IRRIGABLE MEASURES FOR WEIGHTED IRRIGATION PLANS

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ABSTRACT. A model of irrigation network, where lower branches must be thicker in order to support the weight of the higher ones, was recently introduced in [7]. This leads to a countable family of ODEs, describing the thickness of every branch, solved by backward induction. The present paper determines what kind of measures can be irrigated with a finite weighted cost. Indeed, the boundedness of the cost depends on the dimension of the support of the irrigated measure, and also on the asymptotic properties of the ODE which determines the thickness of branches.

1. **Introduction.** In a ramified transport network [1, 2, 14, 15, 16, 17], the Gilbert transport cost along each arc is computed by

$$[\text{length}] \times [\text{flux}]^\alpha \tag{1}$$

for some given $\alpha \in [0, 1]$. When $\alpha < 1$, this accounts for an economy of scale: transporting the same amount of particles is cheaper if these particles travel together along the same arc.

In the recent paper [7], the authors considered an irrigation plan where the cost per unit length is determined by a weight function W. The main motivation behind this model is that, for a free standing structure like a tree, the lower portion of each branch needs to bear the weight of the upper part. Hence, even if the flux of water and nutrients is constant along a branch, the thickness (and hence the cost per unit length) grows as one moves from the tip toward the root. In the variational problems of optimal tree roots and branches[4, 6], this "weighted irrigation cost" is more suitable to model the associated cost for transporting water and nutrients from the roots to the leaves.

In this model, the weights are constructed inductively, starting from the outermost branches and proceeding toward the root. Along each branch, the weight W is determined by solving a suitable ODE, possibly with measure-valued right hand side. This is more conveniently written in the integral form

$$W(s) \;=\; \int_s^\ell f(W(\sigma)) \, d\sigma + m(s), \tag{2}$$

where $s \in [0, \ell]$ is the arc-length parameter along the branch, $s \mapsto m(s)$ is a non-increasing function describing the flux, and f is a non-negative, continuous function. A natural set of assumptions on f is

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(A1) *The function $f : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is continuous on $[0, +\infty[$, twice continuously differentiable for $z > 0$, and satisfies*

$$f(0) \;=\; 0, \quad f'(z) \;>\; 0, \quad f''(z) \;\leq\; 0 \qquad \forall \, z > 0. \tag{3}$$

The main result in [7] established the lower semicontinuity of the weighted irrigation cost, w.r.t. the pointwise convergence of irrigation plans. In particular, for any positive, bounded Radon measure μ, if there is an admissible irrigation plan whose weighted cost is finite, then there exists an irrigation plan for μ with minimum cost.

The goal of the present paper is to understand whether a given Radon measure μ irrigable or not, with respect to the weighted irrigation cost. That is, whether there exists an irrigation plan for μ whose weighted irrigation cost is finite. In the case without weights, i.e., with the classical Gilbert cost (1), this problem has been studied in [8], and further investigated in [3, 9, 10]. The authors in [8] proved that if a measure μ is α-irrigable, then it must be concentrated on a set with Hausdorff dimension $\leq \frac{1}{1-\alpha}$. On the other hand, if $\alpha > 1 - \frac{1}{d}$, every bounded Radon measure with bounded support in $\mathbf{R}^d$ has finite irrigation cost [1, 8].

As shown by our analysis, in the presence of weights the irrigability of a measure μ depends on the dimension of the set where μ is concentrated, on the exponent α, and also on the asymptotic behavior of the function $f(z)$ as $z \to 0^+$.

The remainder of the paper is organized as follows. Section 2 reviews the construction of the weight functions on the various branches of an irrigation plan. In Section 3 we prove our main results on the irrigability of Radon measures.

2. **Review of the weighted irrigation plans.**

2.1. **Weight functions on finitely many branches.** To illustrate the basic idea of the weighted irrigation model, we first consider a network with finitely many branches. As shown on the left of Fig. 1, each directed branch will be denoted by $\gamma_i : [a_i, b_i] \mapsto \mathbf{R}^d, i = 1, \ldots, N$, oriented from the root toward the tip and parameterized by arc-length. Call $P_i = \gamma_i(b_i)$ the ending node of the branch γ_i.
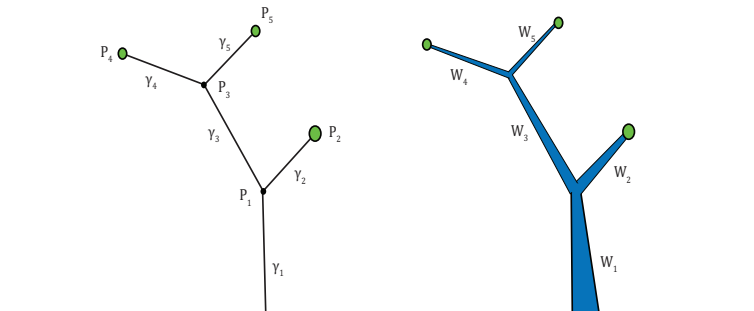

FIGURE 1. Left: A free standing tree with 5 branches. In this example, $\mathcal{O}(1) = \{2, 3\}, \mathcal{O}(3) = \{4, 5\}, \mathcal{O}(2) = \mathcal{O}(4) = \mathcal{O}(5) = \emptyset$. Right: On each branch, the weight decreases as one moves from the lower portion to the tip.

On each branch γ_i, we first prescribe a left-continuous, non-increasing function $m_i : [a_i, b_i] \mapsto \mathbf{R}^+$, which can be interpreted as the "flux" along the branch. Roughly speaking, $m_i(t)$ is the amount of mass transported through the point $\gamma_i(t)$.

Call $\mathcal{O}(i)$ the set of index labelling the branches that originate from the node $P_i = \gamma_i(b_i)$, that is

$$\mathcal{O}(i) \ = \ \Big\{ j \in \{1, \ldots, N\}; \ \gamma_j(a_j) = P_i \Big\}. \tag{4}$$

Moreover, consider the sets of indices inductively defined by

$$\mathcal{I}_1 \ \doteq \ \{ i \in \{1, \ldots, N\}; \ \mathcal{O}(i) = \emptyset \},$$

$$\mathcal{I}_{k+1} \ \doteq \ \{ i \in \{1, \ldots, N\}; \ \mathcal{O}(i) \subseteq \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_k \} \setminus (\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_k). \tag{5}$$

From [7] the weight function $W_i(\cdot)$ on each branch γ_i is defined inductively on $\mathcal{I}_k, k \geq 1$.

(i) For $k = 1$, on each branch $\gamma_i : [a_i, b_i] \mapsto \mathbf{R}^d$ with $i \in \mathcal{I}_1$, the weight $W_i : [a_i, b_i] \mapsto \mathbf{R}_+$ is defined to be the solution of

$$\omega(t) \ = \ \int_t^{b_i} f(\omega(s)) \, ds + m_i(t), \qquad t \in \,]a_i, b_i], \tag{6}$$

where f is a given function, satisfying **(A1)**, and m_i is the flux along the branch.

(ii) Assume the weight functions $W_i(t)$ have already been constructed along all branches $\gamma_j : [a_j, b_j] \mapsto \mathbf{R}^d$ with $j \in \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_{k-1}$.

For $i \in \mathcal{I}_k$, the weight $W_i(t)$ along the i-th branch is defined to be the solution of

$$\omega(t) \ = \ \int_t^{b_i} f(\omega(s)) \, ds + m_i(t) + \overline{\omega}_i, \qquad t \in \,]a_i, b_i], \tag{7}$$

where

$$\overline{\omega}_i \ \doteq \ \sum_{j \in \mathcal{O}(i)} W_j(a_j^+) - \sum_{j \in \mathcal{O}(i)} m_j(a_j^+). \tag{8}$$

2.2. **Irrigation plans for general measures.** Following Maddalena, Morel, and Solimini [15], the transport network for general Radon measure can be described in a Lagrangian way. Let μ be a fixed Radon measure on $\mathbf{R}^d$ with $\mu(\mathbf{R}^d) = M$ and let $\Theta = [0, M]$. We think of $\theta \in \Theta$ as a Lagrangian variable, labelling a water particle. An **irrigation plan for** μ is a function

$$\chi : \Theta \times \mathbf{R}_+ \mapsto \mathbf{R}^d,$$

measurable w.r.t. θ and continuous w.r.t. t, which satisfies the following conditions:

- All particles initially lie at the origin: $\chi(\theta, 0) = 0, \ \forall \, \theta \in \Theta$.
- For a.e. $\theta \in \Theta$ the map $t \mapsto \chi(\theta, t)$ is 1-Lipschitz and constant for t large. Namely, there exists $\tau(\theta) \geq 0$ such that

$$\begin{cases} |\chi(\theta, t) - \chi(\theta, s)| \ \leq \ |t - s| & \text{for all } t, s \geq 0, \\[2mm] \chi(\theta, t) \ = \ \chi(\theta, \tau(\theta)) & \text{for every } t \geq \tau(\theta). \end{cases}$$

Throughout the following, $\tau(\theta)$ will denote the smallest time τ such that $\chi(\theta, \cdot)$ is constant for $t \geq \tau$.

- χ irrigates the measure μ. That is, for each Borel set $V \subseteq \mathbf{R}^d$,

$$\mu(V) \ = \ \text{meas} \left(\{ \theta \in \Theta; \ \chi(\theta, \tau(\theta)) \in V \} \right).$$

One can think of $\chi(\theta, t)$ as the position of particle θ at time t.

To define the flux on χ, which measures the total amount of particles travelling along the same path, we first need an equivalence relation between two Lipschitz maps.

