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RUMOR SPREADING DYNAMICS WITH AN ONLINE RESERVOIR AND ITS ASYMPTOTIC STABILITY

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Abstract. The spread of rumors is a phenomenon that has heavily impacted society for a long time. Recently, there has been a huge change in rumor dynamics, through the advent of the Internet. Today, online communication has become as common as using a phone. At present, getting information from the Internet does not require much effort or time. In this paper, the impact of the Internet on rumor spreading will be considered through a simple SIR type ordinary differential equation. Rumors spreading through the Internet are similar to the spread of infectious diseases through water and air. From these observations, we study a model with the additional principle that spreaders lose interest and stop spreading, based on the SIWR model. We derive the basic reproduction number for this model and demonstrate the existence and global stability of rumor-free and endemic equilibriums.

1. **Introduction.** There are different patterns of rumor spreading depending on the presence or absence of online media [7], for example, the emergence of influential spreaders [2]. Before the development of online media, rumors were transmitted from person to person. With the development of online media such as social network service (SNS), personal broadcasting, blog, and group chatting, rumors can now spread in a variety of ways. In the past, offline media was the starting point and an important means of information delivery. Recently, it has become a social problem that offline media reproduces and delivers rumors from online media. This is a sign that information in online is rapidly being accepted by various social classes. In this paper, we study how the combination of classical interpersonal rumor spreading and online media influences rumor outbreak.

In order to consider the influence of online media, we denote by I the density of people who do not know the rumor but are susceptible, S is the density of people who spread the rumor, and W is the amount of rumor in online generated by the group S, and R is the density of people who know the rumor but are not interested in it or do not believe it. The process of rumor spreading is based on the following assumptions. (1) The group I has an influx rate of b and a natural decay rate of δ_i. (2) Suppose the group I meets S, then I is converted to S with an incidence rate of λ_s, and when the group I encounters the rumor in online media, I is also

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converted to S with a rate of λ_w. (3) We assume that S occurs only from I and if S encounters someone who knows the rumor, then they lose interest or do not believe in the rumor. In this case, let σ_s and σ_r denote the contact rates at which S meets S and R, respectively. (4) S decreases with a natural decay rate of δ_s and becomes R with the transmission rate μ. (5) We assume that an online medium has its own natural decay rate of δ_w and is generated in proportion to the size of S in offline. From the above assumptions, we can derive the following mean field equation.

$$\begin{aligned}\frac{dI}{dt} &= b - \lambda_s IS - \lambda_w IW - \delta_i I,\\ \frac{dS}{dt} &= \lambda_s IS + \lambda_w IW - \sigma_s SS - \sigma_r SR - \mu S - \delta_s S,\\ \frac{dW}{dt} &= \xi S - \delta_w W,\\ \frac{dR}{dt} &= \sigma_s SS + \sigma_r SR + \mu S - \delta_r R.\end{aligned} \tag{1}$$

Remark 1. (1) This rumor spreading process is a relatively short time process. Thus, we do not consider vertical transmission. See [8].

(2) If we take $b = \delta_i = \delta_s = \delta_r$ and $(I + S + R)(0) = 1$, then the total population density $I + S + R$ is conserved. Thus our model is a generalization of the model in [15].

Since the Daley-Kendall model [3], various studies on rumor spreading have been conducted. We briefly state the history of rumor spreading models associated with online media. See [12] for a general rumor spread, and [14] for threshold phenomena for general epidemic models. Since information transmission via online media developed in the late 1990s, intensive researches on rumors and online media began mainly in the early 2000s. In [1], the authors focused on the spread of computer-based rumors and analyzed the spread of rumors via computer-based communication in terms of information transmission. The authors in [7] noted the difference between online-based media and offline media. The study in [17] considered the spread of rumors through online networks by using the SIR model. The fast speed and unprofessional communication of online media is considered in [13]. See also [9]. In [11], a statistical rumor diffusion model is considered for online networks and it contained positive and negative bipolar reinforcement factors. [4, 6, 18] studied a rumor propagation model similar to the European fox rabies SIR model for the situation of changing online community number. In [10], the authors studied the rumor propagation phenomena for a model with two layers: online and offline. See also [19] for the SEIR type online rumor model.

This paper is organized as follows. In Section 2, we present the nonnegativity property of the solution to (1) and the stability of the rumor-free equilibrium. The basic reproduction number $\mathcal{R}_0$ is calculated by using a next-generation matrix. In Section 3, we provide the existence and uniqueness of endemic equilibrium and its global stability. In Section 4, we perform several numerical simulations to verify our analytical results.

2. Elementary properties of the SIWR system and stability for rumor-free equilibrium.

In this section, we consider the conservation of nonnegativity of the densities I, S, W, R and the stability for a rumor-free equilibrium E_0.

2.1. Nonnegativity of I, S, W, R.

Lemma 2.1. *Let (I, S, W, R) be the unique global solution to (1). Assume that σ_s and σ_r are nonnegative constants and the rest of the coefficients are positive. If the initial data $(I(0), S(0), W(0), R(0))$ has only nonnegative components and satisfies*

$$S(0)^2 + W(0)^2 > 0,$$

then the solution is nonnegative for all $t > 0$ and $S(t), W(t) > 0$ for $t > 0$.

Proof. We take any positive $T > 0$. By the continuity of the solution, there is $C(T) > 0$ such that

$$|I(t)|, |S(t)|, |W(t)|, |R(t)| < C(T).$$

By the first equation in (1) and the boundedness, if $I(0) \geq 0$, then for $0 < t < T$,

$$I(t) = I(0)e^{-\int_0^t (\lambda_s S(s)+\lambda_w W(s)+\delta_i)ds} + b\int_0^t e^{-\int_u^t (\lambda_s S(s)+\lambda_w W(s)+\delta_i)ds} du > 0.$$

We first prove that S is nonnegative for $0 < t < T$. Assume not, i.e., there is $t_s^- \in (0, T)$ such that

$$S(t_s^-) < 0.$$

Let $t_0 \in (0, t_s^-)$ be an entering time for S into the negative region such that $S(t) \geq 0$ on $[0, t_0]$ and $S(t) < 0$ on $(t_0, t_0 + \epsilon)$, where $\epsilon > 0$ is a small constant. Note that by the second equation in (1),

$$\begin{aligned} S(t) =& S(0)e^{\int_0^t (\lambda_s I(s)-\sigma_s S(s)-\sigma_r R(s)-\mu-\delta_s)ds} \\ &+ \int_0^t \lambda_w I(u)W(u)e^{\int_u^t (\lambda_s I(s)-\sigma_s S(s)-\sigma_r R(s)-\mu-\delta_s)ds} du. \end{aligned} \tag{2}$$

This and the positivity of I imply that if $W(t) \geq 0$ on $(0, s]$, $S(t)$ is nonnegative on $(0, s]$. Therefore, there is an entering time $t_0' \in (0, t_s^-)$ for W into the negative region such that $W(t) \geq 0$ on $[0, t_0']$ and $W(t) < 0$ on $(t_0', t_0' + \epsilon')$, where $\epsilon' > 0$ is a small constant. If $t_0' < t_0$, then $S(t) \geq 0$ and $W(t) < 0$ on $(t_0', t_0) \cap (t_0', t_0' + \epsilon')$. Similarly, by the third equation in (1),

$$W(t) = W(0)e^{-\delta_w t} + \int_0^t \xi S(u)e^{-\delta_w (t-u)} du. \tag{3}$$

Thus, $W(t)$ is nonnegative on $(0, s]$ if $S \geq 0$ on $(0, s]$. This is a contradiction and we conclude that $t_0 \geq t_0'$. Similarly, we can obtain that $t_0 \leq t_0'$. Thus, $t_0 = t_0'$.

