

VANISHING VISCOSITY FOR A 2×2 SYSTEM MODELING CONGESTED VEHICULAR TRAFFIC

GIUSEPPE MARIA COCLITE

Department of Mechanics, Mathematics and Management
Polytechnic University of Bari
Via E. Orabona 4, 70125 Bari, Italy

NICOLA DE NITTI*

Department of Data Science, Chair in Applied Analysis (Alexander von Humboldt Professorship)
Friedrich-Alexander-Universität Erlangen-Nürnberg
Cauerstr. 11, 91058 Erlangen, Germany

MAURO GARAVELLO

Department of Mathematics and its Applications
University of Milano Bicocca
via R. Cozzi 55, 20125 Milano, Italy

FRANCESCA MARCELLINI

Department of Information Engineering
University of Brescia
via Branze 38, 25123 Brescia, Italy

(Communicated by Paola Goatin)

ABSTRACT. We prove the convergence of the vanishing viscosity approximation for a class of 2×2 systems of conservation laws, which includes a model of traffic flow in congested regimes. The structure of the system allows us to avoid the typical constraints on the total variation and the L^1 norm of the initial data. The key tool is the compensated compactness technique, introduced by Murat and Tartar, used here in the framework developed by Panov. The structure of the Riemann invariants is used to obtain the compactness estimates.

1. Introduction.

1.1. Modeling traffic flow in the congested regime. We consider the Cauchy problem associated with the following 2×2 system of conservation laws in one space dimension:

$$\begin{cases} \partial_t \rho + \partial_x(u\rho f(\rho)) = 0, & t > 0, x \in \mathbb{R}, \\ \partial_t u + \partial_x(u^2 f(\rho)) = 0, & t > 0, x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (1)$$

The functions $\rho : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ and $u : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ represent the vehicular density and the generalized momentum, respectively. The velocity law is given by

2020 *Mathematics Subject Classification.* Primary: 35L65, 35L45, 35B25.

Key words and phrases. Traffic flow, conservation laws, hyperbolic systems, vanishing viscosity, compensated compactness.

* Corresponding author: Nicola De Nitti.

$uf(\rho)$, where the function $f = f(\rho)$ describes the reaction of drivers to the different crowding levels of the road.

System (1) describes the evolution of congested traffic in the second-order macroscopic traffic model introduced in [13] as an extension of the classical first-order Lighthill-Whitham-Richards (LWR) model (see [28, 34]) allowing different drivers to have different maximal speeds. According to the empirical evidence that vehicular traffic behaves differently in the situations of low and high densities, see [24], the model in [13] consists of two different regimes or phases: a free phase, described by a single transport equation, and a congested one, modeled by the 2×2 system (1).

We remark that the well-known second-order Aw-Rascle-Zhang (ARZ) model in its original form [1, Formula (2.10)], i.e.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, & t > 0, x \in \mathbb{R}, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, & t > 0, x \in \mathbb{R}, \end{cases}$$

has some similarities, at least formally, with (1). Indeed the quantity $v + p(\rho)$ in the ARZ model plays an analogous role to that of $\frac{u}{\rho}$ in (1). However, since the pressure term p in the ARZ model depends only on the density ρ , there is no any admissible change of variable, which transforms (1) into the ARZ model and vice-versa.

The original ARZ model does not distinguish between a free and a congested phase, but it was extended in this direction in [20], where Goatin generalized the two-phase model proposed by Colombo in [12], coupling the LWR equation in the free phase with the ARZ model in the congested phase. A peculiar difference between the aforementioned models and the one formulated in [13] is that the two phases are here connected. For other second order macroscopic or two-phase models describing traffic evolution, see [4, 17, 19, 21, 27, 39] and the references therein.

In the present paper, we do not consider phase transitions; we focus on the evolution of traffic in the congested regime given by system (1). Indeed, the more complex and richer dynamics happens in the congested phase. On the other hand, in the free phase the model reduces to a linearly degenerate 2×2 system, where each driver's speed is constantly equal to the maximal one. Our main contribution is a proof that the solutions of the viscous approximations of (1) converge to a weak solution of the hyperbolic system.

1.2. Vanishing viscosity for systems of conservation laws. The vanishing viscosity limit of uniformly parabolic viscous regularizations of scalar conservation laws is a crucial point in Kruřkov's well-posedness theory (see [26] and [5, 14, 23] for a modern exposition). The developments concerning the vanishing viscosity approximation of systems of conservation laws are more recent. DiPerna proved convergence for certain classes of 2×2 genuinely nonlinear systems in [9, 15, 25]. His results were subsequently extended in many directions to more general systems describing gas dynamics or other physical phenomena (e.g. shallow waters, liquid chromatography, etc.) – see, e.g. [10, 22, 31, 40] and references therein. The proofs rely on a compensated compactness argument: the key idea, introduced by Tartar and Murat (see, e.g., [16, Chapter 5] for a survey), is as follows: the invariant region method provides uniform L^∞ bounds on the sequence of viscous approximation, but the weak-star convergence does not allow to pass to limit in the nonlinear terms of the equations; however, the weak limit can be represented in terms of Young measures, which reduce to a Dirac mass (hence giving strong convergence) due to the entropy dissipation mechanism. In [35], Serre proved the global existence of weak

solutions for a 2×2 Temple class systems, that is for systems with either linearly degenerate characteristic fields, or with straight characteristic curves (see also [38]). Coclite, Karlsen, Mishra, Risebro applied an improved compensated compactness result due to Panov (see [33]) to prove convergence for 2×2 triangular systems in [11]. For strictly hyperbolic $n \times n$ systems with small initial total variation, in [3], Bianchini and Bressan managed to develop a theory of vanishing viscosity based a priori BV bounds on solutions. We remark that the general uniqueness results known for systems of conservation laws apply only to BV solutions (see [5, 6, 7, 8, 29, 30]); therefore, the uniqueness of the L^∞ solutions obtained by the compensated compactness method remains a long-standing open problem.

None of the previously known results can be directly applied to our problem: indeed, we do not assume any smallness condition on the initial data and system (1) is neither of Temple class nor genuinely nonlinear nor triangular.

1.3. Outline of the paper. The paper is organized as follows. In Section 2, we introduce the approximate viscous system and we state the main result together with the assumptions on the function f and on the initial data. Section 3 is dedicated to several a priori estimates for the solutions of the viscous system and to the compactness of the family of Riemann invariants, which is a preliminary step in the proof of the main result. Finally, in Section 4, we prove the existence of a solution to (1) by the vanishing viscosity approach. Here the main tool is the version of the compensated compactness proposed by Panov in [33].

2. Main result. Before stating the main result of the paper, Theorem 2.2, we introduce the viscous approximation of (1) and all the required assumptions.

We consider a flux function f that satisfies the following hypothesis:

(F): $f \in C^2([0, 1]; \mathbb{R}^+)$ satisfies $f(1) = 0$ and the function $\rho \mapsto \rho^2 f(\rho)$ is not affine in every nontrivial subinterval of $[0, 1]$.

Assumption **(F)** guarantees that the function $g : [0, 1] \rightarrow \mathbb{R}^+$, defined by

$$g(\rho) = \rho^2 f(\rho), \tag{2}$$

for every $\rho \in [0, 1]$, is genuinely nonlinear.