Definition 2.1. We say that two 1-Lipschitz maps $\gamma : [0, t] \mapsto \mathbf{R}^d$ and $\tilde{\gamma} : [0, \tilde{t}] \mapsto \mathbf{R}^d$ are **equivalent** if they are parametrizations of the same curve, and write it as $\gamma \simeq \tilde{\gamma}$. When we use the arc-length re-parametrization

$$\sigma \mapsto \gamma(s(\sigma)), \qquad \text{where} \qquad \int_0^{s(\sigma)} |\dot{\gamma}(t)|\, dt \;=\; \sigma,$$

then two 1-Lipschitz maps are equivalent means their arc-length re-parametrizations coincide.

Throughout the following, we denote by $\gamma\Big|_{[0,t]}$ the restriction of a map γ to the interval $[0, t]$.

Definition 2.2. Let $\chi : \Theta \times \mathbf{R}_+ \mapsto \mathbf{R}^d$ be an irrigation plan for the measure μ. On the set $\Theta \times \mathbf{R}_+$, we write $(\theta, t) \sim (\theta', t')$ whenever $\chi(\theta, \cdot)\Big|_{[0,t]} \simeq \chi(\theta', \cdot)\Big|_{[0,t']}$. This means that the maps

$$s \;\mapsto\; \chi(\theta, s), \quad s \in [0, t] \qquad \text{and} \qquad s \;\mapsto\; \chi(\theta', s), \quad s \in [0, t']$$

are equivalent in the sense of Definition 2.1.

The **multiplicity** at (θ, t) is then defined as

$$m(\theta, t) \;\doteq\; \mathrm{meas}\Big(\big\{\theta' \in \Theta\,;\; (\theta', t') \sim (\theta, t) \text{ for some } t' > 0\big\}\Big). \tag{9}$$

Given an irrigation plan $\chi : \Theta \times \mathbf{R}_+ \mapsto \mathbf{R}^d$, in order to have finite weighted irrigation cost constructed in the next section, we should always assume the following conditon.

(A2) *For a.e. $\theta \in \Theta$, one has $m(\theta, t) > 0$ for every $0 \le t < \tau(\theta)$.*

2.3. **Weight functions for an irrigation plan.** Given a bounded Radon measure μ in $\mathbf{R}^d$ and an irrigation plan $\chi : \Theta \times \mathbf{R}_+ \mapsto \mathbf{R}^d$ for μ, in this section we review the construction of the weight function $W = W(\theta, t)$ on the irrigation plan. Notice that for an irrigation plan χ of a general Radon measure, for each particle $\theta \in \Theta$, the map $\chi(\theta, \cdot) : \mathbf{R}_+ \mapsto \mathbf{R}^d$ describes a continuous curve in $\mathbf{R}^d$. Thus χ may contain infinitely many branches. To construct the weight function on each branch, the idea is to first compute the weights W^ε on χ^ε, which is the truncation of χ on the branches with multiplicity $\ge \varepsilon$. It turns out that χ^ε only consists of finitely many branches, so that we can compute W^ε as in Section 2.1 . The weight W is then constructed by taking the limit of W^ε, as $\varepsilon \to 0+$.

Definition 2.3. Given an irrigation plan χ, a path $\gamma : [0, \ell] \mapsto \mathbf{R}^d$, parameterized by arc-length, is ε**-good** if and only if

$$\mathrm{meas}\left(\left\{\theta \in \Theta\,;\; \chi(\theta, \cdot)\Big|_{[0,t]} \simeq \gamma \text{ for some } t = t(\theta) > 0\right\}\right) \;\ge\; \varepsilon, \tag{10}$$

where the equivalence relation $\simeq$ is given in Definition 2.1.

In other words, γ is ε-good if there is an amount $\geq \varepsilon$ of particles whose trajectory contains γ as initial portion.

For any given $\varepsilon > 0$, following [7] we define the ε-stopping time $\tau_\varepsilon : \Theta \mapsto \mathbf{R}_+$ by setting

$$\tau_\varepsilon(\theta) \;\doteq\; \max\{t \geq 0;\; m(\theta, t) \geq \varepsilon\}. \tag{11}$$

Define the ε-truncation χ^ε of irrigation plan χ as

$$\chi^\varepsilon(\theta, t) \;\doteq\; \begin{cases} \chi(\theta, t) & \text{if } t < \tau_\varepsilon(\theta) \\[2mm] \chi(\theta, \tau_\varepsilon(\theta)) & \text{if } t \geq \tau_\varepsilon(\theta) \end{cases} \tag{12}$$

In other words, in the ε-truncation χ^ε, only those paths in χ with multiplicity $\geq \varepsilon$ are kept. For any $\theta \in \Theta$, if $\tau_\varepsilon(\theta) > 0$, the ε-good portion $\chi(\theta, \cdot)\big|_{[0, \tau_\varepsilon(\theta)]}$ of the path $t \mapsto \chi(\theta, \cdot)$ is included in χ^ε.

Notice that the family of all curves parameterized by arc-length comes with a natural partial order. Namely, given two maps $\gamma : [0, \ell] \mapsto \mathbf{R}^d$, $\tilde{\gamma} : [0, \tilde{\ell}] \mapsto \mathbf{R}^d$, we write $\gamma \preceq \tilde{\gamma}$ if $\ell \leq \tilde{\ell}$ and $\gamma(s) = \tilde{\gamma}(s)$ for all $s \in [0, \ell]$. In the family of all ε-good paths in the irrigation plan χ, we can thus find the maximal ε-good paths, w.r.t the above partial order. As shown in [7], the total number of maximal ε-good paths in the irrigation plan χ is bounded by $\frac{M}{\varepsilon}$, where M is the total mass of μ. Therefore, the ε-truncation χ^ε is a network with finitely many branches, consisting of all maximal ε-good paths in χ.

For a fixed $\varepsilon > 0$, to compute the weight functions on the ε-truncation χ^ε, we now let $\{\hat{\gamma}_1, \ldots, \hat{\gamma}_\nu\}$ be the set of all maximal ε-good paths. Along each path $\hat{\gamma}_i : [0, \hat{\ell}_i] \mapsto \mathbf{R}^d$ we define the **multiplicity** $\hat{m}_i : [0, \hat{\ell}_i] \mapsto \mathbf{R}_+$ by setting

$$\hat{m}_i(s) \;\doteq\;$$

$$\text{meas}\left(\left\{\theta \in \Theta;\; \text{there exists } t \geq 0 \text{ such that } \chi(\theta, \cdot)\Big|_{[0, t]} \;\simeq\; \hat{\gamma}_i\Big|_{[0, s]}\right\}\right). \tag{13}$$

Since two maximal paths may coincide on the initial portion and bifurcate later, we consider the bifurcation times

$$\tau_{ij} \;=\; \tau_{ji} \;\doteq\; \max\left\{t \geq 0;\; \hat{\gamma}_i(s) = \hat{\gamma}_j(s) \;\forall s \in [0, t]\right\}. \tag{14}$$

For each maximal path $\hat{\gamma}_i$, we split it into several elementary branches γ_k, by the following Path Splitting Algorithm**(PSA)**, which is first introduced in [7].

(PSA) For each $i \in \{1, \ldots, \nu\}$, consider the set

$$\{\tau_{i1}, \ldots, \tau_{i\nu}\} \;=\; \{t_{i,1}, \ldots, t_{i,N(i)}\},$$

where the times

$$0 = t_{i,0} \;<\; t_{i,1} \;<\; t_{i,2} \;<\; \cdots \;<\; t_{i,N(i)} \;<\; \hat{\ell}_i \tag{15}$$

provide an increasing arrangement of the set of times τ_{ij} where the path $\hat{\gamma}_i$ splits apart from other maximal paths. For each $k = 1, \ldots, N(i)$, let $\gamma_{i,k}$ be the restriction of the maximal path $\hat{\gamma}_i$ to the subinterval $[t_{i,k-1}, t_{i,k}]$. The multiplicity function $m_{i,k}$ along this path is defined simply as

$$m_{i,k}(t) \;=\; \hat{m}_i(t), \qquad t \in [t_{i,k-1}, t_{i,k}]. \tag{16}$$

If $\tau_{ij} > 0$, i.e. if the two maximal paths $\hat{\gamma}_i$ and $\hat{\gamma}_j$ partially overlap, it is clear that some of the elementary branches $\gamma_{i,k}$ will coincide with some $\gamma_{j,l}$. To avoid listing multiple times the same branch, we thus remove from our

list all branches $\gamma_{j,l} : [t_{j,l-1}, t_{j,l}] \mapsto \mathbf{R}^d$ such that $t_{j,l} \leq \tau_{ij}$ for some $i < j$. After relabeling all the remaining branches, the algorithm yields a family of elementary branches and corresponding multiplicities

$$\gamma_i : [a_i, b_i] \mapsto \mathbf{R}^d, \qquad m_i : [a_i, b_i] \mapsto \mathbf{R}_+, \qquad i = 1, \ldots, N \qquad (17)$$
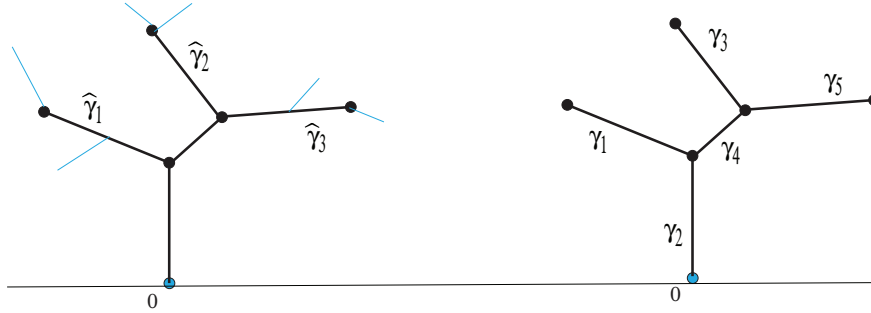
where N is the total number of elementary branches.

