

CONVERGENCE RATES FOR THE HOMOGENIZATION OF THE POISSON PROBLEM IN RANDOMLY PERFORATED DOMAINS

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ABSTRACT. In this paper we provide convergence rates for the homogenization of the Poisson problem with Dirichlet boundary conditions in a randomly perforated domain of \mathbb{R}^d , $d \geq 3$. We assume that the holes that perforate the domain are spherical and are generated by a rescaled marked point process (Φ, \mathcal{R}) . The point process Φ generating the centres of the holes is either a Poisson point process or the lattice \mathbb{Z}^d ; the marks \mathcal{R} generating the radii are unbounded i.i.d random variables having finite $(d - 2 + \beta)$ -moment, for $\beta > 0$. We study the rate of convergence to the homogenized solution in terms of the parameter β . We stress that, for low values of β , the balls generating the holes may overlap with overwhelming probability.

1. Introduction. In this paper we obtain convergence rates for the homogenization of the Poisson problem in a bounded domain of \mathbb{R}^d , $d \geq 3$, that is perforated by many small random holes H^ε . We impose Dirichlet boundary conditions on the boundary of the set and of the holes H^ε . In other words, given $f \in H^{-1}(D)$ and $D \subseteq \mathbb{R}^d$ bounded and regular, we define the perforated set $D^\varepsilon := D \setminus H^\varepsilon$ and study the boundary value problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } D^\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial D^\varepsilon \end{cases} \quad (1.1)$$

We assume that, for $\varepsilon > 0$, the holes H^ε are a union of spherical holes having random centres and radii. Let (Φ, \mathcal{R}) be *marked point process* where Φ is either the lattice \mathbb{Z}^d or a Poisson point process of intensity $\lambda > 0$. We assume that the associated marks $\mathcal{R} = \{\rho_z\}_{z \in \Phi}$ are independent and identically distributed random variables that satisfy the moment condition

$$\mathbb{E}_\rho[\rho^{d-2+\beta}] < +\infty, \quad \beta > 0. \quad (1.2)$$

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Here and below, $\mathbb{E}_\rho[\cdot]$ denotes the expectation under ρ . The set H^ε is thus defined by

$$H^\varepsilon := \bigcup_{z \in \Phi \cap (\frac{1}{\varepsilon}D)} B_{(\varepsilon^{\frac{d}{d-2}}\rho_z) \wedge 1}(\varepsilon z), \quad (\frac{1}{\varepsilon}D) := \{x \in \mathbb{R}^d : \varepsilon x \in D\}. \quad (1.3)$$

As shown in [13], if $\beta = 0$ in (1.2), then for \mathbb{P} -almost every realization of the random set H^ε , the solutions to (1.1) converge weakly in $H_0^1(D)$ to the homogenized problem

$$\begin{cases} -\Delta u + C_0 u = f & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \quad (1.4)$$

The constant $C_0 > 0$ is the limit of the density of harmonic capacity generated by the set H^ε : If S^d denotes the d -dimensional unit sphere and \mathcal{H}^d is the d -dimensional Hausdorff measure, then

$$C_0 := c_d \begin{cases} \mathbb{E}_\rho[\rho^{d-2}] & \text{if } \Phi = \mathbb{Z}^d \\ \lambda \mathbb{E}_\rho[\rho^{d-2}] & \text{if } \Phi = Poi(\lambda) \end{cases}, \quad c_d := (d-2)\mathcal{H}^{d-1}(S^{d-1}). \quad (1.5)$$

In this paper, we strengthen the condition (1.2) on the integrability of the marks \mathcal{R} from $\beta = 0$ to $\beta > 0$ and study the convergence rates of u_ε to the homogenized solution u .

By the Strong Law of Large Numbers, assumption (1.2) with $\beta = 0$ is minimal in order to ensure that for \mathbb{P} -almost every realization of H^ε , its density of capacity admits a finite limit. However, it does not prevent the balls in H^ε from having radii that are much bigger than $\varepsilon^{\frac{d}{d-2}}$. This gives rise to clustering phenomena with overwhelming probability. In particular the expected number of balls of H^ε that intersect, namely such that their radius $\varepsilon^{\frac{d}{d-2}}\rho_z$ is bigger than the typical distance ε between the centres, is of order ε^{-d+2} (over an expected total of ε^{-d} balls). The analogue holds also under assumption (1.2) for values $\beta < \frac{(d-2)^2}{2}$, with the expected number of overlapping balls being of order $\varepsilon^{-d+2+\frac{\beta}{d-2}}$.

The presence of balls that overlap is the main challenge in the proof of the qualitative homogenization statement obtained in [13] and is one of the challenges of the current paper. It requires a careful treatment of the set H^ε to ensure that the presence of long chains of overlapping balls does not destroy the homogenization process. For a more detailed discussion on this issue we refer to the introductory section in [13] and to Subsection 2.2 of the present paper.

The main results contained in this paper provide an *annealed* (i.e. averaged in probability) estimate for the H^1 -norm of the homogenization error $u_\varepsilon - W_\varepsilon u$. The function W_ε is a suitable corrector function that is related to the so-called *oscillating test function* [6, 23]. We assume that Φ is the lattice \mathbb{Z}^d or that it is a Poisson point process in dimension $d = 3$. For a comment on the case $d > 3$, we refer to Remark 2.2. If $\mathbb{E}[\cdot]$ denotes the expectation under the probability measure associated to the process (Φ, \mathcal{R}) , we show that¹

$$\mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2]^{\frac{1}{2}} \leq C \begin{cases} \varepsilon^{\frac{d}{d^2-4}\beta} & \text{if } \beta \leq d-2 \\ \varepsilon^{\frac{d}{d+2}} & \text{if } \beta > d-2 \end{cases} \quad (1.6)$$

¹In the case of Φ being a Poisson point process, there is a factor $\log \varepsilon$ on the right-hand side. We refer to Theorem 2.1 for the precise statement.

We stress that in the case of periodic holes, namely when $\Phi = \mathbb{Z}^d$ and the radii $\rho_z = r$, $r > 0$ are constant and deterministic, the optimal rate on the right-hand side of (1.6) is ε [18]. In the current setting, the unboundedness and randomness of the radii seems to yield slower convergence rates. We refer to Subsection 2.2 for a discussion on the exponents obtained in (1.6) and how they relate to the techniques used in the present paper to treat the case of unbounded and random radii.

The main quantity that governs the decay of the homogenization error $u_\varepsilon - W_\varepsilon u$ is the convergence of the capacity density of H^ε to the constant term C_0 that appears in the homogenized equation (1.4). In the periodic case mentioned in the previous paragraph, the term $C_0 = c_d r^{d-2}$ (c.f. (1.5)) is close to the density of capacity of H^ε already at scale ε . Heuristically, indeed, if $A \subseteq D$ we have

$$\text{Cap}(A \cap H^\varepsilon) \simeq \sum_{z \in (\varepsilon\mathbb{Z})^d \cap A} \text{Cap}(B_{\varepsilon^{\frac{d}{d-2}} r}(z)) \simeq |A| \varepsilon^{-d} c_d (\varepsilon^{\frac{d}{d-2}} r)^{d-2} \stackrel{(1.5)}{=} C_0 |A|,$$

and this chain of identities is true as long as $|A|$ is at least of order ε . On the other hand, in the random setting, this identity is expected to hold at scales that are larger than ε due to the fluctuations of the process (Φ, \mathcal{R}) . We also remark that the threshold $d - 2$ in the parameter β obtained in (1.6) is related to the L^2 -nature of the norm considered for the homogenization error. Roughly speaking, the norm considered in (1.6) requires a control on the expectation of the square of the capacity generated by the balls in H^ε . This quantity depends on the $2(d - 2)$ -moments of the random variable ρ .

Starting with [6] and [21], there is a large amount of literature devoted to the homogenization of (1.1), both for deterministic and random holes H^ε [3, 20]; similar problems have also been studied in the case of the fractional laplacian $(-\Delta)^s$, [2] or for nonlinear elliptic operators [4, 24]. All the models considered in the deterministic case contain assumptions that ensure that, for ε small enough, the holes in H^ε do not to overlap. In the random models mentioned above, a non-overlapping condition is as well imposed, at least for \mathbb{P} -almost every realization and $\varepsilon > 0$ small enough. For a complete and more detailed description of these works, we refer to the introduction of [13].

For what concerns quantitative rates of convergence for (1.1) to (1.4), the first result in the periodic case is contained in [18]. When the holes are randomly distributed, the first quantitative result has been obtained in [9]. In this paper, the authors study the analogue of (1.1) for the operator $-\Delta + \lambda$ in an unbounded domain of \mathbb{R}^3 , that is perforated by m spherical holes of identical radius $\sim m^{-1}$. The centres of the holes are independent and distributed according to a compactly supported and continuous potential V . If u_m denotes the analogue of u_ε , when the massive term λ is big enough compared to the size of V , the authors provide rates of convergence for the L^2 -norm of the difference $u_m - u$ in the limit $m \rightarrow +\infty$. Furthermore, they prove the Gaussianity of the fluctuations of u_m around the homogenized solution u in the CLT-scaling. In [16], this result has been obtained in the same setting of [9] without any constraint on the massive term $\lambda > 0$.

The quantitative estimates developed in this paper are also used in [10] to obtain homogenization results for solutions to (1.1) and the analogous Stokes system in the regimes leading to Darcy's law. In [10], the radii in (1.3) are rescaled by a factor ε^α , $1 < \alpha < \frac{d}{d-2}$ and the random variables $\{\rho_z\}_{z \in \Phi}$ satisfy a suitable moment condition. Like (1.2), also this condition does not prevent the holes from overlapping and give rise to clusters.

We conclude this introduction mentioning that the analogue of (1.1) for a Stokes (and Navier-Stokes) system with no-slip boundary conditions on the holes H^ε has been considered in [1, 22] in the periodic case and then extended to more general configurations of holes (see, e.g., [5, 8, 15]). In the case of the Stokes operator, the limit equation contains an additional zero-th order term similar to C_0 in (1.4). Under the same assumptions of this paper, the analogue of the homogenization result contained in [13] has been proven for a Stokes system in [12, 11]. We believe that techniques similar to the one of this paper may be used to prove the same result of (1.6) also in the case of a Stokes system. However, in this case, we expect that the role played by the pressure would yield a less explicit definition of the corrector function W_ε .

2. Setting and main result. Let $d \geq 3$ and $D \subseteq \mathbb{R}^d$ be a bounded and smooth domain that is star-shaped with respect to the origin. For $\varepsilon > 0$, we define the punctured set $D^\varepsilon = D \setminus H^\varepsilon$, with H^ε as in (1.3). We assume that the union of balls H^ε is generated by a marked point process (Φ, \mathcal{R}) on $\mathbb{R}^d \times \mathbb{R}_+$: We generate the centres of the balls in H^ε via a point process Φ . To each point $z \in \Phi$, we associate a mark $\rho_z \geq 0$ that determines the radius of the ball. We refer to [7, Chapter 9, Definitions 9.1.I - 9.1.IV] for an extensive and rigorous definition of marked point processes and their associated measures on $\mathbb{R}^d \times \mathbb{R}_+$.

We denote by $(\Omega; \mathcal{F}, \mathbb{P})$ the probability space associated to (Φ, \mathcal{R}) , so that the random sets in (1.3) and the random field solving (1.1) may be written as $H^\varepsilon = H^\varepsilon(\omega)$, $D^\varepsilon = D^\varepsilon(\omega)$ and $u_\varepsilon(\omega; \cdot)$, respectively. The set of realizations Ω may be seen as the set of atomic measures $\sum_{n \in \mathbb{N}} \delta_{(z_n, \rho_n)}$ in $\mathbb{R}^d \times \mathbb{R}_+$ or, equivalently, as the set of (unordered) collections $\{(z_n, \rho_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d \times \mathbb{R}_+$.

Throughout this paper we assume that (Φ, \mathcal{R}) satisfies the following conditions:

- (i) Φ is either the lattice \mathbb{Z}^d or $\Phi = Poi(\lambda)$, i.e. a Poisson point process of intensity $\lambda > 0$;
- (ii) The marks $\{\rho_z\}_{z \in \Phi}$ are independent and identically distributed: The marginal of the marks with respect to the process Φ , has n-correlation function, $n \in \mathbb{N}$, that may be written as the product

$$g_n((z_1, \rho_1), \dots, (z_n, \rho_n)) = \prod_{i=1}^n g_1((z_i, \rho_i)), \quad g_1((z, \rho)) = g(\rho).$$

- (iii) The marks \mathcal{R} have finite $(d - 2 + \beta)$ -moment, namely the density function g in (ii) satisfies

$$\mathbb{E}_\rho[\rho^{d-2+\beta}] = \int_0^{+\infty} \rho^{d-2+\beta} g(\rho) d\rho \leq 1, \quad \text{with } \beta > 0. \tag{2.1}$$

We stress that conditions (i)-(ii) yield that (Φ, \mathcal{R}) is stationary. In the case $\Phi = Poi(\lambda)$, the process (Φ, \mathcal{R}) is stationary with respect to the action of the group of translations $\{\tau_x\}_{x \in \mathbb{R}^d}$. This means that the probability measure \mathbb{P} is invariant under the action of the transformation $\tau_x : \Omega \rightarrow \Omega$, $\omega = \{(z_i; \rho_{z_i})\}_{i \in \mathbb{N}} \mapsto \tau_x \omega := \{(z_i + x; \rho_{z_i})\}_{i \in \mathbb{N}}$. In the case $\Phi = \mathbb{Z}^d$ the same holds under the action of the group $\{\tau_z\}_{z \in \mathbb{Z}^d}$.

Notation. When no ambiguity occurs, we skip the argument $\omega \in \Omega$ in the notation for $H^\varepsilon(\omega)$, $D^\varepsilon(\omega)$, $u_\varepsilon(\omega; \cdot)$ and in all the other random objects. We denote by $\mathbb{E}[\cdot]$ and $\mathbb{E}_\Phi[\cdot]$ the expectations under the total probability measure \mathbb{P} and the

probability measure \mathbb{P}_Φ associated to the point process Φ , respectively. For $\varepsilon > 0$ and a set $A \subseteq \mathbb{R}^d$, we define

$$\Phi(A) := \{z \in \Phi : z \in A\}, \quad \Phi_\varepsilon(A) := \{z \in \Phi : \varepsilon z \in A\} \tag{2.2}$$

and the random variables

$$N(A) := \#\Phi(A), \quad N^\varepsilon(A) := \#\Phi^\varepsilon(A). \tag{2.3}$$

For any $\mu \in H^{-1}(D)$, we write $\langle \cdot; \cdot \rangle$ for the duality product with $H_0^1(D)$; we use the notation $\sum_{i \in I}$ for the averaged sum $\#(I)^{-1} \sum_{i \in I}$ and \lesssim and \gtrsim instead of $\leq C$ and $\geq C$ with the constant C depending on the dimension d , the domain D and, in the case of $\Phi = Poi(\lambda)$, the intensity rate λ .

For two sets $A \subseteq B \subseteq \mathbb{R}^d$, we denote by $Cap(A; B)$ the relative harmonic capacity of the set A in B (c.f., for instance, [13][4.17]).

2.1. Main result. Before stating the main results, we need to define a suitable corrector function W_ε that appears in the homogenization error $u_\varepsilon - W_\varepsilon u$. For $x \in \mathbb{R}^d$ we set

$$R_{\varepsilon,x} := \frac{\varepsilon}{4} \min_{\substack{z \in \Phi^\varepsilon(D), \\ z \neq x}} \left\{ |z - x|; 1 \right\} \tag{2.4}$$

Note that, if $\Phi = \mathbb{Z}^d$, then $R_{\varepsilon,x} = \frac{\varepsilon}{4}$ for every $x \in \Phi$. For $\delta > 0$, we denote by $\Phi_\delta^\varepsilon(D) \subseteq \Phi_\varepsilon(D)$ the set

$$\Phi_\delta^\varepsilon(D) := \left\{ z \in \Phi_\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_z \leq \varepsilon^{1+\delta}, R_{\varepsilon,z} \geq 2\sqrt{d} \varepsilon^{\frac{d}{d-2}} \rho_z \vee \varepsilon^2 \right\}, \tag{2.5}$$

where here and through out the paper $a \vee b = \max\{a; b\}$.

For each $z \in \Phi_\delta^\varepsilon(D)$, let $w_{\varepsilon,z} \in H^1(B_{\frac{\varepsilon}{4}}(\varepsilon z))$ be the solution to

$$\begin{cases} -\Delta w_{z,\varepsilon} = 0 & \text{in } B_{R_{\varepsilon,z}}(\varepsilon z) \setminus B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z) \\ w_{z,\varepsilon} = 0 & \text{on } \partial B_{R_{\varepsilon,z}}(\varepsilon z) \\ w_{z,\varepsilon} = 1 & \text{on } \partial B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z). \end{cases} \tag{2.6}$$

We thus define

$$W_\varepsilon(x) = \begin{cases} w_{z,\varepsilon} & \text{if } x \in B_{R_{\varepsilon,z}}(\varepsilon z) \setminus B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z) \\ 0 & \text{if } x \in B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z) \\ 1 & \text{otherwise} \end{cases} \tag{2.7}$$

We stress that (2.5) ensures that definitions (2.6) and (2.7) are well-posed since the set $\{B_{R_{\varepsilon,z}}(\varepsilon z)\}_{z \in \Phi_\delta^\varepsilon(D)}$ is made of disjoint balls and, for every $z \in \Phi_\delta^\varepsilon(D)$, it holds $B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z) \subseteq B_{R_{\varepsilon,z}}(\varepsilon z)$. Note that in the above definition the function $W_\varepsilon \in H^1(D)$ depends on the choice of the parameter δ used to select the subset $\Phi_\delta^\varepsilon(D)$. The optimal parameter δ will be fixed in Theorem 2.1. We finally stress that, in the periodic case $\Phi = \mathbb{Z}^d$ and $\rho_z \equiv r$, for any $\delta > 0$ and ε small enough, the function W_ε coincides with the *oscillating test function* constructed in [6, 18].