Example 1. The affine function $f(\rho) = 1 - \rho$ satisfies assumption **(F)**. Indeed $g''(\rho) = 2 - 6\rho$ is equal to 0 if and only if $\rho = \frac{1}{3}$.

Example 2. Choose $\delta \in (0, 1)$ and define $f \in C^2([0, 1]; \mathbb{R}^+)$ such that $f'(\rho) < 0$ for every $\rho \in (\frac{\delta}{2}, 1)$, $f(\rho) = \frac{2}{\delta} - 1$ for every $\rho \in [0, \frac{\delta}{2}]$, and $f(\rho) = \frac{1}{\rho} - 1$ for $\rho \geq \delta$. This is a typical choice in traffic flow modeling. Note that it is possible to choose f such that assumption **(F)** is satisfied.

On the initial data $\rho_0 : \mathbb{R} \rightarrow \mathbb{R}$ and $u_0 : \mathbb{R} \rightarrow \mathbb{R}$, we assume that there exist two constants $0 < \tilde{w} < \hat{w} < \infty$, such that

$$0 \leq \rho_0 \leq 1, \quad \tilde{w}\rho_0 \leq u_0 \leq \hat{w}\rho_0, \tag{3}$$

$$\ln(\rho_0) \in L^1(\mathbb{R}), \quad \text{TV} \left(\frac{u_0}{\rho_0} \right) < +\infty. \tag{4}$$

Remark 1. Assumptions (3) and (4) on the function ρ_0 imply also that the function $\rho_0 - 1$ belongs to $L^1(\mathbb{R})$.

We use the following definition of weak solution of problem (1).

Definition 2.1 (Weak solutions). Given $\rho_0 \in L^\infty(\mathbb{R}; \mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R}; \mathbb{R})$, we say that the couple (ρ, u) is a weak solution of (1) if the following statements hold:

1. $\rho \in L^\infty((0, +\infty) \times \mathbb{R}; \mathbb{R})$;
2. $u \in L^\infty((0, +\infty) \times \mathbb{R}; \mathbb{R})$;
3. for every $\varphi \in C_c^\infty([0, +\infty) \times \mathbb{R}; \mathbb{R})$,

$$\int_0^{+\infty} \int_{\mathbb{R}} [\rho(t, x) \partial_t \varphi(t, x) + u(t, x) \rho(t, x) f(\rho(t, x)) \partial_x \varphi(t, x)] dx dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) dx = 0;$$

4. for every $\varphi \in C_c^\infty([0, +\infty) \times \mathbb{R}; \mathbb{R})$,

$$\int_0^{+\infty} \int_{\mathbb{R}} [u(t, x) \partial_t \varphi(t, x) + u^2(t, x) f(\rho(t, x)) \partial_x \varphi(t, x)] dx dt + \int_{\mathbb{R}} u_0(x) \varphi(0, x) dx = 0.$$

Let us consider the following viscous approximation of (1):

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x (u_\varepsilon \rho_\varepsilon f(\rho_\varepsilon)) = \varepsilon \partial_{xx}^2 \rho_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \partial_t u_\varepsilon + \partial_x (u_\varepsilon^2 f(\rho_\varepsilon)) = \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \rho_\varepsilon(0, x) = \rho_{0,\varepsilon}(x), & x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}, \end{cases} \quad (5)$$

where $\varepsilon > 0$ and the initial data $\rho_{0,\varepsilon}$ and $u_{0,\varepsilon}$ are smooth approximations of ρ_0 and u_0 , respectively. More precisely we assume:

$$\rho_{0,\varepsilon}, u_{0,\varepsilon} \in C^\infty(\mathbb{R}; \mathbb{R}) \text{ for every } \varepsilon > 0; \quad (6)$$

$$\rho_{0,\varepsilon} \rightarrow \rho_0, u_{0,\varepsilon} \rightarrow u_0 \text{ in } L_{loc}^p(\mathbb{R}), 1 \leq p < \infty, \text{ and a.e. as } \varepsilon \rightarrow 0; \quad (7)$$

$$\|\rho_{0,\varepsilon} - 1\|_{L^1(\mathbb{R})} \leq \|\rho_0 - 1\|_{L^1(\mathbb{R})} \text{ for every } \varepsilon > 0; \quad (8)$$

$$\varepsilon \leq \rho_{0,\varepsilon} \leq 1, \check{w} \rho_{0,\varepsilon} \leq u_{0,\varepsilon} \leq \hat{w} \rho_{0,\varepsilon} \text{ for every } \varepsilon > 0; \quad (9)$$

$$\|\ln(\rho_{0,\varepsilon})\|_{L^1(\mathbb{R})} \leq \|\ln(\rho_0)\|_{L^1(\mathbb{R})}, \left\| \left(\frac{u_{0,\varepsilon}}{\rho_{0,\varepsilon}} \right)' \right\|_{L^1(\mathbb{R})} \leq TV \left(\frac{u_0}{\rho_0} \right) \text{ for all } \varepsilon > 0. \quad (10)$$

The well-posedness of classical solutions of (5) is guaranteed for short time by the Cauchy-Kowaleskaya theorem (see [37]) and for large times by the classical parabolic theory (see [18] or [31, Theorem 1.0.2]), provided uniform L^∞ bounds for ρ_ε and u_ε . These a priori estimates are proved in Lemma 3.1 and imply that ρ_ε is defined for every $t > 0$ and that $\rho_\varepsilon(t, x)$ is strictly positive for every $t > 0$ and $x \in \mathbb{R}$.

A key ingredient for the proof is the analysis of the Riemann invariant

$$w_\varepsilon = \frac{u_\varepsilon}{\rho_\varepsilon}, \quad (11)$$

(see [14, Section 7.3] for a definition of Riemann invariants). From (5), we easily deduce that w_ε satisfies the equation

$$\partial_t w_\varepsilon + \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon = \varepsilon \partial_{xx}^2 w_\varepsilon + 2\varepsilon \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon}. \quad (12)$$

Our main result is the following convergence theorem.

Theorem 2.2 (Convergence of the vanishing viscosity approximation). *Let us suppose that the assumptions (F), (8), (9), and (10) hold. Then, there exists a sequence*

$\{\varepsilon_k\}_{k \in \mathbb{N}} \subset (0, \infty)$, $\varepsilon_k \rightarrow 0$, and a weak solution (ρ, u) of problem (1), in the sense of Definition 2.1, such that

$$\begin{aligned} \rho_{\varepsilon_k} \rightarrow \rho, u_{\varepsilon_k} \rightarrow u & \text{ in } L^p_{loc}((0, \infty) \times \mathbb{R}), 1 \leq p < \infty, \\ & \text{and a.e. in } (0, \infty) \times \mathbb{R} \text{ as } k \rightarrow \infty, \end{aligned} \tag{13}$$

where $(\rho_{\varepsilon_k}, u_{\varepsilon_k})$ is a classical solution of the viscous problem (5).