FIGURE 2. Left: Two finite truncation plans, showing three maximal ε-good paths (thick lines) and six maximal ε'-good paths (thin lines), for $0 < \varepsilon' < \varepsilon$. Right: The three maximal ε-good paths can be partitioned into five elementary branches, by the Path Splitting Algorithm.

On these elementary branches $\gamma_i, i \geq 1$, we can compute the weight function W_i on each γ_i inductively, as in Section 2.1.

On each maximal ε-good path $\hat{\gamma}_i$ with $1 \leq i \leq \nu$, the above construction yields a weight $\hat{W}_{i,k}$ on the restriction of $\hat{\gamma}_i$ to each subinterval $[t_{i,k-1}, t_{i,k}]$. Along the maximal path $\hat{\gamma}_i$, the weight $\hat{W}_i : [0, \hat{\ell}_i] \mapsto \mathbf{R}_+$ is then defined simply by setting

$$\hat{W}_i(t) = \hat{W}_{i,k}(t) \qquad \text{if } t \in [t_{i,k-1}, t_{i,k}]. \qquad (18)$$

Next, on the ε-truncation χ^ε we define the weight function $W^\varepsilon : \Theta \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ by setting

$$W^\varepsilon(\theta, t) \doteq \begin{cases} \hat{W}_i(s) & \text{if} \quad t \leq \tau_\varepsilon(\theta), \quad \chi(\theta, \cdot)\Big|_{[0,t]} \simeq \hat{\gamma}_i\Big|_{[0,s]}, \\ 0 & \text{if} \quad t > \tau_\varepsilon(\theta). \end{cases} \qquad (19)$$

As proved in [7], the map $\varepsilon \mapsto W^\varepsilon(\theta, t)$ is nondecreasing for each (θ, t). This leads to:

Definition 2.4. Let $\chi : \Theta \times \mathbf{R}_+ \mapsto \mathbf{R}^d$ be an irrigation plan satisfying **(A2)**. The **weight function** $W = W(\theta, t)$ for χ is defined as

$$W(\theta, t) \doteq \sup_{\varepsilon > 0} W^\varepsilon(\theta, t). \qquad (20)$$

Once we computed the weight functions on the irrigation plan χ, its weighted irrigation cost $\mathcal{E}^{W,\alpha}$ is defined as follows:

Definition 2.5. Let $f : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be a continuous function, satisfying all the assumptions in **(A1)**. Let χ be an irrigation plan satisfying **(A2)** and let $W \doteq W(\theta, t)$ be the corresponding weight function, as in (20). The weighted cost $\mathcal{E}^{W,\alpha}$ for some $\alpha \in [0, 1]$ is

$$\mathcal{E}^{W,\alpha}(\chi) \ \doteq \ \int_0^M \int_0^{\tau(\theta)} \frac{(W(\theta,t))^\alpha}{m(\theta,t)} |\dot\chi(\theta,t)| \, dt \, d\theta. \tag{21}$$

In the special case where χ consists of only finitely many branches, let W_i be the corresponding weight functions on the branch $\gamma_i : [a_i, b_i] \mapsto \mathbf{R}_+$, by applying the change of variable formula, we have the following identity for the weighted irrigation costs[7]:

$$\mathcal{E}^{W,\alpha}(\chi) \ = \ \sum_{i=1}^N \int_{a_i}^{b_i} [W_i(s)]^\alpha \, ds, \tag{22}$$

where N is the total number of branches.

2.4. **Lower semicontinuity of weighted cost.** In this section we recall the main theorems on the lower semicontinuity of weighted irrigation cost, proved in [7]. Given a sequence of irrigation plans $\chi_n : \Theta \times \mathbf{R}_+ \mapsto \mathbf{R}^d$, we say that χ_n converges to χ pointwise if, for every $\kappa > 0$ and a.e. $\theta \in \Theta$,

$$\lim_{n\to\infty} \|\chi_n(\theta,\cdot) - \chi(\theta,\cdot)\|_{\mathbf{L}^\infty([0,\kappa])} \ = \ 0. \tag{23}$$

Theorem 2.6. *Let $(\chi_n)_{n\geq 1}$ be a sequence of irrigation plans, all satisfying* **(A2)**, *pointwise converging to an irrigation plan χ. Assume that the function f satisfies* **(A1)**. *Then*

$$\mathcal{E}^{W,\alpha}(\chi) \ \leq \ \liminf_{n\to\infty} \mathcal{E}^{W,\alpha}(\chi_n). \tag{24}$$

Given a positive, bounded Radon measure μ on $\mathbf{R}^d$, we define the weighted irrigation cost $\mathcal{I}^{W,\alpha}(\mu)$ of μ as

$$\mathcal{I}^{W,\alpha}(\mu) \ \doteq \ \inf_\chi \mathcal{E}^{W,\alpha}(\chi), \tag{25}$$

where the infimum is taken over all irrigation plans for the measure μ, and $\mathcal{E}^{W,\alpha}$ is defined as in (21). By Theorem 2.6, if there is an irrigation plan for μ with finite weighted irrigation cost, then the infimum in (25) is actually a minimum. That is, there exists an optimal irrigation plan χ^* of μ, such that the weighted irrigation cost $\mathcal{E}^{W,\alpha}(\chi^*)$ is minimum among all admissible irrigation plans, and $\mathcal{I}^{W,\alpha}(\mu) = \mathcal{E}^{W,\alpha}(\chi^*)$.

The next result states the lower semicontinuity of the weighted irrigation cost, w.r.t. weak convergence of measures. For a proof, see Theorem 6.2 in [7].

Theorem 2.7. *Let f satisfies* **(A1)**. *Let $(\mu_n)_{n\geq 1}$ be a sequence of bounded positive Radon measures, with uniformly bounded supports, such that weakly converges to some μ. Then*

$$\mathcal{I}^{W,\alpha}(\mu) \ \leq \ \liminf_{n\to\infty} \mathcal{I}^{W,\alpha}(\mu_n). \tag{26}$$

3. **Irrigability dimension.** When $f = 0$, $\alpha > 1 - \frac{1}{d}$, it is well known that all measures with bounded support and finite mass in $\mathbf{R}^d$ are α-irrigable [1, 8].

Here is a formal computation in this direction. It is obtained by modifying the estimates at p. 113 of [1].

Let μ be a probability measure that supported in $B(0,1) \subseteq \mathbf{R}^d$. For each $j = 1, \ldots, n$, let $\mathcal{P}_j$ be the set of centers of balls of radius $r_j = 2^{-j}$ that cover $Supp(\mu)$. In dimension d, we can assume that the cardinality of this set is

$$\#\mathcal{P}_j \ \leq \ C \, 2^{jd}$$

We can define a map $\gamma_j : \mathcal{P}_j \mapsto \mathcal{P}_{j-1}$ such that

$$|x - \gamma_j(x)| \ \leq \ 3 \cdot 2^{-j}$$

for every $x \in \mathcal{P}_j$, with $\mathcal{P}_0 \doteq \{0\}$.

Consider a probability measure μ_n, supported on $\mathcal{P}_n$. The cost of transporting this measure from $\mathcal{P}_n$ to another measure supported on $\mathcal{P}_{n-1}$ is

$$
\begin{aligned}
E^\alpha(\mathcal{P}_n, \mathcal{P}_{n-1}) \ &\leq \ [\text{number of arcs}] \times [\text{flow}]^\alpha \times [\text{length}] \\
&\leq C\, 2^{nd} \cdot \left(\tfrac{1}{C\, 2^{nd}}\right)^\alpha \cdot 3 \cdot 2^{-n} \ = \ 3C^{1-\alpha} \cdot (2^{\alpha d - d + 1})^{-n}.
\end{aligned}
\tag{27}
$$

Notice that we are here considering the worst possible case, where we have the largest number of arcs and all arcs carry equal flow.

Summing over $j = 1, 2, \ldots, n$ we obtain that the total transportation cost is bounded by

$$E^\alpha \ \leq \ 3C^{1-\alpha} \cdot \sum_{j=1}^{n} (2^{\alpha d - d + 1})^{-j} \ \leq \ \frac{3C^{1-\alpha}}{2^{\alpha d - d + 1} - 1}.
\tag{28}$$

The series $\sum_k 2^{[(d-1) - \alpha d](k+1)}$ converges provided that

$$(d-1) - \alpha d \ < \ 0, \qquad \text{hence} \qquad \alpha \ > \ 1 - \frac{1}{d}$$

To understand what happens in the case where weights are present, we first make an explicit computation in the case of a dyadic irrigation plan [1, 16]. More precisely, as shown in Fig. 3, we now assume

$\mu =$ Radon measure with total mass M, concentrated on a cube Q in $\mathbf{R}^d$. Q is centered at the origin and with edge size $L > 0$.