However, on $t \in [t_0, \infty)$,

$$(I(t), S(t), W(t), R(t)) = \left(I(t_0)e^{-\delta_i (t-t_0)} + \frac{b(1 - e^{-\delta_i (t-t_0)})}{\delta_i}, 0, 0, R(t_0)e^{-\delta_r (t-t_0)} \right)$$

is a solution to (1). By uniqueness of the solution, there is no $t_s^- > 0$ such that $S(t_s^-) < 0$. Therefore, we prove that S is nonnegative.

Similarly, we can also easily obtain that there is no $t_w^- > 0$ such that $W(t_w^-) < 0$. Thus, for all $t > 0$, $I, S, W \geq 0$. By the fourth equation in (1) and nonnegativity of I, S, W, we have that R is also nonnegative on $(0, \infty)$. Therefore, we prove that the solution is nonnegative for all $t > 0$.

Moreover, if $S(0) > 0$, then for all $t > 0$, $S(t) > 0$ by (2). From (3), $W(t) > 0$ on $(0, \infty)$. Similarly, if $W(0) > 0$, then for all $t > 0$, $W(t) > 0$ by (3). By the virtue of the positivity of I and (2), $S(t) > 0$ on $(0, \infty)$. Thus, we conclude that if $S(0) > 0$ or $W(0) > 0$, then $S(t) > 0$ and $W(t) > 0$, $t \in (0, \infty)$. □

2.2. **The basic reproduction number using a next-generation matrix.** In this part, we calculate the basic reproduction number using a next-generation matrix. To consider the asymptotic behavior of the dynamics in (1), we determine the equilibrium point such that

$$\dot{I} = \dot{S} = \dot{W} = \dot{R} = 0. \tag{4}$$

If we assume that there is no rumor ($S = 0$) in system (1) with (4), then the equilibrium point is unique and

$$E_0 = (I_{rf}, S_{rf}, W_{rf}, R_{rf}) = \left(\frac{b}{\delta_i}, 0, 0, 0\right).$$

The basic reproduction number $\mathcal{R}_0$ is generally a measure of the transmission of disease. This is usually expressed in terms of the rate of secondary transmission (or infection) and no transmission. $\mathcal{R}_0$ for more complex systems was calculated in [5, 16] using the next-generation matrix methodology. Here, we follow the method of [16].

For the infected compartments, the next generation matrices at the rumor-free state $E_0 = (b/\delta_i, 0, 0, 0)$ are given by

$$F = \frac{1}{\delta_i}\begin{pmatrix} b\lambda_s & b\lambda_w \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \mu + \delta_s & 0 \\ -\xi & \delta_w \end{pmatrix},$$

and hence

$$V^{-1} = \frac{1}{(\mu + \delta_s)\delta_w}\begin{pmatrix} \delta_w & 0 \\ \xi & \mu + \delta_s \end{pmatrix}.$$

Here, F and V are related to the rate of new infections and transfer individuals, respectively. This yields

$$FV^{-1} = \begin{pmatrix} \frac{b\lambda_s}{(\mu+\delta_s)\delta_i} + \frac{b\lambda_w\xi}{(\mu+\delta_s)\delta_i\delta_w} & \frac{b\lambda_w}{\delta_i\delta_w} \\ 0 & 0 \end{pmatrix}.$$

Therefore, we obtain the following formula for the basic reproduction number:

$$\mathcal{R}_0 = \rho(FV^{-1}) = \frac{b}{\delta_i}\left(\frac{\lambda_s}{\mu + \delta_s} + \frac{\lambda_w\xi}{(\mu + \delta_s)\delta_w}\right).$$

Here, $\rho(A)$ is the spectral radius of a matrix A.

2.3. **Stability for rumor-free equilibrium.** For the linear stability, we consider the Jacobian matrix as follows.

$$J = \begin{pmatrix} -\lambda_s S - \lambda_w W - \delta_i & -\lambda_s I & -\lambda_w I & 0 \\ \lambda_s S + \lambda_w W & \lambda_s I - 2\sigma_s S - \sigma_r R - \mu - \delta_s & \lambda_w I & -\sigma_r S \\ 0 & \xi & -\delta_w & 0 \\ 0 & 2\sigma_s S + \sigma_r R + \mu & 0 & \sigma_r S - \delta_r \end{pmatrix}.$$

Since the rumor-free equilibrium is

$$E_0 = \left(\frac{b}{\delta_i}, 0, 0, 0\right),$$

the Jacobian matrix at the rumor-free equilibrium is given by

$$J_{E_0} = \begin{pmatrix} -\delta_i & -\lambda_s \frac{b}{\delta_i} & -\lambda_w \frac{b}{\delta_i} & 0 \\ 0 & \lambda_s \frac{b}{\delta_i} - \mu - \delta_s & \lambda_w \frac{b}{\delta_i} & 0 \\ 0 & \xi & -\delta_w & 0 \\ 0 & \mu & 0 & -\delta_r \end{pmatrix}.$$

Therefore, the corresponding characteristic equation is

$$p(x) = (x+\delta_r)(x+\delta_i) \times \left(x^2 - \Big(\frac{b\lambda_s}{\delta_i} - \mu - \delta_s - \delta_w\Big)x - \frac{b\delta_w\lambda_s}{\delta_i} + \mu\delta_w + \delta_s\delta_w - \frac{b\lambda_w\xi}{\delta_i}\right).$$

Assume that $\mathcal{R}_0 < 1$. Then by the definition of $\mathcal{R}_0$,

$$\frac{b}{\delta_i}\frac{\lambda_s}{\mu+\delta_s} < \frac{b}{\delta_i}\left(\frac{\lambda_s}{\mu+\delta_s} + \frac{\lambda_w\xi}{(\mu+\delta_s)\delta_w}\right) < 1.$$

Thus,

$$c_1 := -\Big(\frac{b\lambda_s}{\delta_i} - \mu - \delta_s - \delta_w\Big) > \delta_w > 0.$$

Note that

$$\begin{aligned} c_2 &:= -\frac{b\delta_w\lambda_s}{\delta_i} + (\mu+\delta_s)\delta_w - \frac{b\lambda_w\xi}{\delta_i} \\ &= \delta_w(\mu+\delta_s)\left(-\frac{b\lambda_s}{\delta_i(\mu+\delta_s)} - \frac{b\lambda_w\xi}{\delta_i\delta_w(\mu+\delta_s)} + 1\right) \\ &= \delta_w(\mu+\delta_s)(1-\mathcal{R}_0) \\ &> 0. \end{aligned}$$

Clearly, $-\delta_r$ and $-\delta_i$ are the eigenvalues of J_{E_0} and are negative. The rest of the eigenvalues are zeros of the following polynomial.

$$\begin{aligned} p_0(x) &= x^2 - \Big(\frac{b\lambda_s}{\delta_i} - \mu - \delta_s - \delta_w\Big)x - \frac{b\delta_w\lambda_s}{\delta_i} + (\mu+\delta_s)\delta_w - \frac{b\lambda_w\xi}{\delta_i} \\ &= x^2 + c_1 x + c_2. \end{aligned}$$

Since $c_1 > 0$ and $c_2 > 0$, zeros of $p_0(x)$ have only negative real part. Therefore, $E_0 = (b/\delta_i, 0, 0, 0)$ is locally asymptotically stable if $\mathcal{R}_0 < 1$.