Theorem 2.1. *Let (Φ, \mathcal{R}) satisfy conditions (i)-(iii). For $\varepsilon > 0$ and $p > d$, let $f \in W^{1,p}(D)$ with $\|f\|_{W^{1,p}(D)} = 1$ and u_ε and u be as in (1.1) and (1.4), respectively.*

We consider the random field W_ε in (2.7) with

$$\delta = \begin{cases} \frac{4}{d^2-4} & \text{if } \beta \leq d-2 \\ \frac{2}{d-2} - \frac{2d}{(d+2)\beta} & \text{if } \beta > d-2 \end{cases}$$

(a) If $\Phi = \mathbb{Z}^d$, there exists a constant $C = C(d, D, p) > 0$ such that

$$\mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2]^{\frac{1}{2}} \leq C \begin{cases} \varepsilon^{\frac{d}{d^2-4}\beta} & \text{if } \beta \leq d-2 \\ \varepsilon^{\frac{d}{d+2}} & \text{if } \beta > d-2 \end{cases}$$

(b) If $\Phi = \text{Poi}(\lambda)$ with $\lambda > 0$ and $d = 3$, there exists a constant $C = C(\lambda, D, p) > 0$ such that

$$\mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2]^{\frac{1}{2}} \leq C \begin{cases} |\log \varepsilon| \varepsilon^{\frac{3}{5}\beta} & \text{if } \beta \leq 1 \\ |\log \varepsilon| \varepsilon^{\frac{3}{5}} & \text{if } \beta > 1 \end{cases}$$

Remark 2.2. As shown throughout Section 4, the argument of Theorem 2.1 (b) applies also to higher dimensions $d \geq 4$. In this case, the homogenization error decays as ε^α , for an exponent $\alpha = \alpha(d, \beta)$. The exponent α , however, is generally smaller than the one in the periodic case and it is the same only for β small enough ($\beta \leq \frac{2}{d-1}(d-2)$).

We expect that the techniques used to prove Theorem 2.1 do extend to other examples of stationary random distributions of centres. For instance, to point processes that satisfy a finite-range of dependence assumption and for which the expected number of elements in a finite set A scales like its (Lebesgue-)measure. We also believe that Theorem 2.1 may be adapted to sets of holes H_m , $m \in \mathbb{N}$ that are a collection of m balls having centres that are independently distributed according to a fixed density $\Psi \in C_c^\infty(D)$. In this case, the balls have random radii rescaled by $m^{-\frac{1}{d-2}}$ and the homogenization limit is obtained for $m \rightarrow \infty$. For this choice of holes, the deterministic control on the cardinality of the set of centres simplifies some of the estimates.

Remark 2.3. As it becomes apparent in the proof of Theorem 2.1, the choice of W_ε is not unique. The same result holds, for instance, if W_ε is replaced with the oscillating test function w_ε constructed in [13, Section 3] and in Subsection 3.2 of the present paper. The function W_ε , however, has a simpler and more explicit construction that may be implemented numerically with more efficiency. It is, indeed, an oscillating test function basically restricted to the balls of H^ε that do not overlap and have radius smaller than $\varepsilon^{1+\delta}$. We emphasize that the condition on the minimal distance being at least ε^2 is purely technical and may be avoided by either assuming that the radii $\{\rho_z\}_{z \in \Phi}$ are (uniformly) bounded from below, or that their inverse ρ_z^{-1} satisfies a suitable moment condition.

2.2. Ideas of the proofs. The proof of Theorem 2.1 is inspired to the proof of the same result in the case of periodic holes shown in [18]. The latter, in turn, upgrades the result of [6] from the qualitative statement $u_\varepsilon \rightharpoonup u$ in $H_0^1(D)$ to an estimate on the convergence of the homogenization error. Both arguments rely on the construction of suitable oscillating test functions $\{w_\varepsilon\}_{\varepsilon > 0} \subseteq H^1(D)$. In the qualitative statement of [6], these functions allow to pass to the limit $\varepsilon \downarrow 0$ in the weak formulation of (1.1) and infer the homogenized equation (1.4).

The functions $\{w_\varepsilon\}_{\varepsilon > 0}$ may be constructed as W_ε in (2.7), where the set Φ_δ^ε coincides with the whole set $\Phi = \mathbb{Z}^d$ and the functions $\{w_{\varepsilon,z}\}_{z \in \Phi^\varepsilon(D)}$ introduced in

(2.6) satisfy $w_{\varepsilon,z}(\cdot) = w_{\varepsilon,0}(\cdot + z)$. The additional term $C_0 = c_d r^{d-2}$ that appears in the homogenized equation (1.4) is the limit of the measures $-\Delta w_\varepsilon$ when tested against the function $\rho u_\varepsilon \in H_0^1(D^\varepsilon)$, $\rho \in C_0^\infty(D)$. It is not hard to see from (2.7) that, for functions that vanish on the holes H^ε , the action of $-\Delta w_\varepsilon$ reduces the periodic measure

$$\mu_\varepsilon = \sum_{z \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \partial_n w_{\varepsilon,z} \delta_{\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)}, \tag{2.8}$$

that is concentrated on the spheres $\{\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)\}_{z \in \mathbb{Z}^d}$. Here, ∂_n is the outer normal derivative on $\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)$.

In [18], the corrector W_ε is chosen as the oscillating test function w_ε itself. As a first step, it is shown that the decay of $\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}$ boils down to controlling the convergence of the density of capacity of H^ε to its limit C_0 (c.f. (1.5)). The latter is expressed in terms of the decay of the norm $\|\mu_\varepsilon - C_0\|_{H^{-1}(D)}$, with μ_ε as in (2.8). As a second step, the authors appeal to a result of [19] to estimate the decay of $\|\mu_\varepsilon - C_0 \mathbf{1}_D\|_{H^{-1}(D)}$ in terms of the size ε of the periodic cell $C_\varepsilon := [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ of μ_ε . The crucial feature is that, up to a correction of order ε^2 , the measure $\mu_\varepsilon - C_0$ has zero average in C_ε . In other words, we have

$$\int_{\partial B_{\frac{\varepsilon}{4}}(0)} \partial_n w_\varepsilon = \varepsilon^d (C_0 + O(\varepsilon^2)). \tag{2.9}$$

In this paper we adapt to the random setting the previous two-step argument. The first main difference is strictly related to the randomness of the radii in H^ε and needs to be addressed also in the case of bounded radii (i.e. if $\beta = +\infty$ in (2.1)) and periodic centres. In this case, the measure μ_ε is defined as in (2.8) but, on each sphere $\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)$, the term $\partial_n w_\varepsilon$ depends on the random mark ρ_z . Therefore, contrarily to the periodic case, (2.9) may not hold in each cube $\varepsilon z + C_\varepsilon$. Nevertheless, by the Law of Large Numbers, we may expect that the average of $\mu_\varepsilon - C_0$ is close to zero over cubes of size $k\varepsilon$, $k \gg 1$, as the left-hand side in (2.9) turns into an averaged sum of k^d random variables. This motivates the introduction of a partition of the set D into cubes of mesoscopic size $k\varepsilon$ (c.f. Section 3.1) that plays the role of the cells $\varepsilon z + C_\varepsilon$ of the periodic case. This allows us to adapt the result by [19] and obtain

$$\mathbb{E}[\|\mu_\varepsilon - C_0 \mathbf{1}_D\|_{H^{-1}(D)}^2]^{\frac{1}{2}} \lesssim k\varepsilon + \mathbb{E}_\rho \left[\left(\sum_{i=1}^{k^d} \rho_i^{d-2} - \mathbb{E}_\rho[\rho^{d-2}] \right)^2 \right]^{\frac{1}{2}}. \tag{2.10}$$

Here, the last term accounts for the difference between the average of μ_ε in each cube of size $(k\varepsilon)$, $k \in \mathbb{N}$ and the value C_0 . This inequality, implies an estimate of the form:

$$\mathbb{E}[\|\mu_\varepsilon - C_0 \mathbf{1}_D\|_{H^{-1}(D)}^2]^{\frac{1}{2}} \lesssim k\varepsilon + \mathbb{E}_\rho \left[(\rho^{d-2} - \mathbb{E}_\rho[\rho^{d-2}])^2 \right]^{\frac{1}{2}} k^{-\frac{d}{2}}.$$

The optimal choice of k yields the exponent $\frac{d}{d+2}$ of Theorem 2.1. If $\rho_z \equiv r$ for all $z \in \mathbb{Z}^d$, then the second term vanishes and the above estimate with $k = 1$ gives the optimal rate of [18].

In the case of centres distributed according to a Poisson point process, the argument for Theorem 2.1 follows the same ideas sketched above. Although the centres of the holes in H^ε have random positions, their typical distance is still of size ε . This feature gives rise to the additional logarithmic factor in the rate of Theorem

2.1. The main technical challenge is related to the construction of the mesoscopic partition of D that allows to obtain the analogue of (2.10). In contrast with the case $\Phi = \mathbb{Z}^d$, indeed, there are (\mathbb{P} -many) realizations of H^ε where the support of the measure μ_ε defined in (2.8) intersects the boundary of the covering. In other words, the spheres $\{\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)\}_{z \in \Phi_\varepsilon(D)}$ might fall across two cubes of size εk that cover D . This, in particular, implies that to the covering does not correspond to a well-defined partition of the spheres where the measure μ_ε is supported. We tackle this issue by constructing a suitable random covering. We refer to Subsection 3.1 for the precise construction.

A second challenge that arises in the proof of Theorem 2.1 is related to the presence of overlapping holes in the case $\beta < +\infty$ in (2.1). The strategy to deal with this issue is very similar to the one used in [13]: We partition, indeed, the set of holes as $H^\varepsilon = H_b^\varepsilon \cup H_g^\varepsilon$, where the subset H_b^ε contains all the holes that overlap (c.f. Lemma 3.1). As shown in [13], the contribution of H_b^ε to the density of capacity is negligible in the limit $\varepsilon \downarrow 0$. As a consequence, we may modify the estimates of [18], to prove that $\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}$ is controlled by the norm $\|\mu_\varepsilon - C_0 \mathbf{1}_D\|_{H^{-1}(D)}$, where the measure μ_ε is now only related to the union of disjoint balls H_g^ε .

3. Proof of Theorem 2.1, (a).

3.1. Partition of the holes H^ε and mesoscopic covering of D . This subsection contains some technical tools that will be crucial to prove the main result: The first one is an adaptation of [13] and provides a suitable way of dividing the holes H^ε between the ones that may overlap due to the unboundedness of the marks $\{\rho_z\}_{z \in \Phi}$ and the ones that, instead, are disjoint and have radii $\varepsilon^{\frac{d}{d-2}} \rho_z$ much smaller than the distance ε between the centres.

Lemma 3.1. *Let $\delta \in (0, \frac{2}{d-2}]$ be fixed. There exists an $\varepsilon_0 = \varepsilon_0(\delta, d)$ such that for every $\varepsilon \leq \varepsilon_0$ and $\omega \in \Omega$ we may find a partition of the realization of the holes*

$$H^\varepsilon := H_g^\varepsilon \cup H_b^\varepsilon$$

with the following properties:

- There exists a subset of centres $n^\varepsilon(D) \subseteq \Phi^\varepsilon(D)$ such that

$$H_g^\varepsilon := \bigcup_{z \in n^\varepsilon(D)} B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z), \quad \max_{z \in n^\varepsilon(D)} \varepsilon^{\frac{d}{d-2}} \rho_z \leq \varepsilon^{1+\delta}; \tag{3.1}$$

- There exists a set $D_b^\varepsilon \subseteq \{x \in \mathbb{R}^3 : \text{dist}(x, D) \leq 2\}$ satisfying

$$H_b^\varepsilon \subseteq D_b^\varepsilon, \quad \text{Cap}(H_b^\varepsilon, D_b^\varepsilon) \lesssim \varepsilon^d \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2} \tag{3.2}$$

and

$$B_{\frac{\varepsilon}{4}}(\varepsilon z) \cap D_b^\varepsilon = \emptyset, \quad \text{for every } z \in n^\varepsilon(D). \tag{3.3}$$

Proof of Lemma 3.1. The construction for the sets $H_g^\varepsilon, H_b^\varepsilon$ and D_b^ε is the one implemented in the proof of [13, Lemma 2.2]. We fix $\delta \in (0, \frac{2}{d-2}]$ throughout the proof.

We denote by $I_\varepsilon^b \subseteq \Phi_\varepsilon(D)$ the set that generates the holes H_b^ε . We construct it in the following way: We first set $J_b^\varepsilon := \Phi^\varepsilon(D) \setminus \Phi_\delta^\varepsilon(D)$ with Φ_δ^ε as in (2.5). Since $\Phi = \mathbb{Z}^d$, this turns into

$$J_b^\varepsilon = \Phi^\varepsilon(D) \setminus \Phi_\delta^\varepsilon(D) = \left\{ z \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_z \geq \varepsilon^{1+\delta} \right\}. \tag{3.4}$$

Given the holes $\tilde{H}_b^\varepsilon := \bigcup_{z \in J_b^\varepsilon} B_{2(\varepsilon^{\frac{d}{d-2}} \rho_z \wedge 1)}(\varepsilon z)$, we include in I_b^ε also the set of points in $\Phi_\varepsilon(D) \setminus J_b^\varepsilon$ that are “too close” to the set \tilde{H}_b^ε , i.e.

$$\tilde{I}_b^\varepsilon := \left\{ z \in \Phi^\varepsilon(D) \setminus J_b^\varepsilon : \tilde{H}_b^\varepsilon \cap B_{\frac{\varepsilon}{4}}(\varepsilon z) \neq \emptyset \right\}. \tag{3.5}$$

We define

$$\begin{aligned} I_b^\varepsilon &:= \tilde{I}_b^\varepsilon \cup J_b^\varepsilon, & n^\varepsilon(D) &:= \Phi^\varepsilon(D) \setminus I_b^\varepsilon \\ H_b^\varepsilon &:= \bigcup_{z \in I_b^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_z \wedge 1}(\varepsilon z), & H_g^\varepsilon &:= \bigcup_{z_j \in n^\varepsilon(D)} B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z), \\ D_b^\varepsilon &:= \bigcup_{z \in I_b^\varepsilon} B_{2(\varepsilon^{\frac{d}{d-2}} \rho_z \wedge 1)}(\varepsilon z). \end{aligned} \tag{3.6}$$

It remains to show that the sets defined above satisfy properties (3.1)-(3.3). Property (3.1) is an immediate consequence of definition (3.4). The first inclusion in (3.2) follows by the definition of H_b^ε and D_b^ε in (3.6); for the inequality in (3.2) we instead appeal to the subadditivity of the capacity to bound

$$\text{Cap}(H_b^\varepsilon; D_b^\varepsilon) \leq \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \text{Cap}(B_{\varepsilon^{\frac{d}{d-2}} \rho_z \wedge 1}(\varepsilon z); D_b^\varepsilon).$$

Moreover, by the monotonicity property $\text{Cap}(A; C) \leq \text{Cap}(A; B)$ for every $A \subseteq B \subseteq C$, this turns into

$$\begin{aligned} \text{Cap}(H_b^\varepsilon; D_b^\varepsilon) &\leq \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \text{Cap}(B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z); B_{2\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z)) \\ &\lesssim \varepsilon^d \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2}, \end{aligned}$$

i.e. the estimate in (3.2).

To conclude the proof of this lemma, it remains to argue (3.3): By construction (see (3.6)), it holds that

$$D_b^\varepsilon = \tilde{H}_b^\varepsilon \cup \bigcup_{z \in \tilde{I}_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z). \tag{3.7}$$

On the one hand, by (3.5) and the definition of $n^\varepsilon(D)$ in (3.6), for each $z \in n^\varepsilon(D)$ we have that

$$\text{dist}(\varepsilon z; \tilde{H}_b^\varepsilon) \geq \frac{\varepsilon}{4}. \tag{3.8}$$

On the other hand, by (3.4) and (3.5), if $w \in \tilde{I}_b^\varepsilon$, then $4\varepsilon^{\frac{d}{d-2}} \rho_w \leq \varepsilon^{1+\delta}$ so that

$$\text{dist}(\varepsilon z; B_{2\varepsilon^{\frac{d}{d-2}} \rho_w}(\varepsilon w)) \geq \frac{\varepsilon}{2} |z - w| \geq \frac{\varepsilon}{4},$$

whenever ε is such that $\varepsilon^\delta < \frac{1}{4}$. Hence, also

$$\text{dist}(\varepsilon z; \bigcup_{z \in \tilde{I}_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z)) \geq \frac{\varepsilon}{4}.$$

Combining this with (3.8) and (3.7), we infer (3.3). The proof of Lemma 3.1 is complete. \square

We now construct a suitable covering of D that, as explained in Subsection 2.2, plays a fundamental role in the proof of Theorem 2.1. We recall that, by our assumption, the set D is any smooth domain that is star-shaped with respect to the origin.