For each $n \geq 1$, we divide Q into 2nd smaller cubes of equal size, with edge size $L/2^n$. Take $\{Q_i^n\}_{i=1}^{2^{nd}}$ the set of all these closed smaller cubes, call $\mathcal{P}_n \doteq \{x_i^n\}_{i=1}^{2^{nd}}$ the set of centers of these smaller cubes of edge size $L/2^n$. For each $n \geq 1$, define the dyadic approximated measure μ_n

$$\mu_n \ \doteq \ \sum_{x_i^n \in \mathcal{P}_n} m_i^n \, \delta_{x_i^n},
\tag{29}$$

where $\delta_{x_i^n}$ is the Dirac measure at x_i^n, and m_i^n is determined as

$$\hat{Q}_i^n \ \doteq \ Q_i^n \setminus \bigcup_{j < i} Q_j^n, \qquad m_i^n \doteq \mu(\hat{Q}_i^n), \qquad \forall\, 1 \leq i \leq 2^{nd}.$$

It is not hard to show that μ_n weakly converges to μ, see for example [1, 16]. That is, for any bounded continuous function $\phi : \mathbf{R}^d \mapsto \mathbf{R}$, one has $\int \phi \, d\mu_n \to \int \phi \, d\mu$. For each μ_n, we construct an irrigation plan χ_n as follows:

- First, move the particles from the origin (center of Q) to the centers in $\mathcal{P}_1$, with 2^d straight paths connecting the origin and the centers in $\mathcal{P}_1 = \{x_1^1, x_2^1, \ldots, x_{2^d}^1\}$. Each path has length $\frac{\sqrt{d}L}{4}$, on the path that connecting $x_i^1, 1 \leq i \leq 2^d$, the multiplicity is constant m_i^1.
- By induction, at the level $k, 1 < k \leq n$, for the particles arriving at each center x_i^{k-1} in $\mathcal{P}_{k-1}$, where x_i^{k-1} is the center of the cube Q_i^{k-1}, we transport them to the 2^d neighboring centers in $\mathcal{P}_k$, which are all contained in the cube Q_i^{k-1}. Without loss of generality, fixed x_i^{k-1} in $\mathcal{P}_{k-1}$, let $\{x_1^k, \ldots, x_{2^d}^k\}$ be the 2^d neighboring centers around x_i^{k-1}. For each $x_j^k, 1 \leq j \leq 2^d$, we build a straight path connecting x_i^{k-1} to x_j^k, with length $\frac{\sqrt{d}L}{2^{k+1}}$ and constant multiplicity m_j^k.
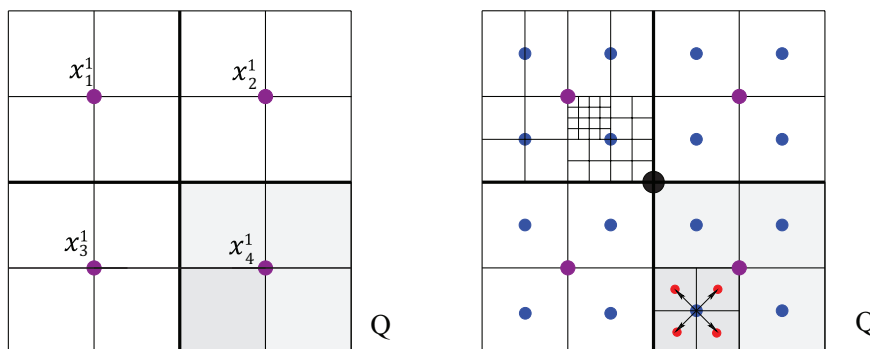
FIGURE 3. Left: The dyadic approxmiated measure μ_1 is supported on the four centers $x_1^1, \ldots, x_4^1$ of small cubes. Right: Dyadic approximated measures corresponding to a family of partitions into dyadic cubes in $\mathbf{R}^2$.

Since the dyadic measure μ_n is supported on the centers in $\mathcal{P}_n$, after n steps we build an irrigation plan for μ_n, which we call the **dyadic irrigation plan** χ_n.

For example, in the case $\mathbf{R}^2$, Fig. 4 shows two dyadic irrigation plans constructed by the preceding procedure.

Given an irrigation plan with finite branches as in Section 2.1, consider the case $f(z) \doteq cz^\beta$, with some constant $c > 0, 0 < \beta < 1$. It is readily to check that f satisfies **(A1)**. With the notions in Section 2.1, consider a measure μ consisting of finitely many point masses $m_i \geq 0$ located at points P_i, where P_i is the ending node of branch $\gamma_i(s) : [0, \ell_i] \mapsto \mathbf{R}^d$. In this case, the multiplicity function on each
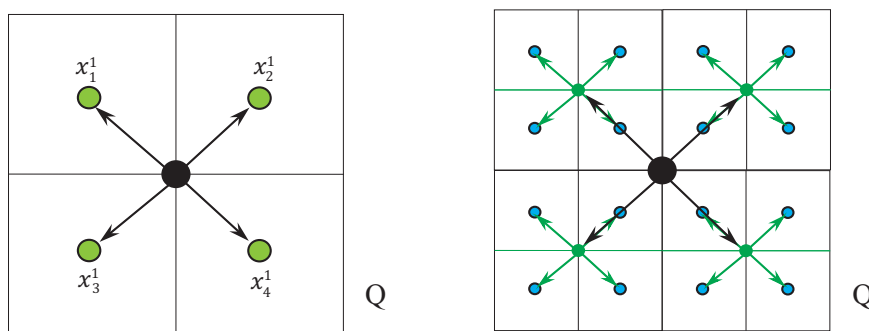

FIGURE 4. The dyadic irrigation plans in $\mathbf{R}^2$. Left: The dyadic irrigation plan χ_1. The multiplicity on each branch equals to the mass on the terminal point. Right: The dyadic irrigation plan χ_2. The particles are first transported to the 4 centers in $\mathcal{P}_1$, then on each center in $\mathcal{P}_1$, the particles are transported to the neighboring 4 centers in $\mathcal{P}_2$.

branch is constant. Then the computation of weights (6)-(8) becomes

$$W_i(s) = \left(\overline{W}_i^{1-\beta} + c(1-\beta)(\ell_i - s) \right)^{\frac{1}{1-\beta}} ,$$

$$\overline{W}_i = m_i + \sum_{j \in \mathcal{O}(i)} \left(\overline{W}_j^{1-\beta} + c(1-\beta)\ell_j \right)^{\frac{1}{1-\beta}} . \tag{30}$$

If $\mathcal{O}(i) = \emptyset$, that is $i \in \mathcal{I}_1$, from (30) we have $\overline{W}_i = m_i$.

The following two lemmas proved that under suitable conditions, the weighted irrigation costs of the dyadic irrigation plans $\{\chi_n\}_{n \geq 1}$ are uniformly bounded. Utilizing this fact and Theorem 2.7, since the dyadic approximated measures μ_n weakly converges to μ, we can conclude the irrigability of μ with weighted cost.

To fix the ideas, we first consider the case that μ is the Lebesgue measure on the unit cube Q.

Lemma 3.1. *Suppose* $1 > \beta > 1 - \frac{1}{d}$, $1 \geq \alpha > 1 - \frac{1}{d}$, *while* μ *is the Lebesgue measure on the unit cube* Q *in* $\mathbf{R}^d$. *Then, in the dyadic irrigation plans* χ_n, *the weight function* W^n *remains uniformly bounded on all branches. Moreover, the irrigation cost* $\mathcal{E}^{W,\alpha}(\chi_n)$ *is uniformly bounded. That is, there exists an uniform constant* $C > 0$, *such that for all dyadic irrigation plan* χ_n,

$$W^n \leq C, \qquad \mathcal{E}^{W,\alpha}(\chi_n) \leq C . \tag{31}$$

Proof. For the dyadic irrigation plan χ_n, since each dyadic irrigation plan has finite branches and μ_n is supported on the centers in $\mathcal{P}_n$, we can use formula (30) to compute the weights W^n. We start from the centers in $\mathcal{P}_n$.

1. From $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$, for each $x_i^{n-1} \in \mathcal{P}_{n-1}$, by the construction of the dyadic irrigation plan χ_n, there are straight paths connecting x_i^{n-1} to the 2^d neighboring centers in $\mathcal{P}_n$. Since μ is the Lebesgue measure on unit cube Q, mass on each center in $\mathcal{P}_n$ is $\frac{1}{2^{nd}}$. The branches connecting x_i^{n-1} to centers in $\mathcal{P}_n$ are identical, with branch length $\sqrt{d}/2^{n+1}$ and constant multiplicity $1/2^{nd}$. We only need to compute the weight on one such branch, and write it as W_n^n, where the superindex n means it is the weight for irrigation plan χ_n, and the subindex n means from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$.

By formula (30), for $s \in [0, \frac{\sqrt{d}}{2^{n+1}}]$,

$$W_n^n(s) = \left((\frac{1}{2^{nd}})^{1-\beta} + c(1-\beta)(\frac{\sqrt{d}}{2^{n+1}} - s) \right)^{\frac{1}{1-\beta}} , \tag{32}$$

$$W_n^n(0) = \left((\frac{1}{2^{nd}})^{1-\beta} + c(1-\beta)\frac{\sqrt{d}}{2^{n+1}} \right)^{\frac{1}{1-\beta}} . \tag{33}$$

2. From $\mathcal{P}_{n-1}$ to $\mathcal{P}_{n-2}$, using formula (30), on each branch we need to first compute the weights $\overline{W}_{n-1}^n$ at the tip. For the dyadic approximated measure μ_n, it is supported on $\mathcal{P}_n$, thus the mass on each center in $\mathcal{P}_k, k \neq n$ is 0. Since each center in $\mathcal{P}_{n-2}$ connects 2^d identical centers in $\mathcal{P}_{n-1}$, we therefore have

$$\overline{W}_{n-1}^n = 2^d W_n^n(0) = 2^d \left((\frac{1}{2^{nd}})^{1-\beta} + c(1-\beta)\frac{\sqrt{d}}{2^{n+1}} \right)^{\frac{1}{1-\beta}} . \tag{34}$$

