NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2021013
©American Institute of Mathematical Sciences
Volume **16**, Number **3**, September **2021** pp. **459–492**

EMERGENT BEHAVIORS OF LOHE HERMITIAN SPHERE PARTICLES UNDER TIME-DELAYED INTERACTIONS

Seung-Yeal Ha

Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826 and
Korea Institute for Advanced Study, Hoegiro 85, Seoul 02455, Republic of Korea

Gyuyoung Hwang* and Hansol Park

Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826, Republic of Korea

(Communicated by Young-Pil Choi)

Abstract. We study emergent behaviors of the Lohe Hermitian sphere(LHS) model with a time-delay for a homogeneous and heterogeneous ensemble. The LHS model is a complex counterpart of the Lohe sphere(LS) aggregation model on the unit sphere in Euclidean space, and it describes the aggregation of particles on the unit Hermitian sphere in $\mathbb{C}^d$ with $d \geq 2$. Recently it has been introduced by two authors of this work as a special case of the Lohe tensor model. When the coupling gain pair satisfies a specific linear relation, namely the Stuart-Landau(SL) coupling gain pair, it can be embedded into the LS model on $\mathbb{R}^{2d}$. In this work, we show that if the coupling gain pair is close to the SL coupling pair case, the dynamics of the LHS model exhibits an emergent aggregate phenomenon via the interplay between time-delayed interactions and nonlinear coupling between states. For this, we present several frameworks for complete aggregation and practical aggregation in terms of initial data and system parameters using the Lyapunov functional approach.

1. **Introduction.**

Emergent dynamics of a many-body system is ubiquitous in classical and quantum systems, e.g., aggregation of bacteria [38, 39], flocking of birds [2], schooling of fish, synchronization of fireflies and neurons [7, 34, 41, 42] and hand clapping of people in a concert hall, etc. For surveys and books, we refer to [1, 2, 4, 16, 19, 35, 36, 40, 42]. In this paper, we continue studies begun in [8, 24] on the emergent dynamics of the LHS model. The LHS model corresponds to the complex counterpart of the Lohe sphere(LS) model which has been extensively studied in previous literature [11, 26, 32, 33, 37, 43]. The LHS model is the first-order aggregation model describing continuous-time dynamics of particle's position on the Hermitian unit sphere $\mathbb{HS}^{d-1} := \left\{ z = ([z]_1, \cdots, [z]_d) \in \mathbb{C}^d : \ \|z\| := \sqrt{\sum_{\alpha=1}^d |[z]_\alpha|^2} = 1 \right\}$ with $d \geq 1$. Here we denote the α-th component of the complex vector $z \in \mathbb{C}^d$ as

2020 *Mathematics Subject Classification.* Primary: 82C10, 82C22, 35Q40.

Key words and phrases. Emergence, Hermitian sphere, tensor, time-delay.

The work of S.-Y. Ha was supported by National Research Foundation of Korea(NRF-2020R1A2C3A01003881). The work of H. Park was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2019R1I1A1A01059585).

* Corresponding author: Gyuyoung Hwang.

$[z]_\alpha$ which is consistent with earlier notation in [23]. As a warmup for our discussion, we briefly introduce the LHS model with time-delayed interactions.

Let $z_j = ([z_j]_1, \cdots, [z_j]_d) \in \mathbb{C}^d$ be a position of the j-th Hermitian Lohe particle on the Hermitian unit sphere, and interaction weight between the j-th and k-th particle is denoted by the real value $a_{jk} \in \mathbb{R}$. Then, the temporal dynamics of z_j is governed by the Cauchy problem to the LHS model with a uniform time-delay $\tau > 0$:

$$\begin{cases} \dot{z}_j = \Omega_j z_j + \dfrac{\kappa_0}{N} \displaystyle\sum_{k \neq j} a_{jk} \big(\langle z_j, z_j \rangle z_k^\tau - \langle z_k^\tau, z_j \rangle z_j \big) \\ \qquad\qquad\qquad + \dfrac{\kappa_1}{N} \displaystyle\sum_{k \neq j} a_{jk} \big(\langle z_j, z_k^\tau \rangle - \langle z_k^\tau, z_j \rangle \big) z_j, \quad t > 0, \\ z_j(t) = \varphi_j(t) \in \mathbb{HS}^{d-1}, \quad -\tau \leq t \leq 0, \quad j \in \mathcal{N} := \{1, \cdots, N\}, \end{cases} \tag{1}$$

where $z_k^\tau(t) := z_k(t-\tau)$, $\varphi_j = \varphi_j(t)$ is a bounded continuous function of t, Ω_j is a $d \times d$ skew-Hermitian matrix and $(a_{ik}) \in \mathbb{R}^{N \times N}$ is a symmetric matrix whose components are all positive. Before we continue further, we introduce

$$\langle w, z \rangle := \sum_{\alpha=1}^d [\bar{w}]_\alpha [z]_\alpha, \quad \|z\| := \sqrt{\langle z, z \rangle}, \quad \bar{w} = (\overline{[w]_1}, \cdots, \overline{[w]_d}).$$

A global well-posedness of system (1) can be done by combining a local well-posedness from the standard Cauchy-Lipschitz theory in [25, 28] and a priori uniform bound in Lemma 2.1. In the absence of time-delay with $\tau = 0$, emergent dynamics of the LHS model was investigated in [8, 24] in which several sufficient frameworks were proposed for complete and practical aggregations. In this paper, we are interested in the following simple question:

> "Under what conditions on system parameters κ_0, κ_1, τ, network topology (a_{ij}) and initial data set $\{\varphi_j\}$, can we verify the emergence of collective behaviors of the LHS with time-delay?"

This question has been already addressed for other low-rank aggregation models (rank-1: vectors or rank-2: matrices), to name a few, the Lohe sphere model [9, 10], the Lohe matrix model [18]. Throughout the paper, we set

$$Z := (z_1, \cdots, z_N), \quad D(Z) := \max_{1 \leq i,j \leq N} \|z_i - z_j\|.$$

Next, we recall several induced concepts on the emergent dynamics of tensors [23, 24] in the following definition.

Definition 1.1. Let $\{z_i\}$ be a global solution to (1).

1. Complete aggregation occurs asymptotically if the ensemble diameter $D(Z)$ tends to zero asymptotically:
$$\lim_{t \to \infty} D(Z(t)) = 0.$$
2. Practical aggregation (with respect to time-delay) occurs asymptotically if the ensemble diameter $D(Z)$ satisfies
$$\lim_{\tau \to 0+} \limsup_{t \to \infty} D(Z(t)) = 0.$$

3. Practical aggregation (with respect to time-delay and coupling strength κ_0) occurs asymptotically if the ensemble diameter $D(Z)$ satisfies

$$\lim_{\kappa_0\to\infty}\lim_{\tau\to 0+}\limsup_{t\to\infty} D(Z(t)) = 0.$$

Then, it is easy to see that complete aggregation implies practical aggregation. In the absence of time-delay $\tau = 0$, emergent dynamics for (1) has been extensively studied in [24] (see Section 2.3). Thus, main point of this paper is to analyze the effect of time-delayed interactions in the emergent dynamics of (1). For notational simplicity, we set

$$\max_{i,j} := \max_{1\le i,j\le N}, \quad \min_{i,j} := \min_{1\le i,j\le N}, \quad \sum_{k\neq j} := \sum_{\substack{k,j=1\\ k\neq j}}^{N}.$$

The main results of this paper are threefold. First, we consider the following setting:

$$a_{ik} \equiv 1, \quad \Omega_j = 0, \quad \forall\ i,k \in \mathcal{N}.$$

In this case, system (1) becomes

$$\begin{cases} \dot z_j = \dfrac{\kappa_0}{N}\displaystyle\sum_{k\neq j}(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j) + \dfrac{\kappa_1}{N}\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j,\ t>0, \\ z_j(t) = \varphi_j(t) \in \mathbb{HS}^{d-1}, \quad -\tau \le t \le 0. \end{cases} \tag{2}$$

When the coupling gain pair (κ_0, κ_1) is close to the SL coupling gain pair, i.e.,

$$\tilde\kappa := \frac{\kappa_0}{2} + \kappa_1, \quad |\tilde\kappa| \ll 1,$$

system (2) can be rewritten as follows (see Section 3):

$$\begin{cases} \dot z_j = \dfrac{\kappa_0}{N}\displaystyle\sum_{k\neq j}\Big(\langle z_j, z_j\rangle z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_j\rangle)z_j\Big) + \dfrac{\tilde\kappa}{N}\sum_{k\neq j}\Big(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\Big)z_j,\ t>0, \\ z_j(t) = \varphi_j(t) \in \mathbb{HS}^{d-1}, \quad -\tau \le t \le 0. \end{cases} \tag{3}$$

Our first set of results is concerned with the complete aggregation of (3) (see Theorem 3.1 and Theorem 3.6). We assume that system parameters and initial data satisfy

$$\kappa_0 > 0, \quad |\tilde\kappa| \ll \kappa_0, \quad \tau \ll 1, \quad N \ge 3, \quad \sup_{-\tau\le t\le 0} D(Z(t)) \ll 1.$$

For the complete aggregation, we introduce a Lyapunov functional:

$$\mathcal{E}_{ij}(t) := \|z_i(t) - z_j(t)\|^2 + \gamma\int_{t-\tau}^{t}\|z_i(s) - z_j(s)\|^2 ds,$$

where γ is a constant to be determined later. Then, in Section 3, we show that $\mathcal{E}_{ij}(t)$ satisfies the energy estimate (see Section 3.2.2):

$$\mathcal{E}_{ij}(t) + \beta\int_0^t \|z_i(s) - z_j(s)\|^2 ds \le \mathcal{E}_{ij}(0), \quad \forall\ t > 0,$$

for some positive constant β. By Barbalat's lemma [3], the above estimate leads to the complete aggregation (see Theorem 3.6):

$$\lim_{t\to\infty}\|z_i(t) - z_j(t)\| = 0.$$

Now, our second set of result deals with the practical aggregation with respect to time-delay (Theorem 4.1). We assume that system parameters and initial data satisfy

$$a_{ij} = 1, \quad \Omega_j = 0, \quad 2|\kappa_1| < \kappa_0, \quad \max_{i,j}\Big(1 - \langle z_i^0, z_j^0\rangle\Big) < 1 - \frac{2|\kappa_1|}{\kappa_0}.$$

Then, the practical aggregation (with respect to the size of time-delay) emerges:

$$\lim_{\tau \searrow 0} \limsup_{t\to\infty} \max_{i,j}\Big(1 - \langle z_i(t), z_j(t)\rangle\Big) = 0.$$

Our final set of result is concerned with the practical aggregation with respect to both time-delay and free flow matrix Ω_j (Theorem 4.7). For system (1), we assume that system parameters satisfy

$$\frac{\sum_{k=1}^N |a_{ik} - a_{jk}|}{\sum_{k=1}^N (a_{ik} + a_{jk})} < \frac{1}{2} \quad \text{and} \quad \max_{i,j} |1 - \langle z_i^0, z_j^0\rangle| < 1 - \frac{2\sum_{k=1}^N |a_{ik} - a_{jk}|}{\sum_{k=1}^N (a_{ik} + a_{jk})}. \tag{4}$$

Note that for any unit complex vectors z_i^0 and z_j^0, we have

$$|1 - \langle z_i^0, z_j^0\rangle| \ge \frac{1}{2}\|z_i^0 - z_j^0\|^2,$$

Thus, relations (4) implies restriction on initial diameter and network structure:

$$D(Z^0) := \max_{i,j\in\mathcal{N}} |z_j^0 - z_i^0| \le \sqrt{2 - \frac{4\sum_{k=1}^N |a_{ik} - a_{jk}|}{\sum_{k=1}^N (a_{ik} + a_{jk})}}. \tag{5}$$

For all-to-all network structure with $a_{ij} = 1$, the R.H.S. of (5) becomes

$$D(Z^0) \le \sqrt{2},$$

which is true for any initial data. Then, the practical aggregation (with respect to time-delay and coupling strength κ_0) emerges:

$$\lim_{\kappa_0\to\infty} \lim_{\tau\searrow 0} \limsup_{t\to\infty} \max_{i,j}\Big(1 - \langle z_i(t), z_j(t)\rangle\Big) = 0.$$

Note that although we imposed the initial condition on $\varphi_j(t)$ for a time-strip $-\tau \le t \le 0$, we require that the initial condition depends on the initial data at $t = 0$ for practical aggregation estimate in Theorem 4.7.

The rest of paper is organized as follows. In Section 2, we present conservation laws for the LHS model with time-delay, its reduction to other aggregation models, and review previous results on the emergent dynamics for the LHS model without time-delay and LS model with a time-delay. In Section 3, we provide a sufficient framework for the complete aggregation when the coupling gain pair is close to that of SL coupling gain pair. In Section 4, we provide a sufficient framework leading to the practical aggregation under a general setting. Finally, Section 5 is devoted to a brief summary of main results and some open problems.

2. **Preliminaries.**

In this section, we discuss two conservation laws of the LHS model with time-delay and its reduction to other aggregation models, and review previous results on the emergent dynamics for the LHS model.

2.1. **Conservation laws.** In this subsection, we study conservation laws associated with (1).

Lemma 2.1. (Conservation of modulus) *Let* $\{z_j\}$ *be a global solution to* (1) *with initial data satisfying* $\|\phi_j(t)\| = 1$ *for* $-\tau \le t \le 0$, $j \in \mathcal{N}$. *Then, the modulus of* z_j *satisfies*

$$\|z_j(t)\| = 1, \quad t \ge 0, \quad j \in \mathcal{N}.$$

i.e., the Hermitian sphere $\mathbb{HS}^{d-1}$ *is a positively invariant set for* (1).

Proof. We use $(a_{jk}) \in \mathbb{R}^{N\times N}$ and sesquilinearity of the inner product to find

$$\begin{aligned}
&\frac{d}{dt}\|z_j\|^2 = \langle \dot z_j, z_j\rangle + \langle z_j, \dot z_j\rangle \\
&= \left\langle \Omega_j z_j + \frac{\kappa_0}{N}\sum_{k\neq j} a_{jk}\big(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j\big) + \frac{\kappa_1}{N}\sum_{k\neq j} a_{jk}\big(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\big) z_j,\ z_j \right\rangle \\
&+ \left\langle z_j,\ \Omega_j z_j + \frac{\kappa_0}{N}\sum_{k\neq j} a_{jk}\big(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j\big) + \frac{\kappa_1}{N}\sum_{k\neq j} a_{jk}\big(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\big) z_j \right\rangle \\
&= \langle \Omega_j z_j, z_j\rangle + \langle z_j, \Omega_j z_j\rangle \\
&+ \frac{\kappa_0}{N}\sum_{k\neq j} a_{jk}\big(\overline{\langle z_j, z_j\rangle}\langle z_k^\tau, z_j\rangle - \overline{\langle z_k^\tau, z_j\rangle}\langle z_j, z_j\rangle\big) + \frac{\kappa_0}{N}\sum_{k\neq j} a_{jk}\big(\langle z_j, z_j\rangle\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\langle z_j, z_j\rangle\big) \\
&+ \frac{\kappa_1}{N}\sum_{k\neq j} a_{jk}\big(\overline{\langle z_j, z_k^\tau\rangle}\langle z_j, z_j\rangle - \overline{\langle z_k^\tau, z_j\rangle}\langle z_j, z_j\rangle\big) + \frac{\kappa_1}{N}\sum_{k\neq j} a_{jk}\big(\langle z_j, z_k^\tau\rangle\langle z_j, z_j\rangle - \langle z_k^\tau, z_j\rangle\langle z_j, z_j\rangle\big) \\
&=: \sum_{i=1}^{6} \mathcal{I}_{1i}.
\end{aligned} \tag{6}$$

Below, we estimate the terms $\mathcal{I}_{1i}$ with $1 \le i \le 6$ one by one.

- Case A (Estimates on $\mathcal{I}_{11} + \mathcal{I}_{12}$): Since Ω_j is skew-Hermitian, we have

$$\mathcal{I}_{11} + \mathcal{I}_{12} = \langle \Omega_j z_j, z_j\rangle + \langle z_j, \Omega_j z_j\rangle = \langle \Omega_j z_j, z_j\rangle + \langle \Omega_j^\dagger z_j, z_j\rangle = \langle \Omega_j z_j, z_j\rangle - \langle \Omega_j z_j, z_j\rangle = 0.$$

- Case B (Estimates on $\mathcal{I}_{13} + \mathcal{I}_{14}$): We use $\langle z_j, z_k^\tau\rangle = \overline{\langle z_k^\tau, z_j\rangle}$ to see that

$$\mathcal{I}_{13} + \mathcal{I}_{14} = 0.$$

- Case C (Estimates on $\mathcal{I}_{15} + \mathcal{I}_{16}$): Similar to Case B, one has

$$\mathcal{I}_{15} + \mathcal{I}_{16} = 0.$$

Finally we combine all the estimates in Cases A, B, and C to obtain

$$\frac{d}{dt}\|z_j(t)\|^2 = 0, \quad \forall\, t > 0, \quad j \in \mathcal{N}.$$

This yields

$$\|z_j(t)\| = \|z_j(0)\| = \|\varphi_j(0)\| = 1.$$

□

Remark 1. Note that the symmetry of a_{ij} plays no role in the conservation of modulus.