Clearly, if $\mathcal{R}_0 = 1$, then one of the eigenvalues has a zero real part. Thus, E_0 is locally stable but not asymptotically stable for the linearized system. Furthermore, if $\mathcal{R}_0 > 1$, then $c_2 < 0$. Therefore, E_0 is linearly unstable. We summarize the above argument to

Theorem 2.2. *The rumor-free equilibrium E_0 of the system in* (1) *is linearly stable if $\mathcal{R}_0 \le 1$ and linearly unstable if if $\mathcal{R}_0 > 1$. Moreover, E_0 is linearly asymptotically stable if $\mathcal{R}_0 < 1$.*

The rumor-free equilibrium E_0 is also a global attractive basin. We can use the standard methodology to obtain the global asymptotical behavior of the solution to (1).

Theorem 2.3. *If $\mathcal{R}_0 < 1$, the rumor-free equilibrium E_0 is globally asymptotically stable on*

$$\{(I,S,R,W) : S > 0 \text{ or } W > 0\} \cap \{(I,S,W,R) : I,S,W,R \ge 0\}.$$

Proof. Let

$$V_0(I,S,W) = \left[I - I_{rf} - I_{rf}\log\frac{I}{I_{rf}}\right] + S + I_{rf}\frac{\lambda_s}{\delta_w}W,$$

where $I_{rf} = b/\delta_i$. Since

$$I - I_{rf} - I_{rf}\log\frac{I}{I_{rf}} > 0, \quad \text{for} \quad I \neq I_{rf},$$

and

$$I - I_{rf} - I_{rf}\log\frac{I}{I_{rf}} = 0, \quad \text{for} \quad I = I_{rf},$$

we note that V_0 is nonnegative and radially unbounded. Then by elementary calculation,

$$\begin{aligned}
\frac{dV_0}{dt} &= (b - \lambda_s IS - \lambda_w IW - \delta_i I) - \frac{b}{\delta_i}\left(\frac{b}{I} - \lambda_s S - \lambda_s W - \delta_i\right) \\
&\quad + \lambda_s IS + \lambda_w IW - \sigma_s SS - \sigma_r SR - (\mu+\delta_s)S + \frac{b\lambda_s}{\delta_i\delta_w}(\xi S - \delta_w W) \\
&= b - \delta_i I - \sigma_s SS - \sigma_r SR - (\mu+\delta_s)S + \frac{b\lambda_s}{\delta_i\delta_w}(\xi S - \delta_w W) \\
&\quad - \frac{b}{\delta_i}\left(\frac{b}{I} - \lambda_s S - \lambda_s W - \delta_i\right) \\
&= -b\left(\frac{b}{\delta_i I} + \frac{\delta_i I}{b} - 2\right) - \sigma_s SS - \sigma_r SR - (\mu+\delta_s)S \\
&\quad + \frac{b\lambda_s}{\delta_i\delta_w}(\xi S - \delta_w W) + \frac{b}{\delta_i}(\lambda_s S + \lambda_s W) \\
&= -b\left(\frac{b}{\delta_i I} + \frac{\delta_i I}{b} - 2\right) - \sigma_s SS - \sigma_r SR \\
&\quad - (\mu+\delta_s)S\left(1 - \frac{b\lambda_s}{(\mu+\delta_s)\delta_i} - \frac{b\lambda_s\xi}{(\mu+\delta_s)\delta_i\delta_w}\right).
\end{aligned}$$

Therefore, we have

$$\frac{dV_0}{dt} \quad = \quad -b\left(\frac{b}{\delta_i I} + \frac{\delta_i I}{b} - 2\right) - \sigma_s SS - \sigma_r SR - (\mu+\delta_s)S(1-\mathcal{R}_0). \qquad (5)$$

Note that by Lemma 2.1 in Section 2, $I, S, R \geq 0$ and $S(t) > 0$ for all $t > 0$. This nonnegativity and (5) imply that if $(I(t), S(t)) \neq (I_{rf}, 0)$ and $\mathcal{R}_0 < 1$, then

$$\frac{dV_0}{dt} < 0.$$

Therefore, $(I(t), S(t))$ converges to $(I_{rf}, 0)$ as t goes to ∞ by Lyapunov stability theorem. By the third equation in (1), $W(t)$ converges to zero as t goes to ∞. Similarly, by the fourth equation in (1), $R(t)$ converges to zero as t goes to ∞.

Therefore, the rumor-free equilibrium E_0 is globally asymptotically stable. □

3. **Stability analysis for endemic states.** In this section, we present the existence and stability of endemic steady states for the rumor spreading model with an online reservoir. Endemic state refers to a nonzero steady state of S, i.e., the rumor is sustained. Since there is an influx b for ignorant I, we can show that the unique endemic state exists as follows.

3.1. **Existence and uniqueness of the endemic equilibrium.** To obtain the endemic equilibrium

$$E_* = (I_*, S_*, W_*, R_*),$$

we consider the following steady state equation:

$$\frac{dI}{dt} = \frac{dS}{dt} = \frac{dW}{dt} = \frac{dR}{dt} = 0.$$

Then the endemic equilibrium $E_* = (I_*, S_*, W_*, R_*)$ satisfies

$$\begin{aligned} 0 &= b - \lambda_s I_* S_* - \lambda_w I_* W_* - \delta_i I_*, \\ 0 &= \lambda_s I_* S_* + \lambda_w I_* W_* - \sigma_s S_* S_* - \sigma_r S_* R_* - \mu S_* - \delta_s S_*, \\ 0 &= \xi S_* - \delta_w W_*, \\ 0 &= \sigma_s S_* S_* + \sigma_r S_* R_* + \mu S_* - \delta_r R_*. \end{aligned}$$

We set

$$U_* = \frac{\delta_w}{\xi} W_*, \ \tilde{I}_* = \delta_i I_*, \ \tilde{S}_* = \delta_s S_*, \ \tilde{R}_* = \delta_r R_*,$$

and

$$\tilde{\mu} = \frac{\mu}{\delta_s}, \ \tilde{\lambda}_s = \frac{\lambda_s \delta_w + \lambda_w \xi}{\delta_i \delta_s \delta_w}, \ \tilde{\sigma}_s = \frac{\sigma_s}{\delta_s^2}, \ \tilde{\sigma}_r = \frac{\sigma_r}{\delta_s \delta_r}.$$

Then $U_* = S_*$ and

$$\begin{aligned} 0 &= b - \tilde{\lambda}_s \tilde{I}_* \tilde{S}_* - \tilde{I}_*, \\ 0 &= \tilde{\lambda}_s \tilde{I}_* \tilde{S}_* - \tilde{\sigma}_s \tilde{S}_* \tilde{S}_* - \tilde{\sigma}_r \tilde{S}_* \tilde{R}_* - \tilde{\mu} \tilde{S}_* - \tilde{S}_*, \\ 0 &= \tilde{\sigma}_s \tilde{S}_* \tilde{S}_* + \tilde{\sigma}_r \tilde{S}_* \tilde{R}_* + \tilde{\mu} \tilde{S}_* - \tilde{R}_*. \end{aligned} \tag{6}$$

Note that the basic reproduction number satisfies

$$\mathcal{R}_0 = \frac{b\tilde{\lambda}_s}{\tilde{\mu} + 1}.$$

To find endemic equilibrium E_*, we set

$$\tilde{S}_* > 0.$$

The sum of all equations in (6) implies that

$$\tilde{R}_* = (b - \tilde{I}_* - \tilde{S}_*). \tag{7}$$

From the second equation in (6),

$$\left(\tilde{\lambda}_s \tilde{S}_* + \tilde{\sigma}_r \tilde{S}_*\right) \tilde{I}_* = \tilde{\sigma}_s \tilde{S}_* \tilde{S}_* - \tilde{\sigma}_r \tilde{S}_* \tilde{S}_* + (\tilde{\mu} + 1)\tilde{S}_* + \tilde{\sigma}_r b \tilde{S}_*. \tag{8}$$