Each branch between $\mathcal{P}_{n-2}$ and $\mathcal{P}_{n-1}$ has length $\frac{\sqrt{d}}{2^n}$. By formula (30), for $s \in [0, \frac{\sqrt{d}}{2^n}]$,

$$W_{n-1}^n(s) = \left(2^{d(1-\beta)} \left[(\frac{1}{2^{nd}})^{1-\beta} + c(1-\beta)\frac{\sqrt{d}}{2^{n+1}} \right] + c(1-\beta)(\frac{\sqrt{d}}{2^n} - s) \right)^{\frac{1}{1-\beta}} , \tag{35}$$

$$W_{n-1}^n(0) = \left((\frac{1}{2^{(n-1)d}})^{1-\beta} + c(1-\beta)\Big[\frac{\sqrt{d}}{2^{n+1-d(1-\beta)}} + \frac{\sqrt{d}}{2^n}\Big]\right)^{\frac{1}{1-\beta}}. \tag{36}$$

3. From $\mathcal{P}_{n-k}$ to $\mathcal{P}_{n-k-1}$, each branch has length $\frac{\sqrt{d}}{2^{n+1-k}}$. Similarly for $s \in [0, \frac{\sqrt{d}}{2^{n+1-k}}]$,

$$\begin{aligned} W_{n-k}^n(s) &= \Big((\frac{1}{2^{(n-k)d}})^{1-\beta} + \\ &\frac{c(1-\beta)\sqrt{d}}{2^{n+1}}\sum_{j=1}^k 2^{(k-j)} + jd(1-\beta) + c(1-\beta)(\frac{\sqrt{d}}{2^{n+1-k}} - s)\Big)^{\frac{1}{1-\beta}}, \end{aligned} \tag{37}$$

$$\begin{aligned} W_{n-k}^n(0) &= \left((\frac{1}{2^{(n-k)d}})^{1-\beta} + \frac{c(1-\beta)\sqrt{d}}{2^{n+1}}\sum_{j=0}^k 2^{(k-j)+jd(1-\beta)}\right)^{\frac{1}{1-\beta}} \\ &= \left((\frac{1}{2^{(n-k)d}})^{1-\beta} + \frac{c(1-\beta)\sqrt{d}}{2^{n+1-k}}\sum_{j=0}^k \frac{1}{2^{[1-d(1-\beta)]j}}\right)^{\frac{1}{1-\beta}}. \end{aligned} \tag{38}$$

4. Since $W_{n-k}^n(s) \leq W_{n-k}^n(0)$, to have an uniform bound on the weight function, we only need to estimate $W_{n-k}^n(0)$, for each $0 \leq k \leq n-1$. Indeed, When $\beta > 1-1/d$, one has $1 - d(1-\beta) > 0$, then for each k, from (38),

$$W_{n-k}^n(0) \leq \left((\frac{1}{2^{(n-k)d}})^{1-\beta} + \frac{c(1-\beta)\sqrt{d}}{2^{n+1-k}} \cdot \frac{1}{1-(\frac{1}{2})^{1-d(1-\beta)}}\right)^{\frac{1}{1-\beta}}. \tag{39}$$

Therefore, we have an uniform bound for the weight function

$$W^n \leq \left(1 + \frac{c(1-\beta)\sqrt{d}}{1-(\frac{1}{2})^{1-d(1-\beta)}}\right)^{\frac{1}{1-\beta}}, \tag{40}$$

which is independent of n.

5. We now estimate the irrigation cost $\mathcal{E}^{W,\alpha}(\chi_n)$ by the formula (22). Fixed the dyadic irrigation plan χ_n, call E_n^n the cost from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$. There are 2nd branches from centers in $\mathcal{P}_n$ to centers in $\mathcal{P}_{n-1}$. On each branch, the weight W_n^n is given by (32). Therefore,

$$\begin{aligned} E_n^n &= 2^{nd} \int_0^{\frac{\sqrt{d}}{2^{n+1}}} \left((\frac{1}{2^{nd}})^{1-\beta} + c(1-\beta)(\frac{\sqrt{d}}{2^{n+1}} - s)\right)^{\frac{\alpha}{1-\beta}} ds \\ &= \frac{2^{nd}}{c(1+\alpha-\beta)}\left(\Big[(\frac{1}{2^{nd}})^{1-\beta} + c(1-\beta)\frac{\sqrt{d}}{2^{n+1}}\Big]^{\frac{1+\alpha-\beta}{1-\beta}} - \Big[(\frac{1}{2^{nd}})^{1-\beta}\Big]^{\frac{1+\alpha-\beta}{1-\beta}}\right). \end{aligned} \tag{41}$$

Similarly, denote E_{n-k}^n the cost from $\mathcal{P}_{n-k}$ to $\mathcal{P}_{n-k-1}$. There are $2^{(n-k)d}$ branches from centers in $\mathcal{P}_{n-k}$ to centers in $\mathcal{P}_{n-k-1}$.

$$E_{n-k}^n = 2^{(n-k)d} \int_0^{\frac{\sqrt{d}}{2^{n+1-k}}} \left(\Big(\overline{W}_{n-k}^n\Big)^{1-\beta} + c(1-\beta)(\frac{\sqrt{d}}{2^{n+1-k}} - s)\right)^{\frac{\alpha}{1-\beta}} ds \tag{42}$$

In the following, we use the same C to denote different constants which only depend on c, α, β and the dimension d. From (39) and the fact that $(1-\beta)d < 1$, for each n and k,

$$\Big(\overline{W}_{n-k}^n\Big)^{1-\beta} + c(1-\beta)\frac{\sqrt{d}}{2^{n+1-k}} \leq \frac{C}{2^{(n-k)(1-\beta)d}}. \tag{43}$$

Consider $x, y \geq 0$,

$$g(x,y) \doteq (x+y)^{\frac{1+\alpha-\beta}{1-\beta}} - x^{\frac{1+\alpha-\beta}{1-\beta}}, \qquad x + y \leq \frac{C}{2^{(n-k)(1-\beta)d}} \tag{44}$$

then, by a first order Taylor expansion,

$$g(x,y) \leq \frac{1+\alpha-\beta}{1-\beta}\left(\frac{C}{2^{(n-k)(1-\beta)d}}\right)^{\frac{\alpha}{1-\beta}} \cdot y \leq \frac{Cy}{2^{(n-k)\alpha d}} \tag{45}$$

Applying (43) and (45) in (42), we obtain

$$E_{n-k}^n \leq 2^{(n-k)d}\frac{C}{2^{(n-k)(\alpha d+1)}} = \frac{C}{2^{(n-k)[(\alpha-1)d+1]}} . \tag{46}$$

When $\alpha > 1 - 1/d$, one has $(\alpha-1)d + 1 > 0$. Then by (46),

$$\mathcal{E}^{W,\alpha}(\chi_n) = \sum_{k=0}^{n-1} E_{n-k}^n \leq \sum_{k=0}^{n-1}\frac{C}{2^{(n-k)[(\alpha-1)d+1]}} \leq \frac{C}{1-(\frac{1}{2})^{(\alpha-1)d+1}}, \tag{47}$$

where C is some constant independent of n. Combining the estimates (40) and (47), we obtain the existence of a constant C, independent of n, such that (31) holds. $\square$

Under the same conditions on α, β, this uniform boundedness result holds for general positive, finite Radon measures.

Lemma 3.2. *Suppose $1 - \frac{1}{d} < \beta < 1$, $1 - \frac{1}{d} < \alpha \leq 1$. μ is a finite measure on the cube Q with edge size L in $\mathbf{R}^d$, denote M the total mass of μ. Then in the dyadic irrigation plan χ_n, the weight function W^n on each branch remains uniformly bounded,*

$$W^n \leq C\left(M^{1-\beta} + L\right)^{\frac{1}{1-\beta}}, \tag{48}$$

Moreover, the irrigation cost $\mathcal{E}^{W,\alpha}(\chi_n)$ is uniformly bounded, namely

$$\mathcal{E}^{W,\alpha}(\chi_n) \leq C\left(M^\alpha L + L^{1+\frac{\alpha}{1-\beta}}\right) \tag{49}$$

where C is some constant independent of n.

Proof. For the dyadic irrigation plan χ_n, to compute the weights W^n, we start from the centers in $\mathcal{P}_n$.

1. From $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$. Let m_i^n be the mass of μ_n on the center x_i^n in $\mathcal{P}_n$. On the branch from x_i^n to any center in $\mathcal{P}_{n-1}$, the arc-length of the branch is $\frac{\sqrt{d}L}{2^{n+1}}$ and the multiplicity is constant m_i^n. Let $W_{n,i}^n$ be the corresponding weights, where the superindex n means we consider the weight function on irrigation plan χ_n, the subindex (n,i) means we consider the weight on the i-th branch from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$. Then by formula (30), for $s \in [0, \frac{\sqrt{d}L}{2^{n+1}}]$,

$$W_{n,i}^n(s) = \left((m_i^n)^{1-\beta} + c(1-\beta)(\frac{\sqrt{d}L}{2^{n+1}} - s)\right)^{\frac{1}{1-\beta}}, \tag{50}$$

$$W_{n,i}^n(0) = \left((m_i^n)^{1-\beta} + c(1-\beta)\frac{\sqrt{d}L}{2^{n+1}}\right)^{\frac{1}{1-\beta}}. \tag{51}$$

2. From $\mathcal{P}_{n-1}$ to $\mathcal{P}_{n-2}$. For each center x_i^{n-1} in $\mathcal{P}_{n-1}$, to compute the weight $W_{n-1,i}^n$ from x_i^{n-1} to any center in $\mathcal{P}_{n-2}$, we first estimate $\overline{W}_{n-1,i}^n$. Each x_i^{n-1} in $\mathcal{P}_{n-1}$ connects 2^d nearby centers in $\mathcal{P}_n$. By (30) and (51) one has,

$$\overline{W}_{n-1,i}^n = \sum_{j\in\mathcal{O}(i)} W_{n,j}^n(0) = \sum_{j\in\mathcal{O}(i)}\left((m_j^n)^{1-\beta} + c(1-\beta)\frac{\sqrt{d}L}{2^{n+1}}\right)^{\frac{1}{1-\beta}}. \tag{52}$$