Lemma 2.2. (Propagation of real-valuedness) *Suppose that* $\{\Omega_j\}$ *and initial data set* $\{\varphi_j\}$ *satisfy the relations:*

$$\Omega_j \in \mathbb{R}^{d\times d}, \quad \Omega_j^T = -\Omega_j, \quad \varphi_j(t) \in \mathbb{R}^d, \quad \|\varphi_j(t)\| = 1$$

for all $j \in \mathcal{N}$ *and* $-\tau \le t \le 0$*, and let* $\{z_j\}$ *be a solution to system* (1)*. Then* z_j *is a real-valued state, i.e.,*

$$\mathrm{Im}([z_j(t)]_\alpha) = 0, \quad \forall t \ge 0,\ \alpha \in \{1, \cdots, d\}, \quad j \in \mathcal{N}.$$

Proof. This follows from the standard uniqueness theory of time-delayed ordinary differential equations [25, 28] □

2.2. **Reduction to aggregation models.** In this subsection, we discuss the reductions of (1) to the Lohe sphere model and the Kuramoto model. Suppose that initial data set $\{\varphi_j\}$ satisfy

$$\varphi_j(t) \in \mathbb{R}^d, \quad \|\varphi_j(t)\| = 1,$$

for all $j \in \mathcal{N}$ and $-\tau \le t \le 0$. Then, it follows from Lemma 2.1 and Lemma 2.2 that

$$z_j(t) \in \mathbb{S}^{d-1} \subset \mathbb{R}^d.$$

In this case, the coupling terms in the R.H.S. of (1) become

$$\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j = \|z_j\|^2 z_k^\tau - \langle z_k^\tau, z_j\rangle z_j, \quad (\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle) z_j = 0.$$

We set

$$x_j(t) := z_j(t), \quad j \in \mathcal{N},\ t \ge 0.$$

Then the real-valued state $x_j \in \mathbb{R}^d$ satisfies the LS model with time-delay [10]:

$$\begin{cases} \dot{x}_j = \Omega_j x_j + \dfrac{\kappa_0}{N} \displaystyle\sum_{k\neq j} (\|x_j\|^2 x_k^\tau - \langle x_k^\tau, x_j\rangle x_j), \\ x_j(t) = \varphi_j(t) \in \mathbb{S}^{d-1} \subset \mathbb{R}^d, \quad -\tau \le t \le 0, \end{cases} \tag{7}$$

where Ω_j is a $d\times d$ skew-symmetric matrix for all j. Emergent dynamics of system (7) has been studied in [9]. To see the reduction to the Kuramoto model, we also set

$$d = 2, \quad x_j := \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}, \quad \varphi_j := \begin{bmatrix} \cos\alpha_j \\ \sin\alpha_j \end{bmatrix}, \quad \Omega_j := \begin{bmatrix} 0 & -\nu_j \\ \nu_j & 0 \end{bmatrix}, \quad \kappa_0 = \kappa. \tag{8}$$

Again, we substitute the ansatz (8) into (6) to derive the Kuramoto model with time-delay [21, 22]:

$$\begin{cases} \dot{\theta}_j = \nu_j + \dfrac{\kappa}{N} \displaystyle\sum_{k\neq j} \sin(\theta_k^\tau - \theta_j), \quad t > 0, \\ \theta_j(t) = \alpha_j(t), \quad -\tau \le t \le 0, \quad j \in \mathcal{N}. \end{cases} \tag{9}$$

The emergent dynamics of (9) has been extensively studied in literature, for example, complete synchronization for the mean-field model [5], complete synchronization [12, 15, 29], critical coupling strength for complete synchronization [17]. In summary, one has the following diagram:

$$\text{LHS model} \quad \xrightarrow[z_j^0 \in \mathbb{S}^{d-1}]{\text{real initial data}} \quad \text{Lohe sphere model} \quad \xrightarrow[d=2]{\text{dimension reduction}} \quad \text{Kuramoto model.}$$

On the other hand, the emergent dynamics of Lohe type matrix models has also been investigated in literature, e.g., sufficient conditions for complete aggregations [6, 14], mean-field approach for quaternion's collective dynamics [13], generalized

Lohe sphere model on Riemannian manifolds [20], a gradient flow approach for the Lohe matrix model [27], conserved quantities and non-abelian generalization of the Kuramoto model [30, 31, 32] etc.

2.3. **Previous results.** In this subsection, we present two results on the emergent dynamics of the LHS model without a time-delay and the Lohe sphere model with time-delay which correspond to the special cases for (1).

First, we consider the LHS model with zero time-delay case with $\tau = 0$ over the complete network with $a_{ik} = 1$ for all i and k. Under these setting, system (1) becomes

$$\begin{cases} \dot{z}_j = \Omega_j z_j + \dfrac{\kappa_0}{N} \displaystyle\sum_{k=1}^N \left(\langle z_j, z_j\rangle z_k - \langle z_k, z_j\rangle z_j\right) + \dfrac{\kappa_1}{N} \displaystyle\sum_{k=1}^N \left(\langle z_j, z_k\rangle - \langle z_k, z_j\rangle\right) z_j, \ t > 0, \\ z_j(0) = z_j^{in} \in \mathbb{HS}^{d-1}, \quad j \in \mathcal{N}. \end{cases} \tag{10}$$

For emergent dynamics of (10), we introduce an order parameter as a modulus of z_c and state diameter:

$$\rho := \|z_c\|, \quad D(Z) := \max_{i,j} \|z_i - z_j\|, \tag{11}$$

where z_c is the centroid of all z_i:

$$z_c := \frac{1}{N} \sum_{k=1}^N z_k.$$

On the other hand, we consider (11) with a zero free flow:

$$\begin{cases} \dot{w}_j = \kappa_0 \Big(w_c \langle w_j, w_j\rangle - w_j \langle w_c . w_j\rangle \Big) + \kappa_1 \Big(\langle w_j, w_c\rangle - \langle w_c, w_j\rangle \Big) w_j, \quad t > 0, \\ w_j(0) = z_j^{in} \in \mathbb{HS}^{d-1}, \quad j \in \mathcal{N}, \end{cases} \tag{12}$$

where $w_c := \dfrac{1}{N} \displaystyle\sum_{k=1}^N w_k$.

Then the emergence of complete aggregation and solution splitting property of (10) can be summarized in the following proposition.

Proposition 1. [24] *Suppose that coupling gains, free flows and initial data satisfy*

$$N \geq 3, \quad 0 < \kappa_1 < \frac{1}{4}\kappa_0, \quad \rho^{in} > \frac{N-2}{N}, \quad \Omega_j \equiv \Omega, \quad j = 1, \cdots, N,$$

where Ω is a skew-Hermitian matrix with size $(d+1) \times (d+1)$. Let $\{z_j\}$ be a global solution to (10). Then, the following assertions hold.

1. *Complete aggregation emerges asymptotically:*
$$\lim_{t \to \infty} D(Z(t)) = 0.$$
2. *Solution splitting property holds:*
$$z_j = e^{\Omega t} w_j, \quad j \in \mathcal{N},$$
where w_j is a solution to (12).

Proof. For a detailed proof, we refer to Theorem 4.1 of [24]. □

Second, we consider the Lohe sphere model on the unit sphere in $\mathbb{R}^d$ under the influence of time-delay:

$$\begin{cases} \dot{x}_j = \Omega x_j + \dfrac{\kappa}{N}\displaystyle\sum_{k\neq j}\left(\|x_j\|^2 x_k^\tau - \langle x_k^\tau, x_j\rangle x_j\right), \quad t>0,\ j\in\mathcal{N}, \\ x_j(t) = \varphi_j(t)\in\mathbb{S}^{d-1}, \quad -\tau\le t\le 0,\ j\in\mathcal{N}, \end{cases} \tag{13}$$

Proposition 2. [10] *Suppose that the system parameters and initial data satisfy*

$$\begin{aligned} &N\ge 3, \quad \kappa>0, \quad \tau<\frac{1}{8(d\|\Omega\|_\infty+2\kappa)}, \\ &\|\varphi_j(t)\|=1, \quad j\in\mathcal{N}, \quad t\in[-\tau,0], \qquad \sup_{-\tau\le t\le 0} D(\varphi(t))<\frac{1}{8}, \end{aligned}$$

where $\|\cdot\|_\infty$ *is defined by* $\|\Omega\|_\infty := \max_{i,j}|\Omega_{ij}|$, *and* $\varphi(t):=(\varphi_1(t),\cdots,\varphi_N(t))$. *Also, let* $\{x_j\}$ *be a global solution to* (13). *Then, we have*

$$\lim_{t\to\infty} D(X(t))=0.$$

Proof. For a proof, we refer to Theorem 3.1 of [10]. □

3. **Emergence of complete aggregation.** In this section, we provide an emergent dynamics of (1) under the following setting:

$$a_{ik}\equiv 1, \quad i,k\in\mathcal{N} \quad \text{and} \quad \Omega=0.$$

Note that this case corresponds to the same free flow and complete network topology. Then system (1) becomes

$$\begin{cases} \dot{z}_j = \dfrac{\kappa_0}{N}\displaystyle\sum_{k\neq j}(\langle z_j,z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j) + \dfrac{\kappa_1}{N}\displaystyle\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j, \\ z_j(t)=\varphi_j(t)\in\mathbb{HS}^{d-1}, \quad -\tau\le t\le 0. \end{cases} \tag{14}$$

In the following two subsections, we study complete aggregation in which coupling gains satisfy the following relations:

$$\begin{aligned} &\kappa_1+\frac{\kappa_0}{2}=0 \quad \text{(Stuart-Landau(SL) coupling gain pair)}, \\ &0<\left|\kappa_1+\frac{\kappa_0}{2}\right|\ll 1 \quad \text{(Close-to-SL coupling gain pair)}. \end{aligned}$$

In Section 2.3 of [8], the authors reduced the vector version of the Stuart-Landau model to the LHS model with the special pair of coupling gains. From this process, Stuart-Landau(SL) coupling gain pair and close-to SL coupling gain pair were naturally obtained. For the convenience of the reader who wants to know how the SL coupling appears from the generalized Stuart-Landau model in $\mathbb{C}^{d+1}$, we added a brief explanation in Appendix A.

3.1. **SL coupling gain pair.** In this subsection, we consider the emergent behavior of (14) for the Stuart-Landau gain pair. In this case, the coupling term can be simplified as follows: on $\mathbb{HS}^{d-1}$,

$$\begin{aligned}
&\kappa_0(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j) + \kappa_1(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j \\
&= \kappa_0\Big[z_k^\tau - \langle z_k^\tau, z_j\rangle z_j - \frac{1}{2}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j\Big] \\
&= \kappa_0\left[z_k^\tau - \frac{1}{2}\Big(\langle z_k^\tau, z_j\rangle + \langle z_j, z_k^\tau\rangle\Big)z_j\right] \\
&= \kappa_0\big(z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_j\rangle)z_j\big).
\end{aligned} \tag{15}$$

Finally, we combine (14) and (15) to get

$$\begin{cases} \dot{z}_j = \dfrac{\kappa_0}{N}\displaystyle\sum_{k\neq j}\big(z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_j\rangle)z_j\big), & t>0, \\ z_j(t) = \varphi_j(t) \in \mathbb{C}^d, & -\tau \le t \le 0. \end{cases} \tag{16}$$

Theorem 3.1. *Suppose system parameters and initial data set φ_j satisfy*

$$\kappa_0 > 0, \quad N \ge 3, \quad j \in \mathcal{N}, \quad \tau < \frac{1}{16\kappa_0}, \quad \|\varphi_j\| = 1, \quad D(\varphi(t)) < \frac{1}{8}, \quad t \in [-\tau, 0],$$

and let $\{z_j\}$ be a global solution to (16). Then, the complete aggregation emerges asymptotically:

$$\lim_{t\to\infty} D(Z(t)) = 0.$$

Proof. We leave its proof in Section 3.1.2. □

Remark 2. (1) The SL coupling gain pair case can be reduced to the Lohe sphere model with time-delay treated in [9, 10].

(2) Due to technical difficulties, we can not obtain analytical results for the exponential decay of the LHS model with $\tau > 0$, however, numerical simulations seem to suggest an exponential aggregation as in the Lohe sphere case without time-delay (see Figure 3.1).

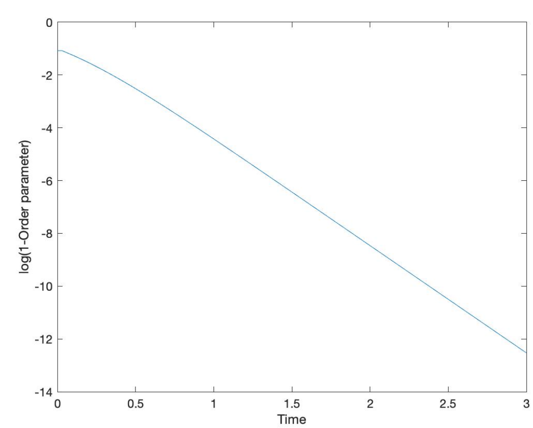

FIGURE 1. Exponential aggregation for $\tau > 0$, $N = 4$ and $d = 2$

3.1.1. *A priori estimates.* In the part, we provide four lemmas for the emergent dynamics of (16) following the strategy in [10].

Lemma 3.2. *Let $\{z_j\}$ be a global solution to (16). Then we have*

$$\begin{aligned}\frac{d}{dt}\|z_i - z_j^s\|^2 \le 2\kappa_0 \mathrm{Re}\langle z_c^\tau - z_c^{\tau+s}, z_i - z_j^s\rangle - \kappa_0\|z_i - z_j^s\|^2(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{\tau+s}, z_j^s\rangle)\\ - \frac{2\kappa_0}{N}\left(\mathrm{Re}\langle z_i - z_j^s, z_i^\tau - z_j^{\tau+s}\rangle - \|z_i - z_j^s\|^2\right),\end{aligned}$$

for all $i, j \in \mathcal{N}$ and $t \ge s + \tau$.

Proof. We set

$$z_j^s(t) = z_j(t-s), \quad j \in \mathcal{N}.$$

Then, it satisfies

$$\dot{z}_j^s = \frac{\kappa_0}{N}\sum_{k\neq j}\left(z_k^{\tau+s} - \mathrm{Re}(\langle z_k^{\tau+s}, z_j^s\rangle)z_j^s\right). \tag{17}$$

It follows from $(16)_1$ and (17) that

$$\begin{aligned}\frac{d}{dt}(z_i - z_j^s) &= \frac{\kappa_0}{N}\left(\sum_{k\neq i}(z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_i\rangle)z_i) - \sum_{k\neq j}(z_k^{\tau+s} - \mathrm{Re}(\langle z_k^{\tau+s}, z_j^s\rangle)z_j^s)\right)\\ &= \kappa_0\left((z_c^\tau - \mathrm{Re}(\langle z_c^\tau, z_i\rangle)z_i) - (z_c^{\tau+s} - \mathrm{Re}(\langle z_c^{\tau+s}, z_j^s\rangle)z_j^s)\right)\\ &\quad - \frac{\kappa_0}{N}\left((z_i^\tau - \mathrm{Re}(\langle z_i^\tau, z_i\rangle)z_i) - (z_j^{\tau+s} - \mathrm{Re}(\langle z_j^{\tau+s}, z_j^s\rangle)z_j^s)\right).\end{aligned} \tag{18}$$

This yields

$$\begin{aligned}&\frac{d}{dt}\|z_i - z_j^s\|^2 = 2\mathrm{Re}\left\langle z_i - z_j^s, \frac{d}{dt}(z_i - z_j^s)\right\rangle\\ &\quad = 2\kappa_0\mathrm{Re}\left\langle z_i - z_j^s, z_c^\tau - \mathrm{Re}(\langle z_c^\tau, z_i\rangle)z_i\right\rangle - 2\kappa_0\mathrm{Re}\left\langle z_i - z_j^s, z_c^{\tau+s} - \mathrm{Re}(\langle z_c^{\tau+s}, z_j^s\rangle)z_j^s\right\rangle\\ &\quad - \frac{2\kappa_0}{N}\mathrm{Re}\left\langle z_i - z_j^s, z_i^\tau - \mathrm{Re}(\langle z_i^\tau, z_i\rangle)z_i\right\rangle + \frac{2\kappa_0}{N}\mathrm{Re}\left\langle z_i - z_j^s, z_j^{\tau+s} - \mathrm{Re}(\langle z_j^{\tau+s}, z_j^s\rangle)z_j^s\right\rangle\\ &\quad = 2\kappa_0\mathrm{Re}\left\langle -z_j^s, z_c^\tau - \mathrm{Re}(\langle z_c^\tau, z_i\rangle)z_i\right\rangle - 2\kappa_0\mathrm{Re}\left\langle z_i, z_c^{\tau+s} - \mathrm{Re}(\langle z_c^{\tau+s}, z_j^s\rangle)z_j^s\right\rangle\\ &\quad - \frac{2\kappa_0}{N}\mathrm{Re}\left\langle -z_j^s, z_i^\tau - \mathrm{Re}(\langle z_i^\tau, z_i\rangle)z_i\right\rangle + \frac{2\kappa_0}{N}\mathrm{Re}\left\langle z_i, z_j^{\tau+s} - \mathrm{Re}(\langle z_j^{\tau+s}, z_j^s\rangle)z_j^s\right\rangle\\ &\quad = 2\kappa_0(\mathrm{Re}\langle z_c^\tau, z_i\rangle\mathrm{Re}\langle z_j^s, z_i\rangle + \mathrm{Re}\langle z_c^{\tau+s}, z_j^s\rangle\mathrm{Re}\langle z_i, z_j^s\rangle - \mathrm{Re}\langle z_j^s, z_c^\tau\rangle - \mathrm{Re}\langle z_i, z_c^{\tau+s}\rangle)\\ &\quad - \frac{2\kappa_0}{N}(\mathrm{Re}\langle z_i^\tau, z_i\rangle\mathrm{Re}\langle z_j^s, z_i\rangle + \mathrm{Re}\langle z_i, z_j^s\rangle\mathrm{Re}\langle z_j^{\tau+s}, z_j^s\rangle - \mathrm{Re}\langle z_j^s, z_i^\tau\rangle - \mathrm{Re}\langle z_i, z_j^{\tau+s}\rangle).\end{aligned} \tag{19}$$

On the other hand, we have

$$\|z_i - z_j^s\|^2 = 2(1 - \mathrm{Re}\langle z_i, z_j^s\rangle), \quad \text{i.e.,} \quad \mathrm{Re}\langle z_i, z_j^s\rangle = 1 - \frac{1}{2}\|z_i - z_j^s\|^2. \tag{20}$$