For $z \in \mathbb{R}^d$, we define the “microscopic cubes”

$$Q_{\varepsilon,z} := \varepsilon z + \varepsilon Q, \quad Q := [-1; 1]^d \tag{3.9}$$

while for $k \in \mathbb{N}$ and any $z \in \mathbb{Z}^d$ we set

$$Q_{k,z} := \varepsilon z + \frac{(2k+1)\varepsilon}{2} Q. \tag{3.10}$$

Let $N_k \subseteq \mathbb{Z}^d$ be such that the collection $\{Q_{k,z}\}_{z \in N_k}$ is an essentially disjoint covering of D . Since D is bounded, we may assume that

$$\#(N_k) \lesssim (\varepsilon k)^{-d}. \tag{3.11}$$

Let

$$\mathring{N}_k := \left\{ z \in N_k : Q_{k,z} \subseteq D, \text{dist}(Q_{k,z}; \partial D) \geq \varepsilon \right\}. \tag{3.12}$$

Since D is smooth and has compact boundary, it is easy to see that there exist $C_1 = C_1(D)$ such that, whenever $k\varepsilon \leq C_1$ it holds

$$\#(N_k \setminus \mathring{N}_k) \lesssim (k\varepsilon)^{d-1}. \tag{3.13}$$

Finally, for each $z \in N_k$ we denote by $N_{k,z} \subseteq \Phi$ the set of points of $\Phi_\delta^\varepsilon(D)$ that, when rescaled, are contained into the cube $Q_{k,z}$, i.e. such that

$$N_{k,z} := \{w \in \Phi_\delta^\varepsilon(D) : \varepsilon w \in Q_{k,z}\} \stackrel{(2.2)}{=} \Phi_\delta^\varepsilon(D) \cap \Phi^\varepsilon(Q_{k,z}). \tag{3.14}$$

Note that, since in this section we assumed that $\Phi = \mathbb{Z}^d$, it follows that for every $z \in N_k$, we have $\bigcup_{w \in N_{k,z}} Q_{\varepsilon,w} \subseteq Q_{k,z}$ where $Q_{\varepsilon,w}$ are defined as in (3.9). Moreover, for every $z \in \mathring{N}_{k,z}$, the collection $\{Q_{\varepsilon,w}\}_{w \in N_{k,z}}$ provides a refinement of $Q_{k,z}$.

3.2. Quenched estimates for the homogenization error. All the results contained in this subsection are quenched, in the sense that they hold for any fixed realization of the holes H^ε . The main result of this section is Lemma 3.2 that allows to control the norm of the homogenization error $u_\varepsilon - W_\varepsilon u$ in terms of suitable averaged sums of the random marks $\{\rho_z\}_{z \in \Phi}$.

Before giving the statement of Lemma 3.2, we recall the construction of the oscillating test function $w_\varepsilon \in H^1(D)$ implemented in [13]. As mentioned in the introduction and in Subsection 2.2, the main feature of this function is to vanish on the holes H^ε and “approximate” the density of the capacity of H^ε . We note that the unboundedness of the marks $\{\rho_z\}_{z \in \Phi}$ implies that the set $\Phi_\delta^\varepsilon(D) \subsetneq \Phi^\varepsilon(D)$ and that the function W_ε in (2.7) does not vanish in all the holes contained in H^ε .

Let $H_g^\varepsilon, H_b^\varepsilon$ and D_b^ε be as in Lemma 3.1. For every $z \in \Phi^\varepsilon(D)$, let²

$$v_\varepsilon := \operatorname{argmin} \left\{ \int_{D_b^\varepsilon} |\nabla u|^2 : u \in H_0^1(D_b^\varepsilon), u = 1 \text{ on } \partial H_b^\varepsilon \right\}.$$

We pick as oscillating test function

$$w_\varepsilon = w_\varepsilon^g \wedge w_\varepsilon^b, \tag{3.15}$$

²We assume that the minimizer exists. If this is not the case, it suffices to take v_ε in the minimizing class such that $\int_{D_b^\varepsilon} |\nabla v_\varepsilon|^2 \leq 2 \operatorname{Cap}(H_b^\varepsilon; D_b^\varepsilon)$.

where w_ε^g and w_ε^b are defined as follows:

$$w_\varepsilon^b := \begin{cases} 1 - v_\varepsilon & \text{in } D_b^\varepsilon \setminus H_b^\varepsilon \\ 0 & \text{in } H_b^\varepsilon \\ 1 & \text{in } \mathbb{R}^3 \setminus D_b^\varepsilon \end{cases} \quad (3.16)$$

and

$$w_\varepsilon^g(x) := \begin{cases} w_{z,\varepsilon} & \text{if } x \in B_{\frac{\varepsilon}{4}}(\varepsilon z) \setminus B_{\frac{\varepsilon}{4} \frac{d}{d-2} \rho_z}(\varepsilon z), \text{ for some } z \in n_\varepsilon(D) \\ 0 & \text{if } x \in B_{\frac{\varepsilon}{4} \frac{d}{d-2} \rho_z}(\varepsilon z), \text{ for some } z \in n_\varepsilon(D) \\ 1 & \text{otherwise} \end{cases} \quad (3.17)$$

For each $z \in n_\varepsilon(D)$, the function $w_{z,\varepsilon}$ is as in (2.6). We remark that each $w_{\varepsilon,z}$ admits the explicit formulation

$$w_{z,\varepsilon}(x) = \frac{(\varepsilon \frac{d}{d-2} \rho_z)^{-(d-2)} - |x - \varepsilon z_i|^{-(d-2)}}{(\varepsilon \frac{d}{d-2} \rho_z)^{-(d-2)} - (\frac{\varepsilon}{4})^{-(d-2)}} \quad \text{in } B_{\frac{\varepsilon}{4}}(\varepsilon z) \setminus B_{\frac{\varepsilon}{4} \frac{d}{d-2} \rho_z}(\varepsilon z). \quad (3.18)$$

For $k \in \mathbb{N}$, let $\{Q_{k,z}\}_{z \in N_k}$ be the covering of D constructed at the end of Subsection 3.1. For every $z \in N_k$, we define the random variables

$$S_{k,z} := \frac{1}{k^d} \sum_{w \in N_{k,z}} Y_{\varepsilon,w} \quad Y_{\varepsilon,w} := \rho_w^{d-2} \frac{1}{1 - 4^{d-2} \varepsilon^2 \rho_w^{d-2}}. \quad (3.19)$$

Lemma 3.2. *Let $\delta \in (0, \frac{2}{d-2}]$ be fixed. Then for every $\varepsilon > 0$ and $k \in \mathbb{N}$ with $k\varepsilon \leq 1$ the following inequality holds: If u_ε, u are as in Theorem 2.1 and W_ε as in (2.7), then*

$$\begin{aligned} & \|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)} \\ & \lesssim \left(\varepsilon^{d+2} \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{d-2} + (k\varepsilon)^2 \varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} + \varepsilon^d \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2} \right)^{\frac{1}{2}} \\ & + \left(\sum_{z \in \check{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2 + (k\varepsilon)^3 \sum_{z \in N_k \setminus \check{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 3.2 relies on the next lemma, that is an adaptation of [18][Theorem 3.2] and shows that controlling the error $u_\varepsilon - W_\varepsilon u$ considered in Theorem 2.1 boils down to controlling the convergence to C_0 of the density of capacity generated by H^ε .

Lemma 3.3. *Let $\delta \in (0; \frac{2}{d-2}]$ be fixed; let $u_\varepsilon, u, w_\varepsilon$ and W_ε be as in Lemma 3.2. Let w_ε be as in (3.15). Then*

$$\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2 \lesssim \|w_\varepsilon - 1\|_{L^2(D)}^2 + \|\nabla(w_\varepsilon - W_\varepsilon)\|_{L^2(\mathbb{R}^d)}^2 + \|\mu_\varepsilon - C_0\|_{H^{-1}(D)}^2,$$

with

$$\mu_\varepsilon := \sum_{z \in \Phi_\delta^\varepsilon(D)} \partial_n w_{\varepsilon,z} \delta_{\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)}. \quad (3.20)$$

Proof of Lemma 3.2. The statement follows from Lemma 3.3, provided that we show that

$$\|\nabla(w_\varepsilon - W_\varepsilon)\|_{L^2(D)}^2 + \|w_\varepsilon - 1\|_{L^2(D)}^2 \lesssim \varepsilon^{d+2} \sum_{z \in n_\varepsilon(D)} \rho_z^{d-2} + \varepsilon^d \sum_{\Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2} \quad (3.21)$$

and that for every $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $k\varepsilon \leq 1$

$$\begin{aligned} \|\mu_\varepsilon - C_0\|_{H^{-1}(D)}^2 &\lesssim (k\varepsilon)^2 \varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \\ &+ \sum_{z \in \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2 + (k\varepsilon)^3 \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2. \end{aligned} \quad (3.22)$$

We first argue (3.21): By definition (3.16) for w_b^ε and Lemma 3.1, we have that

$$\|\nabla w_b^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 \lesssim \varepsilon^d \sum_{\Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2}. \quad (3.23)$$

Since by Lemma 3.1 the sets $\bigcup_{z \in n^\varepsilon(D)} B_{\frac{\varepsilon}{4}}(\varepsilon z)$ and D_b^ε are disjoint, we appeal to (3.15) to estimate

$$\|w_\varepsilon - 1\|_{L^2(D)}^2 = \sum_{z_i \in n^\varepsilon(D)} \|w_\varepsilon^g - 1\|_{L^2(B_{\frac{\varepsilon}{4}}(\varepsilon z_i))}^2 + \|w_\varepsilon^b - 1\|_{L^2(D_b^\varepsilon \cap D)}^2. \quad (3.24)$$

The function $w_\varepsilon^g - 1$ vanishes on $\bigcup_{z \in n^\varepsilon(D)} \partial B_{\frac{\varepsilon}{4}}(\varepsilon z)$: Since the balls $\{B_{\frac{\varepsilon}{4}}(\varepsilon z)\}_{z \in n^\varepsilon(D)}$ are all disjoint, Poincaré's inequality in each ball $B_{\frac{\varepsilon}{4}}(\varepsilon z)$ yields

$$\|w_\varepsilon^g - 1\|_{L^2(D)}^2 \lesssim \varepsilon^2 \sum_{z \in n^\varepsilon(D)} \|\nabla w_\varepsilon^g\|_{L^2(B_{\frac{\varepsilon}{4}}(\varepsilon z))}^2.$$

Using definitions (3.17), (2.6) and property (3.1) of Lemma 3.1, we may rewrite

$$\|w_\varepsilon^g - 1\|_{L^2(D)}^2 \lesssim \varepsilon^{d+2} \sum_{z \in n^\varepsilon(D)} \rho_z^{d-2},$$

and, inserting this into (3.24), also

$$\|w_\varepsilon - 1\|_{L^2(D)}^2 = \varepsilon^{d+2} \sum_{z \in n^\varepsilon(D)} \rho_z^{d-2} + \|w_\varepsilon^b - 1\|_{L^2(D_b^\varepsilon \cap D)}^2. \quad (3.25)$$

To conclude the proof of (3.21) for $w_\varepsilon - 1$, it thus remains to estimate the last term on the right-hand side. By construction (c.f. (3.16)), it holds $w_\varepsilon^b - 1 = 0$ on ∂D_b^ε ; appealing to Lemma 3.1, we also have that $D_b^\varepsilon \subseteq \{x \in \mathbb{R}^3 : \text{dist}(x, D) \leq 2\}$. We thus apply Poincaré's inequality in this set and conclude that

$$\|w_\varepsilon^b - 1\|_{L^2(D_b^\varepsilon \cap D)}^2 \lesssim \|\nabla w_\varepsilon^b\|_{L^2(D_b^\varepsilon)}^2 \stackrel{(3.23)}{\lesssim} \varepsilon^d \sum_{\Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2}.$$

To establish (3.21) for $w_\varepsilon - 1$, it only remains to combine this last inequality with (3.25).

We now argue (3.21) for $\nabla(w_\varepsilon - W_\varepsilon)$: By definition (2.5) and (3.1) of Lemma 3.1, it holds

$$n^\varepsilon(D) \subseteq \Phi_\delta^\varepsilon(D). \quad (3.26)$$

Thanks to definition (3.15) for w_ε and the fact that, by Lemma 3.1 the support of ∇w_g^ε and ∇w_b^ε is disjoint, we use the triangle inequality to infer that

$$\begin{aligned} \|\nabla(w_\varepsilon - W_\varepsilon)\|_{L^2(D)}^2 &\lesssim \|\nabla(w_g^\varepsilon - W_\varepsilon)\|_{L^2(D)}^2 + \|\nabla w_b^\varepsilon\|_{L^2(D)}^2 \\ &\stackrel{(3.23)}{\lesssim} \|\nabla(w_g^\varepsilon - W_\varepsilon)\|_{L^2(D)}^2 + \varepsilon^d \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2}. \end{aligned} \quad (3.27)$$

Comparing definition (3.17) for w_g^ε with definition (2.7) for W_ε and using inclusion (3.26), we observe that

$$\nabla(w_g^\varepsilon - W_\varepsilon) = \sum_{\Phi_\delta^\varepsilon(D) \setminus n^\varepsilon(D)} \nabla W_\varepsilon \mathbf{1}_{B_{\frac{\varepsilon}{4}}(\varepsilon z)}.$$

Since the balls $\{B_{\frac{\varepsilon}{4}}(\varepsilon z)\}_{z \in \Phi_\delta^\varepsilon(D)}$ are disjoint, the previous identity and the triangle inequality imply that

$$\|\nabla(w_g^\varepsilon - W_\varepsilon)\|_{L^2(D)}^2 \lesssim \sum_{\Phi_\delta^\varepsilon(D) \setminus n^\varepsilon(D)} \|\nabla w_{\varepsilon,z}\|_{L^2(B_{\frac{\varepsilon}{4}}(\varepsilon z))}^2 \stackrel{(2.5)-(2.6)}{=} \varepsilon^d \sum_{\Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2}.$$

Inserting this bound into (3.27) yields (3.21) also for the norm of $\nabla(w_\varepsilon - W_\varepsilon)$.

We now turn to (3.22) and claim that we may apply Lemma 5.1 with $M = \mu_\varepsilon$, $\mathcal{Z} = \{\varepsilon w\}_{w \in \Phi_\delta^\varepsilon(D)}$, $\mathcal{X} = \{\varepsilon^{\frac{d}{d-2}} \rho_w\}_{w \in \Phi_\delta^\varepsilon(D)}$ and $r_w \equiv \frac{\varepsilon}{4}$ for every $w \in \Phi_\delta^\varepsilon(D)$. We use as covering $\{K_j\}_{j \in J}$ the sets $\{Q_{k,z}\}_{z \in N_k}$. Conditions (5.1) and (5.3) are satisfied thanks to (2.5) and by construction (see Subsection 3.1), respectively. Appealing to Lemma 5.1, we therefore have that

$$\|\mu_\varepsilon - m_k\|_{H^{-1}(D)}^2 \lesssim (k\varepsilon)^2 \varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)}, \quad m_k \stackrel{(3.19)}{=} c_d \sum_{z \in N_k} S_{k,z} \mathbf{1}_{Q_{k,z}}.$$

By the triangle inequality and the previous estimate, we thus bound

$$\|\mu_\varepsilon - C_0\|_{H^{-1}(D)}^2 \leq (\varepsilon k)^2 \varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} + \|m_k - C_0\|_{H^{-1}(D)}^2 \tag{3.28}$$

so that, to prove (3.22), it only remains to control the last term on the right-hand side above. We do this by observing that, since $\{Q_{k,z}\}_{z \in N_k}$ is a disjoint covering of D , for each $\phi \in H_0^1(D)$ we have

$$|\langle m_k - C_0; \phi \rangle| \simeq \left| \sum_{z \in N_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}]) \int_{Q_{k,z} \cap D} \phi \right|.$$

By the triangle inequality, also

$$\begin{aligned} |\langle m_k - C_0; \phi \rangle| &\stackrel{(3.12)}{\lesssim} \left| \sum_{z \in \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}]) \int_{Q_{k,z}} \phi \right| \\ &\quad + \left| \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}]) \int_{Q_{k,z} \cap D} \phi \right|. \end{aligned} \tag{3.29}$$

We claim that

$$\begin{aligned} &\left| \sum_{z \in \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}]) \int_{Q_{k,z}} \phi \right| \\ &\lesssim \left(\sum_{z \in \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2 \right)^{\frac{1}{2}} \left(\int_D |\nabla \phi|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.30}$$

This is an easy consequence of the properties of the covering $\{Q_{k,z}\}_{z \in N_k}$ of D , (3.11), together with Cauchy-Schwarz's inequality and Poincaré's inequality for ϕ in D .