Notice for fixed $b \geq 0$, $g(x) \doteq (x^{1-\beta} + b)^{\frac{1}{1-\beta}}$ is a concave function of x on $\mathbf{R}_+$. Thus for any N,

$$\frac{1}{N} \sum_{j=1}^N (x_j^{1-\beta} + b)^{\frac{1}{1-\beta}} \leq \left(\left(\frac{\sum_{j=1}^N x_j}{N} \right)^{1-\beta} + b \right)^{\frac{1}{1-\beta}} . \tag{53}$$

For each i, the cardinality of the set $\mathcal{O}(i)$ in (52) is 2^d. From (52)-(53),

$$\overline{W}_{n-1,i}^n \leq 2^d \left[\left(\frac{\sum_{j \in \mathcal{O}(i)} m_j^n}{2^d} \right)^{1-\beta} + c(1-\beta) \frac{\sqrt{d}L}{2^{n+1}} \right]^{\frac{1}{1-\beta}} . \tag{54}$$

Each branch from x_i^{n-1} to $\mathcal{P}_{n-2}$ has length $\frac{\sqrt{d}L}{2^n}$. By the formula (30), for $s \in [0, \frac{\sqrt{d}L}{2^n}]$,

$$
\begin{aligned}
W_{n-1,i}^n(s) &= \left(\left(\overline{W}_{n-1,i}^n \right)^{1-\beta} + c(1-\beta)\left(\frac{\sqrt{d}L}{2^n} - s \right) \right)^{\frac{1}{1-\beta}} \\
&\leq \left(\left(\sum_{j \in \mathcal{O}(i)} m_j^n \right)^{1-\beta} + c(1-\beta) \left[\frac{\sqrt{d}L}{2^{n+1-d(1-\beta)}} + \left(\frac{\sqrt{d}L}{2^n} - s \right) \right] \right)^{\frac{1}{1-\beta}},
\end{aligned}
\tag{55}
$$

$$W_{n-1,i}^n(0) \leq \left(\left(\sum_{j \in \mathcal{O}(i)} m_j^n \right)^{1-\beta} + c(1-\beta) \left[\frac{\sqrt{d}L}{2^{n+1-d(1-\beta)}} + \frac{\sqrt{d}L}{2^n} \right] \right)^{\frac{1}{1-\beta}} . \tag{56}$$

3. From $\mathcal{P}_{n-2}$ to $\mathcal{P}_{n-3}$. For each center x_i^{n-2} in $\mathcal{P}_{n-2}$, according to (56),

$$
\begin{aligned}
\overline{W}_{n-2,i}^n &= \sum_{k \in \mathcal{O}(i)} W_{n-1,k}^n(0) \\
&\leq \sum_{k \in \mathcal{O}(i)} \left(\left(\sum_{j \in \mathcal{O}(k)} m_j^n \right)^{1-\beta} + c(1-\beta) \left[\frac{\sqrt{d}L}{2^{n+1-d(1-\beta)}} + \frac{\sqrt{d}L}{2^n} \right] \right)^{\frac{1}{1-\beta}} .
\end{aligned}
\tag{57}
$$

Using the concavity inequality (53),

$$\overline{W}_{n-2,i}^n \leq 2^d \left[\left(\frac{\sum_{k \in \mathcal{O}(i), j \in \mathcal{O}(k)} m_j^n}{2^d} \right)^{1-\beta} + c(1-\beta) \left[\frac{\sqrt{d}L}{2^{n+1-d(1-\beta)}} + \frac{\sqrt{d}L}{2^n} \right] \right]^{\frac{1}{1-\beta}}$$

In the following, for each center x_i^k in $\mathcal{P}_k$, if there is a concatenated path from x_i^k to center $x_j^n \in \mathcal{P}_n$ in the dyadic irrigation plan χ_n, we say $i \prec j$. With this notation, the above estimate can be written as

$$\overline{W}_{n-2,i}^n \leq 2^d \left[\left(\frac{\sum_{i \prec j} m_j^n}{2^d} \right)^{1-\beta} + c(1-\beta) \left[\frac{\sqrt{d}L}{2^{n+1-d(1-\beta)}} + \frac{\sqrt{d}L}{2^n} \right] \right]^{\frac{1}{1-\beta}} . \tag{58}$$

Each branch from x_i^{n-2} to $\mathcal{P}_{n-3}$ has length $\frac{\sqrt{d}L}{2^{n-1}}$. By the formula (30), for $s \in [0, \frac{\sqrt{d}L}{2^{n-1}}]$,

$$
\begin{aligned}
W_{n-2,i}^n(s) &= \left(\left(\overline{W}_{n-2,i}^n \right)^{1-\beta} + c(1-\beta)\left(\frac{\sqrt{d}L}{2^{n-1}} - s \right) \right)^{\frac{1}{1-\beta}} \\
&\leq \left(\left(\sum_{i \prec j} m_j^n \right)^{1-\beta} + c(1-\beta) \left[\frac{\sqrt{d}L}{2^{n+1-2d(1-\beta)}} + \frac{\sqrt{d}L}{2^{n-d(1-\beta)}} + \left(\frac{\sqrt{d}L}{2^{n-1}} - s \right) \right] \right)^{\frac{1}{1-\beta}}
\end{aligned}
$$

$$W_{n-2,i}^n(0) \leq \left(\left(\sum_{i \prec j} m_j^n \right)^{1-\beta} + c(1-\beta) \left[\frac{\sqrt{d}L}{2^{n+1-2d(1-\beta)}} + \frac{\sqrt{d}L}{2^{n-d(1-\beta)}} + \frac{\sqrt{d}L}{2^{n-1}} \right] \right)^{\frac{1}{1-\beta}}$$

4. From $\mathcal{P}_{n-k}$ to $\mathcal{P}_{n-k-1}$. Similarly we have,

$$\overline{W}_{n-k,i}^n \leq 2^d \left(\left(\frac{\sum_{i \prec j} m_j^n}{2^d} \right)^{1-\beta} + \frac{c(1-\beta)\sqrt{d}L}{2^{n+2-k}} \sum_{l=0}^{k-1} \frac{1}{2^{[1-d(1-\beta)]l}} \right)^{\frac{1}{1-\beta}}, \tag{59}$$

$$W_{n-k,i}^n(s) \leq$$
$$\left(\left(\sum_{i \prec j} m_j^n\right)^{1-\beta} + \frac{c(1-\beta)\sqrt{d}L}{2^{n+1-k}} \sum_{l=1}^{k} \frac{1}{2^{[1-d(1-\beta)]l}} + c(1-\beta)\left(\frac{\sqrt{d}L}{2^{n+1-k}} - s\right)\right)^{\frac{1}{1-\beta}}, \quad (60)$$

$$W_{n-k,i}^n(0) \leq \left(\left(\sum_{i \prec j} m_j^n\right)^{1-\beta} + \frac{c(1-\beta)\sqrt{d}L}{2^{n+1-k}} \sum_{l=0}^{k} \frac{1}{2^{[1-d(1-\beta)]l}}\right)^{\frac{1}{1-\beta}}. \quad (61)$$

5. Since $W_{n-k,i}^n(s) \leq W_{n-k,i}^n(0)$, to have an uniform bound on the weight function, we only need to estimate $W_{n-k,i}^n(0)$, for each $0 \leq k \leq n-1, 1 \leq i \leq 2^{d(n-k)}$. When $\beta > 1 - 1/d$, one has $1 - d(1-\beta) > 0$. From formula (61),

$$W_{n-k,i}^n(0) \leq \left(\left(\sum_{i \prec j} m_j^n\right)^{1-\beta} + \frac{c(1-\beta)\sqrt{d}L}{2^{n+1-k}} \frac{1}{1 - \frac{1}{2^{1-d(1-\beta)}}}\right)^{\frac{1}{1-\beta}}. \quad (62)$$

Since $\sum_{i \prec j} m_j^n \leq M$, if denote W^n the weights on dyadic irrigation plan χ_n, from (62) there is an uniform bound for the weight function

$$W^n \leq \left(M^{1-\beta} + \frac{c(1-\beta)\sqrt{d}L}{1 - \frac{1}{2^{1-d(1-\beta)}}}\right)^{\frac{1}{1-\beta}} \leq C\left(M^{1-\beta} + L\right)^{\frac{1}{1-\beta}} \quad (63)$$

where we use the same C to denote all constants independent of n. This completes the proof of (48).