We combine (19) and (20) to obtain

$$\begin{aligned}
\frac{d}{dt}\|z_i - z_j^s\|^2 &= 2\kappa_0(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{\tau+s}, z_j^s\rangle - \mathrm{Re}\langle z_j^s, z_c^\tau\rangle - \mathrm{Re}\langle z_i, z_c^{\tau+s}\rangle)\\
&- \kappa_0\|z_i - z_j^s\|^2(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{\tau+s}, z_j^s\rangle)\\
&- \frac{2\kappa_0}{N}(\mathrm{Re}\langle z_i^\tau, z_i\rangle + \mathrm{Re}\langle z_j^{\tau+s}, z_j^s\rangle - \mathrm{Re}\langle z_j^s, z_i^\tau\rangle - \mathrm{Re}\langle z_i, z_j^{\tau+s}\rangle)\\
&+ \frac{\kappa_0}{N}\|z_i - z_j^s\|^2(\mathrm{Re}\langle z_i^\tau, z_i\rangle + \mathrm{Re}\langle z_j^{\tau+s}, z_j^s\rangle)\\
&= 2\kappa_0\mathrm{Re}\langle z_c^\tau - z_c^{\tau+s}, z_i - z_j^s\rangle - \kappa_0\|z_i - z_j^s\|^2(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{\tau+s}, z_j^s\rangle)\\
&- \frac{2\kappa_0}{N}\mathrm{Re}\langle z_i - z_j^s, z_i^\tau - z_j^{\tau+s}\rangle + \frac{\kappa_0}{N}\|z_i - z_j^s\|^2(\mathrm{Re}\langle z_i^\tau, z_i\rangle + \mathrm{Re}\langle z_j^{\tau+s}, z_j^s\rangle).
\end{aligned} \tag{21}$$

Finally, relation (21) and $|\langle z, w\rangle| \le \|z\|\cdot\|w\|$ yield desired estimate. $\square$

Lemma 3.3. *Let $\{z_j\}$ be a global solution to (16). Then we have following relation for suitable positive numbers s, u, t:*

$$\begin{aligned}
&\Big|\|z_i(t) - z_j^s(t)\|^2 - \mathrm{Re}\langle z_i^u(t) - z_j^{u+s}(t), z_i(t) - z_j^s(t)\rangle\Big|\\
&\quad\le 2u\kappa_0 \sup_{t-u<v<t}\Big(\|z_i(v) - z_j^s(v)\| + \|z_c^\tau(v) - z_c^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\|\\
&\quad\quad + \frac{2u\kappa_0}{N}\sup_{t-u<v<t}\Big(\|z_i(v) - z_j^s(v)\| + \|z_i^\tau(v) - z_j^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\|.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\big|\|z_i(t) - z_j^s(t)\|^2 - \mathrm{Re}\langle z_i^u(t) - z_j^{u+s}(t), z_i(t) - z_j^s(t)\rangle\big|\\
&\qquad = \Big|\mathrm{Re}\left(\|z_i(t) - z_j^s(t)\|^2 - \langle z_i^u(t) - z_j^{u+s}(t), z_i(t) - z_j^s(t)\rangle\right)\Big|.
\end{aligned}$$

We integrate (18) on the interval $[t-u, t]$ and take the inner product of the resulting relation and $z_i(t) - z_j^s(t)$ as in [10] to find

$$\begin{aligned}
&\Big|\mathrm{Re}\left(\|z_i(t) - z_j^s(t)\|^2 - \langle z_i^u(t) - z_j^{u+s}(t), z_i(t) - z_j^s(t)\rangle\right)\Big|\\
&\quad\le \Big|\|z_i(t) - z_j^s(t)\|^2 - \langle z_i^u(t) - z_j^{u+s}(t), z_i(t) - z_j^s(t)\rangle\Big|\\
&\quad\le 2u\kappa_0 \sup_{t-u<v<t}\Big(\|z_i(v) - z_j^s(v)\| + \|z_c^\tau(v) - z_c^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\|\\
&\quad\quad + \frac{2u\kappa_0}{N}\sup_{t-u<v<t}\Big(\|z_i(v) - z_j^s(v)\| + \|z_i^\tau(v) - z_j^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\|.
\end{aligned}$$

$\square$

For an emergent dynamics, we introduce a modified ensemble diameter as follows:

$$D^{0,\tau}(t) := \max_{i,j}\|z_i(t) - z_j^\tau(t)\|. \tag{22}$$

Lemma 3.4. *Let $\{z_j\}$ be a global solution to (16). Then, the functional (22) satisfies*

$$\begin{aligned}
\frac{d}{dt}D^{0,\tau}(t) &\le \kappa_0\|z_c^\tau - z_c^{2\tau}\| - \frac{\kappa_0 D^{0,\tau}(t)}{2}\left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2}\right)\\
&\qquad + \frac{4\kappa_0^2\tau(N+1)}{N^2}\left(\sup_{t-2\tau<v<t} D^{0,\tau}(v)\right).
\end{aligned}$$

Proof. In Lemma 3.3, we set $s = \tau$ and take $s, u = \tau$. Since inequalities in Lemma 3.3 and Lemma 3.4 are similar to estimates in Lemma 4.1 and 4.2 in [10], we can derive the same result. The only difference is that we have terms involving real parts, but it can be estimated in the same way as [10] since

$$\begin{aligned} 1 - \mathrm{Re}(\langle z_i, z_c^\tau \rangle) &= \mathrm{Re}(1 - \langle z_i, z_c^\tau \rangle) = \frac{1}{N}\sum_{k=1}^N \mathrm{Re}(1 - \langle z_i, z_k^\tau \rangle) \\ &= \frac{1}{N}\sum_{k=1}^N \frac{\|z_i - z_k^\tau\|^2}{2} \le \frac{D^{0,\tau}(t)^2}{2}. \end{aligned}$$

□

We set

$$\Delta_{z_j}^\tau(t) = \|z_j(t) - z_j^\tau(t)\|.$$

In order to control the term $\|z_c^\tau - z_c^{2\tau}\|^2$ appearing in Lemma 3.4, we give the following estimate for $\Delta_{z_j}^\tau$.

Lemma 3.5. *Let $\{z_j\}$ be a global solution to (16). Then, the functional $\Delta_{z_j}^\tau$ satisfies*

$$\Delta_{z_j}^\tau(t) \le 2\kappa_0\tau\left(\frac{N-1}{N}\right).$$

Proof. Note that

$$z_j(t) - z_j^\tau(t) = z_j(t) - z_j(t-\tau) = \int_{t-\tau}^t \dot{z}_j(s)ds,$$

This yields

$$\begin{aligned} \left\|\int_{t-\tau}^t \dot{z}_j(s)ds\right\| &= \left\|\int_{t-\tau}^t \left(\frac{\kappa_0}{N}\sum_{k\neq j}\left(z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_j\rangle)z_j\right)\right)ds\right\| \\ &\le \int_{t-\tau}^t \frac{\kappa_0}{N}\sum_{k\neq j}\|\left(z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_j\rangle)z_j\right)\|ds \\ &\le \int_{t-\tau}^t \frac{\kappa_0}{N}\sum_{k\neq j}\left(\|z_k^\tau\| + |\mathrm{Re}(\langle z_k^\tau, z_j\rangle)|\cdot\|z_j\|\right)ds \\ &\le \int_{t-\tau}^t \frac{\kappa_0}{N}\sum_{k\neq j} 2ds = 2\kappa_0\tau\left(\frac{N-1}{N}\right). \end{aligned}$$

□

Now we are ready to provide a proof of our first main result.

3.1.2. *Proof of Theorem 3.1.* In this part, we present our first result on the complete aggregation by combining all the estimates in Lemma 3.2 - Lemma 3.5 in two steps. We will briefly sketch the proof, since in the next section, we will provide more general statement and its proof.

- Step A (Existence of a trapping set): First, we claim

$$D^{0,\tau}(t) < \frac{1}{2}, \quad t \ge 0.$$

Proof. We first estimate $D^{0,\tau}(t)$ in an interval $[-\tau, 2\tau]$. Next, we define a set

$$\mathcal{T} := \left\{ t \in (2\tau, \infty) : D^{0,\tau}(t) < \frac{1}{2} \right\},$$

and proceed the proof using Lipschitz continuity of $D^{0,\tau}(t)$ in order to show $\sup \mathcal{T} = \infty$.

Note that

$$\begin{aligned}\frac{d}{dt}D^{0,\tau}(t) &\leq \frac{\kappa_0}{8} - \frac{\kappa_0}{2}D^{0,\tau}(t)\left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2}\right) \\ &\quad + \frac{4\kappa_0^2\tau(N+1)}{N^2} \sup_{t-2\tau<v<t} D^{0,\tau}(v) \\ &< \frac{\kappa_0}{8} - \frac{7\kappa_0}{8}D^{0,\tau}(t) + \frac{\kappa_0}{18} < \frac{\kappa_0}{4} - \frac{7\kappa_0}{8}D^{0,\tau}(t).\end{aligned}$$

This yields

$$D^{0,\tau}(t) \leq \max\left(D^{0,\tau}(0), \frac{\frac{\kappa_0}{4}}{\frac{7\kappa_0}{8}}\right) < \frac{1}{2}.$$

• Step B (Key step): We claim

$$\lim_{t\to\infty} \|z_i(t) - z_j(t)\| = 0.$$

For this, we define a Lyapunov functional $\mathcal{E}$: for $Z = (z_1, \cdots, z_N)$,

$$\mathcal{E}_{ij}(t) := \|z_i(t) - z_j(t)\|^2 + \gamma \int_{t-\tau}^{t} \|z_i(s) - z_j(s)\|^2 ds,$$

where γ is a positive constant. Then, one has

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{ij}(t) &= \frac{d}{dt}\|z_i(t) - z_j(t)\|^2 + \gamma\|z_i(t) - z_j(t)\|^2 - \gamma\|z_i^\tau(t) - z_j^\tau(t)\|^2 \\ &\leq -\frac{7\kappa_0}{4}\|z_i - z_j\|^2 + \frac{2\kappa_0}{N}\|z_i - z_j\| \cdot \|z_i^\tau - z_j^\tau\| \\ &\quad + \frac{2\kappa_0}{N}\|z_i - z_j\|^2 + \gamma\|z_i(t) - z_j(t)\|^2 - \gamma\|z_i^\tau(t) - z_j^\tau(t)\|^2.\end{aligned}$$

Here, for the computation of $\frac{d}{dt}\|z_i - z_j\|^2$, we used Lemma 3.2 with $s = 0$, and the fact that $\mathrm{Re}\langle z_i + z_j, z_c^\tau\rangle > \frac{7}{4}$ when $D^{0,\tau}(t) < \frac{1}{2}$(see the proof of Lemma 3.4).

By applying Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{ij}(t) &\leq -\frac{7\kappa_0}{4}\|z_i - z_j\|^2 + \frac{\kappa_0}{N}\|z_i - z_j\|^2 + \frac{\kappa_0}{N}\|z_i^\tau - z_j^\tau\|^2 \\ &\quad + \frac{2\kappa_0}{N}\|z_i - z_j\|^2 + \gamma\|z_i(t) - z_j(t)\|^2 - \gamma\|z_i^\tau(t) - z_j^\tau(t)\|^2.\end{aligned}$$

Now we set $\gamma = \frac{\kappa_0}{N}$ to get

$$\frac{d}{dt}\mathcal{E}_{ij}(t) \leq \left[-\frac{7\kappa_0}{4} + \frac{4\kappa_0}{N}\right]\|z_i - z_j\|^2 \leq -\frac{5\kappa_0}{12}\|z_i - z_j\|^2 \leq 0,$$

since $N \geq 3$. This leads to

$$\frac{5\kappa_0}{12}\int_0^\infty \|z_i(s) - z_j(s)\|^2 ds \leq \mathcal{E}_{ij}(0).$$

Using the boundedness of $\|\dot{z}_j\|$ for all j, we can apply Barbalat's lemma to obtain the desired result. □

3.2. **Close-to-SL coupling gain pair.** In this subsection, we consider the situation in which the coupling gain pair is close to Stuart-Landau coupling gain pair:

$$\tilde{\kappa} := \frac{\kappa_0}{2} + \kappa_1, \quad |\tilde{\kappa}| \ll 1.$$

Note that

$$\begin{aligned}
\dot{z}_j &= \frac{\kappa_0}{N}\sum_{k\neq j}(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j) + \frac{\kappa_1}{N}\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j \\
&= \frac{\kappa_0}{N}\sum_{k\neq j}(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j) + \frac{\kappa_1}{N}\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j \\
&\quad - \frac{\kappa_0}{2N}\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j + \frac{\kappa_0}{2N}\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j \\
&= \frac{\kappa_0}{N}\sum_{k\neq j}\left(z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_j\rangle)z_j\right) + \frac{\tilde{\kappa}}{N}\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j.
\end{aligned} \tag{23}$$

We substitute $\kappa_1 = \tilde{\kappa} - \dfrac{\kappa_0}{2}$ into (23) to get

$$\begin{cases}
\dot{z}_j = \dfrac{\kappa_0}{N}\displaystyle\sum_{k\neq j}(\langle z_j, z_j\rangle z_k^\tau - \mathrm{Re}(\langle z_k^\tau, z_j\rangle)z_j) + \dfrac{\tilde{\kappa}}{N}\displaystyle\sum_{k\neq j}(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle)z_j, \quad t>0, \\
z_j(t) = \varphi_j(t) \in \mathbb{HS}^{d-1}, \quad -\tau \le t \le 0.
\end{cases} \tag{24}$$

By straightforward calculation, one has

$$\begin{aligned}
&\frac{d}{dt}\|z_i - z_j^s\|^2 \le \langle \dot{z}_i - \dot{z}_j{}^s, z_i - z_j^s\rangle + \langle z_i - z_j^s, \dot{z}_i - \dot{z}_j{}^s\rangle = 2\mathrm{Re}\langle \dot{z}_i - \dot{z}_j{}^s, z_i - z_j^s\rangle \\
&= 2\kappa_0 \mathrm{Re}\langle z_c^\tau - z_c^{\tau+s}, z_i - z_j^s\rangle - \kappa_0\|z_i - z_j^s\|^2(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{\tau+s}, z_j^s\rangle) \\
&- \frac{2\kappa_0}{N}\left(\mathrm{Re}\langle z_i - z_j^s, z_i^\tau - z_j^{\tau+s}\rangle - \|z_i - z_j^s\|^2\frac{(\mathrm{Re}\langle z_i^\tau, z_i\rangle + \mathrm{Re}\langle z_j^{\tau+s}, z_j\rangle)}{2}\right) \\
&+ 4\tilde{\kappa}\mathrm{Im}\langle z_i, z_j^s\rangle \mathrm{Im}(\langle z_c^\tau, z_i\rangle - \langle z_c^{\tau+s}, z_j^s\rangle) + \frac{4\tilde{\kappa}}{N}\mathrm{Im}\langle z_i, z_j^s\rangle \mathrm{Im}\left(\langle z_i, z_i^\tau\rangle - \langle z_j^s, z_j^{\tau+s}\rangle\right) \\
&\le 2\kappa_0 \mathrm{Re}\langle z_c^\tau - z_c^{\tau+s}, z_i - z_j^s\rangle - \kappa_0\|z_i - z_j^s\|^2(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{\tau+s}, z_j^s\rangle) \\
&- \frac{2\kappa_0}{N}\left(\mathrm{Re}\langle z_i - z_j^s, z_i^\tau - z_j^{\tau+s}\rangle - \|z_i - z_j^s\|^2\right) + 4|\tilde{\kappa}|\cdot\|z_i - z_j^s\|(\|z_c^\tau - z_c^{\tau+s}\| + \|z_i - z_j^s\|) \\
&+ \frac{4}{N}|\tilde{\kappa}|\cdot\|z_i - z_j^s\|(\|z_i - z_j^s\| + \|z_i^\tau - z_j^{\tau+s}\|).
\end{aligned} \tag{25}$$

For the second inequality (25), we use the triangle inequality,

$$\|z\| = \|w\| = 1 \quad \Longrightarrow \quad |\mathrm{Im}\langle z, w\rangle| = |\mathrm{Im}(\langle z, w\rangle - 1)| = |\mathrm{Im}\langle z, w - z\rangle| \le \|z - w\|,$$

and similar arguments in the proof of Lemma 3.3 to derive

$$
\begin{aligned}
&\Big|\|z_i(t) - z_j^s(t)\|^2 - \mathrm{Re}\langle z_i^u(t) - z_j^{u+s}(t), z_i(t) - z_j^s(t)\rangle\Big| \\
&\quad \le 2u\kappa_0 \sup_{t-u<v<t}\Big(\|z_i(v) - z_j^s(v)\| + \|z_c^\tau(v) - z_c^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\| \\
&\qquad + \frac{2u\kappa_0}{N}\sup_{t-u<v<t}\Big(\|z_i(v) - z_j^s(v)\| + \|z_i^\tau(v) - z_j^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\| \\
&\quad\ 2u|\tilde{\kappa}|\sup_{t-u<v<t}\Big(2\|z_i(v) - z_j^s(v)\| + \|z_c^\tau(v) - z_c^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\| \\
&\qquad + \frac{2u|\tilde{\kappa}|}{N}\sup_{t-u<v<t}\Big(2\|z_i(v) - z_j^s(v)\| + \|z_i^\tau(v) - z_j^{\tau+s}(v)\|\Big)\|z_i(t) - z_j^s(t)\|.
\end{aligned}
\tag{26}
$$

Theorem 3.6. *Suppose system parameters and initial data satisfy*

$$
\begin{aligned}
&\kappa_0 > 0, \quad |\tilde{\kappa}| < \frac{9}{256}\kappa_0, \quad \mathcal{C}_1\tau < \frac{1}{8}, \quad N \ge 3, \\
&\|\varphi_j(t)\| = 1, \quad \sup_{-\tau\le t\le 0} D(Z(t)) < \frac{1}{8}, \quad \Omega_j \equiv 0, \quad j \in \mathcal{N},
\end{aligned}
$$

where $\mathcal{C}_1 := 2\left(\frac{N-1}{N}\right)\cdot(\kappa_0 + |\tilde{\kappa}|)$*, and let* $\{z_j\}$ *be a global solution to* (24)*. Then complete aggregation emerges asymptotically:*

$$
\lim_{t\to\infty}\|z_i(t) - z_j(t)\| = 0, \quad i, j \in \mathcal{N}.
$$

Proof. We leave its detailed proof in Section 3.2.2. □

3.2.1. *Basic estimates.* In this part, we provide several a priori estimates.