By (7)-(8),

$$\tilde{I}_* = \frac{\tilde{\sigma}_s - \tilde{\sigma}_r}{\tilde{\lambda}_s + \tilde{\sigma}_r} \tilde{S}_* + \frac{\tilde{\mu} + 1 + \tilde{\sigma}_r b}{\tilde{\lambda}_s + \tilde{\sigma}_r} := \beta \tilde{S}_* + \gamma.$$

Substituting $\tilde{I}_*$ into the first equation in (6) gives

$$b - \tilde{\lambda}_s (\beta \tilde{S}_* + \gamma) \tilde{S}_* - (\beta \tilde{S}_* + \gamma) = 0.$$

Therefore, we have

$$\beta \tilde{\lambda}_s \tilde{S}_*^2 + (\tilde{\lambda}_s \gamma + \beta) \tilde{S}_* + \gamma - b = 0. \tag{9}$$

If we obtain positive S_*, then by the first and third equations, we can derive I_* and R_* such that

$$\tilde{I}_* = \frac{b}{\tilde{\lambda}_s \tilde{S}_* + 1}$$

and

$$\tilde{R}_* = \frac{\tilde{\sigma}_s \tilde{S}_* \tilde{S}_* + \tilde{\mu}\tilde{S}_*}{1 - \tilde{\sigma}_r \tilde{S}_*}.$$

Thus, if all components are nonnegative,

$$S_* < \frac{1}{\tilde{\sigma}_r}. \tag{10}$$

Theorem 3.1. *If $\mathcal{R}_0 > 1$, then a unique positive endemic state E_* exists, but if $\mathcal{R}_0 \leq 1$, then there is no positive endemic state.*

Proof. Assume that $\mathcal{R}_0 > 1$. Then there are three cases as follows.

- Case 1 ($\tilde{\sigma}_s - \tilde{\sigma}_r = 0$): Since $\beta = 0$, we have

$$\tilde{S}_* = \frac{b-\gamma}{\tilde{\lambda}_s \gamma} = \frac{b}{b\tilde{\sigma}_r + \tilde{\mu} + 1} \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0} < \frac{1}{\tilde{\sigma}_r}.$$

  Condition (10) holds, which implies that a positive endemic state E_* exists and is unique.

- Case 2 ($\tilde{\sigma}_s - \tilde{\sigma}_r > 0$): The equation (9) can be written as

$$\tilde{S}_*^2 + \left(\frac{b\tilde{\sigma}_r + \tilde{\mu} + 1}{\tilde{\sigma}_s - \tilde{\sigma}_r} + \frac{1}{\tilde{\lambda}_s} \right) \tilde{S}_* + \frac{b}{\tilde{\sigma}_s - \tilde{\sigma}_r} \frac{1 - \mathcal{R}_0}{\mathcal{R}_0} = 0.$$

  Since $\mathcal{R}_0 - 1 > 0$ and $\tilde{\sigma}_s - \tilde{\sigma}_r > 0$,

$$\frac{b}{\tilde{\sigma}_s - \tilde{\sigma}_r} \frac{1 - \mathcal{R}_0}{\mathcal{R}_0} < 0.$$

  Therefore, there is a unique positive real root of the equation. To check the condition in (10), let

$$f(x) = x^2 + \left(\frac{b\tilde{\sigma}_r + \tilde{\mu} + 1}{\tilde{\sigma}_s - \tilde{\sigma}_r} + \frac{1}{\tilde{\lambda}_s} \right) x + \frac{b}{\tilde{\sigma}_s - \tilde{\sigma}_r} \frac{1 - \mathcal{R}_0}{\mathcal{R}_0}. \tag{11}$$

  By elementary calculation,

$$f(1/\tilde{\sigma}_r) = \frac{(\tilde{\lambda}_s + \tilde{\sigma}_r)(\tilde{\mu}\tilde{\sigma}_r + \tilde{\sigma}_s)}{\tilde{\lambda}_s \tilde{\sigma}_r^2 (\tilde{\sigma}_s - \tilde{\sigma}_r)}. \tag{12}$$

  Thus, $f(1/\tilde{\sigma}_r) > 0$ and it follows that (10) holds, proving that there is a unique positive endemic equilibrium E_*.

- Case 3 ($\tilde{\sigma}_s - \tilde{\sigma}_r < 0$): Clearly, $\tilde{S}_*$ is a positive root of $f(x)$ in (11). The discriminant is

$$D = \left(\frac{b\tilde{\sigma}_r + \tilde{\mu} + 1}{\tilde{\sigma}_s - \tilde{\sigma}_r} + \frac{1}{\tilde{\lambda}_s} \right)^2 - \frac{4b}{\tilde{\sigma}_s - \tilde{\sigma}_r} \frac{1 - \mathcal{R}_0}{\mathcal{R}_0}.$$

  Since we assume that $\tilde{\sigma}_s - \tilde{\sigma}_r < 0$, we need further analytical calculations to obtain $D > 0$.

Let

$$g(x) = (x - \tilde{\sigma}_r + \tilde{\lambda}_s(1 + \tilde{\mu} + b\tilde{\sigma}_r))^2 - 4\tilde{\lambda}_s(-1 - \tilde{\mu} + b\tilde{\lambda}_s)(\tilde{\sigma}_r - x).$$

Then the discriminant is represented as for $0 \le \tilde{\sigma}_s < \tilde{\sigma}_r$,

$$D = \frac{g(\tilde{\sigma}_s)}{(\tilde{\lambda}_s + \tilde{\sigma}_r)^2}.$$

Note that g is a quadratic function, therefore, g has global minimum value at

$$x = -b\tilde{\sigma}_r\tilde{\lambda}_s + \tilde{\sigma}_r - 2b\tilde{\lambda}_s^2 + \tilde{\mu}\tilde{\lambda}_s + \tilde{\lambda}_s.$$

Since we assume that $\mathcal{R}_0 > 1$,

$$\begin{aligned} -b\tilde{\sigma}_r\tilde{\lambda}_s + \tilde{\sigma}_r - 2b\tilde{\lambda}_s^2 + \tilde{\mu}\tilde{\lambda}_s + \tilde{\lambda}_s &< -(\tilde{\mu}+1)\tilde{\sigma}_r + \tilde{\sigma}_r - 2b\tilde{\lambda}_s^2 + \tilde{\mu}\tilde{\lambda}_s + \tilde{\lambda}_s \\ &= -\tilde{\mu}\tilde{\sigma}_r + \tilde{\lambda}_s(-2b\tilde{\lambda}_s + \tilde{\mu} + 1) \\ &< 0. \end{aligned}$$

Therefore, the minimum value of g on $[0, \tilde{\sigma}_r)$ occurs at $x = 0$, thus, for $\tilde{\sigma}_s \in [0, \tilde{\sigma}_r)$,

$$g(\tilde{\sigma}_s) \ge g(0) = \left(\tilde{\lambda}_s\left(b\tilde{\sigma}_r + \tilde{\mu} + 1\right) - \tilde{\sigma}_r\right)^2 + 4\tilde{\sigma}_r\tilde{\lambda}_s\left(-b\tilde{\lambda}_s + \tilde{\mu} + 1\right) =: h(\tilde{\sigma}_r).$$

We consider $g(0)$ as a function of $\tilde{\sigma}_r$, say $h(\tilde{\sigma}_r)$. Then $h(\tilde{\sigma}_r)$ is also a quadratic function of $\tilde{\sigma}_r$. Thus, $h(\tilde{\sigma}_r)$ has a global minimum as follows:

$$h(\tilde{\sigma}_r) \ge \frac{4b\tilde{\mu}\tilde{\lambda}_s^3\left(b\tilde{\lambda}_s - \tilde{\mu} - 1\right)}{\left(b\tilde{\lambda}_s - 1\right)^2} > 0.$$

Therefore, $D > 0$ and f has two distinct real roots. Note that

$$\frac{b}{\tilde{\sigma}_s - \tilde{\sigma}_r}\frac{1 - \mathcal{R}_0}{\mathcal{R}_0} > 0.$$

Thus, f has two distinct positive roots or two distinct negative roots.