6. We now estimate the irrigation cost $\mathcal{E}^{W,\alpha}(\chi_n)$ by formula (22). In the dyadic irrigation plan χ_n, let E_n^n be the cost from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$, by (50),

$$
\begin{aligned}
E_n^n &= \sum_{x_i^n \in \mathcal{P}_n} \int_0^{\frac{\sqrt{d}L}{2^{n+1}}} \left(W_{n,i}^n(s)\right)^\alpha ds \\
&= \sum_{x_i^n \in \mathcal{P}_n} \int_0^{\frac{\sqrt{d}L}{2^{n+1}}} \left((m_i^n)^{1-\beta} + c(1-\beta)\left(\frac{\sqrt{d}L}{2^{n+1}} - s\right)\right)^{\frac{\alpha}{1-\beta}} ds.
\end{aligned} \quad (64)
$$

Similarly, denote E_{n-k}^n the cost from $\mathcal{P}_{n-k}$ to $\mathcal{P}_{n-k-1}$,

$$E_{n-k}^n = \sum_{x_i^{n-k} \in \mathcal{P}_{n-k}} \int_0^{\frac{\sqrt{d}L}{2^{n+1-k}}} \left(W_{n-k,i}^n(s)\right)^\alpha ds \quad (65)$$

From (61) and the non-decreasing of $W_{n-k,i}^n(s)$,

$$
\begin{aligned}
E_{n-k}^n &\leq \sum_{x_i^{n-k} \in \mathcal{P}_{n-k}} \frac{\sqrt{d}L}{2^{n+1-k}} \left(\left(\sum_{i \prec j} m_j^n\right)^{1-\beta} + \frac{c(1-\beta)\sqrt{d}L}{2^{n+1-k}(1 - \frac{1}{2^{1-d(1-\beta)}})}\right)^{\frac{\alpha}{1-\beta}} \\
&\leq \sum_{x_i^{n-k} \in \mathcal{P}_{n-k}} \left[\frac{CL(\sum_{i \prec j} m_j^n)^\alpha}{2^{n+1-k}} + \frac{CL^{1+\frac{\alpha}{1-\beta}}}{2^{(n+1-k)(\frac{\alpha}{1-\beta}+1)}}\right] \doteq I_{n-k} + J_{n-k}
\end{aligned} \quad (66)
$$

where C is some constant that only depends on α, β, c and on the dimension d. The cardinality of $\mathcal{P}_{n-k}$ is $2^{(n-k)d}$. Therefore

$$J_{n-k} \doteq \sum_{x_i^{n-k} \in \mathcal{P}_{n-k}} \frac{CL^{1+\frac{\alpha}{1-\beta}}}{2^{(n+1-k)(\frac{\alpha}{1-\beta}+1)}} \leq \frac{CL^{1+\frac{\alpha}{1-\beta}}}{2^{(n-k)(1+\frac{\alpha}{1-\beta}-d)}}. \quad (67)$$

On the other hand, $1 \geq \alpha > 0$, by elementary concavity inequality,

$$
\begin{aligned}
I_{n-k} &\doteq \sum_{x_i^{n-k} \in \mathcal{P}_{n-k}} \frac{CL(\sum_{i \prec j} m_j^n)^\alpha}{2^{n+1-k}} \\
&\leq 2^{(n-k)d} \left(\frac{\sum_{x_i^{n-k} \in \mathcal{P}_{n-k}} \sum_{i \prec j} m_j^n}{2^{(n-k)d}}\right)^\alpha \frac{CL}{2^{n+1-k}} \leq \frac{CM^\alpha L}{2^{(n-k)[1-d(1-\alpha)]}}.
\end{aligned} \quad (68)
$$

When $1 \geq \alpha > 1 - 1/d$ and $1 > \beta > 1 - 1/d$, one has

$$1 - d(1-\alpha) > 0, \qquad 1 + \frac{\alpha}{1-\beta} - d > 0. \quad (69)$$

Therefore, using (66)-(68),

$$
\begin{aligned}
\mathcal{E}^{W,\alpha}(\chi_n) &= \sum_{k=0}^{n-1} E_{n-k}^n \ \leq \ \sum_{k=0}^{n-1} \left[I_{n-k} + J_{n-k} \right] \\
&\leq \ C \sum_{j=0}^{n} \left[\frac{LM^\alpha}{2^{[1-d(1-\alpha)]j}} + \frac{L^{1+\frac{\alpha}{1-\beta}}}{2^{(1+\frac{\alpha}{1-\beta}-d)j}} \right] \\
&\leq \ C\left(LM^\alpha + L^{1+\frac{\alpha}{1-\beta}} \right)
\end{aligned}
\tag{70}
$$

where C is some constant independent of n. This completes the proof of (49). $\qquad\square$

By the previous results, when $f(z) \doteq cz^\beta$ in (6)-(8), with the conditions in Lemma 3.2, we have the uniform bounds (49) for the dyadic irrigation plan sequence $\{\chi_n\}_{n\geq 1}$. Since each χ_n is an admissible irrigation plan for μ_n, by the definition (25), we have a uniform bound on all the irrigation costs $\mathcal{I}^{W,\alpha}(\mu_n)$, $n \geq 1$. By the weak convergence $\mu_n \rightharpoonup \mu$ and the lower semicontinuity of the irrigation cost, stated in Theorem 2.7, we conclude $\mathcal{I}^{W,\alpha}(\mu) < +\infty$.

By a comparison argument we can now prove the irrigability for a wide class of functions f and measures μ, with the weighted irrigation cost $\mathcal{I}^{W,\alpha}$ in (25).

Theorem 3.3. *Let μ be a positive, bounded Radon measure in $\mathbf{R}^d$, with total mass $M > 0$ and supported in the cube Q of edge size $L > 0$. Assume $\alpha > 1 - \frac{1}{d}$, f satisfies* **(A1)** *and*

$$
\limsup_{z\to 0^+} z^{-\beta} f(z) \ < \ +\infty
\tag{71}
$$

for some $1 > \beta > 1 - \frac{1}{d}$. Then $\mathcal{I}^{W,\alpha}(\mu) < +\infty$.

Proof. The assumptions (71) and **(A1)** together imply that

$$
\begin{aligned}
f(z) &\leq \ cz^\beta && \forall z \in [0, 2^{\frac{1}{1-\beta}} z_0], \\
f(z) &\leq \ cz && \forall z \in [z_0, \infty),
\end{aligned}
\tag{72}
$$

with some constants $c, z_0 > 0$. We will prove that the weighted irrigation costs of the dyadic approximated measures μ_n, defined as in (29), are uniformly bounded. Since μ_n weakly converges to μ, by Theorem 2.7, this uniform bound implies the boundedness of $\mathcal{I}^{W,\alpha}(\mu)$.

It suffices to prove the uniform bound for dyadic approximated measures $\mu_n = \sum_{x_i^n \in \mathcal{P}_n} m_i^n \delta_{x_i^n}$ with $n \geq n_0$, where n_0 is some fixed integer. Choose n_0 large enough such that in (62),

$$
\frac{c(1-\beta)\sqrt{d}L}{2^{n_0} \cdot \left(1 - \frac{1}{2^{1-d(1-\beta)}}\right)} \ < \ z_0^{1-\beta}.
\tag{73}
$$

In the following, we construct the irrigation plan for μ_n with uniformly bounded weighted cost.

1. Consider first from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$. For those x_i^n such that $m_i^n \geq z_0$, we transport the particles at x_i^n along a straight path directly to the origin. Let $\mathcal{S}_n$ be the set of all such paths. For each path in $\mathcal{S}_n$, the multiplicity is larger than z_0 and bounded by M. The length of path is bounded by $\sqrt{d}L$. Let $W(t)$ be the weight function on these paths, then clearly $W(t) \geq z_0$. By formula (6)-(8) and (72) the weight satisfies

$$
W(t) \ \leq \ \int_t^{\sqrt{d}L} f(W(s))\,ds + M \ \leq \ \int_t^{\sqrt{d}L} cW(s)\,ds + M \ \leq \ e^{c\sqrt{d}L} M.
\tag{74}
$$

On the other hand, for the remaining centers x_i^n, we transport the particles from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$, using the branches of the dyadic irrigation plan χ_n, defined as in Lemma 3.2. Notice on each such branch, $m_i^n < z_0$. Then from (51) and (73), the weight $W_{n,i}^n : [0, \frac{\sqrt{d}L}{2^{n+1}}] \mapsto \mathbf{R}_+$ on the branch γ_i from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$ satisfies

$$W_{n,i}^n(s) = \left((m_i^n)^{1-\beta} + c(1-\beta)\frac{\sqrt{d}L}{2^{n+1}}\right)^{\frac{1}{1-\beta}} \leq (z_0^{1-\beta} + z_0^{1-\beta})^{\frac{1}{1-\beta}} = 2^{\frac{1}{1-\beta}} z_0, \quad (75)$$

where we compute the weight $W_{n,i}^n$ as solution to $\dot{W}_{n,i}^n = c(W_{n,i}^n)^\beta$. Let W_i be the corresponding solution of (6) with $m_i(t)$ replaced by constant multiplicity m_i^n, by (72) and comparision principle from ODE theory,

$$W_i(s) \leq W_{n,i}^n(s). \quad (76)$$

Then clearly the total cost on these dyadic branches from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$ is bounded by E_n^n, given in (64).

2. From $\mathcal{P}_{n-1}$ to $\mathcal{P}_{n-2}$. After removing the point masses transported by branches in $\mathcal{S}_n$, we still denote the remaining measure as μ_n, and transport μ_n to the centers in $\mathcal{P}_{n-1}$, using the branches from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$ of the dyadic irrigation plan χ_n. Notice that after removing the masses transported by branches in $\mathcal{S}_n$, $m_i^n \leq z_0$ for each $1 \leq i \leq 2^{nd}$, with some $m_i^n = 0$.

For each center x_i^{n-1} in $\mathcal{P}_{n-1}$, when

$$\sum_{j \in \mathcal{O}(i)} m_j^n \geq z_0 \quad (77)$$

we then connect x_i^{n-1} to the origin directly by a straight branch. Let $\mathcal{S}_{n-1}$ be the set of all such branches. Similarly as in (74), the weight on each branch in $\mathcal{S}_{n-1}$ is bounded by $e^{c\sqrt{d}L}M$. For the remaining x_i^{n-1}, we transport the flux from $\mathcal{P}_{n-1}$ to $\mathcal{P}_{n-2}$, by the branches of dyadic irrigation plan χ_n. From (62) and (72)-(73), on each dyadic branch γ_i from $\mathcal{P}_{n-1}$ to $\mathcal{P}_{n-2}$,

$$W_i(s) \leq W_{n-1,i}^n(s) < 2^{\frac{1}{1-\beta}} z_0, \qquad s \in [0, \frac{\sqrt{d}L}{2^n}]. \quad (78)$$

Then clearly the total cost on these dyadic branches from $\mathcal{P}_{n-1}$ to $\mathcal{P}_{n-2}$ is bounded by E_{n-1}^n, defined by (65).