We now turn to the second term in (3.29). We note that, by definition (3.12), the set

$$\bigcup_{z \in N_k \setminus \tilde{N}_k} Q_{k,z} \subseteq \{x \in \mathbb{R}^d : \text{dist}(x; \partial D) \leq 4k\varepsilon\}.$$

Since $\phi \in H_0^1(D)$ and D is a smooth and bounded set, we may appeal to Poincaré's inequality in the previous set on the right-hand side to bound

$$\left(\sum_{z \in N_k \setminus \tilde{N}_k} \int_{Q_{k,z}} |\phi|^2 \right)^{\frac{1}{2}} \lesssim (k\varepsilon) \left(\int_D |\nabla \phi|^2 \right)^{\frac{1}{2}}.$$

Appealing once again to Cauchy-Schwarz's inequality and using the above estimate, we control

$$\begin{aligned} & \left| \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}]) \int_{Q_{k,z} \cap D} \phi \right| \\ & \lesssim ((k\varepsilon)^{d+2} \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2)^{\frac{1}{2}} \left(\int_D |\nabla \phi|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, provided $k\varepsilon \lesssim 1$, we may appeal to (3.13) and infer that

$$\begin{aligned} & \left| \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}]) \int_{Q_{k,z} \cap D} \phi \right| \\ & \lesssim ((k\varepsilon)^3 \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2)^{\frac{1}{2}} \left(\int_D |\nabla \phi|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Combining this with (3.30) and (3.29) allows us to infer that for every $\phi \in H_0^1(D)$

$$\begin{aligned} | \langle m_k - C_0; \phi \rangle | & \lesssim ((k\varepsilon)^3 \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2 \\ & + \sum_{z \in \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2)^{\frac{1}{2}} \left(\int_D |\nabla \phi|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

or, equivalently, that

$$\|m_k - C_0\|_{H^{-1}(D)}^2 \lesssim (k\varepsilon)^3 \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2 + \sum_{z \in \tilde{N}_k} (S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2.$$

This, together with (3.28), establishes (3.22). The proof of Lemma 3.2 is complete. \square

Proof of Lemma 3.3. The argument for this lemma is very similar to the one of [18, Theorem 3.1]. Since $f \in W^{1,p}$, $p > d$ and since D is smooth, by standard elliptic regularity we infer that the solution u of (1.4) satisfies $u \in W^{2,\infty}(D)$. By computing the (distributional) Laplacian of $u_\varepsilon - w_\varepsilon u$ we obtain that in D^ε

$$-\Delta(u_\varepsilon - w_\varepsilon u) = (C_0 + \Delta w_\varepsilon)u - 2\nabla \cdot ((1 - w_\varepsilon)\nabla u) + (1 - w_\varepsilon)\Delta u \quad (3.31)$$

We now smuggle the term $(-\Delta w_\varepsilon)u \in H^{-1}(D)$ in the right-hand side so that the previous identity turns into

$$\begin{aligned} & -\Delta(u_\varepsilon - w_\varepsilon u) \\ & = (C_0 + \Delta w_\varepsilon)u - \Delta(W_\varepsilon - w_\varepsilon)u - 2\nabla \cdot ((1 - w_\varepsilon)\nabla u) + (1 - w_\varepsilon)\Delta u \quad \text{in } D_\varepsilon. \end{aligned}$$

We stress that, since $u \in W^{2,+\infty}(D) \cap H_0^1(D)$, $u_\varepsilon \in H_0^1(D^\varepsilon)$, $w_\varepsilon \in H^1(D)$, the above equation holds in the sense that for every $\phi \in H_0^1(D_\varepsilon)$

$$\begin{aligned} \int \nabla \phi \cdot \nabla (u_\varepsilon - w_\varepsilon u) &= \langle C_0 + \Delta W_\varepsilon; u \phi \rangle + \int \nabla (W_\varepsilon - w_\varepsilon) \cdot \nabla (u \phi) \\ &+ 2 \int (1 - w_\varepsilon) \nabla u \cdot \nabla \phi + \int (1 - w_\varepsilon) \Delta u \phi. \end{aligned} \tag{3.32}$$

Since the balls $\{B_{\frac{\varepsilon}{4}}(\varepsilon z)\}_{z \in \Phi_\delta^\varepsilon(D)}$ are all mutually disjoint, by definition (2.7) and equations (2.6) we have that

$$-\Delta W_\varepsilon := \sum_{z \in \Phi_\delta^\varepsilon(D)} \partial_n w_{\varepsilon, z} (\mathbf{1}_{\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)} - \mathbf{1}_{\partial B_{\frac{\varepsilon}{d-2}\rho_z}(\varepsilon z)}).$$

Since $\phi \in H_0^1(D^\varepsilon)$ and therefore it vanishes on the spheres $\{\partial B_{\frac{\varepsilon}{d-2}\rho_z}(\varepsilon z)\}_{z \in \Phi_\delta^\varepsilon(D)}$, the above identity implies that

$$\langle \Delta W_\varepsilon; u \phi \rangle = - \sum_{z \in \Phi_\delta^\varepsilon(D)} \int_{\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)} \partial_n w_{\varepsilon, z} u \phi \stackrel{(3.20)}{=} - \langle \mu_\varepsilon; u \phi \rangle.$$

Inserting this last identity in (3.32), we infer that

$$\begin{aligned} \int \nabla \phi \cdot \nabla (u_\varepsilon - w_\varepsilon u) &= \langle C_0 - \mu_\varepsilon; u \phi \rangle + \int \nabla (W_\varepsilon - w_\varepsilon) \cdot \nabla (u \phi) \\ &+ 2 \int (1 - w_\varepsilon) \nabla u \cdot \nabla \phi + \int (1 - w_\varepsilon) \Delta u \phi. \end{aligned}$$

We now choose $\phi = u_\varepsilon - w_\varepsilon u$ and apply Hölder’s and Poincaré’s inequalities to bound

$$\begin{aligned} &\|u_\varepsilon - w_\varepsilon u\|_{H_0^1(D)}^2 \\ &\lesssim \|u\|_{W^{2,\infty}}^2 (\|w_\varepsilon - 1\|_{L^2(D)}^2 + \|\nabla (w_\varepsilon - W_\varepsilon)\|_{L^2(D)}^2 + \|\mu_\varepsilon - C_0\|_{H^{-1}(D)}^2). \end{aligned}$$

To obtain the claim of Lemma 3.3 it remains to use that, by the triangle inequality and Hölder’s inequality, we have

$$\|\nabla (u_\varepsilon - W_\varepsilon u)\|_{L^2(D)} \leq \|u\|_{W^{2,\infty}} \|W_\varepsilon - w_\varepsilon\|_{H^1(\mathbb{R}^d)} + \|u_\varepsilon - w_\varepsilon u\|_{H_0^1(D)}$$

and that, by definitions (2.7) and (3.15), the difference $W_\varepsilon - w_\varepsilon$ is compactly supported in $\{x \in \mathbb{R}^d : \text{dist}(x; \partial D) \leq 4\}$ (see also Lemma 3.1). \square

3.3. Annealed estimates (Proof of Theorem 2.1, (a)). In this subsection we rely on the quenched estimate of Lemma 3.2 to prove the statement of Theorem 2.1 in the case of periodic centres. The first ingredient is the following annealed bound:

Lemma 3.4. *Let (Φ, \mathcal{R}) satisfy the assumptions of Theorem 2.1, (a). For every $\delta \in (0, \frac{2}{d-2}]$, let $n^\varepsilon(D) \subseteq \Phi^\varepsilon(D)$ be the random subset constructed in Lemma 3.1. Then*

$$\mathbb{E} \left[\varepsilon^d \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2} \right] \lesssim \varepsilon^{(\frac{2}{d-2} - \delta)\beta}.$$

Proof of Theorem 2.1, (a). By the assumptions on D , we may assume that for $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $\varepsilon k \lesssim 1$, the cube $Q_{k,0} \subseteq D$. We restrict to the values of $k \in \mathbb{N}$ satisfying the previous bound.

Combining Lemma 3.2 and Lemma 3.4, we bound for every $\varepsilon > 0$ and $k \in \mathbb{N}$ as above

$$\begin{aligned} \mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2] &\lesssim (k\varepsilon)^2 \mathbb{E}[\varepsilon^d \sum_{z \in \Phi_\varepsilon^\delta(D)} \rho_z^{2(d-2)}] + \varepsilon^2 \mathbb{E}[\varepsilon^d \sum_{z \in \Phi_\varepsilon^\delta(D)} \rho_z^{d-2}] \\ &\quad + \mathbb{E}[\sum_{z \in \dot{N}_k} (S_{k,z} - \mathbb{E}[\rho^{d-2}])^2] \\ &\quad + (\varepsilon k)^3 \mathbb{E}[\sum_{z \in N_k \setminus \dot{N}_k} (S_{k,z} - \mathbb{E}[\rho^{d-2}])^2] + \varepsilon^{(\frac{2}{d-2}-\delta)\beta}. \end{aligned}$$

Since the sets N_k, \dot{N}_k are deterministic and $\{S_{k,z}\}_{z \in \dot{N}_k}$ are identically distributed, we infer that

$$\begin{aligned} \mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2] &\lesssim (k\varepsilon)^2 \mathbb{E}[\varepsilon^d \sum_{z \in \Phi_\varepsilon^\delta(D)} \rho_z^{2(d-2)}] + \varepsilon^2 \mathbb{E}[\varepsilon^d \sum_{z \in \Phi_\varepsilon^\delta(D)} \rho_z^{d-2}] \\ &\quad + \mathbb{E}[(S_{k,0} - \mathbb{E}[\rho^{d-2}])^2] + (\varepsilon k)^3 \sum_{z \in N_k \setminus \dot{N}_k} \mathbb{E}[(S_{k,z} - \mathbb{E}[\rho^{d-2}])^2] \\ &\quad + \varepsilon^{(\frac{2}{d-2}-\delta)\beta}, \end{aligned}$$

We observe that, by (2.5), (2.1) and the inequality $N^\varepsilon(D) \lesssim \varepsilon^{-d}$ (c.f. (2.3)), we have that

$$\mathbb{E}[\varepsilon^d \sum_{z \in \Phi_\varepsilon^\delta(D)} \rho_z^{2(d-2)}] + \mathbb{E}[\varepsilon^d \sum_{z \in \Phi_\varepsilon^\delta(D)} \rho_z^{d-2}] \lesssim \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2}+\delta}}] + 1.$$

Since $k \geq 1$, the previous two displays thus imply

$$\begin{aligned} \mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2] &\lesssim (k\varepsilon)^2 \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2}+\delta}}] + \mathbb{E}[(S_{k,0} - \mathbb{E}[\rho^{d-2}])^2] \\ &\quad + (\varepsilon k)^3 \sum_{z \in N_k \setminus \dot{N}_k} \mathbb{E}[(S_{k,z} - \mathbb{E}[\rho^{d-2}])^2] + \varepsilon^{(\frac{2}{d-2}-\delta)\beta}. \end{aligned} \tag{3.33}$$

We now claim that for every $k \in \mathbb{N}$ such that $\varepsilon k \leq C_1$ with C_1 as in (3.13), then

$$\begin{aligned} \mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2] &\lesssim (k\varepsilon)^2 \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2}+\delta}}] + k^{-d} \text{Var}(Y_{\varepsilon,0} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2}+\delta}}) + \varepsilon^{(\frac{2}{d-2}-\delta)\beta}, \end{aligned} \tag{3.34}$$

where $Y_{\varepsilon,0}$ is defined as in (3.19). We begin by showing how to conclude the proof of the theorem provided (3.34) holds.

Let us first assume that (2.1) holds with $\beta \geq d - 2$; in this case, we have that

$$\mathbb{E}[\rho^{2(d-2)}] + \mathbb{E}[Y_{\varepsilon,0}^2] \lesssim 1$$

and therefore that

$$\mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2] \lesssim (k\varepsilon)^2 + k^{-d} + \varepsilon^{(\frac{2}{d-2}-\delta)\beta}.$$

Estimate of Theorem 2.1 for $\beta \geq d - 2$ follows from this inequality if we minimize the right-hand side above in k , i.e. if we choose $k = \lfloor \varepsilon^{-\frac{2}{d-2}} \rfloor$, and set δ as in Theorem 2.1.

Let us now assume that $\beta < d - 2$ in (2.1): In this case, we bound

$$\text{Var}(Y_{\varepsilon,0} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2}+\delta}}) + \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2}+\delta}}] \lesssim \varepsilon^{-(\frac{2}{d-2}-\delta)(d-2-\beta)}$$

so that (3.34) turns into

$$\mathbb{E}[\|u_\varepsilon - W_\varepsilon u\|_{H_0^1(D)}^2] \lesssim ((k\varepsilon)^2 + k^{-d})\varepsilon^{-(\frac{d}{d-2}-\delta)(d-2-\beta)} + \varepsilon^{(\frac{d}{d-2}-\delta)\beta}.$$

Also in this case, we infer the estimate of Theorem 2.1 by minimizing the right-hand side in k and δ , i.e. choosing $k = \lfloor \varepsilon^{-\frac{2}{d+2}} \rfloor$ and δ as in Theorem 2.1.

To complete the proof of the theorem it only remains to argue (3.34) from (3.33). We first tackle the second term on the right-hand side of (3.33) and show that

$$\mathbb{E}[(S_{k,0} - \mathbb{E}[\rho^{d-2}])^2] \lesssim k^{-d} \text{Var}(Y_{w,\varepsilon} \mathbf{1}_{\rho < \varepsilon^{-\frac{d}{d-2}+\delta}}) + \varepsilon^{(\frac{d}{d-2}-\delta)\beta}. \quad (3.35)$$

This may be done after noticing that the left-hand side may be written, up to an error, as the sum of $(2k+1)^d$ centred and independent random variables: Definitions (2.5) and (3.14) for $\Phi_\delta^\varepsilon(D)$ and $N_{k,z}$ imply that

$$N_{k,z} = \{w \in \mathbb{Z}^d : \varepsilon w \in Q_{k,z} \cap D, \varepsilon^{\frac{d}{d-2}} \rho_w < \varepsilon^{1+\delta}\}. \quad (3.36)$$

Since $0 \in \mathring{N}_k$, this, (3.19) and the triangle inequality allows us to bound

$$\begin{aligned} \mathbb{E}[(S_{k,0} - \mathbb{E}_\rho[\rho^{d-2}])^2] &\lesssim \mathbb{E}[(\sum_{\substack{w \in \mathbb{Z}^d \\ \varepsilon w \in Q_{k,0}}} Y_{w,\varepsilon} \mathbf{1}_{\rho_w < \varepsilon^{-\frac{d}{d-2}+\delta}} - \mathbb{E}_\rho[\rho^{d-2} \mathbf{1}_{\rho < \varepsilon^{-\frac{d}{d-2}+\delta}}])^2] \\ &\quad + \mathbb{E}_\rho[\rho^{d-2} \mathbf{1}_{\rho \geq \varepsilon^{-\frac{d}{d-2}+\delta}}]^2. \end{aligned} \quad (3.37)$$

Thanks to Chebyshev's inequality and assumption (2.1) we have

$$\mathbb{E}_\rho[\rho^{d-2} \mathbf{1}_{\rho \geq \varepsilon^{-\frac{d}{d-2}+\delta}}] \lesssim \varepsilon^{(\frac{d}{d-2}-\delta)\beta}, \quad (3.38)$$

and thus we may rewrite (3.37) as

$$\begin{aligned} \mathbb{E}[(S_{k,0} - \mathbb{E}_\rho[\rho^{d-2}])^2] &\lesssim \mathbb{E}[(\sum_{\substack{w \in \mathbb{Z}^d \\ \varepsilon w \in Q_{k,0}}} Y_{w,\varepsilon} \mathbf{1}_{\rho_w < \varepsilon^{-\frac{d}{d-2}+\delta}} - \mathbb{E}_\rho[\rho^{d-2} \mathbf{1}_{\rho < \varepsilon^{-\frac{d}{d-2}+\delta}}])^2] \\ &\quad + \varepsilon^{2(\frac{d}{d-2}-\delta)\beta}. \end{aligned}$$

Since

$$\begin{aligned} |\mathbb{E}_\rho[Y_{0,\varepsilon} \mathbf{1}_{\rho_0 < \varepsilon^{-\frac{d}{d-2}+\delta}}] - \mathbb{E}_\rho[\rho^{d-2} \mathbf{1}_{\rho < \varepsilon^{-\frac{d}{d-2}+\delta}}]| &\lesssim \varepsilon^2 \mathbb{E}_\rho[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{d}{d-2}+\delta}}] \\ &\stackrel{(2.1)}{\lesssim} \varepsilon^{(\frac{d}{d-2}-\delta)\beta}, \end{aligned}$$

the independence of the random variables $\{\rho_z\}_{z \in \Phi}$, and the fact that $N^\varepsilon(Q_{k,0}) = (2k+1)^d$ (c.f. (2.3)), allows us to obtain (3.35), by means of standard CLT arguments.