Lemma 3.7. *Let* $\{z_j\}$ *be a global solution to* (24)*. For* $t \ge 2\tau$*, we have following inequality:*

$$
\begin{aligned}
\frac{d}{dt}D^{0,\tau}(t) &\le \kappa_0\|z_c^\tau - z_c^{2\tau}\| - \frac{\kappa_0}{2}D^{0,\tau}(t)\left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2}\right) \\
&\quad + \frac{2\kappa_0\tau}{N}\sup_{t-2\tau<v<t}D^{0,\tau}(v)\left(2\kappa_0\left(\frac{N+1}{N}\right) + 3|\tilde{\kappa}|\left(\frac{N+1}{N}\right)\right) \\
&\quad + 4|\tilde{\kappa}|\frac{N+1}{N}\sup_{t-2\tau<v<t}D^{0,\tau}(v).
\end{aligned}
\tag{27}
$$

Proof. We set $s = \tau$ in the inequality (25) to find

$$
\begin{aligned}
\frac{d}{dt}\|z_i - z_j^\tau\|^2 &\le 2\kappa_0\mathrm{Re}\langle z_c^\tau - z_c^{2\tau}, z_i - z_j^\tau\rangle - \kappa_0\|z_i - z_j^\tau\|^2(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{2\tau}, z_j^\tau\rangle) \\
&\quad - \frac{2\kappa_0}{N}\left(\mathrm{Re}\langle z_i - z_j^\tau, z_i^\tau - z_j^{2\tau}\rangle - \|z_i - z_j^\tau\|^2\right) \\
&\quad + 4|\tilde{\kappa}|\|z_i - z_j^\tau\|(\|z_c^\tau - z_c^{2\tau}\| + \|z_i - z_j^\tau\|) \\
&\quad + \frac{4}{N}|\tilde{\kappa}|\|z_i - z_j^\tau\|(\|z_i - z_j^\tau\| + \|z_i^\tau - z_j^{2\tau}\|).
\end{aligned}
$$

In the inequality (26), we set

$$
u = \tau \quad \text{and} \quad s = \tau
$$

to get

$$
\begin{aligned}
&\Big| \|z_i(t) - z_j^\tau(t)\|^2 - \mathrm{Re}\langle z_i^\tau(t) - z_j^{2\tau}(t), z_i(t) - z_j^\tau(t)\rangle \Big| \\
&\quad \le 2\tau\kappa_0 \sup_{t-\tau<v<t} \Big(\|z_i(v) - z_j^\tau(v)\| + \|z_c^\tau(v) - z_c^{2\tau}(v)\| \Big) \|z_i(t) - z_j^\tau(t)\| \\
&\quad + \frac{2\tau\kappa_0}{N} \sup_{t-\tau<v<t} \Big(\|z_i(v) - z_j^\tau(v)\| + \|z_i^\tau(v) - z_j^{2\tau}(v)\| \Big) \|z_i(t) - z_j^\tau(t)\| \\
&\quad + 2\tau|\tilde\kappa| \sup_{t-\tau<v<t} \Big(2\|z_i(v) - z_j^\tau(v)\| + \|z_c^\tau(v) - z_c^{2\tau}(v)\| \Big) \|z_i(t) - z_j^\tau(t)\| \\
&\quad + \frac{2\tau|\tilde\kappa|}{N} \sup_{t-\tau<v<t} \Big(2\|z_i(v) - z_j^\tau(v)\| + \|z_i^\tau(v) - z_j^{2\tau}(v)\| \Big) \|z_i(t) - z_j^\tau(t)\|.
\end{aligned}
$$

For a fixed t, there exist i_t and j_t such that

$$
D^{0,\tau}(t) = \|z_{i_t} - z_{j_t}^\tau\|.
$$

Then, for $t \ge 2\tau$, one has

$$
\begin{aligned}
\frac{d}{dt} D^{0,\tau}(t)^2 &= \frac{d}{dt}\|z_{i_t} - z_{j_t}^\tau\|^2 \\
&\le 2\kappa_0\|z_c^\tau - z_c^{2\tau}\| D^{0,\tau}(t) - \kappa_0 D^{0,\tau}(t)^2 (\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{2\tau}, z_j^\tau\rangle) \\
&+ \frac{2\kappa_0\tau}{N} D^{0,\tau}(t) \sup_{t-2\tau<v<t} D^{0,\tau}(v) \left(4\kappa_0 + \frac{4\kappa_0}{N} + 6|\tilde\kappa| + \frac{6|\tilde\kappa|}{N} \right) \\
&+ 8|\tilde\kappa| D^{0,\tau}(t) \sup_{t-2\tau<v<t} D^{0,\tau}(t) + \frac{8|\tilde\kappa|}{N} D^{0,\tau}(t) \sup_{t-2\tau<v<t} D^{0,\tau}(t).
\end{aligned}
$$

Hence, one has

$$
\begin{aligned}
&\frac{d}{dt} D^{0,\tau}(t) \le \kappa_0\|z_c^\tau - z_c^{2\tau}\| - \frac{\kappa_0}{2} D^{0,\tau}(t) (\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^{2\tau}, z_j^\tau\rangle) \\
&\quad + \frac{\kappa_0\tau}{N} \sup_{t-2\tau<v<t} D^{0,\tau}(v) \left(4\kappa_0 + \frac{4\kappa_0}{N} + 6|\tilde\kappa| + \frac{6|\tilde\kappa|}{N} \right) + \frac{4|\tilde\kappa|(N+1)}{N} \sup_{t-2\tau<v<t} D^{0,\tau}(t) \\
&\le \kappa_0\|z_c^\tau - z_c^{2\tau}\| - \frac{\kappa_0}{2} D^{0,\tau}(t) \left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2} \right) \\
&\quad + \frac{\kappa_0\tau}{N} \sup_{t-2\tau<v<t} D^{0,\tau}(v) \left(4\kappa_0 + \frac{4\kappa_0}{N} + 6|\tilde\kappa| + \frac{6|\tilde\kappa|}{N} \right) + 4|\tilde\kappa| \sup_{t-2\tau<v<t} D^{0,\tau}(v) \\
&\quad + \frac{4|\tilde\kappa|}{N} \sup_{t-2\tau<v<t} D^{0,\tau}(v) \\
&= \kappa_0\|z_c^\tau - z_c^{2\tau}\| - \frac{\kappa_0}{2} D^{0,\tau}(t) \left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2} \right) \\
&\quad + \frac{2\kappa_0\tau}{N} \sup_{t-2\tau<v<t} D^{0,\tau}(v) \left(\frac{N+1}{N} \right) (2\kappa_0 + 3|\tilde\kappa|) + 4|\tilde\kappa| \left(\frac{N+1}{N} \right) \sup_{t-2\tau<v<t} D^{0,\tau}(v).
\end{aligned}
$$

$\square$

Lemma 3.8. *Let $\{z_j\}$ be a global solution to (24). Then, one has*

$$
\|z_i(t) - z_i^\tau(t)\| \le \mathcal{C}_1 \tau.
$$

3.2.2. *Proof of Theorem 3.6.* Suppose system parameters and initial data satisfy

$$\kappa_0 > 0, \quad |\tilde{\kappa}| < \frac{9}{256}\kappa_0, \quad \mathcal{C}_1\tau < \frac{1}{8}, \quad N \geq 3,$$
$$\|\varphi_j(t)\| = 1, \quad \sup_{-\tau\leq t\leq 0} D(Z(t)) < \frac{1}{8}, \quad \Omega_j \equiv 0, \quad j \in \mathcal{N},$$

and let $\{z_j\}$ be a solution of system (24). Then, the proof consists of two steps.

• Step A (Existence of a trapping set): We claim

$$D^{0,\tau}(t) < \frac{1}{2}, \quad t \geq 0. \tag{28}$$

Proof of (28): We follow the same arguments as in [10]. For this, we divide the estimate into three time intervals:

$$0 \leq t \leq \tau, \quad \tau \leq t \leq 2\tau \quad \textit{and} \quad t \geq 2\tau.$$

⋄ Step A.1 (Estimate in the time-interval $[0,\tau]$): By triangle inequality, we have

$$\|z_i(t) - z_j^\tau(t)\| \leq \|z_i(t) - z_i^\tau(t)\| + \|z_i^\tau(t) - z_j^\tau(t)\| \leq \mathcal{C}_1\tau + D(Z(t-\tau)) < \frac{1}{4}.$$

⋄ Step A.2 (Estimate in the time-interval $[\tau, 2\tau]$): Similar to Step A, we use triangular inequality to get

$$\|z_i(t) - z_j^\tau(t)\| \leq \|z_i(t) - z_i^\tau(t)\| + \|z_i^\tau(t) - z_j^\tau(t)\| \leq \mathcal{C}_1\tau + D(Z(t-\tau)).$$

However, since

$$\|z_i(t) - z_j(t)\| \leq \|z_i(t) - z_i^\tau(t)\| + \|z_i^\tau(t) - z_j(t)\| \leq \mathcal{C}_1\tau + \frac{1}{4} < \frac{3}{8},$$

one has

$$D(Z(t-\tau)) < \frac{3}{8}.$$

Therefore, we give

$$\|z_i(t) - z_j^\tau(t)\| < \frac{1}{2}.$$

⋄ Step A.3 (Estimate in the time-interval $[2\tau, \infty)$): By (27), one has

$$\begin{aligned}
\frac{d}{dt}D^{0,\tau}(t) &\leq \kappa_0\|z_c^\tau - z_c^{2\tau}\| - \frac{\kappa_0}{2}D^{0,\tau}(t)\left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2}\right) \\
&\quad + \frac{2\kappa_0\tau}{N}\sup_{t-2\tau<v<t} D^{0,\tau}(v)\left(\frac{N+1}{N}\right)(2\kappa_0 + 3|\tilde{\kappa}|) + 4|\tilde{\kappa}|\left(\frac{N+1}{N}\right)\sup_{t-2\tau<v<t} D^{0,\tau}(v) \\
&\leq \frac{\kappa_0}{8} - \frac{\kappa_0}{2}D^{0,\tau}(t)\left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2}\right) + \frac{3\kappa_0}{4N}\sup_{t-2\tau<v<t} D^{0,\tau}(v) \\
&\quad + 4|\tilde{\kappa}|\frac{N+1}{N}\sup_{t-2\tau<v<t} D^{0,\tau}(v).
\end{aligned}$$

Here, we used

$$\|z_c^\tau - z_c^{2\tau}\| \leq \mathcal{C}_1\tau < \tfrac{1}{8}, \quad \left(\tfrac{N+1}{N}\right)(4\kappa_0 + 6|\tilde{\kappa}|) < \tfrac{6(N+1)}{N}(\kappa_0 + |\tilde{\kappa}|) \leq \tfrac{3(N+1)}{N-1}\mathcal{C}_1 \leq 6\mathcal{C}_1$$

for $N \geq 3$.

Next we claim:

$$D^{0,\tau}(t) < \frac{1}{2}, \quad \forall\, t \geq 2\tau.$$

For a proof, we define a set $\mathcal{T}$ as

$$\mathcal{T} := \left\{ t \in (2\tau, \infty) : D^{0,\tau}(t) < \frac{1}{2} \right\},$$

and proceed the proof using Lipschitz continuity of $D^{0,\tau}(t)$ as in [10]. The only difference is the estimate of $\frac{d}{dt}D^{0,\tau}(t)$. By direct estimates, one has

$$\begin{aligned}\frac{d}{dt}D^{0,\tau}(t) &\le \frac{\kappa_0}{8} - \frac{\kappa_0}{2}D^{0,\tau}(t)\left(2 - \frac{D^{0,\tau}(t)^2}{2} - \frac{D^{0,\tau}(t-\tau)^2}{2}\right) \\ &\quad + \frac{3\kappa_0}{4N}\sup_{t-2\tau<v<t} D^{0,\tau}(v) + \sup_{t-2\tau<v<t} D^{0,\tau}(v)\left(4|\tilde{\kappa}| + \frac{4|\tilde{\kappa}|}{N}\right) \\ &< \frac{\kappa_0}{8} - \frac{7\kappa_0}{8}D^{0,\tau}(t) + \frac{\kappa_0}{8} + \frac{8}{3}|\tilde{\kappa}| = \left(\frac{\kappa_0}{4} + \frac{8}{3}|\tilde{\kappa}|\right) - \frac{7\kappa_0}{8}D^{0,\tau}(t).\end{aligned}$$

Hence, it follows from $|\tilde{\kappa}| < \frac{9}{256}\kappa_0$ that

$$\frac{\frac{\kappa_0}{4} + \frac{8}{3}|\tilde{\kappa}|}{\frac{7\kappa_0}{8}} < \frac{1}{2}, \quad D^{0,\tau}(t) \le \max\left\{D^{0,\tau}(0),\ \frac{\frac{\kappa_0}{4} + \frac{8}{3}|\tilde{\kappa}|}{\frac{7\kappa_0}{8}}\right\} < \frac{1}{2}.$$

In this way, we verified claim (28).

• Step B (Zero convergence of modified diameter): We claim

$$\lim_{t\to\infty} \|z_i(t) - z_j(t)\| = 0. \tag{29}$$

The proof is similar to Theorem 3.1 of [10] with a slight difference. We present main steps that involve such differences. We put $s = 0$ in (25) to get

$$\begin{aligned}\frac{d}{dt}\|z_i - z_j\|^2 &\le -\kappa_0\|z_i - z_j\|^2(\mathrm{Re}\langle z_c^\tau, z_i\rangle + \mathrm{Re}\langle z_c^\tau, z_j\rangle) \\ &\quad - \frac{2\kappa_0}{N}\left(\mathrm{Re}\langle z_i - z_j, z_i^\tau - z_j^\tau\rangle - \|z_i - z_j\|^2\right) + 4|\tilde{\kappa}|\cdot\|z_i - z_j\|^2 \\ &\quad + \frac{4}{N}|\tilde{\kappa}|\cdot\|z_i - z_j\|(\|z_i - z_j\| + \|z_i^\tau - z_j^\tau\|).\end{aligned}$$

Next, we define a Lyapunov functional $\mathcal{E}_{ij}$ for $Z = (z_1, \cdots, z_N)$ and $i, j \in \mathcal{N}$:

$$\mathcal{E}_{ij}(t) := \|z_i(t) - z_j(t)\|^2 + \gamma\int_{t-\tau}^t \|z_i(s) - z_j(s)\|^2 ds,$$

where γ is a positive constant. Then, one has

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{ij}(t) &= \frac{d}{dt}\|z_i(t) - z_j(t)\|^2 + \gamma\|z_i(t) - z_j(t)\|^2 - \gamma\|z_i^\tau(t) - z_j^\tau(t)\|^2 \\ &\le -\frac{7\kappa_0}{4}\|z_i - z_j\|^2 + \frac{2\kappa_0}{N}\|z_i - z_j\|\cdot\|z_i^\tau - z_j^\tau\| \\ &\quad + \frac{2\kappa_0}{N}\|z_i - z_j\|^2 + 4|\tilde{\kappa}|\cdot\|z_i - z_j\|^2 \\ &\quad + \frac{4}{N}|\tilde{\kappa}|\cdot\|z_i - z_j\|(\|z_i - z_j\| + \|z_i^\tau - z_j^\tau\|) \\ &\quad + \gamma\|z_i(t) - z_j(t)\|^2 - \gamma\|z_i^\tau(t) - z_j^\tau(t)\|^2.\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{ij}(t) &\le -\frac{7\kappa_0}{4}\|z_i - z_j\|^2 + \frac{\kappa_0}{N}\|z_i - z_j\|^2 + \frac{\kappa_0}{N}\|z_i^\tau - z_j^\tau\|^2 + \frac{2\kappa_0}{N}\|z_i - z_j\|^2 \\ &\quad + 4|\tilde{\kappa}|\|z_i - z_j\|^2 + 4\frac{|\tilde{\kappa}|}{N}\|z_i - z_j\|^2 + 2\frac{|\tilde{\kappa}|}{N}\left(\|z_i - z_j\|^2 + \|z_i^\tau - z_j^\tau\|^2\right) \\ &\quad + \gamma\|z_i(t) - z_j(t)\|^2 - \gamma\|z_i^\tau(t) - z_j^\tau(t)\|^2 \\ &= \left(-\frac{7}{4}\kappa_0 + \frac{\kappa_0}{N} + \frac{2\kappa_0}{N} + 4|\tilde{\kappa}| + \frac{4|\tilde{\kappa}|}{N} + \frac{2|\tilde{\kappa}|}{N} + \gamma\right)\|z_i - z_j\|^2 \\ &\quad + \left(\frac{\kappa_0}{N} - \gamma + \frac{2|\tilde{\kappa}|}{N}\right)\|z_i^\tau - z_j^\tau\|^2.\end{aligned}$$

Now, we set

$$\gamma = \frac{\kappa_0}{N} + \frac{2|\tilde{\kappa}|}{N}.$$

Then, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{ij}(t) &\le \left(-\frac{7}{4}\kappa_0 + \frac{\kappa_0}{N} + \frac{2\kappa_0}{N} + 4|\tilde{\kappa}| + \frac{4|\tilde{\kappa}|}{N} + \frac{2|\tilde{\kappa}|}{N} + \gamma\right)\|z_i - z_j\|^2 \\ &= \left(-\frac{7}{4}\kappa_0 + \frac{4\kappa_0}{N} + 4|\tilde{\kappa}| + \frac{8|\tilde{\kappa}|}{N}\right)\|z_i - z_j\|^2.\end{aligned}$$

For $N \ge 3$ and $|\tilde{\kappa}| < \frac{1}{16}\kappa_0$, we have

$$-\frac{7}{4}\kappa_0 + \frac{4\kappa_0}{N} + 4|\tilde{\kappa}| + \frac{8|\tilde{\kappa}|}{N} < 0.$$

Here we set

$$\beta = -\left(-\frac{7}{4}\kappa_0 + \frac{4\kappa_0}{N} + 4|\tilde{\kappa}| + \frac{8|\tilde{\kappa}|}{N}\right)$$

to obtain

$$\frac{d}{dt}\mathcal{E}_{ij}(t) \le -\beta\|z_i - z_j\|^2.$$

This yields

$$\mathcal{E}_{ij}(t) - \mathcal{E}_{ij}(0) \le -\beta\int_0^t \|z_i(s) - z_j(s)\|^2 ds$$

which is equivalent to

$$\mathcal{E}_{ij}(t) + \beta\int_0^t \|z_i(s) - z_j(s)\|^2 ds \le \mathcal{E}_{ij}(0).$$

It follows from definition of $\mathcal{E}_{ij}$ that

$$\mathcal{E}_{ij} \ge 0.$$

Finally, we have

$$\beta\int_0^t \|z_i(s) - z_j(s)\|^2 ds \le \mathcal{E}_{ij}(0).$$

By letting $t \to \infty$, one has

$$\beta\int_0^\infty \|z_i(s) - z_j(s)\|^2 ds \le \mathcal{E}_{ij}(0).$$

It follows from the boundedness of $\|\dot{z}_j\|$ for all j that

$$\sup_{0\le t<\infty}\left|\frac{d}{dt}\|z_i(t) - z_j(t)\|^2\right| < \infty.$$

This means $\|z_i(s)-z_j(s)\|$ is uniformly continuous. Hence, we can apply Barbalat's lemma to obtain the desired estimate (29).