By (12) and $\tilde{\sigma}_s - \tilde{\sigma}_r < 0$,

$$f(1/\tilde{\sigma}_r) = \frac{(\tilde{\lambda}_s + \tilde{\sigma}_r)(\tilde{\mu}\tilde{\sigma}_r + \tilde{\sigma}_s)}{\tilde{\lambda}_s\tilde{\sigma}_r^2(\tilde{\sigma}_s - \tilde{\sigma}_r)} < 0.$$

Therefore, f has two distinct positive roots and one is less than $1/\tilde{\sigma}_r$ and one is greater than $1/\tilde{\sigma}_r$. For small root, $\tilde{R}_*$ is positive and for large root, $\tilde{R}_*$ is negative.

For any case, we conclude that if $\mathcal{R}_0 > 1$, then a unique positive endemic state E_* exists.

For the remaining part, we assume that $\mathcal{R}_0 \le 1$. Similar to the previous proof, we have three cases.

- Case $1'$ ($\tilde{\sigma}_s - \tilde{\sigma}_r = 0$): From $\beta = 0$ and (9), it follows that

$$\tilde{S}_* = \frac{b - \gamma}{\tilde{\lambda}_s\gamma} = \frac{b}{b\tilde{\sigma}_r + \tilde{\mu} + 1}\frac{\mathcal{R}_0 - 1}{\mathcal{R}_0} \le 0.$$

Thus there is no positive endemic state E_*.

- Case $2'$ ($\tilde{\sigma}_s - \tilde{\sigma}_r > 0$): Note that $\tilde{S}_*$ is a positive root of $f(x)$ in (11). For $0 \le \tilde{\sigma}_r < \tilde{\sigma}_s$, the discriminant is
$$D = \frac{g(\tilde{\sigma}_s)}{(\tilde{\lambda}_s + \tilde{\sigma}_r)^2}$$
and g has a global minimum value of $4b\lambda_s^2(\lambda_s + \sigma_s)(-b\lambda_s + \mu + 1)$.
  Since we assume that $\mathcal{R}_0 < 1$, the minimum value is positive, which yields that $D > 0$. Moreover,
$$\frac{b}{\tilde{\sigma}_s - \tilde{\sigma}_r} \frac{1 - \mathcal{R}_0}{\mathcal{R}_0} \ge 0,$$
this implies that if $\mathcal{R}_0 < 1$, f has two distinct positive roots or two distinct negative roots, and if $\mathcal{R}_0 = 1$, 0 is a root of f.
  Note that f has a global minimum value at
$$x = -\frac{b\sigma_r\lambda_s + \mu\lambda_s + \lambda_s + (\sigma_s - \sigma_r)}{2\lambda_s(\sigma_s - \sigma_r)} < 0.$$
Thus, f has no positive root and there is no positive endemic equilibrium E_*.

- Case $3'$ ($\tilde{\sigma}_s - \tilde{\sigma}_r < 0$): Note that
$$\tilde{S}_*^2 + \left(\frac{b\tilde{\sigma}_r + \tilde{\mu} + 1}{\tilde{\sigma}_s - \tilde{\sigma}_r} + \frac{1}{\tilde{\lambda}_s}\right)\tilde{S}_* + \frac{b}{\tilde{\sigma}_s - \tilde{\sigma}_r}\frac{1 - \mathcal{R}_0}{\mathcal{R}_0} = 0.$$
Since $\mathcal{R}_0 - 1 \le 0$ and $\tilde{\sigma}_s - \tilde{\sigma}_r < 0$, we have
$$\frac{b}{\tilde{\sigma}_s - \tilde{\sigma}_r}\frac{1 - \mathcal{R}_0}{\mathcal{R}_0} \ge 0.$$
Therefore, there is at most one positive real root of the equation. However,
$$f(1/\tilde{\sigma}_r) = \frac{(\tilde{\lambda}_s + \tilde{\sigma}_r)(\tilde{\mu}\tilde{\sigma}_r + \tilde{\sigma}_s)}{\tilde{\lambda}_s\tilde{\sigma}_r^2(\tilde{\sigma}_s - \tilde{\sigma}_r)} < 0.$$
Thus, (10) does not hold. This implies that there is no positive endemic equilibrium E_*.

Therefore, we conclude that if $\mathcal{R}_0 \le 1$, then there is no positive endemic state. □

3.2. **Stability for endemic equilibrium.** In this part, we consider asymptotic stability for the endemic state E_*. Since the endemic state E_* exists only for $\mathcal{R}_0 > 1$, we consider the case of $\mathcal{R}_0 > 1$.

Theorem 3.2. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium E_* is globally asymptotically stable on*
$$\{(I,S,R,W) : S > 0 \text{ or } W > 0\} \cap \{(I,S,W,R) : I,S,W,R \ge 0\}.$$

Proof. Let
$$\begin{aligned} V_*(I,S,W,R) = &\left[I - I_* - I_* \log\frac{I}{I_*}\right] + \left[S - S_* - S_* \log\frac{S}{S_*}\right] \\ &+\frac{\lambda_w}{\delta_w}I_*\left[W - W_* - W_*\log\frac{W}{W_*}\right] + \frac{R_*\sigma_r}{\mu + R_*\sigma_r}\left[R - R_* - R_*\log\frac{R}{R_*}\right] \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$
In the same manner as Theorem 2.3, note that V_* is nonnegative and radially unbounded.

We claim that if $(I(t),S(t),W(t),R(t)) \neq (I_*,S_*,W_*,R_*)$ and $S(t)>0$, then

$$\frac{dV_*}{dt} < 0.$$

Since (I_*,S_*,W_*,R_*) is the steady state of (1),

$$(b-\lambda_s I_* S_* - \lambda_w I_* W_* - \delta_i I_*) = 0.$$

Therefore,

$$\begin{aligned}\frac{dJ_1}{dt} &= (b-\lambda_s IS - \lambda_w IW - \delta_i I)\left(1-\frac{I_*}{I}\right)\\ &\quad + (b-\lambda_s I_* S_* - \lambda_w I_* W_* - \delta_i I_*)\left(1-\frac{I}{I_*}\right)\\ &= b\left(2-\frac{I_*}{I}-\frac{I}{I_*}\right) - \lambda_s\left(I_*-I\right)\left(S_*-S\right) - \lambda_w\left(I_*-I\right)\left(W_*-W\right).\end{aligned}$$

Similarly,

$$(\lambda_s I_* S_* + \lambda_w I_* W_* - \sigma_s S_* S_* - \sigma_r S_* R_* - \mu S_*) = 0.$$