We now turn to the remaining term in (3.33) and argue that

$$(\varepsilon k)^3 \sum_{z \in N_k \setminus \mathring{N}_k} \mathbb{E}[(S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2] \lesssim (k\varepsilon)^2 \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{d}{d-2}+\delta}}]. \quad (3.39)$$

By the triangle inequality and assumption (2.1), the left-hand side is bounded by

$$(\varepsilon k)^3 \sum_{z \in N_k \setminus \mathring{N}_k} \mathbb{E}[(S_{k,z} - \mathbb{E}_\rho[\rho^{d-2}])^2] \lesssim (\varepsilon k)^3 + (\varepsilon k)^3 \sum_{z \in N_k \setminus \mathring{N}_k} \mathbb{E}[S_{k,z}^2]. \quad (3.40)$$

To establish (3.39) from this it suffices to remark that, by (3.19) and (3.36), we have

$$S_{k,z}^2 \lesssim \sum_{\substack{w \in \mathbb{Z}^d \\ \varepsilon w \in Q_{k,z} \cap D}} \rho_w^{2(d-2)} \mathbf{1}_{\rho_w < \varepsilon^{-\frac{d}{d-2}+\delta}}$$

so that this, and the fact that the random variables $\{\rho_z\}_{z \in \Phi^\varepsilon(D)}$ are identically distributed, yields

$$\sum_{z \in N_k \setminus \dot{N}_k} \mathbb{E}[S_{k,z}^2] \lesssim \mathbb{E}[\mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \rho^{2(d-2)}].$$

Inserting this into (3.40) implies (3.39). To establish (3.34) it remains to combine (3.39), (3.35) and (3.33). The proof of Theorem 2.1, (a) is complete. \square

Proof of Lemma 3.4. We resort to the construction of the set $n^\varepsilon(D)$ implemented in Lemma 3.1: By (3.6), (3.4) and (3.5) in the proof of Lemma 3.1 we decompose

$$\varepsilon^d \sum_{\Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_w^{d-2} = \varepsilon^d \sum_{z \in J_b^\varepsilon} \rho_z^{d-2} + \varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2}. \quad (3.41)$$

and prove the statement of Lemma 3.4 for each one of the two sums. We begin with the first one: Using (3.4) we write

$$\varepsilon^d \sum_{z \in J_b^\varepsilon} \rho_z^{d-2} = \varepsilon^d \sum_{z \in \Phi^\varepsilon(D)} \rho_z^{d-2} \mathbf{1}_{\rho_z \geq \varepsilon^{-\frac{2}{d-2} + \delta}}.$$

Taking the expectation and using that $\{\rho_z\}_{\Phi^\varepsilon(D)}$ are identically distributed and that $N^\varepsilon(D) \lesssim \varepsilon^{-d}$, we immediately bound

$$\varepsilon^d \mathbb{E} \left[\sum_{z \in J_b^\varepsilon} \rho_z^{d-2} \right] \lesssim \mathbb{E} \rho [\rho^{d-2} \mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}}] \stackrel{(3.38)}{\lesssim} \varepsilon^{(d-2-\delta)\beta}, \quad (3.42)$$

i.e. the claim of Lemma 3.4 for the first sum in (3.41).

We now turn to the second sum in (3.41): By definition (3.5), if $z \in \tilde{I}_b^\varepsilon$, then $\rho_z \leq \varepsilon^{-\frac{2}{d-2} + \delta}$ and there exists an element $w \in J_b^\varepsilon$ such that $\varepsilon|z-w| \lesssim \varepsilon + (\varepsilon^{\frac{d}{d-2}} \rho_w \wedge 1)$. This allows us to bound

$$\begin{aligned} \varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2} &\leq \varepsilon^d \sum_{w \in J_b^\varepsilon} \sum_{\substack{z \in \Phi^\varepsilon(D) \setminus \{w\}, \\ \varepsilon|z-w| \lesssim \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho_w \wedge 1}} \rho_z^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \\ &= \varepsilon^d \sum_{w \in \Phi^\varepsilon(D)} \mathbf{1}_{\rho_w \geq \varepsilon^{-\frac{2}{d-2} + \delta}} \sum_{\substack{z \in \Phi^\varepsilon(D) \setminus \{w\}, \\ \varepsilon|z-w| \lesssim \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho_w \wedge 1}} \rho_z^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}}. \end{aligned}$$

We now take the expectation and use that $\Phi = \mathbb{Z}^d$ and that $\{\rho_z\}_{z \in \Phi}$ are independent and identically distributed: This implies that

$$\begin{aligned} \mathbb{E} \left[\varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2} \right] &\lesssim \mathbb{E} \left[\varepsilon^d \sum_{w \in \Phi^\varepsilon(D)} \mathbf{1}_{\rho_w \geq \varepsilon^{-\frac{2}{d-2} + \delta}} \#\{z \in \Phi^\varepsilon(D) \setminus \{w\} : \varepsilon|z-w| \lesssim \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho_w \wedge 1\} \right]. \end{aligned}$$

Since for every $w \in J_b^\varepsilon$, the set

$$\#\{z \in \Phi^\varepsilon(D) \setminus \{w\} : \varepsilon|z-w| < \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho_w \wedge 1\} \lesssim 1 + \rho_w^{d-2},$$

we obtain that

$$\begin{aligned} \mathbb{E}\left[\varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2}\right] &\lesssim \mathbb{E}\left[\varepsilon^d \sum_{w \in \Phi^\varepsilon(D)} \rho_w^{d-2} \mathbf{1}_{\rho_w \geq \varepsilon^{-\frac{2}{d-2} + \delta}}\right] \stackrel{(3.4)}{=} \mathbb{E}\left[\varepsilon^d \sum_{w \in J_b^\varepsilon} \rho_w^{d-2}\right] \\ &\stackrel{(3.42)}{\lesssim} \varepsilon^{(\frac{2}{d-2} - \delta)\beta}. \end{aligned}$$

This, together with identities (3.41) and (3.42), establishes Lemma 3.4. □

4. Proof of Theorem 2.1, (b). In this section we adapt the argument of the previous section to Theorem 2.1 in case (b). As mentioned in Subsection 2.2, the main challenge is related to the construction of a mesoscopic covering $\{K_{k,z}\}_{z \in N_k}$ that plays the same role of $\{Q_{k,z}\}_{z \in N_k}$ of Subsection 3.1. In the present case, the random positions of the centres imply that (with positive probability) there are configurations in which some of the spheres $\{\partial B_{\frac{\varepsilon}{4}}(\varepsilon z)\}_{z \in \Phi}$ intersect the boundary of $\{Q_{k,z}\}_{z \in N_k}$. This prevents us from appealing to Lemma 5.1 as condition (5.3) is violated.

All the results contained in this section besides hold for any dimension $d \geq 3$. However, in the proof of Theorem 2.1, (b) we obtain the same decay rate of case (a) only in $d = 3$. In higher dimensions we obtain a slower (but still algebraic) rate. In order to best appreciate this dimensional constraint, in the whole section we work in a general dimension $d \geq 3$.

Throughout this section we set δ as in Theorem 2.1 and define the parameters

$$k := \lfloor \varepsilon^{-\frac{2}{d+2}} \rfloor, \quad \kappa := \frac{2}{(d-1)(d+2)}. \tag{4.1}$$

4.1. Partition of the holes and mesoscopic covering of D . This subsection contains an adaptation to the case of random centres of Lemma 3.1 and of the sets $\{Q_{k,z}\}_{z \in N_k}$.

Lemma 4.1. *Let δ be as in Theorem 2.1. We recall the definition (2.4) of $R_{\varepsilon,z}$. For $\omega \in \Omega$, we consider a realization of the marked point process $(\Phi; \mathcal{R})$ and of the associated set of holes H^ε . Then, there exists a partition*

$$H^\varepsilon := H_g^\varepsilon \cup H_b^\varepsilon,$$

with the following properties:

- There exists a subset of centres $n^\varepsilon(D) \subseteq \Phi^\varepsilon(D)$ such that

$$H_g^\varepsilon := \bigcup_{z \in n^\varepsilon(D)} B_{\varepsilon^{\frac{d}{d-2}} \rho_z}(\varepsilon z), \quad \min_{z \in n^\varepsilon(D)} R_{\varepsilon,z} \geq \varepsilon^2, \quad \max_{z \in n^\varepsilon(D)} \varepsilon^{\frac{d}{d-2}} \rho_z \leq \varepsilon^{1+\delta},$$

and such that $2\sqrt{d}\varepsilon^{\frac{d}{d-2}} \rho_z \leq R_{\varepsilon,z}$, for every $z \in n^\varepsilon(D)$.

- There exists a set $D_b^\varepsilon(\omega) \subseteq \{x \in \mathbb{R}^d : \text{dist}(x, D) \leq 2\}$ satisfying

$$H_b^\varepsilon \subseteq D_b^\varepsilon, \quad \text{Cap}(H_b^\varepsilon, D_b^\varepsilon) \lesssim \varepsilon^{\frac{d}{d-2}} \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2}$$

and for which $B_{R_{\varepsilon,z}}(\varepsilon z) \cap D_b^\varepsilon = \emptyset$, for every $z \in n^\varepsilon(D)$.

Proof of Lemma 4.1. The construction for the sets $H_g^\varepsilon, H_b^\varepsilon$ and D_b^ε is analogous to the one implemented in the proof of Lemma 3.1, with the only difference that in this case, we set

$$J_b^\varepsilon = \Phi^\varepsilon(D) \setminus \Phi_\delta^\varepsilon(D) = \left\{ z \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_z \geq \varepsilon^{1+\delta} \text{ OR } R_{\varepsilon,z} \leq \varepsilon^2 \vee \varepsilon^{\frac{d}{d-2}} \rho_z \right\}$$

and in the definition (3.5) of \tilde{I}_b^ε the ball $B_{\frac{\varepsilon}{4}}(\varepsilon z)$ is replaced by $B_{R_{\varepsilon,z}}(\varepsilon z)$. □

For k as in (4.1), let $\{Q_{k,z}\}_{z \in N_k}$ be as in Subsection 3.1. For every $z \in N_k$ we define the sets $N_{k,z}$ as in (3.14). We stress that, in this case, (3.14) is ill-defined for the realizations of Φ such that there are points in $\Phi_\delta^\varepsilon(D)$ that fall on the boundary of the cubes $\{Q_{k,z}\}_{z \in N_k}$. This issue may be easily solved by fixing a deterministic rule to assign these points to a particular cube that shares the boundary considered. We stress that all the arguments of this section do not depend on this rule since the set of the boundaries of the covering $\{Q_{k,z}\}_{z \in N_k}$ has zero (Lebesgue)-measure.

For $z \in N_k$ and $w \in N_{k,z}$, we define the modification of the minimal distance $R_{\varepsilon,w}$:

$$\tilde{R}_{\varepsilon,w} := \begin{cases} R_{\varepsilon,w} & \text{if } \varepsilon w \in Q_{z,k-1} \\ \varepsilon^{1+\kappa} \wedge R_{\varepsilon,w} & \text{if } \text{dist}(\varepsilon w; \partial Q_{z,k}) \leq \varepsilon^{1+\kappa} \\ (2^{n-1} \varepsilon^{1+\kappa}) \wedge R_{\varepsilon,w} & \text{if } \varepsilon w \notin Q_{z,k-1}, \quad 2^{n-1} \varepsilon^{1+\kappa} \leq \text{dist}(\varepsilon w; \partial Q_{z,k}) \\ & \leq 2^n \varepsilon^{1+\kappa}. \end{cases} \tag{4.2}$$

We aim at obtaining a (random) collection of disjoint sets $\{K_{k,z}\}_{z \in N_k}$ having size $\simeq \varepsilon(2k+1)$ and such that for every $z \in N_k$ and $w \in \Phi_\delta^\varepsilon(D)$

$$B_{\tilde{R}_{\varepsilon,z}}(\varepsilon w) \cap K_{k,z} = \emptyset \quad \text{OR} \quad B_{2\tilde{R}_{\varepsilon,z}}(\varepsilon w) \subseteq K_{k,z}.$$

We modify $\{Q_{k,z}\}_{z \in N_k}$ as follows: For κ as in (4.1), any $z \in N_k$ and $w \in N_{k,z}$, we consider the cubes

$$\tilde{Q}_{\varepsilon,w} := \varepsilon w + 2[-\tilde{R}_{\varepsilon,z}; \tilde{R}_{\varepsilon,z}].$$

Note that, by definition (2.4), all the cubes above are disjoint. For every $z \in N_k$, we thus set (see Figure (1))

$$K_{k,z} := (Q_{k,z} \bigcup_{w \in N_{k,z}} \tilde{Q}_{\varepsilon,w}) \setminus \bigcup_{w \in \Phi_\delta^\varepsilon(D) \setminus N_{k,z}} \tilde{Q}_{\varepsilon,w}. \tag{4.3}$$

Since the cubes $\{Q_{\varepsilon,z}\}_{z \in \Phi_\delta^\varepsilon(D)}$ are disjoint we have that

$$\begin{aligned} \bigcup_{z \in N_k} K_{k,z} \supseteq D, \quad |diam(K_{k,z})| \lesssim k\varepsilon, \\ (2k+1 - \varepsilon^\kappa)^d \varepsilon^d \leq |K_{k,z}| \leq (2k+1 + \varepsilon^\kappa)^d \varepsilon^d. \end{aligned} \tag{4.4}$$

We emphasize that the previous properties hold for every realization of the point process Φ . The introduction of the modified random variable $\tilde{R}_{\varepsilon,z}$ is needed to ensure that the second property in (4.4) holds with ε^κ instead of 1. This yields that the difference between the volume of the set $K_{k,z}$ and the cube $Q_{k,z}$ is of order $\varepsilon^{d+\kappa} k^{d-1}$ instead of $\varepsilon^d k^{d-1}$. This condition plays a crucial role in the proof of the theorem (see (4.14)) and is the main term that forces the dimensional constraint $d = 3$ in the rates of convergence.

4.2. Quenched estimates for the homogenization error. In this section we adapt Lemma 3.2 to the current setting. As in the case of Lemma 3.2, the next result relies on a variation of Lemma 3.3 that allows us to replace in the definition (3.20) of μ_ε the radii $\frac{\varepsilon}{4}$ with $\tilde{R}_{\varepsilon,z}$ defined in (4.2).

We define the oscillating test function $w_\varepsilon \in H^1(D)$ as done in Subsection 3.2, this time using the sets $H_b^\varepsilon, H_g^\varepsilon$ and D_b^ε of Lemma 4.1 with δ as in Theorem 2.1, and $R_{\varepsilon,z}$ instead of $\frac{\varepsilon}{4}$ in (3.17). We also define the analogues of (3.19), this time

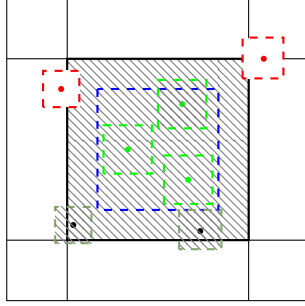


FIGURE 1. The construction of $K_{\epsilon, z}$ from the cube $Q_{k, \epsilon}$. The dashed grey area corresponds to the set $K_{\epsilon, z}$, while $Q_{\epsilon, z}$ is the square bounded by the thick black line. The green dots are the points of Φ_{δ}^{ϵ} that fall inside the set $Q_{k-1, z}$ (here bounded by the dashed blue line). The red dots are the points that are outside of $Q_{k, z}$ but whose associated cube intersects $\partial Q_{k, z}$. The black dots are the points that are in $Q_{k, z} \setminus Q_{k-1, z}$. Note that the cubes associated to the black and red dots are typically smaller than the ones associated to the green dots due to the cut-off $\tilde{R}_{\epsilon, z}$.

associated to the covering $\{K_{k, z}\}_{z \in N_k}$ constructed in the previous subsection: For every $z \in N_k$ we indeed set

$$S_{k, z} := \frac{\epsilon^d}{|K_{k, z}|} \sum_{w \in N_{k, z}} Y_{\epsilon, w}, \quad Y_{\epsilon, w} := \rho_w^{d-2} \frac{\tilde{R}_{\epsilon, w}^{d-2}}{\tilde{R}_{\epsilon, w}^{d-2} - \epsilon^d \rho_w^{d-2}}. \quad (4.5)$$

Lemma 4.2. *Let W_{ϵ} be as in (2.7) and let u_{ϵ}, u as in Theorem 2.1. Then, for every $\epsilon > 0$ and $k \in \mathbb{N}$ such that $\epsilon k \lesssim 1$ we have that*

$$\begin{aligned} & \|u_{\epsilon} - W_{\epsilon} u\|_{H_0^1(D)} \\ & \lesssim \left(\epsilon^{d+2} \sum_{z \in n_{\epsilon}(D)} \rho_z^{d-2} + (k\epsilon)^2 \epsilon^d \sum_{z \in \Phi_{\delta}^{\epsilon}(D)} \rho_z^{2(d-2)} \left(\frac{\epsilon}{\tilde{R}_{\epsilon, z}}\right)^d + \epsilon^d \sum_{z \in \Phi^{\epsilon}(D) \setminus n_{\epsilon}(D)} \rho_z^{d-2} \right)^{\frac{1}{2}} \\ & + \left(\sum_{z \in \tilde{N}_k} (S_{k, z} - \lambda \mathbb{E}[\rho^{d-2}])^2 + (k\epsilon)^3 \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k, z} - \lambda \mathbb{E}_{\rho}[\rho^{d-2}])^2 \right)^{\frac{1}{2}} \\ & + \left(\epsilon^{d+\frac{2d}{d+2}} \sum_{z \in N_k} \sum_{\substack{w \in N_{k, z} \\ \epsilon w \in Q_{k, z} \setminus Q_{k-1, z}}} \rho_w^{2(d-2)} \right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 4.3. *Let u_{ϵ}, u and W_{ϵ} be as in Lemma 3.2 and let w_{ϵ} be as defined above. Then*

$$\begin{aligned} \|u_{\epsilon} - W_{\epsilon} u\|_{H_0^1(D)}^2 & \lesssim \|w_{\epsilon} - 1\|_{L^2(D)}^2 + \|\nabla(w_{\epsilon} - W_{\epsilon})\|_{L^2(\mathbb{R}^d)}^2 \\ & + \|\nabla(\tilde{W}_{\epsilon} - W_{\epsilon})\|_{L^2(\mathbb{R}^d)}^2 + \|\mu_{\epsilon} - C_0\|_{H^{-1}(D)}^2, \end{aligned}$$

where \tilde{W}_ε is defined as in (2.7) with $R_{\varepsilon,z}$ substituted by $\tilde{R}_{\varepsilon,z}$. Furthermore, in this case

$$\mu_\varepsilon := \sum_{z \in \Phi_\delta^\varepsilon(D)} \partial_n \tilde{w}_{\varepsilon,z} \delta_{\partial B_{\tilde{R}_{\varepsilon,z}}(\varepsilon z)},$$

with $\tilde{w}_{\varepsilon,z}$ as in (2.6) with $\tilde{R}_{\varepsilon,z}$ instead of $R_{\varepsilon,z}$.