3. A priori estimates and compactness results. In this section, we obtain several a priori estimates on the functions $\rho_\varepsilon, u_\varepsilon$, solutions of (5), and on the function w_ε , defined in (11). For the sake of simplicity, throughout this section, we use c to denote various constants, which are independent of the parameter ε and of the time t .

Lemma 3.1 (L^∞ estimates on $\rho_\varepsilon, u_\varepsilon, w_\varepsilon$). *Let us assume that (F) and (9) hold. For every $t > 0$ and $x \in \mathbb{R}$, we have that*

$$\begin{aligned} 0 < c_\varepsilon(t) \leq \rho_\varepsilon(t, x) \leq 1, \\ \check{w}\rho_\varepsilon(t, x) \leq u_\varepsilon(t, x) \leq \hat{w}\rho_\varepsilon(t, x), \\ \check{w} \leq w_\varepsilon(t, x) \leq \hat{w}, \end{aligned} \tag{14}$$

where $c_\varepsilon(\cdot)$ is positive and continuous function defined in $[0, \infty)$.

Proof. Due to (F) and (9), the functions $r = \rho_\varepsilon$ and $r = 1$ are a solution and a supersolution (respectively) of the Cauchy problem

$$\begin{cases} \partial_t r + \partial_x(u_\varepsilon r f(r)) = \varepsilon \partial_{xx}^2 r, & t > 0, x \in \mathbb{R}, \\ r(0, x) = \rho_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases} \tag{15}$$

Therefore, the third inequality of the first line follows from the comparison principle for parabolic equations (see [18]). Moreover, the functions $r = \rho_\varepsilon$ and $r = 0$ are a solution and a subsolution of the Cauchy problem (15), respectively; hence, the comparison principle implies that $0 \leq \rho_\varepsilon(t, x)$.

Due to (9), the functions $r = u_\varepsilon - \check{w}\rho_\varepsilon$ and $r = 0$ are respectively a solution and a subsolution of the Cauchy problem

$$\begin{cases} \partial_t r + \partial_x(r u_\varepsilon f(\rho_\varepsilon)) = \varepsilon \partial_{xx}^2 r, & t > 0, x \in \mathbb{R}, \\ r(0, x) = u_{0,\varepsilon}(x) - \check{w}\rho_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases}$$

Using the comparison principle for parabolic equations (see [18]), we gain $\check{w}\rho_\varepsilon \leq u_\varepsilon$. An analogous argument proves that $u_\varepsilon \leq \hat{w}\rho_\varepsilon$.

For the proof that $\rho_\varepsilon(t, x) \geq c_\varepsilon(t)$ for every $t > 0$ and $x \in \mathbb{R}$, we use the same argument as in [31, Theorem 1.0.2]. The first equation in (5), with the change of variable $q_\varepsilon = \log(\rho_\varepsilon)$, can be written as

$$\partial_t q_\varepsilon = \varepsilon \partial_{xx}^2 q_\varepsilon + \varepsilon \left(\partial_x q_\varepsilon - \frac{u_\varepsilon f(e^{q_\varepsilon})}{2\varepsilon} \right)^2 - \frac{u_\varepsilon^2 (f(e^{q_\varepsilon}))^2}{4\varepsilon} - \partial_x(u_\varepsilon f(e^{q_\varepsilon})),$$

so that, using the heat kernel $K_\varepsilon(t, x) = \frac{1}{2\sqrt{\pi\varepsilon t}} e^{-\frac{x^2}{4\varepsilon t}}$ and (9),

$$\begin{aligned} q_\varepsilon(t, x) &= \int_{\mathbb{R}} K_\varepsilon(t, x - y) \log(\rho_{0,\varepsilon}(y)) \, dy \\ &+ \varepsilon \int_0^t \int_{\mathbb{R}} K_\varepsilon(t - \tau, x - y) \varepsilon \left(\partial_y q_\varepsilon(\tau, y) - \frac{u_\varepsilon(\tau, y) f(e^{q_\varepsilon(\tau, y)})}{2\varepsilon} \right)^2 \, dy \, d\tau \end{aligned}$$

$$\begin{aligned}
& - \int_0^t \int_{\mathbb{R}} K_\varepsilon(t-\tau, x-y) \frac{u_\varepsilon^2(\tau, y) f^2(e^{q_\varepsilon(\tau, y)})}{4\varepsilon} dy d\tau \\
& - \int_0^t \int_{\mathbb{R}} K_\varepsilon(t-\tau, x-y) \partial_y \left(u_\varepsilon(\tau, y) f(e^{q_\varepsilon(\tau, y)}) \right) dy d\tau \\
& \geq \log(\varepsilon) - \int_0^t \int_{\mathbb{R}} K_\varepsilon(t-\tau, x-y) \frac{u_\varepsilon^2(\tau, y) f^2(e^{q_\varepsilon(\tau, y)})}{4\varepsilon} dy d\tau \\
& + \int_0^t \int_{\mathbb{R}} \partial_y K_\varepsilon(t-\tau, x-y) u_\varepsilon(\tau, y) f(e^{q_\varepsilon(\tau, y)}) dy d\tau.
\end{aligned}$$

From **(F)** and the estimate $0 \leq u_\varepsilon \leq \hat{w}\rho_\varepsilon \leq \hat{w}$, we deduce that

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}} K_\varepsilon(t-\tau, x-y) \frac{u_\varepsilon^2(\tau, y) f^2(e^{q_\varepsilon(\tau, y)})}{4\varepsilon} dy d\tau \\
& \leq \frac{\hat{w}^2 \|f\|_{L^\infty(\mathbb{R})}^2}{4\varepsilon} \int_0^t \int_{\mathbb{R}} K_\varepsilon(t-\tau, x-y) dy d\tau = \frac{\hat{w}^2 \|f\|_{L^\infty(\mathbb{R})}^2}{4\varepsilon} t.
\end{aligned}$$

Moreover, using again **(F)** and the estimate $0 \leq u_\varepsilon \leq \hat{w}\rho_\varepsilon \leq \hat{w}$, we get

$$\begin{aligned}
& \left| \int_0^t \int_{\mathbb{R}} \partial_y K_\varepsilon(t-\tau, x-y) u_\varepsilon(\tau, y) f(e^{q_\varepsilon(\tau, y)}) dy d\tau \right| \\
& \leq \hat{w} \|f\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} |\partial_y K_\varepsilon(t-\tau, x-y)| dy d\tau \leq \frac{2}{\sqrt{\varepsilon\pi}} t \hat{w} \|f\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Therefore, for $x \in \mathbb{R}$ and $t > 0$, we conclude that

$$q_\varepsilon(t, x) \geq \log(\varepsilon) - \frac{\hat{w}^2 \|f\|_{L^\infty(\mathbb{R})}^2}{4\varepsilon} t - \frac{2}{\sqrt{\varepsilon\pi}} \sqrt{t} \hat{w} \|f\|_{L^\infty(\mathbb{R})}.$$

So, for every $x \in \mathbb{R}$ and $t > 0$,

$$\rho_\varepsilon(t, x) \geq c_\varepsilon(t) > 0$$

with

$$c_\varepsilon(t) = \varepsilon \exp \left(- \frac{\hat{w}^2 \|f\|_{L^\infty}^2}{4\varepsilon} t - \frac{2}{\sqrt{\varepsilon\pi}} \sqrt{t} \hat{w} \|f\|_{L^\infty(\mathbb{R})} \right).$$

This proves the first inequality in the first line of **(14)**.