This implies that

$$\begin{aligned}\frac{dJ_2}{dt} &= (\lambda_s IS + \lambda_w IW - \sigma_s SS - \sigma_r SR - (\mu+\delta_s)S)\left(1-\frac{S_*}{S}\right)\\ &\quad + (\lambda_s I_* S_* + \lambda_w I_* W_* - \sigma_s S_* S_* - \sigma_r S_* R_* - (\mu+\delta_s)S_*)\left(1-\frac{S}{S_*}\right)\\ &= \lambda_s(I_*-I)\left(S_*-S\right) - \sigma_s(S_*-S)\left(S_*-S\right) - \sigma_r(R_*-R)\left(S_*-S\right)\\ &\quad + \lambda_w IW\left(1-\frac{S_*}{S}\right) + \lambda_w I_* W_*\left(1-\frac{S}{S_*}\right).\end{aligned}$$

Note that

$$\frac{dJ_3}{dt} = \frac{\lambda_w}{\delta_w} I_*\left(\xi S - \delta_w W\right)\left(1-\frac{W_*}{W}\right).$$

We add the derivatives of J_1,J_2, and J_3 to obtain

$$\begin{aligned}&\frac{d(J_1+J_2+J_3)}{dt} - b\left(2-\frac{I_*}{I}-\frac{I}{I_*}\right) + \sigma_s(S_*-S)\left(S_*-S\right) + \sigma_r(R_*-R)\left(S_*-S\right)\\ &\quad = -\lambda_w\left(I_*-I\right)\left(W_*-W\right) + \lambda_w IW\left(1-\frac{S_*}{S}\right) + \lambda_w I_* W_*\left(1-\frac{S}{S_*}\right)\\ &\qquad + \frac{\lambda_w}{\delta_w} I_*\left(\xi S - \delta_w W\right)\left(1-\frac{W_*}{W}\right)\\ &\quad = \lambda_w I_* W + \lambda_w I W_* - \lambda_w I\frac{S_*}{S}W - \lambda_w I_*\frac{S}{S_*}W_* + \frac{\lambda_w}{\delta_w} I_*\left(\xi S - \delta_w W\right)\left(1-\frac{W_*}{W}\right).\end{aligned}$$

Using $\delta_w W_* = \xi S_*$,

$$\begin{aligned}&\frac{d(J_1+J_2+J_3)}{dt} - b\left(2-\frac{I_*}{I}-\frac{I}{I_*}\right)\\ &\quad + \sigma_s(S_*-S)\left(S_*-S\right) + \sigma_r(R_*-R)\left(S_*-S\right)\\ &\qquad = \lambda_w\left(IW_* - I\frac{S_*}{S}W - \frac{\xi}{\delta_w} I_*\frac{W_*}{W}S + I_* W_*\right).\end{aligned} \tag{13}$$

Using $\delta_w W_* = \xi S_*$ again,

$$
\begin{aligned}
&IW_* - I\frac{S_*}{S}W - \frac{\xi}{\delta_w}I_*\frac{W_*}{W}S + I_*W_* \\
&\quad = \left(IW_* + I_*\frac{I_*}{I}W_* - 2I_*W_*\right) \\
&\quad\quad - \left(\frac{\xi}{\delta_w}I_*S\frac{W_*}{W} + I\frac{S_*}{S}W + I_*\frac{I_*}{I}W_* - 3I_*W_*\right) \\
&\quad = I_*W_*\left(\frac{I}{I_*} + \frac{I_*}{I} - 2\right) - \frac{\xi}{\delta_w}I_*W_*\left(\frac{S}{W} + \frac{\delta_w^2}{\xi^2}\frac{IW}{I_*S} + \frac{\delta_w}{\xi}\frac{I_*}{I} - 3\frac{\delta_w}{\xi}\right).
\end{aligned}
\tag{14}
$$

Combining (13) and (14) with $\delta_w W_* = \xi S_*$,

$$
\begin{aligned}
\frac{d(J_1+J_2+J_3)}{dt} &= \left(\frac{\lambda_w\xi}{\delta_w}I_*S_* - b\right)\left(\frac{I}{I_*} + \frac{I_*}{I} - 2\right) \\
&\quad - \frac{\lambda_w\xi^2}{\delta_w^2}I_*S_*\left(\frac{S}{W} + \frac{\delta_w^2}{\xi^2}\frac{IW}{I_*S} + \frac{\delta_w}{\xi}\frac{I_*}{I} - 3\frac{\delta_w}{\xi}\right) \\
&\quad - \sigma_s(SS + S_*S_* - 2SS_*) - \sigma_r(SR + S_*R_* - S_*R - R_*S).
\end{aligned}
$$

Since

$$
\sigma_s S_* S_* + \sigma_r S_* R_* + \mu S_* - \delta_r R_* = 0,
$$

we have

$$
\begin{aligned}
\frac{\mu + R_*\sigma_r}{R_*\sigma_r}\frac{dJ_4}{dt} &= \frac{dR}{dt}\left(1 - \frac{R_*}{R}\right) \\
&= (\sigma_s SS + \sigma_r SR + \mu S - \delta_r R)\left(1 - \frac{R_*}{R}\right) \\
&\quad + (\sigma_s S_*S_* + \sigma_r S_*R_* + \mu S_* - \delta_r R_*)\left(1 - \frac{R}{R_*}\right) \\
&= \sigma_s\left(SS + S_*S_* - \frac{R_*}{R}SS - \frac{R}{R_*}S_*S_*\right) \\
&\quad + \sigma_r\left(SR + S_*R_* - \frac{R_*}{R}RS - \frac{R}{R_*}R_*S_*\right) \\
&\quad + \mu\left(S + S_* - \frac{R_*}{R}S - \frac{R}{R_*}S_*\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{dV_*}{dt} &= \left(\frac{\lambda_w\xi}{\delta_w}I_*S_* - b\right)\left(\frac{I}{I_*} + \frac{I_*}{I} - 2\right) \\
&\quad - \frac{\lambda_w\xi^2}{\delta_w^2}I_*S_*\left(\frac{S}{W} + \frac{\delta_w^2}{\xi^2}\frac{IW}{I_*S} + \frac{\delta_w}{\xi}\frac{I_*}{I} - 3\frac{\delta_w}{\xi}\right) \\
&\quad - \sigma_s(SS + S_*S_* - 2SS_*) - \sigma_r(SR + S_*R_* - S_*R - R_*S) \\
&\quad + \frac{R_*\sigma_r}{\mu + R_*\sigma_r}\sigma_s\left(SS + S_*S_* - \frac{R_*}{R}SS - \frac{R}{R_*}S_*S_*\right) \\
&\quad + \frac{R_*\sigma_r}{\mu + R_*\sigma_r}\sigma_r\left(SR + S_*R_* - \frac{R_*}{R}RS - \frac{R}{R_*}R_*S_*\right)
\end{aligned}
$$

$$+\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\mu\left(S+S_*-\frac{R_*}{R}S-\frac{R}{R_*}S_*\right).$$

Note that

$$\begin{aligned}
&-\sigma_s(SS+S_*S_*-2SS_*)+\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\sigma_s\left(SS+S_*S_*-\frac{R_*}{R}SS-\frac{R}{R_*}S_*S_*\right)\\
&=-\frac{\mu}{\mu+R_*\sigma_r}\sigma_s(SS+S_*S_*-2SS_*)\\
&\quad-\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\sigma_sS_*S\left(\frac{R_*}{R}\frac{S}{S_*}+\frac{R}{R_*}\frac{S_*}{S}-2\right)
\end{aligned}$$

and

$$\begin{aligned}
&-\sigma_r(SR+S_*R_*-S_*R-R_*S)\\
&\quad+\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\sigma_r\left(SR+S_*R_*-\frac{R_*}{R}RS-\frac{R}{R_*}R_*S_*\right)\\
&\quad+\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\mu\left(S+S_*-\frac{R_*}{R}S-\frac{R}{R_*}S_*\right)\\
&\qquad=-\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\mu S\left(\frac{R}{R_*}+\frac{R_*}{R}-2\right).
\end{aligned}$$