4. **Emergence of practical aggregation.** In this section, we study the practical aggregation of the LHS model.

4.1. **Complete network topology.** In this subsection, we set

$$a_{ij} \equiv 1, \quad \Omega_j \equiv 0, \quad i,j \in \mathcal{N}.$$

In this case, system (1) becomes

$$\begin{cases} \dot{z}_j = \dfrac{\kappa_0}{N} \displaystyle\sum_{k\neq j} \big(\langle z_j, z_j \rangle z_k^\tau - \langle z_k^\tau, z_j \rangle z_j \big) + \dfrac{\kappa_1}{N} \displaystyle\sum_{k\neq j} \big(\langle z_j, z_k^\tau \rangle - \langle z_k^\tau, z_j \rangle \big) z_j, \quad t>0, \\ z_j(t) = \varphi_j(t) \in \mathbb{C}^d, \quad -\tau \le t \le 0, \ j \in \mathcal{N}. \end{cases} \tag{30}$$

For handy notation, we set

$$G_{ij} := \langle z_i, z_j \rangle, \qquad G_{ij}^\tau := \langle z_i^\tau, z_j \rangle, \qquad L_{ij} := 1 - G_{ij}, \quad L_{ij}^\tau := 1 - G_{ij}^\tau. \tag{31}$$

Our third main result is concerned with the practical aggregation. Recall that

$$L(t) = \max_{i,j} |1 - \langle z_i(t), z_j(t) \rangle|.$$

Theorem 4.1. *Suppose coupling gains and initial data satisfy*

$$2|\kappa_1| < \kappa_0, \quad L(0) < 1 - \frac{2|\kappa_1|}{\kappa_0},$$

and let $\{z_j\}$ be a global solution to (24). Then, system (24) exhibits the practical synchronization:

$$\lim_{\tau \searrow 0} \limsup_{t\to\infty} L(t) = 0.$$

Proof. We leave its proof in Section 4.1.2. □

4.1.1. *Basic estimates.* In this part, we provide several lemmas to be crucially used in the proof of Theorem 4.1.

Lemma 4.2. *Let $\{z_j\}$ be a global solution to (31). Then, G_{ij} satisfies*

$$\begin{aligned} \frac{d}{dt} G_{ij} &= \frac{\kappa_0}{N} \sum_{k\neq i} (G_{kj}^\tau - \overline{G}_{ki}^\tau G_{ij}) + \frac{\kappa_0}{N} \sum_{k\neq j} (\overline{G}_{ki}^\tau - G_{kj}^\tau G_{ij}) \\ &\quad + \frac{\kappa_1}{N} \sum_{k\neq i} (G_{ki}^\tau - \overline{G}_{ki}^\tau) G_{ij} + \frac{\kappa_1}{N} \sum_{k\neq j} (\overline{G}_{kj}^\tau - G_{kj}^\tau) G_{ij}. \end{aligned}$$

Proof. By direct calculation, one has

$$
\begin{aligned}
\frac{d}{dt}G_{ij} &= \langle \dot{z}_i, z_j\rangle + \langle z_i, \dot{z}_j\rangle \\
&= \left\langle \frac{\kappa_0}{N}\sum_{k\neq i}\big(\langle z_i, z_i\rangle z_k^\tau - \langle z_k^\tau, z_i\rangle z_i\big) + \frac{\kappa_1}{N}\sum_{k\neq i}\big(\langle z_i, z_k^\tau\rangle - \langle z_k^\tau, z_i\rangle\big) z_i, z_j \right\rangle \\
&\quad + \left\langle z_i, \frac{\kappa_0}{N}\sum_{k\neq j}\big(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j\big) + \frac{\kappa_1}{N}\sum_{k\neq j}\big(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\big) z_j \right\rangle \\
&= \frac{\kappa_0}{N}\sum_{k\neq i}\big(\langle z_k^\tau, z_j\rangle - \overline{\langle z_k^\tau, z_i\rangle}\langle z_i, z_j\rangle\big) + \frac{\kappa_1}{N}\sum_{k\neq i}\big(\overline{\langle z_i, z_k^\tau\rangle} - \overline{\langle z_k^\tau, z_i\rangle}\big)\langle z_i, z_j\rangle \\
&\quad + \frac{\kappa_0}{N}\sum_{k\neq j}\big(\langle z_i, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\langle z_i, z_j\rangle\big) + \frac{\kappa_1}{N}\sum_{k\neq j}\big(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\big)\langle z_i, z_j\rangle \\
&= \frac{\kappa_0}{N}\sum_{k\neq i}(G_{kj}^\tau - \overline{G}_{ki}^\tau G_{ij}) + \frac{\kappa_0}{N}\sum_{k\neq j}(\overline{G}_{ki}^\tau - G_{kj}^\tau G_{ij}) + \frac{\kappa_1}{N}\sum_{k\neq i}(G_{ki}^\tau - \overline{G}_{ki}^\tau)G_{ij} \\
&\quad + \frac{\kappa_1}{N}\sum_{k\neq j}(\overline{G}_{kj}^\tau - G_{kj}^\tau)G_{ij}.
\end{aligned}
$$

□

Lemma 4.3. *Let $A \in \mathbb{C}^{d\times d}$ and $v \in \mathbb{C}^d$ be given matrix and vector, respectively. Then, one has*

$$\|Av\| \le \|A\|_F \cdot \|v\|,$$

where $\|\cdot\|$ is a vector norm in $\mathbb{C}^d$ and $\|\cdot\|_F$ is the Frobenius norm defined by $\|A\|_F := \sqrt{tr(A^\dagger A)}$.

Proof. We set the componentwise form of A and v as follows:

$$A := [A]_{\alpha\beta} \quad \text{and} \quad v := [v]_\gamma,$$

where $1 \le \alpha, \beta, \gamma \le d$. By the Cauchy-Schwarz inequality, we have

$$|[Av]_\alpha| = \left|\sum_{\beta=1}^d [A]_{\alpha\beta}[v]_\beta\right| \le \sqrt{\sum_{\beta=1}^d [\bar{A}]_{\alpha\beta}[A]_{\alpha\beta}} \cdot \sqrt{\sum_{\beta=1}^d [\bar{v}]_\beta [v]_\beta},$$

Thus, one has

$$\|Av\|^2 = \sum_{\alpha=1}^d |[Av]_\alpha|^2 \le \left(\sum_{\alpha,\beta=1}^d [\bar{A}]_{\alpha\beta}[A]_{\alpha\beta}\right)\cdot\left(\sum_{\beta=1}^d [\bar{v}]_\beta[v]_\beta\right) = \|A\|_F^2 \cdot \|v\|^2,$$

and this yields the desired result. □

Lemma 4.4. *Let $\{z_j\}$ be a global solution to (30). Then L_{ij} in (31) satisfies*

$$|L_{ij}(t) - L_{ij}^\tau(t)| \le \tau\mathcal{C}_2,$$

where the positive constant $\mathcal{C}_2$ is given by

$$\mathcal{C}_2 := \frac{2(N-1)}{N}\left(\kappa_0 + |\kappa_1|\right).$$

Proof. By the Cauchy-Schwarz inequality, we have

$$|L_{ij}(t) - L^\tau_{ij}(t)| = |\langle z_i - z_i^\tau, z_j\rangle| \le \|z_i - z_i^\tau\| \cdot \|z_j\|.$$

Note that $\|z_j\| = 1$. By Lemma 3.8, we have

$$\|z_i - z_i^\tau\| \le \tau\Big(2\frac{N-1}{N}(\kappa_0 + |\kappa_1|)\Big).$$

□

Lemma 4.5. *Let $\{z_j\}$ be a global solution to (30). Then, $|L_{ij}|$ satisfies*

$$\begin{aligned}\frac{d}{dt}|L_{ij}|^2 \le &-\frac{2\kappa_0}{N}\sum_{k=1}^N |L_{ij}|^2(\mathrm{Re}(\langle z_k^\tau, z_i + z_j\rangle) + 4|\kappa_1|\cdot|L_{ij}|\cdot(|L_{ci}| + |L_{cj}| + 2\mathcal{C}_2\tau)\\ &+ |L_{ij}|\frac{8\mathcal{C}_2\tau}{N}(\kappa_0 + |\kappa_1|) + |L_{ij}|^2\frac{4\kappa_0\mathcal{C}_2\tau}{N}.\end{aligned}$$

Proof. We use (30) to get

$$\begin{aligned}\frac{d}{dt}G_{ij} &= \frac{\kappa_0}{N}\sum_{k\ne i}(G^\tau_{kj} - \overline{G}^\tau_{ki}G_{ij}) + \frac{\kappa_0}{N}\sum_{k\ne j}(\overline{G}^\tau_{ki} - G^\tau_{kj}G_{ij}) + \frac{\kappa_1}{N}\sum_{k\ne i}(G^\tau_{ki}G_{ij} - \overline{G}^\tau_{ki}G_{ij})\\ &+ \frac{\kappa_1}{N}\sum_{k\ne j}(\overline{G}^\tau_{kj}G_{ij} - G^\tau_{kj}G_{ij})\\ &= \frac{\kappa_0}{N}\sum_{k=1}^N(G^\tau_{kj} - \overline{G}^\tau_{ki}G_{ij} + \overline{G}^\tau_{ki} - G^\tau_{kj}G_{ij}) + \frac{\kappa_1}{N}\sum_{k=1}^N(G^\tau_{ki}G_{ij} - \overline{G}^\tau_{ki}G_{ij} + \overline{G}^\tau_{kj}G_{ij} - G^\tau_{kj}G_{ij})\\ &- \frac{\kappa_0}{N}(G^\tau_{ij} - \overline{G}^\tau_{ii}G_{ij} + \overline{G}^\tau_{ji} - G^\tau_{jj}G_{ij}) - \frac{\kappa_1}{N}(G^\tau_{ii}G_{ij} - \overline{G}^\tau_{ii}G_{ij} + \overline{G}^\tau_{jj}G_{ij} - G^\tau_{jj}G_{ij})\\ &= \frac{\kappa_0}{N}\sum_{k=1}^N(2 - L^\tau_{kj} - \overline{L}^\tau_{ki})L_{ij} + \frac{2\mathrm{i}\kappa_1}{N}\sum_{k=1}^N(\mathrm{Im}L^\tau_{kj} - \mathrm{Im}L^\tau_{ki})(1 - L_{ij})\\ &- \frac{\kappa_0}{N}(2L_{ij} - L^\tau_{ij} - \overline{L}^\tau_{ji} + L^\tau_{jj} + \overline{L}^\tau_{ii} - \overline{L}^\tau_{ii}L_{ij} - L^\tau_{jj}L_{ij}) - \frac{2\mathrm{i}\kappa_1}{N}(1 - L_{ij})(\mathrm{Im}L^\tau_{jj} - \mathrm{Im}L^\tau_{ii})\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{d}{dt}|L_{ij}|^2 &= \frac{d}{dt}(L_{ij}\bar{L}_{ij}) = \dot{L}_{ij}\overline{L}_{ij} + L_{ij}\dot{\overline{L}}_{ij} = -\frac{d}{dt}\langle z_i, z_j\rangle(1 - \langle z_j, z_i\rangle) + (1 - \langle z_i, z_j\rangle)(-\frac{d}{dt}\langle z_j, z_i\rangle)\\ &= -L_{ji}\Big(\frac{\kappa_0}{N}\sum_{k=1}^N(2 - L^\tau_{kj} - \overline{L}^\tau_{ki})L_{ij} + \frac{2\mathrm{i}\kappa_1}{N}\sum_{k=1}^N(\mathrm{Im}L^\tau_{kj} - \mathrm{Im}L^\tau_{ki})(1 - L_{ij})\\ &- \frac{\kappa_0}{N}(2L_{ij} - L^\tau_{ij} - \overline{L}^\tau_{ji} + L^\tau_{jj} + \overline{L}^\tau_{ii} - \overline{L}^\tau_{ii}L_{ij} - L^\tau_{jj}L_{ij}) - \frac{2\mathrm{i}\kappa_1}{N}(1 - L_{ij})(\mathrm{Im}L^\tau_{jj} - \mathrm{Im}L^\tau_{ii})\Big)\\ &- L_{ij}\Big(\frac{\kappa_0}{N}\sum_{k=1}^N(2 - L^\tau_{ki} - \overline{L}^\tau_{kj})L_{ji} + \frac{2\mathrm{i}\kappa_1}{N}\sum_{k=1}^N(\mathrm{Im}L^\tau_{ki} - \mathrm{Im}L^\tau_{kj})(1 - L_{ji})\\ &- \frac{\kappa_0}{N}(2L_{ji} - L^\tau_{ji} - \overline{L}^\tau_{ij} + L^\tau_{ii} + \overline{L}^\tau_{jj} - \overline{L}^\tau_{jj}L_{ji} - L^\tau_{ii}L_{ji}) - \frac{2\mathrm{i}\kappa_1}{N}(1 - L_{ji})(\mathrm{Im}L^\tau_{ii} - \mathrm{Im}L^\tau_{jj})\Big)\\ &= \frac{\kappa_0}{N}\sum_{k=1}^N L_{ij}L_{ji}(L^\tau_{kj} + \overline{L}^\tau_{kj} + L^\tau_{ki} + \overline{L}^\tau_{ki} - 4) + \frac{2\mathrm{i}\kappa_1}{N}\sum_{k=1}^N(L_{ij} - L_{ji})(\mathrm{Im}L^\tau_{kj} - \mathrm{Im}L^\tau_{ki})\\ &+ \frac{\kappa_0}{N}(4L_{ij}L_{ji} + (-L^\tau_{ij} - \overline{L}^\tau_{ji} + L^\tau_{jj} + \overline{L}^\tau_{ii})L_{ji} + (-L^\tau_{ji} - \overline{L}^\tau_{ij} + L^\tau_{ii} + \overline{L}^\tau_{jj})L_{ij}\\ &- (\overline{L}^\tau_{ii} + L^\tau_{jj} + \overline{L}^\tau_{jj} + L^\tau_{ii})L_{ij}L_{ji}) + \frac{2\mathrm{i}\kappa_1}{N}(L_{ij} - L_{ji})(\mathrm{Im}L^\tau_{ii} - \mathrm{Im}L^\tau_{jj}).\end{aligned}$$

Note that

$$L_{ij}L_{ji} = (1 - \langle z_i, z_j\rangle)(1 - \langle z_j, z_i\rangle) = |L_{ij}|^2 \quad \text{and} \quad \overline{L_{ij}} = L_{ji}.$$

So we have

$$\begin{aligned}\frac{d}{dt}|L_{ij}|^2 &= \frac{2\kappa_0}{N}\sum_{k=1}^{N}|L_{ij}|^2(\mathrm{Re}L^\tau_{ki}+\mathrm{Re}L^\tau_{kj}-2)+\frac{4\kappa_1}{N}\sum_{k=1}^{N}\mathrm{Im}L_{ij}(\mathrm{Im}L^\tau_{ki}-\mathrm{Im}L^\tau_{kj})\\ &\quad+\frac{\kappa_0}{N}\big(4|L_{ij}|^2+2\mathrm{Re}((-L^\tau_{ji}-\overline{L}^\tau_{ij}+L^\tau_{ii}+\overline{L}^\tau_{jj})L_{ij})-2(\mathrm{Re}(L^\tau_{ii}+L^\tau_{jj})|L_{ij}|^2)\big)\\ &\quad-\frac{4\kappa_1}{N}\mathrm{Im}L_{ij}\mathrm{Im}(L^\tau_{ii}-L^\tau_{jj}).\end{aligned}$$

Note that the last two terms of the right hand side in above equation goes to zero as τ goes to 0.

- Step A: Note that

$$\|z_i-z_j\|^2=|2(1-\mathrm{Re}(\langle z_i,z_j\rangle)|\le 2|L_{ij}|,$$

By Lemma 4.4, we have for any i,

$$|L^\tau_{ii}|\le\tau\mathcal{C}_2.$$

Since $L_{ii}=0$, one has

$$\begin{aligned}\left|\frac{4\kappa_1}{N}\mathrm{Im}L_{ij}\mathrm{Im}(L^\tau_{ii}-L^\tau_{jj})\right| &= \frac{4|\kappa_1|}{N}|L_{ij}|\mathrm{Im}(\langle z_i,z_i^\tau\rangle-\langle z_j,z_j^\tau\rangle)|\\ &= \frac{4|\kappa_1|}{N}|L_{ij}|\mathrm{Im}(\langle z_i-z_j,z_i^\tau\rangle+\langle z_j,z_i^\tau-z_j^\tau\rangle)|\\ &\le \frac{4|\kappa_1|}{N}|L_{ij}|(|L^\tau_{ii}|+|L^\tau_{jj}|)\le\frac{8\mathcal{C}_2|\kappa_1|\tau}{N}|L_{ij}|.\end{aligned}$$

- Step B: Next, we analyze the term

$$\begin{aligned}A &:= \frac{\kappa_0}{N}\big(4|L_{ij}|^2+2\mathrm{Re}((-L^\tau_{ji}-\overline{L}^\tau_{ij}+L^\tau_{ii}+\overline{L}^\tau_{jj})L_{ij})-2(\mathrm{Re}(L^\tau_{ii}+L^\tau_{jj})|L_{ij}|^2)\\ &= \frac{\kappa_0}{N}\Big(\underbrace{4|L_{ij}|^2+2\mathrm{Re}((-L^\tau_{ji}L_{ij}-\overline{L}^\tau_{ij}L_{ij})}_{=:A_1}+\underbrace{2\mathrm{Re}(L^\tau_{ii}L_{ij}+\overline{L}^\tau_{jj}L_{ij})-2(\mathrm{Re}(L^\tau_{ii}+L^\tau_{jj})|L_{ij}|^2}_{=:A_2}\Big).\end{aligned}\tag{32}$$

In the sequel, we estimate $A_i,\ i=1,2$ as follows.