Proof of Lemma 4.2. Analogously to the proof of Lemma 3.2, we appeal to Lemma 4.3 and reduce to showing that

$$\begin{aligned} \|\nabla(W_\varepsilon - w_\varepsilon)\|_{L^2(D)}^2 + \|w_\varepsilon - 1\|_{L^2(D)}^2 & \quad (4.6) \\ & \lesssim \varepsilon^{d+2} \sum_{z \in n_\varepsilon(D)} \rho_z^{d-2} + \varepsilon^d \sum_{\Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2}, \end{aligned}$$

$$\|\nabla(\tilde{W}_\varepsilon - W_\varepsilon)\|_{L^2(\mathbb{R}^d)}^2 \lesssim \varepsilon^{d+\frac{2d}{d+2}} \sum_{z \in \tilde{N}_k} \sum_{\substack{w \in N_{k,z}, \\ \varepsilon w \in Q_{k,z} \setminus Q_{k-1,z}}} \rho_w^{2(d-2)} \quad (4.7)$$

and

$$\begin{aligned} \|\mu_\varepsilon - C_0\|_{H^{-1}(D)}^2 & \lesssim (k\varepsilon)^2 \varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \\ & + (k\varepsilon)^3 \sum_{z \in N_k \setminus \tilde{N}_k} (S_{k,z} - \lambda \mathbb{E}_\rho[\rho^{d-2}])^2 + \sum_{z \in \tilde{N}_k} (S_{k,z} - \lambda \mathbb{E}_\rho[\rho^{d-2}])^2. \end{aligned} \quad (4.8)$$

Inequality (4.6) may be argued exactly as done for (3.21) in the proof of Lemma 3.2, this time appealing to Lemma 4.1 instead of Lemma 3.1.

We thus turn to (4.7). We begin by remarking that \tilde{W}_ε is well-defined: Indeed, by definition (2.4) and (4.2), we have that $\tilde{R}_{\varepsilon,z} \leq R_{\varepsilon,z} \leq \frac{\varepsilon}{4}$ for every $z \in \Phi_\delta^\varepsilon(D)$. Furthermore, since $\kappa < \delta$ (c.f. (4.1) and Theorem 2.1), it follows from (2.5) that $2\varepsilon^{\frac{d}{d-2}} \rho_z \leq \tilde{R}_{\varepsilon,z}$, for every $z \in \Phi_\delta^\varepsilon(D)$. Therefore, comparing the two definitions of W_ε and \tilde{W}_ε , we use (4.2) to bound:

$$\|\nabla(\tilde{W}_\varepsilon - W_\varepsilon)\|_{L^2}^2 = \sum_{z \in \tilde{N}_k} \sum_{\substack{w \in N_{k,w}, \\ \varepsilon w \in Q_k \setminus Q_{k-1}}} \|\nabla(\tilde{W}_\varepsilon - W_\varepsilon)\|_{L^2(B_{R_{\varepsilon,w}}(\varepsilon w))}^2. \quad (4.9)$$

Since, if $R_{\varepsilon,w} \neq \tilde{R}_{\varepsilon,w}$, then $\varepsilon^{1+\kappa} \leq \tilde{R}_{\varepsilon,w} \leq R_{\varepsilon,w}$, we have that

$$\begin{aligned} \|\nabla(\tilde{W}_\varepsilon - W_\varepsilon)\|_{L^2(B_{R_{\varepsilon,w}}(\varepsilon w))}^2 & \leq \int_{B_{R_{\varepsilon,w}}(\varepsilon w) \setminus B_{\varepsilon^{1+\kappa}}(\varepsilon w)} |\nabla W_\varepsilon|^2 \\ & + \int_{B_{\tilde{R}_{\varepsilon,w}}(\varepsilon w) \setminus B_{\varepsilon^{\frac{d}{d-2}} \rho_w}(\varepsilon w)} |\nabla(W_\varepsilon - \tilde{W}_\varepsilon)|^2. \end{aligned}$$

Appealing to (2.7), (2.6) and the adaptation of (3.18) for both \tilde{W}_ε and W_ε , the previous integrals may be bounded by

$$\|\nabla(\tilde{W}_\varepsilon - W_\varepsilon)\|_{L^2(B_{R_{\varepsilon,w}}(\varepsilon w))}^2 \lesssim \varepsilon^d \rho_w^{2(d-2)} \varepsilon^{2-(d-2)\kappa} + \varepsilon^d \rho_w^{3(d-2)} \varepsilon^{2-(d-2)\kappa}$$

Since $w \in \Phi_\delta^\varepsilon(D)$, we have that $\rho_w \leq \varepsilon^{-\frac{2}{d-2}+\delta}$ so that

$$\begin{aligned} \|\nabla(\tilde{W}_\varepsilon - W_\varepsilon)\|_{L^2(B_{R\varepsilon,w}(\varepsilon w))}^2 &\lesssim \varepsilon^d \rho_w^{2(d-2)} \varepsilon^{2-(d-2)\kappa} + \varepsilon^d \rho_w^{2(d-2)} \varepsilon^{2-2(d-2)\kappa+(d-2)\delta} \\ &\lesssim_{\delta>\kappa} \varepsilon^d \rho_w^{2(d-2)} \varepsilon^{2-(d-2)\kappa}. \end{aligned}$$

Inserting this into (4.9) and appealing to (4.1) for κ yields (4.7).

We finally tackle (4.8): As done for (3.22) of Lemma 3.2, we aim at applying Lemma 5.1. We thus pick $\mathcal{Z} = \Phi_\delta^\varepsilon(D)$ and $\mathcal{X} = \{\varepsilon^{\frac{d}{d-2}} \rho_z\}_{z \in \mathcal{Z}}$, $\mathcal{R} := \{\tilde{R}_{\varepsilon,z}\}_{z \in \mathcal{Z}}$. As shown above before the argument for (4.7), condition (5.1) is satisfied. Moreover, thanks to (4.3), the collection $\{K_{k,z}\}_{z \in N_k}$ satisfies (5.3). Hence, by Lemma 5.1, we have that

$$\begin{aligned} \|\mu_\varepsilon - m_k\|_{H^{-1}} &\lesssim \max_{z \in N_k} (\text{diam}(K_{k,z})) \left(\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right)^{\frac{1}{2}} \\ &\stackrel{(4.4)}{\lesssim} \varepsilon k \left(\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right)^{\frac{1}{2}} \end{aligned}$$

where, thanks to (4.5), we have that $m_k = c_d \sum_{z \in N_k} S_{k,z} \mathbf{1}_{K_{k,z}}$. By the triangle inequality it thus only remains to control the norm $\|m_k - C_0\|_{H^{-1}(D)}$. Using (4.4), this may be done exactly as in the proof of Lemma 3.2. The proof of Lemma 4.2 is complete. \square

Proof of Lemma 4.3. This lemma may be argued as done for Lemma 3.3. The only difference is that, in (3.31), we smuggle in $-\Delta \tilde{W}_\varepsilon$ instead of $-\Delta W_\varepsilon$ and apply the triangle inequality to bound $\|\nabla(\tilde{W}_\varepsilon - w_\varepsilon)\|_{L^2} \leq \|\nabla(W_\varepsilon - w_\varepsilon)\|_{L^2} + \|\nabla(\tilde{W}_\varepsilon - W_\varepsilon)\|_{L^2}$. \square

4.3. Annealed estimates (Proof of Theorem 2.1, (b)). As in case (a), the next lemma provides annealed bounds for some of the quantities appearing in the right-hand side of Lemma 4.2.

Lemma 4.4. *Let $n^\varepsilon(D) \subseteq \Phi^\varepsilon(D)$ the (random) subset constructed in Lemma 4.1. Then*

$$\mathbb{E} \left[\varepsilon^d \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z^{d-2} \right] \lesssim \varepsilon^{(\frac{2}{d-2}-\delta)\beta} + \varepsilon^2.$$

Proof of Theorem 2.1, (b). We recall that k satisfies (4.1). Combining Lemma 4.2 and Lemma 4.4, we bound

$$\begin{aligned} \mathbb{E} [\|u_\varepsilon - \tilde{W}_\varepsilon u\|_{H_0^1(D)}^2] &\lesssim (k\varepsilon)^2 \mathbb{E} \left[\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right] \\ &\quad + \mathbb{E} \left[\sum_{z \in \hat{N}_k} (S_{k,z} - \lambda \mathbb{E}[\rho^{d-2}])^2 \right] \\ &\quad + (k\varepsilon)^3 \mathbb{E} \left[\sum_{z \in N_k \setminus \hat{N}_k} (S_{k,z} - \lambda \mathbb{E}_\rho[\rho^{d-2}])^2 \right] \\ &\quad + \varepsilon^{(\frac{2}{d-2}-\delta)\beta} + \varepsilon^2 + \varepsilon^{d+\frac{2d}{d+2}} \mathbb{E} \left[\sum_{z \in N_k} \sum_{\substack{w \in N_{k,z} \\ \varepsilon w \in Q_{k,z} \setminus Q_{k-1,z}}} \rho_w^{2(d-2)} \right] \end{aligned}$$

As done in the proof of Theorem 2.1 (a), this also turns into

$$\begin{aligned}
\mathbb{E}[\|u_\varepsilon - \tilde{W}_\varepsilon u\|_{H_0^1(D)}^2] &\lesssim (k\varepsilon)^2 \mathbb{E}[\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d] \\
&\quad + \mathbb{E}[(S_{k,0} - \lambda \mathbb{E}[\rho^{d-2}])^2] \\
&\quad + (k\varepsilon)^3 \sum_{z \in N_k \setminus \dot{N}_k} \mathbb{E}[(S_{k,z} - \lambda \mathbb{E}_\rho[\rho^{d-2}])^2] \\
&\quad + \varepsilon^{(\frac{2}{d-2}-\delta)\beta} + \varepsilon^2 + \varepsilon^{d+\frac{2d}{d+2}} \mathbb{E}\left[\sum_{z \in \dot{N}_k} \sum_{\substack{w \in N_{k,z}, \\ \varepsilon w \in Q_{k,z} \setminus Q_{k-1,z}}} \rho_w^{2(d-2)}\right].
\end{aligned} \tag{4.10}$$

We now claim that, thanks to (4.1), the previous estimate reduces to

$$\begin{aligned}
\mathbb{E}[\|u_\varepsilon - \tilde{W}_\varepsilon u\|_{H_0^1(D)}^2] &\lesssim (|\log \varepsilon| (k\varepsilon)^2 + k^{-d}) \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}}] \\
&\quad + \varepsilon^{(\frac{2}{d-2}-\delta)\beta} + \varepsilon^2 + k^{-2} \varepsilon^{\frac{4}{(d+2)(d-1)}}.
\end{aligned} \tag{4.11}$$

If the previous estimate holds, by the choice of δ and (4.1), we infer that

$$\mathbb{E}[\|u_\varepsilon - \tilde{W}_\varepsilon u\|_{H_0^1(D)}^2] \lesssim |\log \varepsilon| \varepsilon^{\frac{2d}{d-4}\beta \wedge (d-2)} + \varepsilon^{\frac{2}{(d+2)}(\frac{2}{d-1}+2)},$$

which establishes Theorem 2.1, (b) if $d = 3$.

To conclude the proof, we only need to obtain (4.11) from (4.10). The sum over $z \in N_k \setminus \dot{N}_k$ may be treated as in (3.39). We thus obtain inequality (4.11) provided that

$$\varepsilon^{d+\frac{2d}{d+2}} \mathbb{E}\left[\sum_{z \in \dot{N}_k} \sum_{\substack{w \in N_{k,z}, \\ \varepsilon w \in Q_{k,z} \setminus Q_{k-1,z}}} \rho_w^{2(d-2)}\right] \lesssim (\varepsilon k)^2 \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}}], \tag{4.12}$$

$$\mathbb{E}\left[\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] \lesssim |\log \varepsilon| \mathbb{E}[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}}] \tag{4.13}$$

and

$$\mathbb{E}[(S_{k,0} - \lambda \mathbb{E}[\rho])^2] \lesssim k^{-d} \mathbb{E}[\rho^2 \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}}] + \varepsilon^{(\frac{2}{d-2}-\delta)\beta} + k^{-2} \varepsilon^{\frac{4}{(d+2)(d-1)}} \tag{4.14}$$

We argue (4.13): Recalling the definition of the covering $\{Q_{k,z}\}_{z \in N_k}$, we decompose

$$D \subseteq \bigcup_{z \in N_k} (Q_{k-1,z} \cup (Q_{k,z} \setminus Q_{k-1,z}))$$

and rewrite

$$\begin{aligned}
&\mathbb{E}\left[\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] \\
&\leq \mathbb{E}\left[\varepsilon^d \sum_{w \in N_k} \left(\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k-1,w}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d + \sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k,w} \setminus Q_{k-1,w}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right)\right].
\end{aligned}$$

Since the process (Φ, \mathcal{R}) is stationary, we bound

$$\begin{aligned} \mathbb{E}\left[\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] \\ \stackrel{(3.11)}{\lesssim} k^{-d} \mathbb{E}\left[\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d + \sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k,0} \setminus Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right]. \end{aligned} \tag{4.15}$$

Let us partition the cube $Q_{k,0}$ into $(2k+1)^d$ cubes of size ε and let Q be as in (3.10); the definitions of $\Phi_\delta^\varepsilon(D)$ and $\tilde{R}_{\varepsilon,z}$ (c.f. (2.5), (4.2)) and the stationarity of (Φ, \mathcal{R}) imply that

$$k^{-d} \mathbb{E}\left[\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right] \lesssim \mathbb{E}\left[\sum_{z \in \Phi(Q)} \rho_z^{2(d-2)} \mathbf{1}_{\rho_z \leq \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,z} \geq \varepsilon^2} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right]. \tag{4.16}$$

We now apply Lemma 5.2 with $G((x, \rho); \omega) = \left(\frac{\varepsilon}{R_{\varepsilon,x}}\right)^d \mathbf{1}_{R_{\varepsilon,x} \geq \varepsilon^2} \rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}}$ to infer that

$$k^{-d} \mathbb{E}\left[\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] \lesssim \mathbb{E}_\rho[\rho^{2(d-2)} \mathbf{1}_{\rho \leq \varepsilon^{-\frac{2}{d-2} + \delta}}] \mathbb{E}_\Phi\left[\left(\frac{\varepsilon}{R_{\varepsilon,0}}\right)^d \mathbf{1}_{R_{\varepsilon,0} \geq \varepsilon^2}\right]. \tag{4.17}$$

We rewrite

$$\begin{aligned} \mathbb{E}_\Phi\left[\left(\frac{\varepsilon}{R_{\varepsilon,0}}\right)^d \mathbf{1}_{R_{\varepsilon,0} \geq \varepsilon^2}\right] &= \mathbb{E}_\Phi\left[\left(\frac{\varepsilon}{R_{\varepsilon,0}}\right)^d \mathbf{1}_{R_{\varepsilon,0} \geq \varepsilon}\right] + \mathbb{E}_\Phi\left[\left(\frac{\varepsilon}{R_{\varepsilon,0}}\right)^d \mathbf{1}_{\varepsilon^2 \leq R_{\varepsilon,0} < \varepsilon}\right] \\ &\leq 1 + \mathbb{E}_\Phi\left[\left(\frac{\varepsilon}{R_{\varepsilon,0}}\right)^d \mathbf{1}_{\varepsilon^2 \leq R_{\varepsilon,0} < \varepsilon}\right]. \end{aligned}$$

We claim that

$$\mathbb{E}_\Phi\left[\left(\frac{\varepsilon}{R_{\varepsilon,0}}\right)^d \mathbf{1}_{\varepsilon^2 \leq R_{\varepsilon,0} < \varepsilon}\right] \lesssim |\log \varepsilon|.$$

This inequality, indeed, is obtained by decomposing the indicator function above into

$$\mathbf{1}_{\varepsilon^2 < R_{0,\varepsilon} < \varepsilon} \simeq \sum_{n=0}^{\lceil 2 \log \varepsilon \rceil} \mathbf{1}_{\varepsilon^2 2^n < R_{0,\varepsilon} < \varepsilon^2 2^{n+1}}$$

and using that, by definition (2.4), it holds

$$\mathbb{P}(\varepsilon^2 2^n < R_{0,\varepsilon} < \varepsilon^2 2^{n+1}) \leq \mathbb{P}(\Phi(B_{2^{n+1}\varepsilon}(0)) \geq 1) \lesssim 2^{d(n+1)} \varepsilon^d,$$

where $B_r(0)$ is the ball of radius $r > 0$ centred at the origin. Hence,

$$\mathbb{E}_\Phi\left[\left(\frac{\varepsilon}{R_{\varepsilon,0}}\right)^d \mathbf{1}_{R_{\varepsilon,0} \geq \varepsilon^2}\right] \lesssim |\log \varepsilon| + 1. \tag{4.18}$$

Inserting this into (4.17) we obtain that for $\varepsilon \lesssim 1$

$$k^{-d} \mathbb{E}\left[\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \right] \lesssim |\log \varepsilon| \mathbb{E}_\rho[\rho^{2(d-2)} \mathbf{1}_{\rho \leq \varepsilon^{-\frac{2}{d-2} + \delta}}].$$