3. By backward induction we construct the irrigation plan until to the level $\mathcal{P}_{n_0}$. For each $k > n_0$, from $\mathcal{P}_k$ to $\mathcal{P}_{k-1}$, there are two types of paths, one is the branches in $\mathcal{S}_k$, and the other one is the dyadic branches of χ_n. Clearly we have

$$\#\left(\bigcup_{k>n_0}^n \mathcal{S}_k\right) \leq \frac{M}{z_0} \quad (79)$$

where M is the total mass of μ. Indeed, from our construction, each branch in $\cup_{k>n_0}^n \mathcal{S}_k$ will transport distinct groups of particles with mass $\geq z_0$, the total mass of μ_n is M, thus we have the upper bound in (79). For each branch in $\mathcal{S}_k$, there is an uniform bound (74) on the weight $W(t)$, and the length of each branch is bounded by $\sqrt{d}L$, thus the total cost J on branches in $\mathcal{S}_k, k > n_0$ is bounded by

$$J \leq \frac{M}{z_0} \cdot \left(e^{c\sqrt{d}LM}\right)^\alpha \sqrt{d}L \doteq \kappa_0 \quad (80)$$

On the other hand, the total cost I on the dyadic branches is bounded by

$$I \ \leq \ \sum_{k>n_0}^{n} E_k^n \ \leq \ C\Big(M^\alpha L + L^{1+\frac{\alpha}{1-\beta}}\Big) \ \doteq \ \kappa_1 \tag{81}$$

where the last inequality comes from (49).

Notice the bounds in (80)-(81) are independent of n, therefore, there exists a uniform constant $C > 0$, such that for each dyadic approximation μ_n, we have $\mathcal{I}^{W,\alpha}(\mu_n) \leq C$. Thanks to Theorem 2.7, we conclude that $\mathcal{I}^{W,\alpha}(\mu) \ \leq \ C$. $\qquad\square$

3.1. **Examples of non-irrigable measures.** In the following we show some cases for measures μ with infinite weighted irrigation cost $\mathcal{I}^{W,\alpha}$.

Definition 3.4. Let μ be a positive, bounded measure in $\mathbf{R}^d$. If there exists $\gamma > 0$ and a constant $C \geq 1$ such that

$$\frac{1}{C}r^\gamma \ \leq \ \mu(B(x,r)) \ \leq \ Cr^\gamma, \qquad \forall x \in \text{supp}(\mu), \ r \in [0,1], \tag{82}$$

then we say μ is Ahlfors regular in dimension γ. Here $\text{supp}(\mu)$ is the support of μ, $B(x,r)$ is the ball of radius r that centered at x.

Remark 1. If a measure μ is Ahlfors regular in dimension γ, then one can prove $supp(\mu)$ has Hausdorff dimension γ. Indeed, consider any covering $\cup_{i=1}^\infty B(x_i, r_i)$ of $supp(\mu)$, consists of closed balls with radius less than 1. From the second inequality in (82) one has

$$\sum_{i\geq 1}(r_i)^\gamma \ \geq \ \sum_{i\geq 1}\frac{\mu(B(x_i,r_i))}{C} \ \geq \ \frac{M}{C} \ > \ 0,$$

which implies $supp(\mu)$ has Hausdorff dimension $\geq \gamma$. On the other hand, by the Vitali's Convering Theorem[11], there exists a countable subcollection of disjoint $B(x_i, r_i)$, which we still denote as $\sum_{i=1}^\infty B(x_i, r_i)$, such that $supp(\mu) \subseteq \cup_{i=1}^\infty B(x_i, 5r_i)$. Then from the first inequality in (82), since $B(x_i, r_i)$ are disjoint,

$$\sum_{i\geq 1}(5r_i)^\gamma \ = \ 5^\gamma \sum_{i\geq 1} r_i^\gamma \ \leq \ 5^\gamma C \sum_{i\geq 1} \mu(B(x_i,r_i)) \ \leq \ 5^\beta CM,$$

and it implies the Hausdorff dimension of $supp(\mu) \leq \gamma$.

For the irrigation cost $\mathcal{I}^\alpha(\cdot)$ without weights that defined in [15], we recall the following theorem. For a proof, see Theorem 1.2 in [15].

Theorem 3.5. *Let μ be a finite α-irrigable measure, with $\alpha \in (0,1)$. That is, $\mathcal{I}^\alpha(\mu) < \infty$. Then there is a Borel set $E \subseteq \mathbf{R}^d, \mu(\mathbf{R}^d \setminus E) = 0$, such that for any $s > \frac{1}{1-\alpha}$,*

$$\mathcal{H}^s(E) \ = \ 0,$$

where $\mathcal{H}^s(E)$ is the s-Hausdorff measure of the set E. In other words, if μ is α-irrigable, then μ is concentrated on a set E with Hausdorff dimension $\leq \frac{1}{1-\alpha}$.

Remark 2. As mentioned in [7], for any bounded Radon measure μ, we always have $\mathcal{I}^{W,\alpha}(\mu) \geq \mathcal{I}^\alpha(\mu)$. Therefore, if $\mathcal{I}^{W,\alpha}(\mu) < +\infty$, from Theorem 3.5, μ is concentrated on a set E with Hausdorff dimension $\leq \frac{1}{1-\alpha}$.

Lemma 3.6. *If μ is a bounded Radon measure as in Theorem 3.3 and let χ be an irrigation plan of μ with finite weighted irrigation cost $\mathcal{E}^{W,\alpha}(\chi) < \infty$. Then for any $r > 0$,*

$$\mu(\mathbf{R}^d \setminus B(0,r)) \ \leq \ \left(\frac{\mathcal{E}^{W,\alpha}(\chi)}{r}\right)^{\frac{1}{\alpha}}. \tag{83}$$

Proof. The function

$$x \ \mapsto \ \left(x^{1-\beta} + c(1-\beta)(r-t)\right)^{\frac{1}{1-\beta}}, \qquad x \in \mathbf{R}_+$$

is concave. Let $m_r \doteq \mu(\mathbf{R}^d \setminus B(0,r))$, then by definition (21) and (30) we have

$$\int_0^r \left(m_r^{1-\beta} + c(1-\beta)(r-t)\right)^{\frac{\alpha}{1-\beta}} dt \ \leq \ \mathcal{E}^{W,\alpha}(\chi). \tag{84}$$

Since $r - t \geq 0$, it implies that

$$\left(m_r^{1-\beta}\right)^{\frac{\alpha}{1-\beta}} \cdot r \ = \ m_r^\alpha \cdot r \ \leq \ \mathcal{E}^{W,\alpha}(\chi),$$

which completes the proof of (83). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Theorem 3.7. *Let μ be a positive, bounded Radon measure in $\mathbf{R}^d$ and Ahlfors regular in dimension d. Let f satisfy **(A1)**.*

If either $\alpha < 1 - \frac{1}{d}$ or

$$\liminf_{z \to 0^+} \ z^{-\beta} f(z) \ > \ 0 \tag{85}$$

for some $\beta < 1 - \frac{1}{d-1}$, then $\mathcal{I}^{W,\alpha}(\mu) = +\infty$.

Proof. **Case 1:** If $\alpha < 1 - \frac{1}{d}$, then $\frac{1}{1-\alpha} < d$. Suppose $\mathcal{I}^{W,\alpha}(\mu) < +\infty$, by Remark 2, μ is concentrated on a set E with Hausdorff dimension $\leq \frac{1}{1-\alpha} < d$, which is a contradiction to the assumption that μ is Ahlfors regular in dimension d (see Remark 3.5). Thus, we have $\mathcal{I}^{W,\alpha}(\mu) = +\infty$.

Case 2: The assumption (85) implies that, for some constants $c, z_0 > 0$,

$$f(z) \ \geq \ c z^\beta \qquad\qquad \forall z \in [0, z_0]. \tag{86}$$

Since μ is Ahlfors regular in dimension d, then for each irrigation plan χ and any $\delta > 0$, there are $\mathcal{O}(\frac{1}{\delta^d})$ disjoint cubes with diameter δ and each of them has measure $\approx \delta^d$. In each cube, the lower bound for the cost is

$$\int_0^\delta \left(\delta^{d(1-\beta)} + c(1-\beta)(\delta-t)\right)^{\frac{\alpha}{1-\beta}} dt \tag{87}$$

and the total number of such disjoint cubes is $\frac{1}{\delta^d}$.

$$\begin{aligned}
\mathcal{E}^{W,\alpha}(\chi) \ &\geq \ \tfrac{1}{\delta^d} \int_0^\delta \left(\delta^{d(1-\beta)} + c(1-\beta)(\delta-t)\right)^{\frac{\alpha}{1-\beta}} dt \\[2mm]
&\geq \ \tfrac{1}{\delta^d} \int_0^\delta \left(c(1-\beta)(\delta-t)\right)^{\frac{\alpha}{1-\beta}} dt \\[2mm]
&\geq \ C\delta^{\frac{1+\alpha-\beta}{1-\beta}-d}
\end{aligned} \tag{88}$$

where C is some constant independent of δ. Since $1 \geq \alpha > 1 - \frac{1}{d}, 1 - \frac{1}{d-1} > \beta > 0$, we have $\frac{1+\alpha-\beta}{1-\beta} < d$. Sending δ to $0+$, the right hand side in (88) goes to $+\infty$. Thus, for any irrigation plan χ of μ, $\mathcal{E}^{W,\alpha}(\chi) = +\infty$ and we conclude $\mathcal{I}^{W,\alpha}(\mu) = +\infty$. $\quad\square$

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