 0\\ 0 & 0 & Q^U_m\end{pmatrix},$$
$$D(\mathfrak{A}) = \left\{ \begin{pmatrix} f\\ \varphi\\ \psi\end{pmatrix} \in D(A_m) \times W^{1,p}([-r,0],X) \times D(Q^U) : \begin{pmatrix} f(1)=c^{-1}(1)\mathbb{B}c(0)f(0)\\ \varphi(0)=f+D\varphi+K_1\psi\end{pmatrix}\right\}.$$

generates a strongly continuous semigroups $(\mathfrak{U}(t))_{t\geq 0}$ *on the following product space*

$$X \times L^p([-r,0];X) \times L^p([-r,0];\mathbb{C}^{n_0}).$$

In this case, for $1 \in \rho(\mathbb{A}_\mu) \cap \rho(De_\mu)$, we have

$$\Xi(\mu) = (I - De_\mu - R(\mu,A)Le_\mu - D_\mu(I_{\mathbb{C}^n} - \mathbb{A}_\mu)^{-1}MR(\mu,A)Le_\mu)^{-1}.$$

The fact that the transport network system of neutral type (35) is approximately controllable follows from the following result:

Corollary 7. *Let* $1 \in \rho(\mathbb{A}_\mu) \cap \rho(De_\mu)$, *then the system* (35) *is approximately controllable if and only if the following matrix is of rank* d_i

$$\sum_{j=1}^m \begin{pmatrix} \left\langle \Xi(\mu)\left(\mathrm{b}_{1j}+\mathsf{k}_{1j}e_\mu 1\right),\varphi_i^1\right\rangle & \cdots & \left\langle \Xi(\mu)\left(\mathrm{b}_{1j}+\mathsf{k}_{1j}e_\mu 1\right),\varphi_i^{d_i}\right\rangle\\ \left\langle \Xi(\mu)\left(\mathrm{b}_{2j}+\mathsf{k}_{2j}e_\mu 1\right),\varphi_i^1\right\rangle & \cdots & \left\langle \Xi(\mu)\left(\mathrm{b}_{2j}+\mathsf{k}_{2j}e_\mu 1\right),\varphi_i^{d_i}\right\rangle\\ \vdots & & \vdots\\ \left\langle \Xi(\mu)\left(\mathrm{b}_{nj}+\mathsf{k}_{nj}e_\mu 1\right),\varphi_i^1\right\rangle & \ldots & \left\langle \Xi(\mu)\left(\mathrm{b}_{nj}+\mathsf{k}_{nj}e_\mu 1\right),\varphi_i^{d_i}\right\rangle, \end{pmatrix}, \quad i = 1,\ldots,n.$$

Here d_i *denote the dimension of*

$$\left(\sum_{l=1}^{n_0} \Xi(\mu)D_\mu(I_{\mathbb{C}^n} - \mathbb{A}_\mu)^{-1}\mathrm{k}_{il}\right)^\perp,$$

and $(\varphi_i^1, \varphi_i^2, \ldots, \varphi_i^{d_i})$ *denote the associated basis, for* $i = 1,\ldots,n$.

Proof. The statements follows from Theorem 4.4 with

$$\Pi_i(\mu) = \sum_{j=1}^{m} \Xi(\mu)\mathsf{k}_{ij}, \qquad \Pi_i'\mu) = \sum_{j=1}^{m} \Xi(\mu)\mathrm{b}_{ij},$$

$$\Upsilon_i(\mu) = \left(\sum_{l=1}^{n_0} \Xi(\mu) D_\mu (I_{\mathbb{C}^n} - \mathbb{A}_\mu)^{-1} \mathrm{k}_{il} \right)^{\perp}.$$

□

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