Thus, inequality (4.15) turns into

$$\begin{aligned} \mathbb{E}\left[\varepsilon^d \sum_{z \in \Phi_\delta^\varepsilon(D)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] &\lesssim |\log \varepsilon| \mathbb{E}_\rho[\rho_z^{2(d-2)} \mathbf{1}_{\rho_z \leq \varepsilon^{-\frac{d-2}{2}+\delta}}] \\ &\quad + k^{-d} \mathbb{E}\left[\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k,0} \setminus Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right]. \end{aligned}$$

This yields (4.13) provided that

$$k^{-d} \mathbb{E}\left[\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k,0} \setminus Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] \lesssim \mathbb{E}_\rho[\rho_z^{2(d-2)} \mathbf{1}_{\rho_z \leq \varepsilon^{-\frac{d-2}{2}+\delta}}]. \quad (4.19)$$

Let Q_r be the cube of size $r > 0$ centred at the origin. Using (2.5) we bound

$$\begin{aligned} k^{-d} \mathbb{E}\left[\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k,0} \setminus Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] \\ \leq k^{-d} \mathbb{E}\left[\sum_{z \in \Phi(Q_k \setminus Q_{k-1})} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \mathbf{1}_{R_{\varepsilon,z} \geq \varepsilon^2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{d-2}{2}+\delta}}\right]. \end{aligned}$$

Since we may decompose the set $Q_k \setminus Q_{k-1}$ into $\lesssim k^{d-1}$ unitary cubes, we use again the stationarity of $(\Phi; \mathcal{R})$ and infer that

$$\begin{aligned} k^{-d} \mathbb{E}\left[\sum_{\substack{z \in \Phi_\delta^\varepsilon(D), \\ \varepsilon z \in Q_{k,0} \setminus Q_{k-1,0}}} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d\right] \\ \leq k^{-1} \mathbb{E}\left[\sum_{z \in \Phi(Q_1)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \mathbf{1}_{R_{\varepsilon,z} \geq \varepsilon^2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{d-2}{2}+\delta}}\right], \end{aligned} \quad (4.20)$$

where Q_1 is any unitary cube that is contained in $Q_k \setminus Q_{k-1}$ ³. We now decompose $Q_1 = \sum_{n=1}^{\lceil -\kappa \log \varepsilon \rceil} A_n$ with

$$\begin{aligned} A_n &:= \{x \in Q_1 : 2^n \varepsilon^\kappa \leq \text{dist}(x; \partial Q_k) \leq 2^{n+1} \varepsilon^\kappa\}, \\ A_0 &:= \{x \in Q_1 : \text{dist}(x; \partial Q_k) \leq \varepsilon^\kappa\} \end{aligned}$$

and use (4.2) to rewrite

$$\begin{aligned} \mathbb{E}\left[\sum_{z \in \Phi(Q_1)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \mathbf{1}_{R_{\varepsilon,z} \geq \varepsilon^2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{d-2}{2}+\delta}}\right] \\ \lesssim \sum_{n=0}^{\lceil -\kappa \log \varepsilon \rceil} \mathbb{E}\left[\sum_{z \in \Phi(A_n)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{R_{\varepsilon,z} \wedge 2^n \varepsilon^{1+\kappa}}\right)^d \mathbf{1}_{R_{\varepsilon,z} \geq \varepsilon^2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{d-2}{2}+\delta}}\right] \end{aligned}$$

We now appeal again to Lemma 5.2 as for (4.16) to reduce to

$$\begin{aligned} \mathbb{E}\left[\sum_{z \in \Phi(Q_1)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,z}}\right)^d \mathbf{1}_{R_{\varepsilon,z} \geq \varepsilon^2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{d-2}{2}+\delta}}\right] \\ \lesssim \sum_{n=0}^{\lceil -\kappa \log \varepsilon \rceil} \mathbb{E}_\rho[\rho_z^{2(d-2)} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{d-2}{2}+\delta}}] |A_n| \mathbb{E}\left[\left(\frac{\varepsilon}{R_{\varepsilon,0} \wedge 2^n \varepsilon^{1+\kappa}}\right)^d \mathbf{1}_{R_{\varepsilon,0} \geq \varepsilon^2}\right]. \end{aligned}$$

³In this last step one should distinguish between unitary cubes according to the number of faces that they share with ∂Q_k . However, the argument shown below is immediately adapted to any of the previous cubes.

Arguing as for (4.18) and using the stationarity of Φ we infer that

$$\begin{aligned} \mathbb{E}\left[\sum_{z \in \Phi(Q_1)} \rho_z^{2(d-2)} \left(\frac{\varepsilon}{R_{\varepsilon,z}}\right)^d \mathbf{1}_{R_{\varepsilon,z} \geq \varepsilon} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}}\right] \\ \lesssim \mathbb{E}_\rho \left[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right] \sum_{n=0}^{\lceil -\kappa \log \varepsilon \rceil} 2^n \varepsilon^\kappa (2^{-dn} \varepsilon^{-d\kappa} - \log \varepsilon) \\ \lesssim \mathbb{E}_\rho \left[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right] \varepsilon^{-(d-1)\kappa}. \end{aligned}$$

To establish (4.19) it only remains to combine the previous inequality with (4.20) and use (4.1). The proof of (4.13) is therefore complete.

Inequality (4.12) may be obtained in a similar way as to that of (4.13): Since we may decompose the set $\bigcup_{z \in N_k} Q_{k,z} \setminus Q_{k-1,z}$ into $n \lesssim (\varepsilon k)^{-d} k^{d-1}$ disjoint cubes $\{Q_{\varepsilon,i}\}_{i=1}^n$ of size ε , we use definition (2.5) and the stationarity of (Φ, \mathcal{R}) to bound

$$\varepsilon^{d + \frac{2d}{d+2}} \mathbb{E} \left[\sum_{z \in N_k} \sum_{\substack{w \in N_{k,z}, \\ \varepsilon w \in Q_{k,z} \setminus Q_{k-1,z}}} \rho_w^{2(d-2)} \right] \lesssim k^{-1} \varepsilon^{\frac{2d}{d+2}} \mathbb{E} \left[\sum_{w \in \Phi(Q)} \rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right]$$

so that, again by Lemma 5.2, we obtain

$$\varepsilon^{d + \frac{2d}{d+2}} \mathbb{E} \left[\sum_{z \in N_k} \sum_{\substack{w \in N_{k,z}, \\ \varepsilon w \in Q_{k,z} \setminus Q_{k-1,z}}} \rho_w^{2(d-2)} \right] \lesssim k^{-1} \varepsilon^{\frac{2d}{d+2}} \mathbb{E} \left[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right]$$

We establish (4.12) after observing that, thanks to (4.1), it holds $k^{-1} \varepsilon^{\frac{2d}{d+2}} \leq (\varepsilon k)^2$.

We now tackle (4.14): As in the proof of case (a), we may assume that for $\varepsilon \lesssim 1$ the set $Q_{0,k} \subseteq D$. By construction (see definition (4.3)), the (random) set $K_{\varepsilon,0}$ satisfies

$$\{w \in \Phi_\delta^\varepsilon(D) : \varepsilon w \in K_{k,0}\} = \{w \in \Phi_\delta^\varepsilon(D) : \varepsilon w \in Q_{k,0}\} = \Phi(Q_k),$$

where Q_k is, as above, the cube of size $2k + 1$ centred at the origin. Hence, decomposing $Q_k = \bigcup_{i=1}^{(2k+1)^d} Q_i$ into unitary cubes, definitions (4.5) and (2.5) allow us to rewrite

$$S_{k,0} - \lambda \mathbb{E}[\rho^{d-2}] = \frac{\varepsilon^d}{|K_{k,z}|} \sum_{i=1}^{(2k+1)^d} Z_i - \lambda \mathbb{E}[\rho]$$

with

$$Z_i := \sum_{\Phi(Q_i)} Y_{\varepsilon,z} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,z} \geq 2\varepsilon^{\frac{d}{d-2}} \rho_z}, \quad i = 1, \dots, k^d. \quad (4.21)$$

We rewrite

$$S_{k,0} - \lambda \mathbb{E}[\rho^{d-2}] = \frac{\varepsilon^d}{|K_{k,z}|} \sum_{i=1}^{(2k+1)^d} (Z_i - \lambda \mathbb{E}[\rho^{d-2}]) + \lambda \left(\frac{\varepsilon^d (2k+1)^d}{|K_{k,z}|} - 1 \right) \mathbb{E}_\rho[\rho^{d-2}]$$

so that the triangle inequality, assumption (2.1) and the quenched bounds in (4.4) yield

$$(S_{k,0} - \lambda \mathbb{E}[\rho^{d-2}])^2 \lesssim \left(\sum_{i=1}^{(2k+1)^d} (Z_i - \lambda \mathbb{E}[\rho^{d-2}]) \right)^2 + k^{-2} \varepsilon^{2\kappa}.$$

Appealing to definitions (4.21), (4.5) and (2.4), we observe that Z_i and Z_j are independent whenever i, j are such that Q_j and Q_i are not adjacent. We emphasize that they are not identically distributed as in definition (4.21) the random variables $Y_{\varepsilon, z}$ contain the modified radii $\tilde{R}_{\varepsilon, z}$ (see (4.5) and (4.2)). Hence, by taking the expectation in the previous inequality, we estimate

$$\begin{aligned} \mathbb{E} \left[(S_{k,0} - \lambda \mathbb{E}[\rho])^2 \right] &\lesssim (2k+1)^{-d} \sum_{i=1}^{(2k+1)^d} \mathbb{E} \left[(Z_i - \lambda \mathbb{E}[\rho^{d-2}])^2 \right] \\ &+ \sum_{i=1}^{(2k+1)^d} \mathbb{E} \left[Z_i - \lambda \mathbb{E}[\rho^{d-2}] \right]^2 + k^{-2} \varepsilon^{2\kappa}. \end{aligned} \quad (4.22)$$

To establish (4.14) from (4.22) it suffices to bound

$$\mathbb{E} \left[(Z_i - \lambda \mathbb{E}[\rho^{d-2}])^2 \right] \lesssim \mathbb{E} \left[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right] + 1, \quad (4.23)$$

$$\mathbb{E} \left[Z_i - \lambda \mathbb{E}[\rho^{d-2}] \right] \lesssim \varepsilon^{(\frac{2}{d-2} - \delta)\beta}. \quad (4.24)$$

We emphasize that in the first bound the right-hand side may be bounded by $\lesssim \mathbb{E} \left[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right]$.

Inequality (4.23) follows from Cauchy-Schwarz's inequality, the triangle inequality and definitions (4.21) and (4.5). We thus turn to (4.24) and fix $i = 1, \dots, (2k+1)^d$. Since by definition (4.5) it holds

$$Y_{\varepsilon, z} = \rho_z^{d-2} + \varepsilon^d \frac{\rho_z^{2(d-2)}}{\tilde{R}_{\varepsilon, w}^{d-2} - \varepsilon^d \rho_w^{d-2}},$$

we use this and (4.21) to rewrite

$$\begin{aligned} \mathbb{E} \left[Z_i - \lambda \mathbb{E}[\rho^{d-2}] \right] &= \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \rho_z^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2 \vee \varepsilon^{\frac{d}{d-2}} \rho_z} \right] - \lambda \mathbb{E}[\rho^{d-2}] \\ &+ \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \varepsilon^d \frac{\rho_z^{2(d-2)}}{\tilde{R}_{\varepsilon, z}^{d-2} - \varepsilon^d \rho_z^{d-2}} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2 \vee \varepsilon^{\frac{d}{d-2}} \rho_z} \right]. \end{aligned}$$

Observing that $\mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \rho_z^{d-2} \right] = \lambda \mathbb{E}[\rho^{d-2}]$, and writing

$$\mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2 \vee \varepsilon^{\frac{d}{d-2}} \rho_z} = 1 - \mathbf{1}_{\rho_z \geq \varepsilon^{-\frac{2}{d-2} + \delta}} - \mathbf{1}_{\rho_z \leq \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \leq 2\varepsilon^{\frac{d}{d-2}} \rho_z \vee \varepsilon^2},$$

we infer that

$$\begin{aligned} \left| \mathbb{E} \left[Z_i - \lambda \mathbb{E}[\rho^{d-2}] \right] \right| &\lesssim \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \rho_z^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \leq \varepsilon^{\frac{d}{d-2}} \rho_z \vee \varepsilon^2} \right] \\ &+ \mathbb{E} \left[\rho^{d-2} \mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}} \right] \\ &+ \varepsilon^d \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\rho_z^2}{\tilde{R}_{\varepsilon, z}} \right)^{d-2} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^{\frac{d}{d-2}} \rho_z \vee \varepsilon^2} \right]. \end{aligned}$$

We now appeal to Lemma 5.2 with $G((x, \rho); \omega) = \rho \mathbf{1}_{\rho \leq \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, x} \leq \varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2}$ to infer that

$$\begin{aligned} \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \rho_z^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \leq \varepsilon^{\frac{d}{d-2}} \rho_z \vee \varepsilon^2} \right] \\ \lesssim \mathbb{E}_\rho \left[\rho^{d-2} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbb{E}_\Phi \left[\mathbf{1}_{R_{\varepsilon, 0} < \varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \right] \right]. \end{aligned}$$

By the properties of the Poisson point process and definition (2.4) this yields

$$\begin{aligned} \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \rho_z^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \leq \varepsilon^{\frac{d}{d-2}} \rho_z \vee \varepsilon^2} \right] \\ \lesssim \mathbb{E}_\rho \left[\rho^{d-2} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} (\varepsilon^{\frac{2}{d-2}} \rho \vee \varepsilon)^d \right] \\ \lesssim \varepsilon^d \mathbb{E}_\rho \left[\rho^{d-2} \right] + \varepsilon^{\frac{2d}{d-2}} \mathbb{E}_\rho \left[\rho^{2d-2} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right] \\ \stackrel{(2.1)}{\lesssim} \varepsilon^d + \varepsilon^{(\frac{2}{d-2} - \delta)\beta}. \end{aligned}$$

Hence

$$\begin{aligned} |\mathbb{E}[(Z_i - \lambda \mathbb{E}[\rho])]| &\lesssim \varepsilon^d + \mathbb{E}[\rho \mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}}] \\ &\quad + \varepsilon^d \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\rho_z^2}{\tilde{R}_{\varepsilon, z}} \right)^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^{\frac{d}{d-2}} \rho_z \vee \varepsilon^2} \right] \\ &\stackrel{(3.38)}{\lesssim} \varepsilon^d + \varepsilon^{(\frac{2}{d-2} - \delta)\beta} + \varepsilon^d \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\rho_z^2}{\tilde{R}_{\varepsilon, z}} \right)^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2} \right] \end{aligned} \tag{4.25}$$

To establish (4.24) it remains to bound the last term above by $\varepsilon^{(\frac{2}{d-2} - \delta)\beta}$: By stationarity we have

$$\begin{aligned} \varepsilon^d \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\rho_z^2}{\tilde{R}_{\varepsilon, z}} \right)^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2} \right] \\ \lesssim \mathbb{E}_\rho \left[\rho^{2(d-2)} \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \right] \varepsilon^2 \mathbb{E}_\rho \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon, z}} \right)^{d-2} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2} \right] \\ \stackrel{(3.38)}{\lesssim} \varepsilon^{(\frac{2}{d-2} - \delta)\beta} \varepsilon^2 \mathbb{E}_\rho \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\varepsilon}{\tilde{R}_{\varepsilon, z}} \right)^{d-2} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2} \right]. \end{aligned}$$

We now observe that if $Q_i \in Q_{k-1}$, then by (4.2) we have that $\tilde{R}_{\varepsilon, z} = R_{\varepsilon, z}$: In this case, the expectation on right-hand side above may be bounded similarly to (4.18) so that

$$\varepsilon^d \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\rho_z^2}{\tilde{R}_{\varepsilon, z}} \right)^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon, z} \geq \varepsilon^2} \right] \lesssim \varepsilon^{2 + (\frac{2}{d-2} - \delta)\beta} \lesssim \varepsilon^{(\frac{2}{d-2} - \delta)\beta}. \tag{4.26}$$

If, otherwise, $Q_i \in Q_k \setminus Q_{k-1}$, then (4.2) and a decomposition of Q_i similar to the one performed in (4.20) implies that

$$\begin{aligned} \varepsilon^d \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\rho_z^2}{\tilde{R}_{\varepsilon,z}} \right)^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,z} > \varepsilon^2} \right] &\lesssim \varepsilon^{2 + (\frac{2}{d-2} - \delta)\beta} \varepsilon^2 \mathbb{E}_\rho \left[\left(\frac{\varepsilon}{\tilde{R}_{\varepsilon,0}} \right)^{d-2} \mathbf{1}_{R_{\varepsilon,0} > \varepsilon^2} \right] \\ &+ \varepsilon^2 \begin{cases} |\log \varepsilon| & \text{if } d = 3 \\ \varepsilon^{-\kappa(d-3)} & \text{if } d > 3 \end{cases} \end{aligned}$$

Using that, if $d > 3$, we have $\kappa < \frac{2}{d-3}$ (c.f. (4.1)) and dealing with the remaining expectation as done in (4.26), yields that

$$\varepsilon^d \mathbb{E} \left[\sum_{z \in \Phi(Q_i)} \left(\frac{\rho_z^2}{\tilde{R}_{\varepsilon,z}} \right)^{d-2} \mathbf{1}_{\rho_z < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,z} > \varepsilon^2} \right] \lesssim \varepsilon^{(\frac{2}{d-2} - \delta)\beta}. \quad (4.27)$$

Combining (4.27) and (4.26) with (4.25) implies (4.24) and, in turn, (4.14). The proof of Theorem 2.1 is complete. \square