Finally, the third line of **(14)** follows from the second one, the definition of w_ε given in **(11)**, and the fact that $\rho_\varepsilon > 0$. \square

Lemma 3.2 (L^1 estimates on $\rho_\varepsilon - 1$). *Let us assume that **(F)**, **(8)**, and **(9)** hold. For every $t \geq 0$, we have that*

$$\|\rho_\varepsilon(t, \cdot) - 1\|_{L^1(\mathbb{R})} \leq \|\rho_0 - 1\|_{L^1(\mathbb{R})}. \quad (16)$$

Proof. Lemma **3.1** implies that $1 - \rho_\varepsilon$ is positive. Therefore, using **(5)** and observing

$$\lim_{x \rightarrow \pm\infty} \rho_\varepsilon(t, x) f(\rho_\varepsilon(t, x)) = f(1) = 0, \quad \lim_{x \rightarrow \pm\infty} \partial_x \rho_\varepsilon(t, x) = 0,$$

due to **(14)**, we deduce that

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} |\rho_\varepsilon - 1| dx &= \frac{d}{dt} \int_{\mathbb{R}} (1 - \rho_\varepsilon) dx = - \int_{\mathbb{R}} \partial_t \rho_\varepsilon dx \\
&= - \int_{\mathbb{R}} \partial_x (\varepsilon \partial_x \rho_\varepsilon - u_\varepsilon \rho_\varepsilon f(\rho_\varepsilon)) dx = 0.
\end{aligned}$$

An integration over $(0, t)$ and assumption **(8)** give the claim. \square

Lemma 3.3 (BV estimate on w_ε). *Let us assume that (10) holds. We have that*

$$\|\partial_x w_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq TV\left(\frac{u_0}{\rho_0}\right), \tag{17}$$

for every $t \geq 0$.

Proof. Differentiating (12) with respect to x , we get

$$\partial_{tx}^2 w_\varepsilon + \partial_x(\rho_\varepsilon f(\rho_\varepsilon)w_\varepsilon \partial_x w_\varepsilon) = \varepsilon \partial_{xxx}^3 w_\varepsilon + 2\varepsilon \partial_x \left(\frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} \right).$$

In light of [2, Lemma 2],

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |\partial_x w_\varepsilon| dx &= \int_{\mathbb{R}} \partial_{tx}^2 w_\varepsilon \operatorname{sign}(\partial_x w_\varepsilon) dx \\ &= \varepsilon \int_{\mathbb{R}} \partial_{xxx}^3 w_\varepsilon \operatorname{sign}(\partial_x w_\varepsilon) dx + 2\varepsilon \int_{\mathbb{R}} \partial_x \left(\frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} \right) \operatorname{sign}(\partial_x w_\varepsilon) dx \\ &\quad - \int_{\mathbb{R}} \partial_x(\rho_\varepsilon f(\rho_\varepsilon)w_\varepsilon \partial_x w_\varepsilon) \operatorname{sign}(\partial_x w_\varepsilon) dx \\ &= -\varepsilon \underbrace{\int_{\mathbb{R}} (\partial_{xx}^2 w_\varepsilon)^2 \delta_{\{\partial_x w_\varepsilon=0\}} dx}_{\leq 0} \\ &\quad - 2\varepsilon \underbrace{\int_{\mathbb{R}} \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} \partial_{xx}^2 w_\varepsilon \delta_{\{\partial_x w_\varepsilon=0\}} dx}_{=0} \\ &\quad + \underbrace{\int_{\mathbb{R}} \rho_\varepsilon f(\rho_\varepsilon)w_\varepsilon \partial_x w_\varepsilon \partial_{xx}^2 w_\varepsilon \delta_{\{\partial_x w_\varepsilon=0\}} dx}_{=0} \leq 0, \end{aligned}$$

where $\delta_{\{\partial_x w_\varepsilon=0\}}$ is the Dirac delta measure concentrated on the set $\{\partial_x w_\varepsilon = 0\}$. An integration over $(0, t)$ and assumption (10) give the claim. \square

Lemma 3.4 (L^1 estimate on $\ln(\rho_\varepsilon)$). *Assume (F), (8), (9), and (10) hold. We have that*

$$\begin{aligned} \|\ln(\rho_\varepsilon(t, \cdot))\|_{L^1(\mathbb{R})} + \varepsilon \int_0^t \left\| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ \leq \|\ln(\rho_0)\|_{L^1(\mathbb{R})} + t TV\left(\frac{u_0}{\rho_0}\right) \int_0^1 |f(\xi)| d\xi, \end{aligned} \tag{18}$$

for every $t \geq 0$.

Proof. Using the definition of w_ε (see (11)) in (5), we get

$$\partial_t \rho_\varepsilon + \partial_x(w_\varepsilon \rho_\varepsilon^2 f(\rho_\varepsilon)) = \varepsilon \partial_{xx}^2 \rho_\varepsilon. \tag{19}$$

Let us consider the function $F : (0, +\infty) \rightarrow \mathbb{R}$ defined, for every $\xi > 0$, by

$$F(\xi) = \int_1^\xi f(s) ds.$$

Thanks to (14) and (17), we have that

$$\frac{d}{dt} \int_{\mathbb{R}} |\ln(\rho_\varepsilon)| dx = - \frac{d}{dt} \int_{\mathbb{R}} \ln(\rho_\varepsilon) dx = - \int_{\mathbb{R}} \frac{\partial_t \rho_\varepsilon}{\rho_\varepsilon} dx$$

$$\begin{aligned}
&= -\varepsilon \int_{\mathbb{R}} \frac{\partial_{xx}^2 \rho_\varepsilon}{\rho_\varepsilon} dx + \int_{\mathbb{R}} \frac{\partial_x (w_\varepsilon \rho_\varepsilon^2 f(\rho_\varepsilon))}{\rho_\varepsilon} dx \\
&= -\varepsilon \int_{\mathbb{R}} \frac{(\partial_x \rho_\varepsilon)^2}{\rho_\varepsilon^2} dx + \int_{\mathbb{R}} w_\varepsilon \underbrace{f(\rho_\varepsilon) \partial_x \rho_\varepsilon}_{\partial_x F(\rho_\varepsilon)} dx \\
&= -\varepsilon \int_{\mathbb{R}} \frac{(\partial_x \rho_\varepsilon)^2}{\rho_\varepsilon^2} dx - \int_{\mathbb{R}} \partial_x w_\varepsilon F(\rho_\varepsilon) dx \\
&\leq -\varepsilon \int_{\mathbb{R}} \frac{(\partial_x \rho_\varepsilon)^2}{\rho_\varepsilon^2} dx + \|F\|_{L^\infty(0,1)} \int_{\mathbb{R}} |\partial_x w_\varepsilon| dx.
\end{aligned}$$

An integration over $(0, t)$ and (17) give the claim. \square

Lemma 3.5 (L^2_{loc} estimate on w_ε). *Let us assume that the assumptions (F), (8), (9), and (10) hold. Let $\chi \in C_c^\infty(\mathbb{R})$ be a non negative cut-off function with compact support. Then there exists a positive constant c , possibly depending on the function χ , such that*

$$\|w_\varepsilon(t, \cdot) \sqrt{\chi}\|_{L^2(\mathbb{R})}^2 + \varepsilon \int_0^t \|\partial_x w_\varepsilon(s, \cdot) \sqrt{\chi}\|_{L^2(\mathbb{R})}^2 ds \leq c(t+1), \quad (20)$$

for every $t \geq 0$.