- (Estimate of A_2): By direct estimate, one has

$$\begin{aligned}A_2 &\le 2|L^\tau_{ii}L_{ij}|+2|\overline{L}^\tau_{jj}L_{ij}|+2\big((|L^\tau_{ii}|+|L^\tau_{jj}|)|L_{ij}|^2\big)\\ &\le 2|L_{ij}|(|L^\tau_{ii}|+|\overline{L}^\tau_{jj}|)+2|L_{ij}|^2(|L^\tau_{ii}|+|L^\tau_{jj}|)\\ &\le 4\mathcal{C}_2\tau|L_{ij}|+4\mathcal{C}_2\tau|L_{ij}|^2.\end{aligned}\tag{33}$$

On the other hand, note that

$$\begin{aligned}|L_{ij}|^2-\mathrm{Re}L^\tau_{ji}L_{ij} &= \mathrm{Re}|L_{ij}|^2-\mathrm{Re}L^\tau_{ji}L_{ij}=\mathrm{Re}\big((L_{ji}-L^\tau_{ji})L_{ij}\big)\\ &\le |\big((L_{ji}-L^\tau_{ji})L_{ij}\big)|=|L_{ji}-L^\tau_{ji}||L_{ij}|\le\mathcal{C}_2\tau|L_{ij}|\end{aligned}$$

- (Estimate of A_1): Similarly, by Lemma 4.4, one has

$$\begin{aligned}|L_{ij}|^2-\mathrm{Re}L_{ij}\overline{L}^\tau_{ij} &= \mathrm{Re}(|L_{ij}|^2-L_{ij}\overline{L}^\tau_{ij})=\mathrm{Re}(L_{ij}L_{ji}-L_{ij}\overline{L}^\tau_{ij})\\ &\le |L_{ij}||L_{ji}-\overline{L}^\tau_{ij}|\le\mathcal{C}_2\tau|L_{ij}|.\end{aligned}$$

Thus, we have

$$\begin{aligned} A_1 &= 4|L_{ij}|^2 + 2\mathrm{Re}((-L^\tau_{ji}L_{ij} - \overline{L}^\tau_{ij}L_{ij}) \\ &= 2(|L_{ij}|^2 + |L_{ij}|^2 - \mathrm{Re}L^\tau_{ji}L_{ij} - \mathrm{Re}\overline{L}^\tau_{ij}L_{ij}) \le 4\mathcal{C}_2\tau|L_{ij}|. \end{aligned} \tag{34}$$

In (32), we combine all the estimate (33) and (34) to find

$$A = \frac{\kappa_0}{N}(A_1 + A_2) \le \frac{4\kappa_0\mathcal{C}_2\tau}{N}(2|L_{ij}| + |L_{ij}|^2),$$

and

$$\begin{aligned} &\frac{\kappa_0}{N}\big(4|L_{ij}|^2 + 2\mathrm{Re}((-L^\tau_{ji} - \overline{L}^\tau_{ij} + L^\tau_{ii} + \overline{L}^\tau_{jj})L_{ij}) \\ &\qquad - 2(\mathrm{Re}(L^\tau_{ii} + L^\tau_{jj})|L_{ij}|^2)\big) - \frac{4\kappa_1}{N}\mathrm{Im}L_{ij}\mathrm{Im}(L^\tau_{ii} - L^\tau_{jj}) \\ &\qquad \le |L_{ij}|\frac{8\mathcal{C}_2|\kappa_1|\tau + 8\mathcal{C}_2\kappa_0\tau}{N} + |L_{ij}|^2\frac{4\kappa_0\mathcal{C}_2\tau}{N}. \end{aligned}$$

• Step C: Finally, we analyze the term

$$\frac{2\kappa_0}{N}\sum_{k=1}^N |L_{ij}|^2(\mathrm{Re}L^\tau_{ki} + \mathrm{Re}L^\tau_{kj} - 2) + \frac{4\kappa_1}{N}\sum_{k=1}^N \mathrm{Im}L_{ij}(\mathrm{Im}L^\tau_{ki} - \mathrm{Im}L^\tau_{kj}).$$

By direct calculation, we have

$$\begin{aligned} &\frac{2\kappa_0}{N}\sum_{k=1}^N |L_{ij}|^2(\mathrm{Re}L^\tau_{ki} + \mathrm{Re}L^\tau_{kj} - 2) + \frac{4\kappa_1}{N}\sum_{k=1}^N \mathrm{Im}L_{ij}(\mathrm{Im}L^\tau_{ki} - \mathrm{Im}L^\tau_{kj}) \\ &= \frac{2\kappa_0}{N}\sum_{k=1}^N |1 - \langle z_i, z_j\rangle|^2(\mathrm{Re}(1 - \langle z^\tau_k, z_i\rangle + 1 - \langle z^\tau_k, z_j\rangle - 2) \\ &+ \frac{4\kappa_1}{N}\sum_{k=1}^N \mathrm{Im}(1 - \langle z_i, z_j\rangle)(\mathrm{Im}(1 - \langle z^\tau_k, z_i\rangle - 1 + \langle z^\tau_k, z_j\rangle) \\ &= -\frac{2\kappa_0}{N}\sum_{k=1}^N |1 - \langle z_i, z_j\rangle|^2(\mathrm{Re}(\langle z^\tau_k, z_i + z_j\rangle) + \frac{4\kappa_1}{N}\sum_{k=1}^N \mathrm{Im}(\langle z_i, z_j\rangle)(\mathrm{Im}(\langle z^\tau_k, z_i - z_j\rangle). \end{aligned}$$

Note that

$$\begin{aligned} &\frac{4\kappa_1}{N}\sum_{k=1}^N \mathrm{Im}(\langle z_i, z_j\rangle)(\mathrm{Im}(\langle z^\tau_k, z_i - z_j\rangle) = 4\kappa_1\mathrm{Im}(\langle z_i, z_j\rangle)(\mathrm{Im}(\langle z^\tau_c, z_i - z_j\rangle) \\ &\qquad \le 4|\kappa_1|\cdot|L_{ij}|\cdot(|L^\tau_{ci}| + |L^\tau_{cj}|) \le 4|\kappa_1|\cdot|L_{ij}|\cdot(|L_{ci}| + |L_{cj}| + 2\mathcal{C}_2\tau), \end{aligned}$$

and so

$$\begin{aligned} &-\frac{2\kappa_0}{N}\sum_{k=1}^N |1 - \langle z_i, z_j\rangle|^2(\mathrm{Re}(\langle z^\tau_k, z_i + z_j\rangle) + \frac{4\kappa_1}{N}\sum_{k=1}^N \mathrm{Im}(\langle z_i, z_j\rangle)(\mathrm{Im}(\langle z^\tau_k, z_i - z_j\rangle) \\ &\qquad \le -\frac{2\kappa_0}{N}\sum_{k=1}^N |1 - \langle z_i, z_j\rangle|^2(\mathrm{Re}(\langle z^\tau_k, z_i + z_j\rangle) + 4|\kappa_1|\cdot|L_{ij}|\cdot(|L_{ci}| + |L_{cj}| + 2\mathcal{C}_2\tau). \end{aligned}$$

• Step D: We collect all the estimates in Step A - Step C to find

$$
\begin{aligned}
\frac{d}{dt}|L_{ij}|^2 &= \frac{2\kappa_0}{N}\sum_{k=1}^{N}|L_{ij}|^2(\mathrm{Re}L_{ki}^\tau + \mathrm{Re}L_{kj}^\tau - 2) + \frac{4\kappa_1}{N}\sum_{k=1}^{N}\mathrm{Im}L_{ij}(\mathrm{Im}L_{ki}^\tau - \mathrm{Im}L_{kj}^\tau) \\
&\quad + \frac{\kappa_0}{N}(4|L_{ij}|^2 + 2\mathrm{Re}((-L_{ji}^\tau - \overline{L}_{ij}^\tau + L_{ii}^\tau + \overline{L}_{jj}^\tau)L_{ij}) - 2(\mathrm{Re}(L_{ii}^\tau + L_{jj}^\tau)|L_{ij}|^2) \\
&\quad - \frac{4\kappa_1}{N}\mathrm{Im}L_{ij}\mathrm{Im}(L_{ii}^\tau - L_{jj}^\tau) \\
&\le -\frac{2\kappa_0}{N}\sum_{k=1}^{N}|L_{ij}|^2(\mathrm{Re}(\langle z_k^\tau, z_i + z_j\rangle) + 4|\kappa_1|\cdot|L_{ij}|\cdot(|L_{ci}| + |L_{cj}| + 2\mathcal{C}_2\tau) \\
&\quad + |L_{ij}|\frac{8\mathcal{C}_2|\kappa_1|\tau + 8\mathcal{C}_2\kappa_0\tau}{N} + |L_{ij}|^2\frac{4\kappa_0\mathcal{C}_2\tau}{N}.
\end{aligned}
$$

□

We set

$$L(t) = \max_{i,j}|L_{ij}|.$$

Then, for each time t, there exists i_t and j_t by which the maximum is attained, i.e.

$$L(t) = |1 - \langle z_{i_t}, z_{j_t}\rangle|. \tag{35}$$

Now we want to obtain the dynamics of $L(t)$.

Lemma 4.6. *Let $\{z_j\}$ be a global solution to (24). Then, the functional $L(t)$ in (35) satisfies*

$$
\begin{aligned}
\frac{d}{dt}L(t) &\le 2\kappa_0 L(t)^2 + \left(-2\kappa_0 + 2\mathcal{C}_2\kappa_0\tau + 4|\kappa_1| + \frac{2\mathcal{C}_2\kappa_0\tau}{N}\right)L(t) \\
&\quad + \left(4\mathcal{C}_2|\kappa_1|\tau + \frac{4\mathcal{C}_2\tau}{N}(\kappa_0 + |\kappa_1|)\right).
\end{aligned}
$$

Proof. It follows from Lemma 4.5 and the fact that $|L_{ci}| \le L(t)$, we have

$$
\begin{aligned}
\frac{d}{dt}L(t)^2 &\le -2\kappa_0 L(t)^2(\mathrm{Re}(\langle z_c^\tau, z_i + z_j\rangle) + 4|\kappa_1|\cdot L(t)\cdot(2L(t) + 2\mathcal{C}_2\tau) \\
&\quad + L(t)\frac{8\mathcal{C}_2|\kappa_1|\tau + 8\mathcal{C}_2\kappa_0\tau}{N} + L(t)^2\frac{4\kappa_0\mathcal{C}_2\tau}{N} \\
&= \left(-2\kappa_0\mathrm{Re}(\langle z_c^\tau, z_i + z_j\rangle) + 8|\kappa_1| + \frac{4\mathcal{C}_2\kappa_0\tau}{N}\right)L(t)^2 \\
&\quad + \left(8\mathcal{C}_2|\kappa_1|\tau + \frac{8\mathcal{C}_2|\kappa_1|\tau + 8\mathcal{C}_2\kappa_0\tau}{N}\right)L(t).
\end{aligned}
$$

Note that

$$\mathrm{Re}(\langle z_c^\tau, z_i + z_j\rangle) = -\mathrm{Re}(L_{ci}^\tau + L_{cj}^\tau) + 2.$$

Since $|L_{ci}^\tau| \le |L_{ci}| + \mathcal{C}_2\tau \le L(t) + \mathcal{C}_2\tau$, we have

$$
\begin{aligned}
\frac{d}{dt}L(t)^2 &\le \Big(-2\kappa_0\mathrm{Re}(2 - L_{ci}^\tau - L_{cj}^\tau) + 8|\kappa_1| + \frac{4\mathcal{C}_2\kappa_0\tau}{N}\Big)L(t)^2 \\
&\quad + \left(8\mathcal{C}_2|\kappa_1|\tau + \frac{8\mathcal{C}_2|\kappa_1|\tau + 8\mathcal{C}_2\kappa_0\tau}{N}\right)L(t)
\end{aligned}
$$

$$\leq 4\kappa_0 L(t)^3 + \left(-4\kappa_0 + 4\mathcal{C}_2\kappa_0\tau + 8|\kappa_1| + \frac{4\mathcal{C}_2\kappa_0\tau}{N}\right) L(t)^2 + \left(8\mathcal{C}_2|\kappa_1|\tau + \frac{8\mathcal{C}_2|\kappa_1|\tau + 8\mathcal{C}_2\kappa_0\tau}{N}\right) L(t).$$

Hence we obtain the desired estimate:

$$\frac{d}{dt}L(t) \leq 2\kappa_0 L(t)^2 + \left(-2\kappa_0 + 2\mathcal{C}_2\kappa_0\tau + 4|\kappa_1| + \frac{2\mathcal{C}_2\kappa_0\tau}{N}\right) L(t) + \left(4\mathcal{C}_2|\kappa_1|\tau + \frac{4\mathcal{C}_2\tau}{N}(\kappa_0 + |\kappa_1|)\right).$$

□

4.1.2. *Proof of Theorem 4.1.* Consider a quadratic polynomial

$$\begin{aligned} f(x) &= 2\kappa_0 x^2 + \left(-2\kappa_0 + 2\mathcal{C}_2\kappa_0\tau + 4|\kappa_1| + \frac{2\mathcal{C}_2\kappa_0\tau}{N}\right) x + \left(4\mathcal{C}_2|\kappa_1|\tau + \frac{4\mathcal{C}_2\tau}{N}(\kappa_0 + |\kappa_1|)\right) \\ &= 2\kappa_0\left(x^2 - \left(1 - \frac{2|\kappa_1|}{\kappa_0} - \mathcal{C}_2\tau\left(\frac{N+1}{N}\right)\right) x + \mathcal{C}_2\tau\left(\frac{2|\kappa_1|}{\kappa_0}\left(1 + \frac{1}{N}\right) + \frac{2}{N}\right)\right). \end{aligned}$$

Now we study the practical aggregation as $\tau \searrow 0$. Here we fix the other variables. Let us assume that $2|\kappa_1| < \kappa_0$. Then for a sufficiently small τ, there are two roots of $f(x) = 0$.

Note that the discriminant D_τ of $f(x)$ is given explicitly as

$$D_\tau :== \mathcal{C}_2^2\left(\frac{N+1}{N}\right)^2 \tau^2 - 2\mathcal{C}_2\left(\left(1 + \frac{2|\kappa_1|}{\kappa_0}\right)\frac{N+1}{N} + \frac{4}{N}\right)\tau + \left(1 - \frac{2|\kappa_1|}{\kappa_0}\right)^2.$$

Note that the constant term $(1 - \frac{2|\kappa_1|}{\kappa_0})^2$ is positive by the assumption so that

$$D_0 > 0,$$

so D_τ is positive as τ tends to 0. Now, for $\tau \ll 1$, we denote two roots by $x_-(\tau)$ and $x_+(\tau)$ with $x_-(\tau) \leq x_+(\tau)$. Then we have following property from the phase portrait:

$$L(0) < x_+(\tau) \quad \Rightarrow \quad \limsup_{t\to\infty} L(t) \leq x_-(\tau).$$

We also obtain

$$\lim_{\tau\searrow 0} x_-(\tau) = 0, \quad \lim_{\tau\searrow 0} x_+(\tau) = 1 - \frac{2|\kappa_1|}{\kappa_0}.$$

4.2. **General network topology.** In this subsection, we will study practical aggregation of the system (1). Here we set network topology as $\{a_{ij}\}$ with $a_{ij} \geq 0$, for all $i, j \in \mathcal{N}$.

$$\begin{cases} \dot{z}_j = \Omega_j z_j + \dfrac{\kappa_0}{N}\displaystyle\sum_{k\neq j} a_{jk}\big(\langle z_j, z_j\rangle z_k^\tau - \langle z_k^\tau, z_j\rangle z_j\big) + \dfrac{\kappa_1}{N}\displaystyle\sum_{k\neq j} a_{jk}\big(\langle z_j, z_k^\tau\rangle - \langle z_k^\tau, z_j\rangle\big) z_j, \\ z_j(t) = \varphi_j(t) \in \mathbb{C}^d, \quad -\tau \leq t \leq 0, \end{cases} \tag{36}$$

Theorem 4.7. *Let $\{z_j\}$ be a global solution to (36) with the following initial condition:*

$$L(0) < 1 - \frac{2\sum_{k=1}^N |a_{ik} - a_{jk}|}{\sum_{k=1}^N (a_{ik} + a_{jk})}.$$

Then, system (36) *exhibits the practical aggregation:*

$$\lim_{\kappa_0 \to \infty} \lim_{\tau \searrow 0} \limsup_{t \to \infty} L(t) = 0.$$

Proof. We leave its proof in Section 4.2.2 $\square$

4.2.1. *Basic estimates.* In this part, we provide several a priori estimates to be used in the proof of Theorem 4.7.