Proof of Lemma 4.4. The proof of this lemma follows the same lines of the argument for Lemma 3.4. We resort to the construction made in Lemma 4.1 (c.f. (3.6)) to decompose

$$\varepsilon^d \sum_{z \in \Phi^\varepsilon(D) \setminus n^\varepsilon(D)} \rho_z = \varepsilon^d \sum_{z \in J_b^\varepsilon} \rho_z^{d-2} + \varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2}. \quad (4.28)$$

The expectation of the first sum may be bounded by $\varepsilon^{(\frac{2}{d-2} - \delta)\beta} + \varepsilon^d$ by arguing in a way analogue to the one for (3.42) in Lemma 3.4. In this case, besides (2.1), we also appeal to assumption (ii) and to the properties of the Poisson point process. Hence, it only remains to estimate the last sum in (4.28). As done for the same sum in (3.41), we use the definition of $n^\varepsilon(D)$ and the stationarity of (Φ, \mathcal{R}) to rewrite

$$\begin{aligned} &\mathbb{E} \left[\varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2} \right] \\ &\lesssim \mathbb{E} \left[\sum_{w \in \Phi(Q_i)} \left(\mathbf{1}_{\rho_w \geq \varepsilon^{-\frac{2}{d-2} + \delta}} + \mathbf{1}_{\rho_w < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,w} \leq 2\varepsilon^{\frac{d}{d-2}} \rho_w \vee \varepsilon^2} \right) \sum_{\substack{z \in \Phi^\varepsilon(D) \setminus \{w\} \\ \varepsilon|w-z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho_w \wedge 1}} \rho_z^{d-2} \right]. \end{aligned}$$

By Lemma 5.2 applied to

$$G((x, \rho), \omega) = \left(\mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}} + \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,x} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \right) \sum_{\substack{z \in \Phi^\varepsilon(D) \\ \varepsilon|x-z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho \wedge 1}} \rho_z^{d-2},$$

we infer that

$$\begin{aligned} &\mathbb{E} \left[\varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2} \right] \\ &\lesssim \mathbb{E}_\rho \left[\mathbb{E} \left[\left(\mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}} + \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,0} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \right) \sum_{\substack{z \in \Phi^\varepsilon(D) \setminus \{0\} \\ \varepsilon|z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho \wedge 1}} \rho_z^{d-2} \right] \right]. \end{aligned}$$

Since the marks $\{\rho_z\}$ are identically distributed and independent, we use (2.1) to bound

$$\begin{aligned} \mathbb{E}\left[\varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2}\right] &\lesssim \mathbb{E}_\rho \left[\mathbb{E}_\Phi \left[\left(\mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}} + \mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,0} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \right) \right. \right. \\ &\quad \left. \left. \times \#\{z \in \Phi^\varepsilon(D) \setminus \{0\} : \varepsilon|z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho \wedge 1\} \right] \right] dx. \end{aligned} \quad (4.29)$$

Since $\mathbb{E}_\Phi[\#\{z \in \Phi^\varepsilon(D) \setminus \{0\} : \varepsilon|z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho \wedge 1\}] \lesssim (\rho^{d-2} + 1)$, the term corresponding to the first sum on the right-hand side above is easily bounded by

$$\begin{aligned} \mathbb{E}_\rho \left[\mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbb{E}_\Phi[\#\{z \in \Phi^\varepsilon(D) \setminus \{0\} : \varepsilon|z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho \wedge 1\}] \right] \\ \lesssim \mathbb{E}_\rho \left[\rho^{d-2} \mathbf{1}_{\rho \geq \varepsilon^{-\frac{2}{d-2} + \delta}} \right] \stackrel{(3.38)}{\lesssim} \varepsilon^{(\frac{2}{d-2} - \delta)\beta}. \end{aligned} \quad (4.30)$$

We now turn to the second term in (4.29): Since this term reduces to the values $\rho < \varepsilon^{-\frac{2}{d-2} + \delta}$, we have that

$$\begin{aligned} \mathbb{E}_\rho \left[\mathbb{E}_\Phi \left[\mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,0} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \#\{z \in \Phi^\varepsilon(D) \setminus \{0\} : \varepsilon|z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho \wedge 1\} \right] \right] \\ \leq \mathbb{E}_\rho \left[\mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbb{E}_\Phi \left[\mathbf{1}_{R_{\varepsilon,0} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \#\{z \in \Phi^\varepsilon(D) \setminus \{0\} : |z| \leq 4\} \right] \right] \end{aligned}$$

Using Hölder's inequality with exponents $\frac{d}{d-1}$ and d in the inner expectation, definition (2.4) and the fact that Φ is a Poisson point process, this implies that

$$\begin{aligned} \mathbb{E}_\rho \left[\mathbb{E}_\Phi \left[\mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbf{1}_{R_{\varepsilon,0} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \#\{z \in \Phi^\varepsilon(D) \setminus \{0\} : \varepsilon|z| \leq \varepsilon + \varepsilon^{\frac{d}{d-2}} \rho \wedge 1\} \right] \right] \\ \lesssim \mathbb{E}_\rho \left[\mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbb{E}_\Phi \left[\mathbf{1}_{R_{\varepsilon,0} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \right]^{\frac{d-1}{d}} \right]. \end{aligned} \quad (4.31)$$

We control the last term by

$$\begin{aligned} \mathbb{E}_\rho \left[\mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \mathbb{E}_\Phi \left[\mathbf{1}_{R_{\varepsilon,0} \leq 2\varepsilon^{\frac{d}{d-2}} \rho \vee \varepsilon^2} \right]^{\frac{d-1}{d}} \right] &\leq \mathbb{E}_\rho \left[\mathbf{1}_{\rho < \varepsilon^{-\frac{2}{d-2} + \delta}} \left(2\varepsilon^{\frac{2}{d-2}} \rho \vee \varepsilon \right)^{d-1} \right] \\ &\stackrel{(2.1)}{\lesssim} \varepsilon^{d-1} + \varepsilon^{2\frac{d-1}{d-2} - (\frac{2}{d-2} - \delta)(1-\beta)_+} \end{aligned}$$

Since $d \geq 3$ and $\delta \leq \frac{2}{d-2}$, the last term on the right-hand side above is bounded by ε^2 . Combining this with (4.31), (4.29) and (4.30) yields $\mathbb{E}[\varepsilon^d \sum_{z \in \tilde{I}_b^\varepsilon} \rho_z^{d-2}] \lesssim \varepsilon^{(\frac{2}{d-2} - \delta)\beta} + \varepsilon^2$. This concludes the proof of Lemma 4.4. \square

5. Auxiliary results. Let $\mathcal{Z} := \{z_i\}_{i \in I} \subseteq D$ be a collection of points and let $\mathcal{X} := \{X_i\}_{i \in I}$, $\mathcal{R} := \{r_i\}_{i \in I} \subseteq \mathbb{R}_+$. We assume that

$$2X_i < r_i < \min_{\substack{z_j \in \mathcal{Z}, \\ z_j \neq z_i}} \{|z_j - z_i|\}, \quad \text{for every } z_i \in \mathcal{Z}. \quad (5.1)$$

We define the measure

$$M := \sum_{i \in I} \partial_n v_i \delta_{\partial B_{r_i}(z_i)} \in H^{-1}(D), \quad (5.2)$$

where each $v_i \in H^1(B_{r_i}(z_i))$ solves (2.6) with $B_{\varepsilon\rho_z}(\varepsilon z)$ and $B_{R_{\varepsilon,z}}(\varepsilon z)$ replaced by $B_{X_i}(z_i)$ and $B_{r_i}(z_i)$, respectively.

The next lemma is a generalization of the result by [19] used in [18] to show the analogue of Theorem 2.1 in the case of periodic holes H_ε .

Lemma 5.1. *Let \mathcal{Z} , \mathcal{X} and \mathcal{R} be as above and let M be as in (5.2). Then, there exists a constant $C = C(d) < +\infty$ such that for every Lipschitz and (essentially) disjoint covering $\{K_j\}_{j \in J}$ of D such that*

$$B_{2r_i}(z_i) \subseteq K_j \quad \text{OR} \quad B_{r_i}(z_i) \cap K_j = \emptyset \quad \text{for every } i \in I, j \in J \quad (5.3)$$

we have that

$$\|M - m\|_{H^{-1}(D)} \leq C \max_{j \in J} \text{diam}(K_j) \left(\sum_{i \in I} X_i^{2(d-2)} r_i^{-d} \right)^{\frac{1}{2}},$$

with

$$m := c_d \sum_{j \in J} \left(\frac{1}{|K_j|} \sum_{\substack{i \in I, \\ z_i \in K_j}} \frac{X_i^{d-2} r_i^{d-2}}{r_i^{d-2} - X_i^{d-2}} \right) \mathbf{1}_{K_j}. \quad (5.4)$$

Here, the constant c_d is as in (1.5).

The next result is a consequence of the assumptions (i)-(iii) on the marked point process (Φ, \mathcal{R}) . Since it is used extensively in the proof of Theorem 2.1, in the sake of a self-contained presentation, we give below the statement and its brief proof. Let $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying probability space for (Φ, \mathcal{R}) . Let $G : \mathbb{R}^d \times \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a stationary random field. This means that for almost every $(x, \rho) \in \mathbb{R}^d \times \mathbb{R}_+$, the expectation $\mathbb{E}[G((x, \rho); \omega)] = \mathbb{E}[G((0, \rho); \omega)]$.

Lemma 5.2. *Let $A \subseteq \mathbb{R}^d$ be bounded and such that $0 \in A$. Let $(\Phi; \mathcal{R})$ satisfy (i)-(iii) with $\Phi = \text{Poi}(\lambda)$. For every $\varepsilon > 0$ and $x \in \mathbb{R}^d$, let $R_{\varepsilon,z}$ be as in (2.4). Then for every stationary $G : \mathbb{R}^d \times \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ it holds*

$$\mathbb{E} \left[\sum_{z \in \Phi(A)} G((z, \rho_z); \omega \setminus \{(z, \rho_z)\}) \right] = \lambda |A| \mathbb{E}_\rho \left[\mathbb{E}[G((0, \rho); \omega)] \right].$$

Proof of Lemma 5.1. With no loss of generality, we give the proof for $d = 3$. We start by remarking that, thanks to (5.3), we may rewrite the measure M in (5.2) as

$$M = \sum_{j \in J} M_j, \quad M_j := \sum_{\substack{i \in I, \\ z_i \subseteq K_j}} \partial_n v_i \delta_{\partial B_{r_i}(z_i)}. \quad (5.5)$$

By the definition of the capacity functions $\{v_i\}_{i \in I}$ (see also (3.18)), m in (5.4) satisfies

$$m = \sum_{j \in J} m_j, \quad m_j = \left(\frac{1}{|K_j|} \sum_{\substack{i \in I, \\ z_i \subseteq K_j}} \int_{\partial B_{r_i}(z_i)} \partial_n v_i \right) \mathbf{1}_{K_j} \quad (5.6)$$

For every $j \in J$, we thus define $q_j \in H^1(K_j)$ as the (weak) solution to

$$\begin{cases} -\Delta q_j = M_j - m_j & \text{in } K_j \\ \partial_n q_j = 0 & \text{on } \partial K_j \end{cases}, \quad \int_{K_j} q_j = 0, \quad (5.7)$$

We stress that q_j exists since K_j is a Lipschitz domain and, thanks to (5.3) and (5.5)-(5.6), the compatibility condition $\langle M_j; 1 \rangle - \int_{K_j} m_j = 0$ is satisfied.

By (5.7) and (5.5)-(5.6), we thus have that

$$\|M - m\|_{H^{-1}(D)} \leq \left(\sum_{j \in J} \int_{K_j} |\nabla q_j|^2 \right)^{\frac{1}{2}}.$$

The statement of Lemma 5.1 holds, provided that we show that for each $j \in J$

$$\left(\int_{K_j} |\nabla q_j|^2 \right)^{\frac{1}{2}} \lesssim \text{diam}(K_j) \left(\sum_{\substack{i \in I, \\ z_i \in K_j}} X_i^2 r_i^{-3} \right)^{\frac{1}{2}}. \tag{5.8}$$

We argue (5.8) as follows: testing (5.7) with q_j itself and using that $\int_{K_j} q_j = 0$, we obtain

$$\int_{K_j} |\nabla q_j|^2 = \sum_{\substack{i \in I, \\ z_i \in K_j}} \int_{\partial B_{r_i}(z_i)} \partial_n v_i q_z.$$

By Cauchy-Schwarz's inequality and by the definition of v_i (see also (3.18)), this implies that

$$\int_{K_j} |\nabla q_j|^2 \lesssim \sum_{\substack{i \in I, \\ z_i \in K_j}} r_i^{-1} \left(\frac{X_i r_i}{r_i - X_i} \right) \left(\int_{\partial B_{r_i}(z_i)} |q_j|^2 \right)^{\frac{1}{2}} \stackrel{(5.1)}{\lesssim} \sum_{\substack{i \in I, \\ z_i \in K_j}} r_i^{-1} X_i \left(\int_{\partial B_{r_i}(z_i)} |q_j|^2 \right)^{\frac{1}{2}}. \tag{5.9}$$

By the trace estimate for functions $u \in H^1(B_r)$, $r > 0$

$$\int_{\partial B_r} |u|^2 \lesssim r^{-1} \left(\int_{B_r} |u|^2 + r^2 \int_{B_r} |\nabla u|^2 \right),$$

inequality (5.9) turns into

$$\begin{aligned} \int_{K_j} |\nabla q_j|^2 &\lesssim \sum_{\substack{i \in I, \\ z_i \in K_j}} r_i^{-1} \left(\frac{X_i r_i}{r_i - X_i} \right) \left(\int_{\partial B_{r_i}(z_i)} |q_j|^2 \right)^{\frac{1}{2}} \\ &\stackrel{(5.1)}{\lesssim} \sum_{\substack{i \in I, \\ z_i \in K_j}} r_i^{-\frac{3}{2}} X_i \left(\int_{B_{r_i}(z_i)} |q_j|^2 + r_i^2 \int_{B_{r_i}(z_i)} |\nabla q_j|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since by (5.3) we have $r_i \leq \text{diam}(K_j)$, we infer that

$$\int_{K_j} |\nabla q_j|^2 \lesssim \sum_{\substack{i \in I, \\ z_i \in K_j}} r_i^{-\frac{3}{2}} X_i \left(\int_{B_{r_i}(z_i)} |q_j|^2 + \text{diam}(K_j)^2 \int_{B_{r_i}(z_i)} |\nabla q_j|^2 \right)^{\frac{1}{2}}.$$

This, Cauchy-Schwarz's inequality, and (5.1) further yield

$$\int_{K_j} |\nabla q_j|^2 \lesssim \left(\sum_{\substack{i \in I, \\ z_i \in K_j}} X_i^2 r_i^{-3} \right)^{\frac{1}{2}} \left(\int_{K_j} |q_z|^2 + \text{diam}(K_j)^2 \int_{K_j} |\nabla q_z|^2 \right)^{\frac{1}{2}}$$

Since by (5.7) the function q_j has zero mean, we may apply Poincaré-Wirtinger's inequality to conclude that

$$\int_{K_j} |\nabla q_j|^2 \lesssim \text{diam}(K_j) \left(\sum_{\substack{i \in I, \\ z_i \in K_j}} X_i^2 r_i^{-3} \right)^{\frac{1}{2}} \left(\int_{K_j} |\nabla q_j|^2 \right)^{\frac{1}{2}}.$$

This establishes (5.8) and, in turn, concludes the proof of Lemma 5.1. □

Proof of Lemma 5.2. Without loss of generality we assume that $|A| = 1$. By the assumption (i)-(ii) on (Φ, \mathcal{R}) and by symmetry we have that

$$\begin{aligned} \mathbb{E}\left[\sum_{z \in \Phi(A)} G((z, \rho_z), \omega \setminus (z, \rho_z))\right] &= \lambda e^{-\lambda} \sum_{n \geq 1} \frac{\lambda^{n-1}}{(n-1)!} \\ &\times \int_{(A \times \mathbb{R}_+)^n} \mathbb{E}[G((x_1, \rho_1), \omega \setminus \{(x_1, \rho_1)\}) | \Phi(A) \\ &= n, \{\rho_z\}_{\Phi(A)}] f(\rho_1) d\rho_1 dx_1 \cdots f(\rho_n) d\rho_n dx_n. \end{aligned}$$

Appealing to Fubini's theorem and relabelling the elements $\{(x_i, \rho_i)\}_{i=1}^n$, this implies

$$\begin{aligned} \mathbb{E}\left[\sum_{z \in \Phi(A)} G((z, \rho_z), \omega \setminus \{(z, \rho_z)\})\right] &= \lambda \int_{A \times \mathbb{R}_+} \left(e^{-\lambda} \sum_{n \geq 0} \frac{\lambda^n}{n!} \right. \\ &\times \left. \int_{(A \times \mathbb{R}_+)^n} \mathbb{E}[G((x, \rho), \omega) | \Phi(A) = n, \{\rho_z\}_{\Phi(A)}] f(\rho_1) d\rho_1 dx_1 \cdots f(\rho_n) d\rho_n dx_n \right) f(\rho) dx, \end{aligned}$$

i.e. $\mathbb{E}\left[\sum_{z \in \Phi(A)} G((z, \rho_z), \omega \setminus \{(z, \rho_z)\})\right] = \lambda \int_A \mathbb{E}_\rho \left[\mathbb{E}[G((x, \rho), \omega)] \right] dx$. Since G is stationary, the above equality immediately implies Lemma 5.2. \square

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