Proof. Thanks to (12), (14), and (17), we have that

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \frac{w_\varepsilon^2}{2} \chi(x) dx &= \int_{\mathbb{R}} \partial_t w_\varepsilon w_\varepsilon \chi(x) dx \\
&= \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 w_\varepsilon w_\varepsilon \chi(x) dx + 2\varepsilon \int_{\mathbb{R}} \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} w_\varepsilon \chi(x) dx \\
&\quad - \int_{\mathbb{R}} \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon^2 \partial_x w_\varepsilon \chi(x) dx \\
&= -\varepsilon \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 \chi(x) dx - \varepsilon \int_{\mathbb{R}} \partial_x w_\varepsilon w_\varepsilon \chi'(x) dx \\
&\quad + 2\varepsilon \int_{\mathbb{R}} \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} w_\varepsilon \chi(x) dx - \int_{\mathbb{R}} \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon^2 \partial_x w_\varepsilon \chi(x) dx \\
&\leq -\frac{\varepsilon}{2} \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 \chi(x) dx \\
&\quad + 4\varepsilon \int_{\mathbb{R}} \left(\frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon} \right)^2 w_\varepsilon^2 \chi(x) dx + c \int_{\mathbb{R}} |\partial_x w_\varepsilon| dx \\
&\leq -\frac{\varepsilon}{2} \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 \chi(x) dx + c\varepsilon \int_{\mathbb{R}} \left(\frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon} \right)^2 dx + c.
\end{aligned}$$

Integrating over $(0, t)$ and using (10) and (18), we deduce that

$$\begin{aligned}
&\|w_\varepsilon(t, \cdot) \sqrt{\chi}\|_{L^2(\mathbb{R})}^2 + \varepsilon \int_0^t \|\partial_x w_\varepsilon(s, \cdot) \sqrt{\chi}\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \left\| \frac{u_{0,\varepsilon}}{\rho_{0,\varepsilon}} \sqrt{\chi} \right\|_{L^2(\mathbb{R})}^2 + c\varepsilon \int_0^t \left\| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + ct \\
&\leq c(t+1),
\end{aligned}$$

where we used assumption (9) and Lemma 3.4 in the last line. This concludes the proof. \square

3.1. Compactness of w_ε . This subsection deals with the compactness of $\{w_\varepsilon\}_{\varepsilon>0}$, which is a preliminary step for the proof of Theorem 2.2. We use the following result, due to Murat (see [32] or [14, Lemma 17.2.2]).

Theorem 3.6 (Murat’s compact embedding). *Let Ω be a bounded and open subset of \mathbb{R}^N with $N \geq 2$. Assume $\{\mathcal{L}_n\}_{n \in \mathbb{N}}$ is a bounded sequence of distributions in $W^{-1,\infty}(\Omega)$. Suppose also that, for every $n \in \mathbb{N}$, there exists a decomposition*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n \in \mathbb{N}}$ lies in a compact subset of $H_{loc}^{-1}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n \in \mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n \in \mathbb{N}}$ belongs to a compact subset of $H_{loc}^{-1}(\Omega)$.

The following result about the compactness of w_ε holds.

Lemma 3.7 (Compactness of $\{w_\varepsilon\}_{\varepsilon>0}$). *Let us assume that the assumptions (F), (8), (9), and (10) hold. Then, there exist a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}} \subset (0, \infty)$, $\varepsilon_k \rightarrow 0$, and a function*

$$w \in L^\infty((0, \infty) \times \mathbb{R}) \cap L^\infty(0, \infty; BV(\mathbb{R})),$$

such that

$$\begin{aligned} w_{\varepsilon_k} &\rightarrow w \quad \text{in } L_{loc}^p((0, \infty) \times \mathbb{R}), \quad 1 \leq p < \infty, \\ &\text{and a.e. in } (0, \infty) \times \mathbb{R}, \end{aligned} \tag{21}$$

as $k \rightarrow +\infty$.

Proof. Note that the equation (12) for w_ε can be rewritten in the form

$$\partial_t w_\varepsilon = \partial_x(\sqrt{\varepsilon}(\sqrt{\varepsilon}\partial_x w_\varepsilon)) + 2\varepsilon \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} - \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon. \tag{22}$$

Thanks to Lemma 3.1,

$$\{\partial_t w_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } W^{-1,\infty}((0, \infty) \times \mathbb{R}). \tag{23}$$

Observing that $\{\sqrt{\varepsilon}\partial_x w_\varepsilon\}_{\varepsilon>0}$ is bounded in $L_{loc}^2((0, \infty) \times \mathbb{R})$ (see Lemma 3.5), we gain

$$\{\partial_x(\sqrt{\varepsilon}(\sqrt{\varepsilon}\partial_x w_\varepsilon))\}_{\varepsilon>0} \quad \text{compact in } H_{loc}^{-1}((0, \infty) \times \mathbb{R}). \tag{24}$$

Using Lemmas 3.4 and 3.5,

$$\left\{ \varepsilon \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} \right\}_{\varepsilon>0} \quad \text{bounded in } L_{loc}^1((0, \infty) \times \mathbb{R}). \tag{25}$$

Finally, Lemmas 14 and 3.3 guarantee that

$$\{-\rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L_{loc}^1((0, \infty) \times \mathbb{R}). \tag{26}$$

Therefore, in light of Theorem 3.6, we deduce that

$$\{\partial_t w_\varepsilon\}_{\varepsilon>0} \quad \text{is compact in } H_{loc}^{-1}((0, \infty) \times \mathbb{R}). \tag{27}$$

On the other hand, from Lemma 3.3, we deduce that $\{\partial_x w_\varepsilon\}_{\varepsilon>0}$ is bounded in $L_{loc}^1((0, \infty) \times \mathbb{R})$ and, by Lemma 3.1, it is bounded in $W^{-1,\infty}((0, \infty) \times \mathbb{R})$. Therefore, Theorem 3.6 yields that

$$\{\partial_x w_\varepsilon\}_{\varepsilon>0} \quad \text{is compact in } H_{loc}^{-1}((0, \infty) \times \mathbb{R}). \tag{28}$$

This concludes the proof. □

4. Proof of the main theorem. In this section, we prove Theorem 2.2. To this end, first we state – in our setting – a compensated compactness result due to Panov (see [33, Theorem 2.5 and Remark 1]), which improves the classical compensated compactness theorem by Tartar (see [36] or [14, Lemma 17.4.1]).