In conclusion, we have

$$\begin{aligned}
\frac{dV_*}{dt}&=\left(\frac{\lambda_w\xi}{\delta_w}I_*S_*-b\right)\left(\frac{I}{I_*}+\frac{I_*}{I}-2\right)\\
&\quad-\frac{\lambda_w\xi^2}{\delta_w^2}I_*S_*\left(\frac{S}{W}+\frac{\delta_w^2}{\xi^2}\frac{IW}{I_*S}+\frac{\delta_w}{\xi}\frac{I_*}{I}-3\frac{\delta_w}{\xi}\right)\\
&\quad-\frac{\mu}{\mu+R_*\sigma_r}\sigma_s(SS+S_*S_*-2SS_*)\\
&\quad-\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\sigma_sS_*S\left(\frac{R_*}{R}\frac{S}{S_*}+\frac{R}{R_*}\frac{S_*}{S}-2\right)\\
&\quad-\frac{R_*\sigma_r}{\mu+R_*\sigma_r}\mu S\left(\frac{R}{R_*}+\frac{R_*}{R}-2\right).
\end{aligned}$$

From the first equation in (1), it follows that

$$b=\lambda_sI_*S_*+\lambda_wI_*W_*+\delta_iI_*=\lambda_sI_*S_*+\frac{\lambda_w\xi}{\delta_w}I_*S_*+\delta_iI_*.$$

This implies that

$$\frac{\lambda_w\xi}{\delta_w}I_*S_*-b=-\lambda_sI_*S_*-\delta_iI_*<0.$$

By the relationship between arithmetic and geometric means, if

$$(I(t),S(t),W(t),R(t))\neq(I_*,S_*,W_*,R_*)\text{ and }S(t)>0,$$

then

$$\frac{dV_*}{dt}<0.$$

If we assume that $S(0)>0$ or $W(0)>0$, then by the result in Section 2, $S(t)>0$ for $t>0$. Therefore, by Lyapunov stability theorem, we conclude that if $S(0)>0$

or $W(0) > 0$, then $(I(t), S(t), W(t), R(t))$ converges to E_* as t goes to ∞ and this implies that the endemic equilibrium E_* is globally asymptotically stable on

$$\{(I, S, R, W) : S > 0 \text{ or } W > 0\} \cap \{(I, S, W, R) : I, S, W, R \geq 0\}.$$

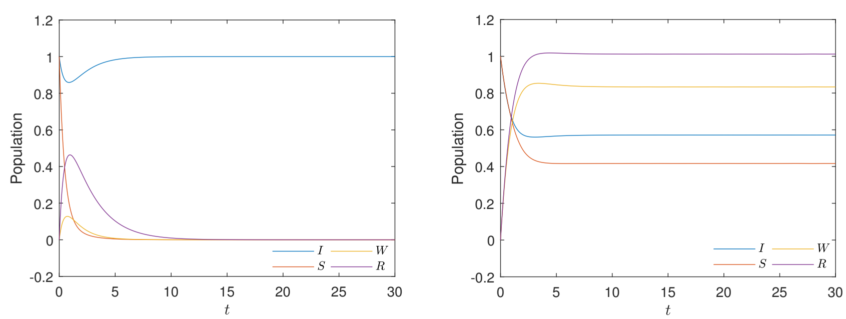
□

4. **Numerical simulation.** In this section, we carry out some numerical simulations to verify the theoretical results. We use the fourth-order Runge-Kutta method with time step size $\Delta t = 0.01$. We assume that the initial condition is $(I(0), S(0), W(0), R(0)) = (1, 1, 0, 0)$.

As shown before, the basic reproduction number is

$$\mathcal{R}_0 = \rho(FV^{-1}) = \frac{b}{\delta_i}\left(\frac{\lambda_s}{\mu + \delta_s} + \frac{\lambda_w \xi}{(\mu + \delta_s)\delta_w}\right).$$

We assume that the influx b is 1. Since the parameters σ_s, σ_r, and δ_r are not involved in the basic reproduction number $\mathcal{R}_0$, we fix these parameters as $\sigma_s = \sigma_r = \delta_r = 0.5$. If we take $\lambda_s = \lambda_w = \xi = 0.5$ and $\delta_i = \delta_s = \delta_w = \mu = 1$, then $\mathcal{R}_0 = 0.375$. In this case, as we proved in Theorem 2.3, the rumor-free equilibrium is $b/\delta_i = 1$ and is globally asymptotically stable. As seen in Figure 1(A), the population density of ignorants $I(t)$ converges to 1 and the other densities $S(t)$, $W(t)$, and $R(t)$ converge to zero. On the other hand, if we set $\delta_i = \delta_s = \delta_w = \mu = 0.5$ and $\lambda_s = \lambda_w = \xi = 1$, then $\mathcal{R}_0 = 6$. As we proved in Theorem 3.2, the endemic equilibrium exists and is asymptotically stable. See Figure 1(B).

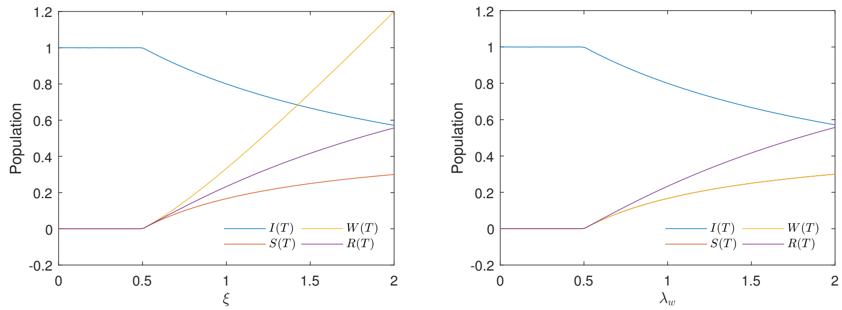


(A) $\mathcal{R}_0 = 0.375$ (B) $\mathcal{R}_0 = 6$

FIGURE 1. Numerical simulations when $b = 1$, $\sigma_s = 0.5$, and $\sigma_r = 0.5$

Now, we investigate the influence of an online reservoir. We change ξ from 0 to 2 and all other parameters are fixed as $\lambda_s = \lambda_w = \sigma_s = \sigma_r = \delta_i = \delta_w = \delta_r = \mu = 0.5$ and $\delta_i = 1$. The final densities $I(T)$, $S(T)$, $W(T)$, and $R(T)$ when the final time $T = 10^3$ are given in Figure 2(A). We observe the phase transition when $\mathcal{R}_0 = 1$. That is, $\xi = 0.5$. We change λ_w from 0 to 2 and all the other parameters are the same as the previous case. If we set $\xi = 0.5$, then we observe the effect of the trust rate λ_w in Figure 2(B). In this case, the densities $S(t)$ and $R(t)$ are the same since $\xi = \delta_w$. As ξ and λ_w increase, the densities of the spreaders and stiflers also increase. Therefore, as online reservoirs become more active, rumors are more expansively spread.