Lemma 4.8. *Let* $\{z_j\}$ *be a global solution to* (36)*. Then* G_{ij} *in* (31) *satisfies*

$$\begin{aligned}\frac{d}{dt}G_{ij} &= \langle (\Omega_i - \Omega_j) z_i, z_j \rangle + \frac{\kappa_0}{N} \sum_{k \neq i} a_{ik}(G^\tau_{kj} - \overline{G}^\tau_{ki} G_{ij}) + \frac{\kappa_0}{N} \sum_{k \neq j} a_{jk}(\overline{G}^\tau_{ki} - G^\tau_{kj} G_{ij}) \\ &\quad + \frac{\kappa_1}{N} \sum_{k \neq i} a_{ik}(G^\tau_{ki} - \overline{G}^\tau_{ki}) G_{ij} + \frac{\kappa_1}{N} \sum_{k \neq j} a_{jk}(\overline{G}^\tau_{kj} - G^\tau_{kj}) G_{ij}.\end{aligned}$$

Proof. By definition of $G_{ij} = \langle z_i, z_j \rangle$, one has

$$\begin{aligned}
&\frac{d}{dt}G_{ij} = \langle \dot z_i, z_j \rangle + \langle z_i, \dot z_j \rangle \\
&= \left\langle \Omega_i z_i + \frac{\kappa_0}{N} \sum_{k \neq i} a_{ik}\big(z^\tau_k - \langle z^\tau_k, z_i \rangle z_i\big) + \frac{\kappa_1}{N} \sum_{k \neq i} a_{ik}\big(\langle z_i, z^\tau_k \rangle - \langle z^\tau_k, z_i \rangle\big) z_i, z_j \right\rangle \\
&+ \left\langle z_i, \Omega_j z_j + \frac{\kappa_0}{N} \sum_{k \neq j} a_{jk}\big(z^\tau_k - \langle z^\tau_k, z_j \rangle z_j\big) + \frac{\kappa_1}{N} \sum_{k \neq j} a_{jk}\big(\langle z_j, z^\tau_k \rangle - \langle z^\tau_k, z_j \rangle\big) z_j \right\rangle \\
&= \langle \Omega_i z_i, z_j \rangle + \frac{\kappa_0}{N} \sum_{k \neq i} a_{ik}\big(\langle z^\tau_k, z_j \rangle - \overline{\langle z^\tau_k, z_i \rangle} \langle z_i, z_j \rangle\big) + \frac{\kappa_1}{N} \sum_{k \neq i} a_{ik}\big(\overline{\langle z_i, z^\tau_k \rangle} - \overline{\langle z^\tau_k, z_i \rangle}\big) \langle z_i, z_j \rangle \\
&+ \langle z_i, \Omega_j z_j \rangle + \frac{\kappa_0}{N} \sum_{k \neq j} a_{jk}\big(\langle z_i, z^\tau_k \rangle - \langle z^\tau_k, z_j \rangle \langle z_i, z_j \rangle\big) + \frac{\kappa_1}{N} \sum_{k \neq j} a_{jk}\big(\langle z_j, z^\tau_k \rangle - \langle z^\tau_k, z_j \rangle\big) \langle z_i, z_j \rangle \\
&= \langle (\Omega_i - \Omega_j) z_i, z_j \rangle + \frac{\kappa_0}{N} \sum_{k \neq i} a_{ik}\big(\langle z^\tau_k, z_j \rangle - \overline{\langle z^\tau_k, z_i \rangle} \langle z_i, z_j \rangle\big) + \frac{\kappa_1}{N} \sum_{k \neq i} a_{ik}\big(\overline{\langle z_i, z^\tau_k \rangle} - \overline{\langle z^\tau_k, z_i \rangle}\big) \langle z_i, z_j \rangle \\
&+ \frac{\kappa_0}{N} \sum_{k \neq j} a_{jk}\big(\langle z_i, z^\tau_k \rangle - \langle z^\tau_k, z_j \rangle \langle z_i, z_j \rangle\big) + \frac{\kappa_1}{N} \sum_{k \neq j} a_{jk}\big(\langle z_j, z^\tau_k \rangle - \langle z^\tau_k, z_j \rangle\big) \langle z_i, z_j \rangle \\
&= \langle (\Omega_i - \Omega_j) z_i, z_j \rangle + \frac{\kappa_0}{N} \sum_{k \neq i} a_{ik}(G^\tau_{kj} - \overline{G}^\tau_{ki} G_{ij}) + \frac{\kappa_0}{N} \sum_{k \neq j} a_{jk}(\overline{G}^\tau_{ki} - G^\tau_{kj} G_{ij}) \\
&+ \frac{\kappa_1}{N} \sum_{k \neq i} a_{ik}(G^\tau_{ki} - \overline{G}^\tau_{ki}) G_{ij} + \frac{\kappa_1}{N} \sum_{k \neq j} a_{jk}(\overline{G}^\tau_{kj} - G^\tau_{kj}) G_{ij}.
\end{aligned}$$

which yields the desired estimate. $\square$

Lemma 4.9. *Let* $\{z_j\}$ *be a global solution to* (36)*. Then, one has*

$$|L_{ij}(t) - L^\tau_{ij}(t)| \le \tau \mathcal{C}_3,$$

where the positive constant $\mathcal{C}_3$ *is given by*

$$\mathcal{C}_3 := \max_i \|\Omega_i\|_\infty + \frac{2\|A\|_\infty (N-1)(\kappa_0 + |\kappa_1|)}{N}.$$

Proof. By the Cauchy-Schwarz inequality, we have

$$|L_{ij}(t) - L^\tau_{ij}(t)| = |\langle z_i - z^\tau_i, z_j \rangle| \le \|z_i - z^\tau_i\| \cdot \|z_j\| = \|z_i - z^\tau_i\|.$$

On the other hand, we have

$$\begin{aligned}
\|z_i(t)-z_i^\tau(t)\| &= \left\|\int_{t-\tau}^t \dot{z}_i(s)ds\right\| \\
&\le \int_{t-\tau}^t \left\|\Omega_i z_i + \frac{\kappa_0}{N}\sum_{k\neq i} a_{ik}\big(\langle z_i, z_i\rangle z_k^\tau - \langle z_k^\tau, z_i\rangle z_i\big) + \frac{\kappa_1}{N}\sum_{k\neq i} a_{ik}\big(\langle z_i, z_k^\tau\rangle - \langle z_k^\tau, z_i\rangle\big) z_i\right\| ds \\
&\le \tau\left(\|\Omega_i\|_\infty + \sum_{k\neq i}\frac{2a_{ik}(\kappa_0+|\kappa_1|)}{N}\right).
\end{aligned}$$

Now, we set

$$\|A\|_\infty := \max_{i,j} a_{ij}.$$

Then, we have

$$|L_{ij}(t)-L_{ij}^\tau(t)| \le \tau\left(\max_i \|\Omega_i\|_\infty + \frac{2\|A\|_\infty (N-1)(\kappa_0+|\kappa_1|)}{N}\right).$$

□

We set

$$D(\Omega) := \max_{i,j}\|\Omega_i - \Omega_j\|_\infty.$$

Then by direct calculation, we get the following lemma.

Lemma 4.10. *Let $\{z_j\}$ be a global solution to (36). Then $|L_{ij}|$ satisfies*

$$\begin{aligned}
\frac{d}{dt}|L_{ij}|^2 &\le 2|L_{ij}|D(\Omega) + \frac{2\kappa_0}{N}\sum_{k=1}^N |a_{ik}-a_{jk}|\Big(|L_{ki}|+|L_{kj}|+2\tau\mathcal{C}_3\Big)|L_{ij}| \\
&\quad + \frac{4\mathcal{C}_3|\kappa_1|\tau}{N}(a_{ii}+a_{jj})|L_{ij}| - 2\frac{\kappa_0}{N}\sum_{k=1}^N (a_{ik}+a_{jk})|L_{ij}|^2 \\
&\quad + \frac{2\kappa_0}{N}\sum_{k=1}^N\Big(a_{ik}(|L_{ki}|+\tau\mathcal{C}_3) + a_{jk}(|L_{kj}|+\tau\mathcal{C}_3)\Big)|L_{ij}|^2 \\
&\quad + \frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})|L_{ij}| + \frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})(|L_{ij}|^2+|L_{ij}|) \\
&\quad + \frac{4|\kappa_1|}{N}\sum_{k=1}^N |L_{ij}|\Big(a_{ik}(|L_{ki}|+\tau\mathcal{C}_3) + a_{jk}(|L_{kj}|+\tau\mathcal{C}_3)\Big).
\end{aligned} \tag{37}$$

Proof. We use (36) to find

$$\begin{aligned}
\frac{d}{dt}\langle z_i, z_j\rangle &= \langle(\Omega_i-\Omega_j)z_i, z_j\rangle + \frac{\kappa_0}{N}\sum_{k\neq i} a_{ik}(G_{kj}^\tau - \overline{G}_{ki}^\tau G_{ij}) + \frac{\kappa_0}{N}\sum_{k\neq j} a_{jk}(\overline{G}_{ki}^\tau - G_{kj}^\tau G_{ij}) \\
&+ \frac{\kappa_1}{N}\sum_{k\neq i} a_{ik}(G_{ki}^\tau - \overline{G}_{ki}^\tau)G_{ij} + \frac{\kappa_1}{N}\sum_{k\neq j} a_{jk}(\overline{G}_{kj}^\tau - G_{kj}^\tau)G_{ij} \\
&= \langle(\Omega_i-\Omega_j)z_i, z_j\rangle + \frac{\kappa_0}{N}\sum_{k=1}^N\Big(a_{ik}(G_{kj}^\tau - \overline{G}_{ki}^\tau G_{ij}) + a_{jk}(\overline{G}_{ki}^\tau - G_{kj}^\tau G_{ij})\Big) \\
&+ \frac{\kappa_1}{N}\sum_{k=1}^N\Big(a_{ik}(G_{ki}^\tau G_{ij} - \overline{G}_{ki}^\tau G_{ij}) + a_{jk}(\overline{G}_{kj}^\tau G_{ij} - G_{kj}^\tau G_{ij})\Big) \\
&- \frac{\kappa_0}{N}\Big(a_{ii}(G_{ij}^\tau - \overline{G}_{ii}^\tau G_{ij}) + a_{jj}(\overline{G}_{ji}^\tau - G_{jj}^\tau G_{ij})\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\kappa_1}{N}\bigg(a_{ii}(G^\tau_{ii}G_{ij} - \overline{G}^\tau_{ii}G_{ij}) + a_{jj}(\overline{G}^\tau_{jj}G_{ij} - G^\tau_{jj}G_{ij}) \bigg) \\
&= \langle (\Omega_i - \Omega_j)z_i, z_j\rangle \\
&+ \frac{\kappa_0}{N}\sum_{k=1}^N \Big[(a_{jk} - a_{ik})L^\tau_{kj} + (a_{ik} - a_{jk})\overline{L}^\tau_{ki} + (a_{ik} + a_{jk} - a_{ik}\overline{L}^\tau_{ki} - a_{jk}L^\tau_{kj})L_{ij} \Big] \\
&+ \frac{2i\kappa_1}{N}\sum_{k=1}^N \big(a_{jk}\mathrm{Im}L^\tau_{kj} - a_{ik}\mathrm{Im}L^\tau_{ki})(1 - L_{ij})\big) - \frac{2\mathrm{i}\kappa_1}{N}(a_{jj}\mathrm{Im}L^\tau_{jj} - a_{ii}\mathrm{Im}L^\tau_{ii})(1 - L_{ij}) \\
&- \frac{\kappa_0}{N}\left(a_{jj}L^\tau_{jj} - a_{ii}L^\tau_{ij} + a_{ii}\overline{L}^\tau_{ii} - a_{jj}\overline{L}^\tau_{ji} + (a_{ii} + a_{jj})L_{ij} - a_{ii}\overline{L}^\tau_{ii}L_{ij} - a_{jj}L^\tau_{jj}L_{ij} \right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{d}{dt}|L_{ij}|^2 = \frac{d}{dt}(L_{ij}\overline{L}_{ij}) = 2\mathrm{Re}(\dot{L_{ij}}\overline{L}_{ij}) = -2\mathrm{Re}\left(\frac{d}{dt}\langle z_i, z_j\rangle(1 - \langle z_j, z_i\rangle)\right) \\
&= -2\mathrm{Re}\bigg(L_{ji}\bigg[\langle (\Omega_i - \Omega_j)z_i, z_j\rangle + \frac{\kappa_0}{N}\sum_{k=1}^N \big\{ (a_{jk} - a_{ik})L^\tau_{kj} \\
&+ (a_{ik} - a_{jk})\overline{L}^\tau_{ki} + (a_{ik} + a_{jk} - a_{ik}\overline{L}^\tau_{ki} - a_{jk}L^\tau_{kj})L_{ij} \big\} \\
&+ \frac{2\mathrm{i}\kappa_1}{N}\sum_{k=1}^N \big(a_{jk}\mathrm{Im}L^\tau_{kj} - a_{ik}\mathrm{Im}L^\tau_{ki})(1 - L_{ij})\big) - \frac{2\mathrm{i}\kappa_1}{N}(a_{jj}\mathrm{Im}L^\tau_{jj} - a_{ii}\mathrm{Im}L^\tau_{ii})(1 - L_{ij}) \\
&- \frac{\kappa_0}{N}\left(a_{jj}L^\tau_{jj} - a_{ii}L^\tau_{ij} + a_{ii}\overline{L}^\tau_{ii} - a_{jj}\overline{L}^\tau_{ji} + (a_{ii} + a_{jj})L_{ij} - a_{ii}\overline{L}^\tau_{ii}L_{ij} - a_{jj}L^\tau_{jj}L_{ij} \right) \bigg] \bigg) \\
&\le 2|L_{ij}|D(\Omega) - 2\mathrm{Re}\frac{\kappa_0}{N}\sum_{k=1}^N \Big\{ (a_{ik} + a_{jk} - a_{ik}\overline{L}^\tau_{ki} - a_{jk}L^\tau_{kj})|L_{ij}|^2 + (a_{ik} - a_{jk})(\overline{L}^\tau_{ki} - L^\tau_{kj})L_{ji} \Big\} \\
&+ \mathrm{Re}\frac{2\kappa_0}{N}((a_{ii} + a_{jj} - a_{ii}\overline{L}^\tau_{ii} - a_{jj}L^\tau_{jj})|L_{ij}|^2 + (a_{jj}L^\tau_{jj} - a_{ii}L^\tau_{ij} + a_{ii}\overline{L}^\tau_{ii} - a_{jj}\overline{L}^\tau_{ji})L_{ji}) \\
&+ \frac{4\kappa_1}{N}\sum_{k=1}^N (\mathrm{Im}(a_{ik}L^\tau_{ki} - a_{jk}L^\tau_{kj})\mathrm{Im}L_{ij} - \frac{4\kappa_1}{N}\mathrm{Im}(a_{ii}L^\tau_{ii} - a_{jj}L^\tau_{jj})\mathrm{Im}L_{ij}.
\end{aligned}
$$

In the sequel, we prove each term in the R.H.S. of the above relation.

• Step A: By direct calculation, one has

$$
\begin{aligned}
\left| \frac{4\kappa_1}{N}\mathrm{Im}(a_{ii}L^\tau_{ii} - a_{jj}L^\tau_{jj})\mathrm{Im}L_{ij} \right| &= \frac{4|\kappa_1|}{N}|L_{ij}||\mathrm{Im}(a_{ii}L^\tau_{ii} - a_{jj}L^\tau_{jj})| \\
&\le \frac{4|\kappa_1|}{N}|L_{ij}|(a_{ii}|L^\tau_{ii}| + a_{jj}|L^\tau_{jj}|) \le \frac{4\mathcal{C}_3|\kappa_1|\tau}{N}(a_{ii} + a_{jj})|L_{ij}|.
\end{aligned}
$$

• Step B: We set

$$
A := \frac{2\kappa_0}{N}\mathrm{Re}\bigg((a_{ii} + a_{jj} - a_{ii}\overline{L}^\tau_{ii} - a_{jj}L^\tau_{jj})|L_{ij}|^2 + (a_{jj}L^\tau_{jj} - a_{ii}L^\tau_{ij} + a_{ii}\overline{L}^\tau_{ii} - a_{jj}\overline{L}^\tau_{ji})L_{ji} \bigg).
$$

Then, one has

$$
\begin{aligned}
A &\le \frac{2\kappa_0}{N}\mathrm{Re}\left(a_{ii}(L_{ij} - L^\tau_{ij})L_{ji} + a_{jj}(L_{ij} - \overline{L}^\tau_{ji})L_{ji} \right) + \frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii} + a_{jj})(|L_{ij}|^2 + |L_{ij}|) \\
&\le \frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii} + a_{jj})|L_{ij}| + \frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii} + a_{jj})(|L_{ij}|^2 + |L_{ij}|).
\end{aligned}
$$

• Step C: We set

$$\begin{aligned}
B &= -2\mathrm{Re}\frac{\kappa_0}{N}\sum_{k=1}^{N}\Big\{(a_{ik}+a_{jk}-a_{ik}\overline{L}_{ki}^{\tau}-a_{jk}L_{kj}^{\tau})|L_{ij}|^2+(a_{ik}-a_{jk})(\overline{L}_{ki}^{\tau}-L_{kj}^{\tau})L_{ji}\Big\} \\
&\le -2\frac{\kappa_0}{N}\sum_{k=1}^{N}(a_{ik}+a_{jk})|L_{ij}|^2+\frac{2\kappa_0}{N}\sum_{k=1}^{N}\Big(a_{ik}(|L_{ki}|+\tau\mathcal{C}_3)+a_{jk}(|L_{kj}|+\tau\mathcal{C}_3)\Big)|L_{ij}|^2 \\
&+\frac{2\kappa_0}{N}\sum_{k=1}^{N}|a_{ik}-a_{jk}|(|L_{ki}|+|L_{kj}|+2\tau\mathcal{C}_3)|L_{ij}|.
\end{aligned}$$

• Step D: Note that

$$\frac{4\kappa_1}{N}\sum_{k=1}^{N}(\mathrm{Im}(a_{ik}L_{ki}^{\tau}-a_{jk}L_{kj}^{\tau})\mathrm{Im}L_{ij}\le\frac{4|\kappa_1|}{N}\sum_{k=1}^{N}|L_{ij}|(a_{ik}(|L_{ki}|+\tau\mathcal{C}_3)+a_{jk}(|L_{kj}|+\tau\mathcal{C}_3)).$$

• Step E: We collect all the estimates in Step A - Step D to get

$$\begin{aligned}
\frac{d}{dt}|L_{ij}|^2 &\le 2|L_{ij}|D(\Omega)+\frac{2\kappa_0}{N}\sum_{k=1}^{N}|a_{ik}-a_{jk}|(|L_{ki}|+|L_{kj}|+2\tau\mathcal{C}_3)|L_{ij}| \\
&+\frac{4\mathcal{C}_3|\kappa_1|\tau}{N}(a_{ii}+a_{jj})|L_{ij}|-\frac{2\kappa_0}{N}\sum_{k=1}^{N}(a_{ik}+a_{jk})|L_{ij}|^2 \\
&+\frac{2\kappa_0}{N}\sum_{k=1}^{N}\Big(a_{ik}(|L_{ki}|+\tau\mathcal{C}_3)+a_{jk}(|L_{kj}|+\tau\mathcal{C}_3)\Big)|L_{ij}|^2 \\
&+\frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})|L_{ij}|+\frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})(|L_{ij}|^2+|L_{ij}|) \\
&+\frac{4|\kappa_1|}{N}\sum_{k=1}^{N}|L_{ij}|(a_{ik}(|L_{ki}|+\tau\mathcal{C}_3)+a_{jk}(|L_{kj}|+\tau\mathcal{C}_3)).
\end{aligned}$$