Theorem 4.1 (Panov’s compensated compactness). *Let $\{v_\nu\}_{\nu>0}$ be a family of functions defined on $(0, \infty) \times \mathbb{R}$ and w the limit function introduced in Lemma 3.7. If $\{v_\nu\}_{\nu \in \mathbb{N}}$ lies in a bounded set of $L^\infty_{loc}((0, \infty) \times \mathbb{R})$ and if, for every constant $c \in \mathbb{R}$, the family*

$$\{\partial_t |v_\nu - c| + \partial_x(\text{sign}(v_\nu - c)(g(v_\nu) - g(c))w)\}_{\nu>0},$$

where g is a genuinely nonlinear function, lies in a compact set of $H^{-1}_{loc}((0, \infty) \times \mathbb{R})$, then there exist a sequence $\{\nu_k\}_{k \in \mathbb{N}} \subset (0, \infty)$, $\nu_k \rightarrow 0$, and a map $v \in L^\infty((0, \infty) \times \mathbb{R})$ such that

$$v_{\nu_k} \rightarrow v \quad \text{in } L^p_{loc}((0, \infty) \times \mathbb{R}), \quad 1 \leq p < \infty,$$

and a.e. in $(0, \infty) \times \mathbb{R}$,

as $k \rightarrow \infty$.

Proof of Theorem 2.2. We begin by proving the compactness of $\{\rho_\varepsilon\}_{\varepsilon>0}$. Let $c \in \mathbb{R}$ be fixed. We claim that the family

$$\{\partial_t |\rho_{\varepsilon_k} - c| + \partial_x [\text{sign}(\rho_{\varepsilon_k} - c)(g(\rho_{\varepsilon_k}) - g(c))w]\}_{k \in \mathbb{N}}$$

is compact in $H^{-1}_{loc}((0, +\infty) \times \mathbb{R})$, where g is the function defined in (2), which is genuinely nonlinear due to assumption (F). For simplicity, we introduce the following notations:

$$\begin{aligned} \eta_0(\xi) &= |\xi - c| - |c|, \\ q_0(\xi) &= \text{sign}(\xi - c)(g(\xi) - g(c)) + \text{sign}(-c)g(c). \end{aligned}$$

Let us remark that

$$\begin{aligned} \eta_0(0) &= q_0(0) = 0, \\ \partial_t |\rho_{\varepsilon_k} - c| + \partial_x [\text{sign}(\rho_{\varepsilon_k} - c)(g(\rho_{\varepsilon_k}) - g(c))w] & \\ &= \partial_t \eta_0(\rho_{\varepsilon_k}) + \partial_x(q_0(\rho_{\varepsilon_k})w) - \text{sign}(-c)g(c)\partial_x w. \end{aligned} \tag{29}$$

Let $\{(\eta_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ be a family of maps such that

$$\begin{aligned} \eta_\varepsilon &\in C^2(\mathbb{R}), \quad q_\varepsilon \in C^2(\mathbb{R}), \\ q'_\varepsilon &= g'\eta'_\varepsilon, \quad \eta''_\varepsilon \geq 0, \\ \|\eta_\varepsilon - \eta_0\|_{L^\infty(0,1)} &\leq \varepsilon, \quad \|\eta'_\varepsilon - \eta'_0\|_{L^1(0,1)} \leq \varepsilon, \\ \|\eta'_\varepsilon\|_{L^\infty(0,1)} &\leq 1, \quad \eta_\varepsilon(0) = q_\varepsilon(0) = 0, \end{aligned} \tag{30}$$

for every $\varepsilon > 0$.

Using (2), (5), (11), and (30), we deduce that

$$\begin{aligned} &\partial_t \eta_0(\rho_{\varepsilon_k}) + \partial_x(q_0(\rho_{\varepsilon_k})w) \\ &= \partial_t \eta_{\varepsilon_k}(\rho_{\varepsilon_k}) + \partial_x(q_{\varepsilon_k}(\rho_{\varepsilon_k})w_{\varepsilon_k}) + \underbrace{\partial_t (\eta_0(\rho_{\varepsilon_k}) - \eta_{\varepsilon_k}(\rho_{\varepsilon_k}))}_{I_{4,k}} \\ &\quad + \underbrace{\partial_x((q_0(\rho_{\varepsilon_k}) - q_{\varepsilon_k}(\rho_{\varepsilon_k}))w)}_{I_{5,k}} + \underbrace{\partial_x(q_{\varepsilon_k}(\rho_{\varepsilon_k})(w - w_{\varepsilon_k}))}_{I_{6,k}} \\ &= \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\partial_t \rho_{\varepsilon_k} + q'_{\varepsilon_k}(\rho_{\varepsilon_k})w_{\varepsilon_k}\partial_x \rho_{\varepsilon_k} + q_{\varepsilon_k}(\rho_{\varepsilon_k})\partial_x w_{\varepsilon_k} + I_{4,k} + I_{5,k} + I_{6,k} \end{aligned}$$

$$\begin{aligned}
&= \varepsilon_k \eta'_{\varepsilon_k}(\rho_{\varepsilon_k}) \partial_{xx}^2 \rho_{\varepsilon_k} - \eta'_{\varepsilon_k}(\rho_{\varepsilon_k}) \partial_x (w_{\varepsilon_k} g(\rho_{\varepsilon_k})) + g'(\rho_{\varepsilon_k}) \eta'_{\varepsilon_k}(\rho_{\varepsilon_k}) w_{\varepsilon_k} \partial_x \rho_{\varepsilon_k} \\
&\quad + q_{\varepsilon_k}(\rho_{\varepsilon_k}) \partial_x w_{\varepsilon_k} + I_{4,k} + I_{5,k} + I_{6,k} \\
&= \underbrace{\varepsilon_k \partial_{xx}^2 \eta_{\varepsilon_k}(\rho_{\varepsilon_k})}_{I_{2,k}} - \underbrace{\varepsilon_k \eta''_{\varepsilon_k}(\rho_{\varepsilon_k}) (\partial_x \rho_{\varepsilon_k})^2}_{I_{3,k}} - \eta'_{\varepsilon_k}(\rho_{\varepsilon_k}) g(\rho_{\varepsilon_k}) \partial_x w_{\varepsilon_k} \\
&\quad - \eta'_{\varepsilon_k}(\rho_{\varepsilon_k}) g'(\rho_{\varepsilon_k}) w_{\varepsilon_k} \partial_x \rho_{\varepsilon_k} + \eta'_{\varepsilon_k}(\rho_{\varepsilon_k}) g'(\rho_{\varepsilon_k}) w_{\varepsilon_k} \partial_x \rho_{\varepsilon_k} \\
&\quad + q_{\varepsilon_k}(\rho_{\varepsilon_k}) \partial_x w_{\varepsilon_k} + I_{4,k} + I_{5,k} + I_{6,k} \\
&= - \underbrace{(\eta'_{\varepsilon_k}(\rho_{\varepsilon_k}) g(\rho_{\varepsilon_k}) - q_{\varepsilon_k}(\rho_{\varepsilon_k})) \partial_x w_{\varepsilon_k}}_{I_{1,k}} + I_{2,k} + I_{3,k} + I_{4,k} + I_{5,k} + I_{6,k}.
\end{aligned}$$

By Lemma 3.1, Lemma 3.3, and (30), there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\|I_{1,k}\|_{L^1((0,T) \times \mathbb{R})} \leq c_1 \int_0^T \|\partial_x w_{\varepsilon_k}(s)\|_{L^1(\mathbb{R})} ds \leq c_2 T,$$

proving that $I_{1,k}$ is bounded in $L^1((0,T) \times \mathbb{R})$ for every $T > 0$.