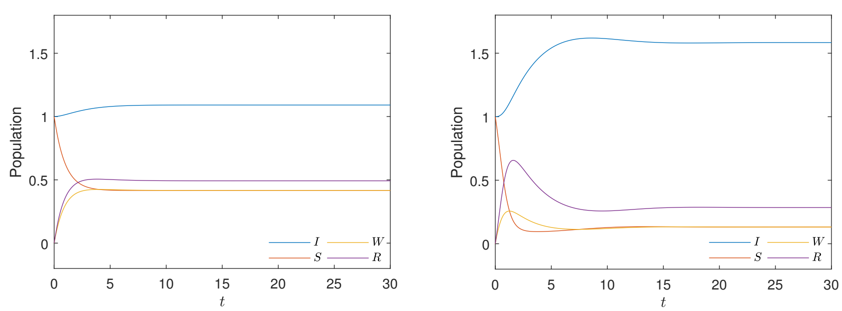
Even though the contact rates σ_s and σ_r are not involved in $\mathcal{R}_0$, they affect the behavior of the solutions. See Figure 3. We choose σ_s and σ_r between 0.1 and 2. All other parameters are fixed to 0.5. That is, $\lambda_s = \lambda_w = \delta_i = \delta_s = \mu = \xi = \delta_w =$



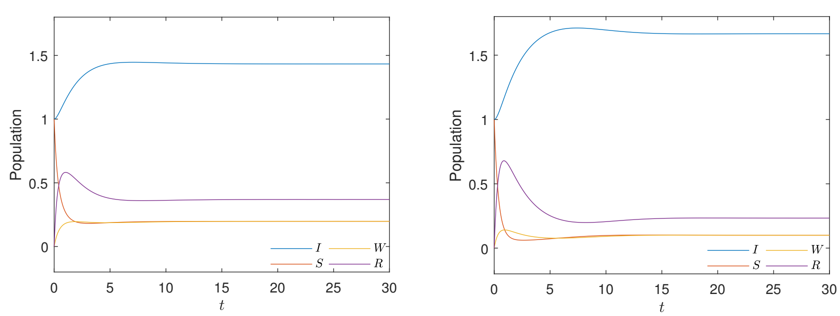
(A) Final densities when $T = 10^3$ (B) Final densities when $T = 10^3$

FIGURE 2. Final densities $I(T)$, $S(T)$, $W(T)$, and $R(T)$ with $T = 10^3$

$\delta_r = 0.5$ and hence $\mathcal{R}_0 = 2$. Since $\mathcal{R}_0 > 1$, the final time $T = 30$ is large enough. The bigger contact rates lead to a sharp reduction in the density of spreaders in a short time. Therefore, the final densities of the spreaders and stiflers are low if the contact rates are high. Furthermore, we confirm that the aggressive activity of the stiflers has an immense influence on the spread of a rumor. In Figure 4, we change (σ_s, σ_r) on $[0, 1] \times [0, 1]$ and display the final densities $I(T)$, $S(T)$, $W(T)$, and $R(T)$ with $T = 30$.

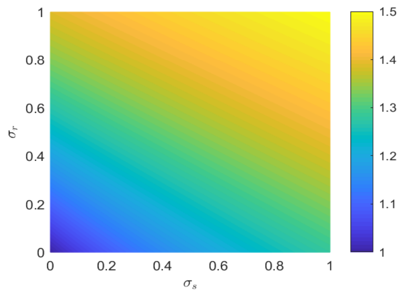


(A) $\sigma_s = \sigma_r = 0.1$ (B) $\sigma_s = 0.1$ and $\sigma_r = 2$

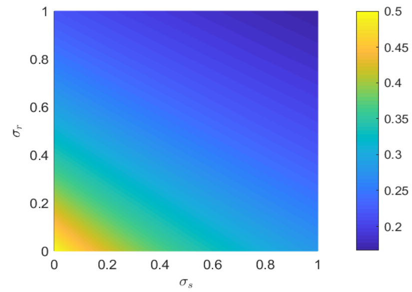


(C) $\sigma_s = 2$ and $\sigma_r = 0.1$ (D) $\sigma_s = \sigma_r = 2$

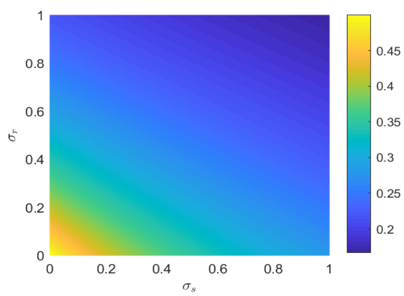
FIGURE 3. Evolution of the solution with different parameters σ_s and σ_r



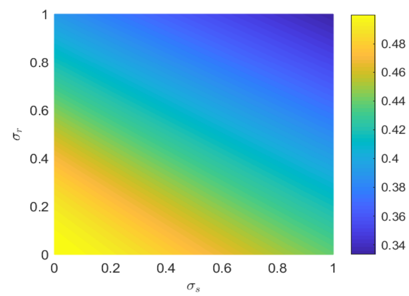
(A) Final density of $I(T)$



(B) Final density of $S(T)$



(C) Final density of $W(T)$



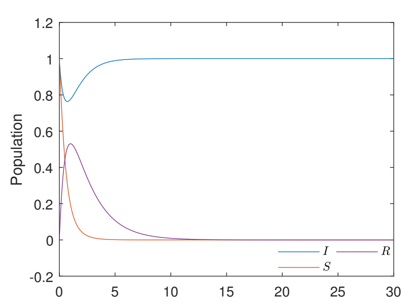
(B) Final density of $R(T)$

FIGURE 4. Final densities $I(T)$, $S(T)$, $W(T)$, and $R(T)$ with $T = 30$

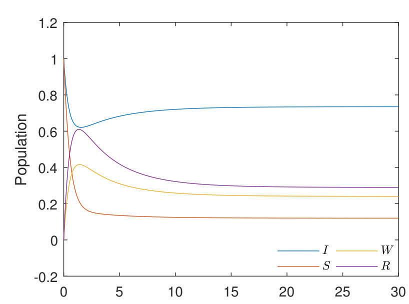
We next compare the SIR and SIWR models. Without online an reservoir, the basic reproduction number of the SIR model is given by

$$\mathcal{R}_0^{SIR} = \frac{b\lambda_s}{(\mu + \delta_s)\delta_i}.$$

If we fix the parameters such as $\sigma_s = \sigma_r = \delta_r = 0.5$ and $\lambda_s = \delta_i = \delta_s = \mu = 1$, then $\mathcal{R}_0^{SIR} = 0.5$. However, we introduce an online reservoir with $\delta_w = 0.5$ and $\lambda_w = \xi = 1$, then $\mathcal{R}_0 = 1.5$. Therefore, we conclude that an online reservoir promotes the spread of a rumor. The comparison of SIR and SIWR is given in Figure 5.



(A) SIR model



(B) SIWR model

FIGURE 5. Comparison of the SIR and SIWR models

5. **Conclusion.** In this paper, we consider a rumor spreading model with an online reservoir. By using a next-generation matrix, we calculated the basic reproduction number $\mathcal{R}_0$. We proved that a unique rumor-free equilibrium E_0 exists and if $\mathcal{R}_0 \leq 1$, then E_0 is linearly stable and if $\mathcal{R}_0 > 1$, then E_0 is linearly unstable. For the asymptotic behavior, E_0 is globally asymptotically stable if $\mathcal{R}_0 < 1$. For the endemic equilibrium, if $\mathcal{R}_0 > 1$, then there is a unique positive endemic state E_* and if $\mathcal{R}_0 \leq 1$, then there is no positive endemic state. Moreover, for a rumor spreading dynamics with an online reservoir, the endemic equilibrium E_* is globally asymptotically stable if $\mathcal{R}_0 > 1$. The presence of σ_s and σ_r does not affect the basic reproduction number and the asymptotic behaviors of steady states. We also investigated that the reproduction number $\mathcal{R}_0$ increases by the effect of the online reservoir. Thus, the development of online media promotes rumor propagation.

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