□

4.2.2. *Proof of Theorem 4.7.* Recall that

$$L(t):=\max_{i,j}L_{ij}.$$

Then, for each time t, there exists i_t and j_t by which the maximum is attained:

$$L(t)=|1-\langle z_{i_t},z_{j_t}\rangle|.$$

Then by Lemma 4.10, one has

$$\begin{aligned}
&\frac{d}{dt}L(t)^2\le 2L(t)D(\Omega)+\frac{4\kappa_0}{N}\sum_{k=1}^{N}|a_{ik}-a_{jk}|(L(t)+\tau\mathcal{C}_3)L(t)+\frac{4\mathcal{C}_3|\kappa_1|\tau}{N}(a_{ii}+a_{jj})L(t) \\
&-\frac{2\kappa_0}{N}\sum_{k=1}^{N}(a_{ik}+a_{jk})L(t)^2+\frac{2\kappa_0}{N}\sum_{k=1}^{N}\Big(a_{ik}(L(t)+\tau\mathcal{C}_3)+a_{jk}(L(t)+\tau\mathcal{C}_3)\Big)L(t)^2 \\
&+\frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})L(t)+\frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})(L(t)^2+L(t)) \\
&+\frac{4|\kappa_1|}{N}\sum_{k=1}^{N}L(t)\Big(a_{ik}(L(t)+\tau\mathcal{C}_3)+a_{jk}(L(t)+\tau\mathcal{C}_3)\Big) \\
&\le\frac{2\kappa_0}{N}\sum_{k=1}^{N}(a_{ik}+a_{jk})L(t)^3
\end{aligned}$$

$$
\begin{aligned}
&+\Big(-\frac{2\kappa_0}{N}\sum_{k=1}^N(a_{ik}+a_{jk})+\frac{4\kappa_0}{N}\sum_{k=1}^N|a_{ik}-a_{jk}|+\frac{4\mathcal{C}_3\kappa_0\tau}{N}\sum_{k=1}^N(a_{ik}+a_{jk})\\
&+\frac{2\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})+\frac{4|\kappa_1|}{N}\sum_{k=1}^N(a_{ik}+a_{jk})\Big)L(t)^2\\
&+\Big(2D(\Omega)+\frac{4\mathcal{C}_3\kappa_0\tau}{N}\sum_{k=1}^N|a_{ik}-a_{jk}|+(a_{ii}+a_{jj})\frac{4\mathcal{C}_3\tau}{N}(\kappa_0+|\kappa_1|)+\frac{4\mathcal{C}_3|\kappa_1|\tau}{N}\sum_{k=1}^N(a_{ik}+a_{jk})\Big)L(t).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\frac{d}{dt}L(t)\le{}&\frac{\kappa_0}{N}\sum_{k=1}^N(a_{ik}+a_{jk})L(t)^2\\
&+\Big(-\frac{\kappa_0}{N}\sum_{k=1}^N(a_{ik}+a_{jk})+\frac{2\kappa_0}{N}\sum_{k=1}^N|a_{ik}-a_{jk}|+\frac{2\mathcal{C}_3\kappa_0\tau}{N}\sum_{k=1}^N(a_{ik}+a_{jk})\\
&+\frac{\mathcal{C}_3\kappa_0\tau}{N}(a_{ii}+a_{jj})+\frac{2|\kappa_1|}{N}\sum_{k=1}^N(a_{ik}+a_{jk})\Big)L(t)\\
&+\Big(D(\Omega)+\frac{2\mathcal{C}_3\kappa_0\tau}{N}\sum_{k=1}^N|a_{ik}-a_{jk}|+\frac{2\mathcal{C}_3\tau}{N}(a_{ii}+a_{jj})(\kappa_0+|\kappa_1|)+\frac{2\mathcal{C}_3|\kappa_1|\tau}{N}\sum_{k=1}^N(a_{ik}+a_{jk})\Big).
\end{aligned}
$$

Now, we set

$$
\begin{aligned}
&\mathcal{A}_1:=\frac{1}{N}\sum_{k=1}^N(a_{ik}+a_{jk}),\\
&\mathcal{A}_2:=\frac{2}{N}\sum_{k=1}^N|a_{ik}-a_{jk}|+\frac{2\mathcal{C}_3\tau}{N}\sum_{k=1}^N(a_{ik}+a_{jk})+\frac{\mathcal{C}_3\tau}{N}(a_{ii}+a_{jj})+\frac{2|\kappa_1|}{N\kappa_0}\sum_{k=1}^N(a_{ik}+a_{jk}),\\
&\mathcal{A}_3:=\frac{D(\Omega)}{\kappa_0}+\frac{2\mathcal{C}_3\tau}{N}\sum_{k=1}^N|a_{ik}-a_{jk}|+\frac{2\mathcal{C}_3\tau}{N}(a_{ii}+a_{jj})\left(1+\frac{|\kappa_1|}{\kappa_0}\right)+\frac{2\mathcal{C}_3|\kappa_1|\tau}{N\kappa_0}\sum_{k=1}^N(a_{ik}+a_{jk}).
\end{aligned}
$$

This yields

$$
\frac{d}{dt}L\le\kappa_0\Big(\mathcal{A}_1L^2-(\mathcal{A}_1-\mathcal{A}_2)L+\mathcal{A}_3\Big).
$$

If we impose following conditions:

$$
\tau\searrow 0\quad\text{and after that}\quad\kappa_0\to\infty, \tag{38}
$$

we obtain

$$
\lim_{\kappa_0\to\infty}\lim_{\tau\searrow 0}\mathcal{A}_2=\frac{2}{N}\sum_{k=1}^N|a_{ik}-a_{jk}|,\qquad\lim_{\kappa_0\to\infty}\lim_{\tau\searrow 0}\mathcal{A}_3=0.
$$

Since the roots of

$$
\mathcal{A}_1x^2-(\mathcal{A}_1-\mathcal{A}_2)x+\mathcal{A}_3=0
$$

can be expressed as

$$
x_\pm=\frac{\mathcal{A}_1-\mathcal{A}_2\pm\sqrt{(\mathcal{A}_1-\mathcal{A}_2)^2-4\mathcal{A}_1\mathcal{A}_3}}{2\mathcal{A}_1}.
$$

If we combine this expression, under the condition (38) we have

$$
\lim_{\kappa_0\to\infty}\lim_{\tau\searrow 0}x_+=1-\frac{2\sum_{k=1}^N|a_{ik}-a_{jk}|}{\sum_{k=1}^N(a_{ik}+a_{jk})},\qquad\lim_{\kappa_0\to\infty}\lim_{\tau\searrow 0}x_-=0.
$$

By similar arguments with previous result, we have desired estimate. □

5. **Conclusion.** In this paper, we have proposed several sufficient frameworks leading to complete aggregation and practical aggregation in terms of initial data, coupling gains and size of time-delay for the LHS model with time-delayed interactions. The LHS model is a complex counterpart of the LS model, and it describes the continuous-time dynamics of the Lohe Hermitian sphere particles on the unit Hermitian sphere. For the SL coupling gain pair with $\kappa_1 = -\frac{\kappa_0}{2}$, the LHS model on $\mathbb{C}^d$ can be rewritten as the LS model on $\mathbb{R}^{2d}$. When the coupling gain pair is close to that of the SL coupling gain pair and the corresponding linear flows are the same, we show that the LHS flow with a time-delay tends to complete aggregation asymptotically for some admissible class of initial data and system parameters. For a general network, we also provided a sufficient framework for practical aggregation to the LHS model with respect to time-delay. Even for the LS model on the unit sphere in Euclidean space, the complete aggregation is not known in previous literature, except some weak aggregation estimates such as practical aggregation. We leave this interesting issue for a future work.

Appendix A. **Derivation of the SL coupling.** We begin with the generalized Stuart-Landau model on $\mathbb{C}^{d+1}$:

$$\frac{dz_j}{dt} = \big((1-\|z_j\|^2)I_{d+1} + \Omega\big)z_j + \frac{\kappa}{N}\sum_{k=1}^{N}(z_k - z_j), \tag{39}$$

where $z_j \in \mathbb{C}^{d+1}$ for all $j \in \mathcal{N}$, Ω is a skew-Hermitian matrix with the size $(d+1)\times(d+1)$ and I_{d+1} is the identity matrix with the size $(d+1)\times(d+1)$. Now, we substitute the ansatz:

$$z_j = r_j w_j, \quad r_j = \|z_j\| \quad \text{and} \quad w_j = \frac{z_j}{\|z_j\|}, \quad \forall\, j \in \mathcal{N}$$

into (39) to see

$$\dot r_j w_j + r_j \dot w_j = (1-r_j^2)r_j w_j + r_j\Omega w_j + \frac{\kappa}{N}\sum_{k=1}^{N}(r_k w_k - r_j w_j). \tag{40}$$

Then, $\langle w_j, (40)\rangle$ implies

$$\dot r_j + r_j\langle w_j, \dot w_j\rangle = (1-r_j^2)r_j + r_j\langle w_j, \Omega w_j\rangle + \frac{\kappa}{N}\sum_{k=1}^{N}(r_k\langle w_j, w_k\rangle - r_j), \tag{41}$$

where we used the fact that $\|w_j\| = 1$.

If we take the real part of (41), one has

$$\dot r_j = (1-r_j^2)r_j + \frac{\kappa}{N}\sum_{k=1}^{N}(r_k \mathrm{Re}(\langle w_j, w_k\rangle) - r_j), \tag{42}$$

where we used the relations:

$$0 = \langle w_j, \dot w_j\rangle + \langle \dot w_j, w_j\rangle = \langle w_j, \dot w_j\rangle + \overline{\langle w_j, \dot w_j\rangle} = 2\mathrm{Re}\ \langle w_j, \dot w_j\rangle, \quad \langle w_j, \Omega w_j\rangle = 0.$$

Now, we combine (40) and (42) to get

$$\dot w_j = \Omega w_j + \frac{\kappa}{N}\sum_{k=1}^{N}\frac{r_k}{r_j}\Big(w_k - \mathrm{Re}(\langle w_k, w_j\rangle)w_j\Big). \tag{43}$$

Similarly, we impose $r_j \equiv 1$ on (43) to obtain

$$\begin{aligned}\dot{w}_j &= \Omega w_j + \frac{\kappa}{N}\sum_{k=1}^{N}(w_k - \mathrm{Re}(\langle w_k, w_j\rangle)w_j)\\ &= \Omega w_j + \frac{\kappa}{N}\sum_{k=1}^{N}\Big[w_k - \frac{1}{2}\Big(\langle w_k, w_j\rangle + \langle w_j, w_k\rangle\Big)w_j\Big]\\ &= \Omega w_j + \frac{\kappa}{N}\sum_{k=1}^{N}(w_k - \langle w_k, w_j\rangle w_j) - \frac{\kappa}{2N}\sum_{k=1}^{N}(\langle w_j, w_k\rangle - \langle w_k, w_j\rangle)w_j.\end{aligned}$$

If we put $\kappa_0 = \kappa$ and $\kappa_1 = -\frac{\kappa}{2}$ into the LHS model, then we get the above system. Therefore, the SL coupling gain pair $(\kappa_0, \kappa_1) = \big(\kappa, -\frac{\kappa}{2}\big)$ can be obtained by the generalized Stuart-Landau model.

REFERENCES

[1] J. A. Acebron, L. L. Bonilla, C. J. P. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, *Rev. Mod. Phys.*, **77** (2005), 137–185.

[2] G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, J. Kim, L. Pareschi, D. Poyato and J. Soler, Vehicular traffic, crowds and swarms: From kinetic theory and multiscale methods to applications and research perspectives, *Math. Models Methods Appl. Sci.*, **29** (2019), 1901–2005.

[3] I. Barbalat, *Systemes dequations differentielles d oscillations non lineaires*, *Rev. Math. Pures Appl.*, **4** (1959), 267–270.

[4] N. Bellomo and S.-Y. Ha, A quest toward a mathematical theory of the dynamics of swarms, *Math. Models Methods Appl. Sci.*, **27** (2017), 745–770.

[5] D. Benedetto, E. Caglioti and U. Montemagno, On the complete phase synchronization for the Kuramoto model in the mean-field limit, *Commun. Math. Sci.*, **13** (2015), 1775–1786.

[6] J. Bronski, T. Carty and S. Simpson, A matrix-valued Kuramoto model, *J. Stat. Phys.*, **178** (2020), 595–624.

[7] J. Buck and E. Buck, Biology of synchronous flashing of fireflies, *Nature*, **211** (1966), 562–564.

[8] J. Byeon, S.-Y. Ha and H. Park, Asymptotic interplay of states and adapted coupling gains in the Lohe Hermitian sphere model, Submitted.

[9] S.-H. Choi and S.-Y. Ha, Time-delayed interactions and synchronization of identical Lohe oscillators, *Quart. Appl. Math.*, **74** (2016), 297–319.

[10] S.-H. Choi and S.-Y. Ha, Large-time dynamics of the asymptotic Lohe model with a small time-delay, *J. Phys. A*, **48** (2015), 425101 34 pp.

[11] S.-H. Choi and S.-Y. Ha, Complete entrainment of Lohe oscillators under attractive and repulsive couplings, *SIAM. J. Appl. Dyn. Syst.*, **13** (2014), 1417–1441.

[12] N. Chopra and M. W. Spong, On exponential synchronization of Kuramoto oscillators, *IEEE Trans. Automatic Control*, **54** (2009), 353–357.

[13] P. Degond, A. Frouvelle, S. Merino-Aceituno and A. Trescases, Quaternions in collective dynamics, *Multiscale Model. Simul.*, **16** (2018), 28–77.

[14] L. DeVille, Synchronization and stability for quantum Kuramoto, *J. Stat. Phys.*, **174** (2019), 160–187.

[15] J.-G. Dong and X. Xue, Synchronization analysis of Kuramoto oscillators, *Commun. Math. Sci.*, **11** (2013), 465–480.

[16] F. Dörfler and F. Bullo, Synchronization in complex networks of phase oscillators: A survey, *Automatica J. IFAC*, **50** (2014), 1539–1564.

[17] F. Dörfler and F. Bullo, On the critical coupling for Kuramoto oscillators, *SIAM. J. Appl. Dyn. Syst.*, **10** (2011), 1070–1099.

[18] S.-Y. Ha, D. Kim, D. Kim, H. Park and W. Shim, Emergent dynamics of the Lohe matrix ensemble on a network under time-delayed interactions, *J. Math. Phys.*, **61** (2020), 012702.

[19] S.-Y. Ha, D. Ko, J. Park and X. Zhang, Collective synchronization of classical and quantum oscillators, *EMS Surv. Math. Sci.*, **3** (2016), 209–267.

[20] S.-Y. Ha, D. Ko and S. W. Ryoo, On the relaxation dynamics of Lohe oscillators on some Riemannian manifolds, *J. Stat. Phys.*, **172** (2018), 1427–1478.
[21] S.-Y. Ha, J.Lee, Z. Li and J. Park, Emergent dynamics of Kuramoto oscillators with adaptive couplings: Conservation law and fast learning, *SIAM J. Appl. Dyn. Syst.*, **17** (2018), 1560–1588.
[22] S.-Y. Ha, S. E. Noh and J. Park, Synchronization of Kuramoto oscillators with adaptive couplings, *SIAM J. Appl. Dyn. Syst.*, **15** (2016), 162–194.
[23] S.-Y. Ha and H. Park, Emergent behaviors of Lohe tensor flocks, *J. Stat. Phys.*, **178** (2020), 1268–1292.
[24] S.-Y. Ha and H. Park, From the Lohe tensor model to the Lohe Hermitian sphere model and emergent dynamics, *SIAM J. Appl. Dyn. Syst.*, **19** (2020), 1312–1342.
[25] J. Hale, *Theory of Functional Differential Equations*, 2nd ed., Springer-Verlag, New York-Heidelberg, 1977.
[26] V. Jaćimović and A. Crnkić, Low-dimensional dynamics in non-Abelian Kuramoto model on the 3-sphere, *Chaos*, **28** (2018), 083105.
[27] D. Kim, State-dependent dynamics of the Lohe matrix ensemble on the unitary group under the gradient flow, *SIAM J. Appl. Dyn. Syst.*, **19** (2020), 1080–1123.
[28] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Inc. Boston, MA, 1993.
[29] Y. Kuramoto, *Self-Entrainment of a Population of Coupled Non-Linear Oscillators*, International Symposium on Mathematical Problems in Mathematical Physics, Lecture Notes in Phys., Vol. 39, Springer, Berlin, 1975, 420–422.
[30] M. A. Lohe, Systems of matrix Riccati equations, linear fractional transformations, partial integrability and synchronization, *J. Math. Phys.*, **60** (2019), 072701.
[31] M. A. Lohe, Quantum synchronization over quantum networks, *J. Phys. A*, **43** (2010), 465301.
[32] M. A. Lohe, Non-abelian Kuramoto models and synchronization, *J. Phys. A*, **42** (2009), 395101.
[33] J. Markdahl, J. Thunberg and J. Goncalves, Almost global consensus on the n-sphere, *IEEE Trans. Automat. Control*, **63** (2018), 1664–1675.
[34] C. S. Peskin, *Mathematical Aspects of Heart Physiology*, Courant Institute of Mathematical Sciences, New York University, New York, 1975.
[35] A. Pikovsky, M. Rosenblum and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2001.
[36] S. H. Strogatz, From Kuramoto to Crawford: Exploring the Onset of Synchronization in Populations of Coupled Oscillators, *Phys. D*, **143** (2000), 1–20.
[37] J. Thunberg, J. Markdahl, F. Bernard and J. Goncalves, A lifting method for analyzing distributed synchronization on the unit sphere, *Automatica J. IFAC*, **96** (2018), 253–258.
[38] C. M. Topaz, A. L. Bertozzi and M. A. Lewis, A nonlocal continuum model for biological aggregation, *Bull. Math. Biol.*, **68** (2006), 1601–1623.
[39] C. M. Topaz and A. L. Bertozzi, Swarming patterns in a two-dimensional kinematic model for biological groups, *SIAM J. Appl. Math.*, **65** (2004), 152–174.
[40] T.Vicsek and A. Zefeiris, Collective motion, *Phys. Rep.*, **517** (2012), 71–140.
[41] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, *J. Theor. Biol.*, **16** (1967), 15–42.
[42] A. T. Winfree, *The Geometry of Biological Time*, Springer-Verlag, Berlin-New York, 1980.
[43] J. Zhu, Synchronization of Kuramoto model in a high-dimensional linear space, *Phys. Lett. A*, **377** (2013), 2939–2943.

Received January 2021; 1st revision March 2021; 2nd revision April 2021.

E-mail address: syha@snu.ac.kr
E-mail address: hgy0407@snu.ac.kr
E-mail address: hansol960612@snu.ac.kr