By Lemma 3.1, Lemma 3.4, and (30), we deduce that there exist $c_1 > 0$ and $c_2 > 0$ such that, for every $T > 0$,

$$\begin{aligned}
\varepsilon_k^2 \int_0^T \int_{\mathbb{R}} |\partial_x \eta_{\varepsilon_k}(\rho_{\varepsilon_k})|^2 dx dt &= \varepsilon_k^2 \int_0^T \int_{\mathbb{R}} |\rho_{\varepsilon_k}^2 \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})|^2 \left| \frac{\partial_x \rho_{\varepsilon_k}}{\rho_{\varepsilon_k}} \right|^2 dx dt \\
&\leq c_1 \varepsilon_k^2 \int_0^T \left\| \frac{\partial_x \rho_{\varepsilon_k}(t, \cdot)}{\rho_{\varepsilon_k}} \right\|_{L^2(\mathbb{R})}^2 dt \\
&\leq \varepsilon_k c_1 c_2 (1 + T),
\end{aligned}$$

proving that $I_{2,k} \rightarrow 0$ as $k \rightarrow +\infty$ in $H^{-1}((0,T) \times \mathbb{R})$.

By Lemma 3.1 and Lemma 3.4, there exists $c > 0$ such that, for every $T > 0$,

$$\begin{aligned}
\varepsilon_k \int_0^T \int_{\mathbb{R}} |\eta''_{\varepsilon_k}(\rho_{\varepsilon_k})| |\partial_x \rho_{\varepsilon_k}|^2 dx dt &= \varepsilon_k \int_0^T \int_{\mathbb{R}} |\rho_{\varepsilon_k}^2 \eta''_{\varepsilon_k}(\rho_{\varepsilon_k})| \left| \frac{\partial_x \rho_{\varepsilon_k}}{\rho_{\varepsilon_k}} \right|^2 dx dt \\
&\leq c(1 + T),
\end{aligned}$$

proving that $I_{3,k}$ is bounded in $L^1_{loc}((0, \infty) \times \mathbb{R})$.

By Lemma 3.1 and (30), there exists $c > 0$ such that

$$\begin{aligned}
\|\eta_0(\rho_{\varepsilon_k}) - \eta_{\varepsilon_k}(\rho_{\varepsilon_k})\|_{L^\infty((0, \infty) \times \mathbb{R})} &\leq \|\eta_0 - \eta_{\varepsilon_k}\|_{L^\infty(0,1)} \leq \varepsilon_k, \\
\|(q_0(\rho_{\varepsilon_k}) - q_{\varepsilon_k}(\rho_{\varepsilon_k}))w\|_{L^\infty((0, \infty) \times \mathbb{R})} &\leq \|q_0 - q_{\varepsilon_k}\|_{L^\infty(0,1)} \hat{w} \\
&\leq \hat{w} \|g'\|_{L^\infty(0,1)} \|\eta'_{\varepsilon_k} - \eta'_0\|_{L^1(0,1)} \leq c \varepsilon_k,
\end{aligned}$$

proving that both $I_{4,k} \rightarrow 0$ and $I_{5,k} \rightarrow 0$ as $k \rightarrow +\infty$ in $H^{-1}_{loc}((0, \infty) \times \mathbb{R})$.

Finally, (30) implies that, for every $\xi \in (0, 1)$,

$$|q_{\varepsilon_k}(\xi)| \leq \int_0^1 |g'(s)| |\eta'_{\varepsilon_k}(s)| ds \leq \int_0^1 |g'(s)| ds \leq c,$$

for a suitable constant $c > 0$. By Lemma 3.1 and Lemma 3.7, for every set K which is compactly embedded in $(0, \infty) \times \mathbb{R}$, we get

$$\begin{aligned}
\|q_{\varepsilon_k}(\rho_{\varepsilon_k})(w - w_{\varepsilon_k})\|_{L^2(K)} &\leq \|q_{\varepsilon_k}(\rho_{\varepsilon_k})\|_{L^\infty(K)} \|w - w_{\varepsilon_k}\|_{L^2(K)} \\
&\leq c \|w - w_{\varepsilon_k}\|_{L^2(K)},
\end{aligned}$$

and so

$$I_{6,k} \rightarrow 0 \quad \text{in} \quad H^{-1}_{loc}((0, \infty) \times \mathbb{R}).$$

Having proved that the family

$$\{\partial_t |\rho_{\varepsilon_k} - c| + \partial_x [\text{sign}(\rho_{\varepsilon_k} - c)(g(\rho_{\varepsilon_k}) - g(c))w]\}_{k \in \mathbb{N}}$$

is compact in $H_{loc}^{-1}((0, +\infty) \times \mathbb{R})$, the compactness of $\{\rho_\varepsilon\}_{\varepsilon > 0}$ follows from Theorem 4.1. This, together with the compactness of $\{w_\varepsilon\}_{\varepsilon > 0}$ established in Lemma 3.7, yields the compactness of $\{u_\varepsilon\}_{\varepsilon > 0}$ since $u_\varepsilon = w_\varepsilon \rho_\varepsilon$ (see (11)).

In conclusion, we have proved that there exists $(u, \rho) \in L^\infty((0, \infty) \times \mathbb{R}; \mathbb{R})$ such that

$$\begin{aligned} \rho_{\varepsilon_k} &\rightarrow \rho, u_{\varepsilon_k} \rightarrow u && \text{in } L_{loc}^p((0, \infty) \times \mathbb{R}), 1 \leq p < \infty, \\ &&& \text{and a.e. in } (0, \infty) \times \mathbb{R} \text{ as } k \rightarrow \infty. \end{aligned}$$

By Lebesgue's dominated convergence theorem, we conclude that (ρ, u) is a weak solution of (1) in the sense of Definition 2.1. \square

Acknowledgments. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). G. M. Coclite has been partially supported by the Research Project of National Relevance "Multiscale Innovative Materials and Structures" granted by the Italian Ministry of Education, University and Research (MIUR Prin 2017, project code 2017J4EAYB and the Italian Ministry of Education, University and Research under the Programme Department of Excellence Legge 232/2016 (Grant No. CUP - D94I18000260001). N. De Nitti is partially supported by the Alexander von Humboldt foundation and by the TRR-154 project of the Deutsche Forschungsgemeinschaft (DFG). We thank M. Gugat and E. Zuazua for several helpful conversations. We also thank the Referees for their valuable comments, which have contributed to improve the paper.

REFERENCES

- [1] A. Aw and M. Rascle, [Resurrection of "second order" models of traffic flow](#), *SIAM J. Appl. Math.*, **60** (2000), 916–938.
- [2] C. Bardos, A. Y. le Roux and J.-C. Nédélec, [First order quasilinear equations with boundary conditions](#), *Comm. Partial Differential Equations*, **4** (1979), 1017–1034.
- [3] S. Bianchini and A. Bressan, [Vanishing viscosity solutions of nonlinear hyperbolic systems](#), *Ann. of Math. (2)*, **161** (2005), 223–342.
- [4] S. Blandin, D. Work, P. Goatin, B. Piccoli and A. Bayen, [A general phase transition model for vehicular traffic](#), *SIAM J. Appl. Math.*, **71** (2011), 107–127.
- [5] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem*, Oxford Lecture Series in Mathematics and its Applications, Vol. 20, Oxford University Press, Oxford, 2000.
- [6] A. Bressan and R. M. Colombo, [The semigroup generated by \$2 \times 2\$ conservation laws](#), *Arch. Rational Mech. Anal.*, **133** (1995), 1–75.
- [7] A. Bressan, G. Crasta and B. Piccoli, [Well-posedness of the Cauchy problem for \$n \times n\$ systems of conservation laws](#), *Mem. Amer. Math. Soc.*, **146** (2000).
- [8] A. Bressan, T.-P. Liu and T. Yang, [L¹ stability estimates for \$n \times n\$ conservation laws](#), *Arch. Ration. Mech. Anal.*, **149** (1999), 1–22.
- [9] G.-Q. Chen, [Remarks on R. J. DiPerna's paper: "Convergence of the viscosity method for isentropic gas dynamics"](#), *Proc. Amer. Math. Soc.*, **125** (1997), 2981–2986.
- [10] G.-Q. Chen and H. Frid, [Vanishing viscosity limit for initial-boundary value problems for conservation laws](#), in *Nonlinear Partial Differential Equations*, Contemp. Math., Vol. 238, Amer. Math. Soc., Providence, RI, 1999, 35–51.
- [11] G. M. Coclite, K. H. Karlsen, S. Mishra and N. H. Risebro, [Convergence of vanishing viscosity approximations of \$2 \times 2\$ triangular systems of multi-dimensional conservation laws](#), *Boll. Unione Mat. Ital. (9)*, **2** (2009), 275–284.

- [12] R. M. Colombo, [Hyperbolic phase transitions in traffic flow](#), *SIAM J. Appl. Math.*, **63** (2002), 708–721.
- [13] R. M. Colombo, F. Marcellini and M. Rascle, [A 2-phase traffic model based on a speed bound](#), *SIAM J. Appl. Math.*, **70** (2010), 2652–2666.
- [14] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 4th edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 325, Springer-Verlag, Berlin, 2016.
- [15] R. J. DiPerna, [Convergence of the viscosity method for isentropic gas dynamics](#), *Comm. Math. Phys.*, **91** (1983), 1–30.
- [16] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations*, CBMS Regional Conference Series in Mathematics, Vol. 74, Conference Board of the Mathematical Sciences, American Mathematical Society, Providence, RI, 1990.
- [17] S. Fan, M. Herty and B. Seibold, [Comparative model accuracy of a data-fitted generalized Aw-Rascle-Zhang model](#), *Netw. Heterog. Media*, **9** (2014), 239–268.
- [18] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
- [19] M. Garavello and F. Marcellini, [The Riemann problem at a junction for a phase transition traffic model](#), *Discrete Contin. Dyn. Syst.*, **37** (2017), 5191–5209.
- [20] P. Goatin, [The Aw-Rascle vehicular traffic flow model with phase transitions](#), *Math. Comput. Modelling*, **44** (2006), 287–303.
- [21] J. M. Greenberg, A. Klar and M. Rascle, [Congestion on multilane highways](#), *SIAM J. Appl. Math.*, **63** (2003), 818–833.
- [22] F. Gu, Y.-g. Lu and Q. Zhang, [Global solutions to one-dimensional shallow water magneto-hydrodynamic equations](#), *J. Math. Anal. Appl.*, **401** (2013), 714–723.
- [23] H. Holden and N. H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, 2nd edition, Applied Mathematical Sciences, Vol. 152, Springer, Heidelberg, 2015.
- [24] B. S. Kerner, *The Physics of Traffic: Empirical Freeway Pattern Features, Engineering Applications, and Theory*, Springer, Berlin, New York, 2004.
- [25] C. Klingenberg and Y.-g. Lu, [The vacuum case in Diperna’s paper](#), *J. Math. Anal. Appl.*, **225** (1998), 679–684.
- [26] S. N. Kružkov, First order quasilinear equations with several independent variables, *Mat. Sb. (N.S.)*, **81** (1970), 228–255.
- [27] J. P. Lebacque, X. Louis, S. Mammari, B. Schnetzler and H. Haj-Salem, [Modélisation du trafic autoroutier au second ordre](#), *C. R. Math. Acad. Sci. Paris*, **346** (2008), 1203–1206.
- [28] M. J. Lighthill and G. B. Whitham, [On kinematic waves. II. A theory of traffic flow on long crowded roads](#), *Proc. Roy. Soc. London. Ser. A.*, **229** (1955), 317–345.
- [29] T.-P. Liu and T. Yang, [L₁ stability for 2 × 2 systems of hyperbolic conservation laws](#), *J. Amer. Math. Soc.*, **12** (1999), 729–774.
- [30] T.-P. Liu and T. Yang, [L₁ stability of conservation laws with coinciding Hugoniot and characteristic curves](#), *Indiana Univ. Math. J.*, **48** (1999), 237–247.
- [31] Y. Lu, *Hyperbolic Conservation Laws and the Compensated Compactness Method*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, Vol. 128, Chapman & Hall/CRC, Boca Raton, FL, 2003.
- [32] F. Murat, L’injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl. (9)*, **60** (1981), 309–322.
- [33] E. Panov, [On weak completeness of the set of entropy solutions to a scalar conservation law](#), *SIAM J. Math. Anal.*, **41** (2009), 26–36.
- [34] P. I. Richards, [Shock waves on the highway](#), *Operations Res.*, **4** (1956), 42–51.
- [35] D. Serre, [Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation](#), *J. Differential Equations*, **68** (1987), 137–168.
- [36] L. Tartar, Compensated compactness and applications to partial differential equations, in *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV*, Res. Notes in Math., Vol. 39, Pitman, Boston, MA, London, 1979, 136–212.
- [37] M. E. Taylor, *Partial Differential Equations I. Basic Theory*, 2nd edition, Applied Mathematical Sciences, Vol. 115, Springer, New York, 2011.
- [38] B. Temple, [Systems of conservation laws with invariant submanifolds](#), *Trans. Amer. Math. Soc.*, **280** (1983), 781–795.

- [39] G. Wong and S. Wong, [A multi-class traffic flow model - an extension of LWR model with heterogeneous drivers](#), *Transportation Research Part A: Policy and Practice*, **36** (2002), 827–841.
- [40] D.-y. Zheng, Y.-g. Lu, G.-q. Song and X.-z. Lu, [Global existence of solutions for a nonstrictly hyperbolic system](#), *Abstr. Appl. Anal.* (2014), Art. ID 691429, 7 pp.

Received October 2020; 1st revision January 2021; final revision March 2021.

E-mail address: giuseppemaria.coclite@poliba.it

E-mail address: nicola.de.nitti@fau.de

E-mail address: mauro.garavello@unimib.it

E-mail address: francesca.marcellini@unibs.it