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SIMULTANEOUS OBSERVABILITY OF INFINITELY MANY STRINGS AND BEAMS

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Abstract. We investigate the simultaneous observability of infinite systems of vibrating strings or beams having a common endpoint where the observation is taking place. Our results are new even for finite systems because we allow the vibrations to take place in independent directions. Our main tool is a vectorial generalization of some classical theorems of Ingham, Beurling and Kahane in nonharmonic analysis.

1. **Introduction.** In this paper we are investigating finite and infinite systems of strings or beams having a common endpoint, whose transversal vibrations may take place in different planes. We are interested in conditions ensuring their simultaneous observability and in estimating the sufficient observability time.

There have been many results during the last twenty years on the simultaneous observability and controllability of systems of strings and beams, see e.g., [1]-[6], [11]-[12], [20], [25]. In all earlier papers the vibrations were assumed to take place in a common vertical plane. Here, we still assume that each string or beam is vibrating in some plane, but these planes may differ from one another. This leads to important new difficulties, requiring vectorial generalizations of clasical Ingham type theorems. Our approach also allows us to consider infinite systems of strings or beams, which requires a deeper study of the overall density of the union of all corresponding eigenfrequencies.

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For a general introduction to the controllability of PDE's we refer to [22, 23] or [18]. The approach of the present paper is based on some classical results of Ingham [14], Beurling [9] and Kahane [17] on nonharmonic analysis. Some of the first applications to control thery were given in the papers of Ball and Slemrod [7] and Haraux [13]. We refer to [19] for a general introduction.

The paper is organized as follows. Section 2 is devoted to the statement of our main results. In Section 3 we briefly recall out harmonic analysis tools on which the proofs of our main theorems are based. The remaining part of the paper is devoted to the proofs of the results. In particular, the theorems concerning the observability of string systems (Theorems 2.1, 2.2 and 2.3) are proved in Sections 4 and 5. In Section 6 we prove Theorem 2.4 on the observability of infinite beam systems under some algebraic conditions on the lengths of the beams. Finally in Section 7 we prove Proposition 1 providing many examples where the hypotheses of Theorem 2.4 are satisfied.

2. Main results.

In what follows we state our main results on the simultaneous observability of string and beam systems.

2.1. Simultaneous observability of string systems.

We consider a system of $J < \infty$ vibrating strings of length ℓ_j, $1 \le j \le J$. The problem is set in the spherical coordinate system $(r, \varphi, \theta) \in [0, \infty) \times \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2}]$. The direction of the j-th string is $d_j := (\ell_j, \varphi_j, \theta_j)$: this, together with an unit vector v_j in $\mathbb{R}^3$ orthogonal to d_j, is fixed by initial data. The transversal displacement of the j-th string at time t and at the point (r, φ_j, θ_j) for $r \in [0, \ell_j]$ is denoted by $u_j(t, r, \varphi_j, \theta_j)v_j$.

We consider the following uncoupled system:

$$\begin{cases} u_{j,tt} - u_{j,rr} = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t, 0, \varphi_j, \theta_j) = u_j(t, \ell_j, \varphi_j, \theta_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0, r, \varphi_j, \theta_j) = u_{j0}(r) \quad \text{and} \quad u_{j,t}(0, r, \varphi_j, \theta_j) = u_{j1}(r) & \text{for} \quad r \in (0, \ell_j), \\ j = 1, \dots, J. \end{cases} \tag{1}$$

(As usual, the subscripts t and r denote differentiations with respect to these variables.) It is well-known that the system is well posed for

$$u_{j0} \in H_0^1(0, \ell_j) \quad \text{and} \quad u_{j1} \in L^2(0, \ell_j), \quad j = 1, \dots, J,$$

and the corresponding functions $t \mapsto u_{j,r}(t, 0, \varphi_j, \theta_j)$ are locally square integrable ("hidden regularity").

We seek conditions ensuring that the linear map

$$(u_{10}, u_{11}, \dots, u_{J0}, u_{J1}) \mapsto \sum_{j=1}^{J} u_{j,r}(\cdot, 0, \varphi_j, \theta_j)v_j \tag{2}$$

of the Hilbert space $H := \prod_{j=1}^{J} H_0^1(0, \ell_j) \times L^2(0, \ell_j)$ into $L^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^3)$ is one-to-one. If this is so, then we would also like to obtain more precise, quantitative norm estimates.

Setting

$$\omega_{j,k} := \frac{k\pi}{\ell_j}$$

for brevity, the solutions of (1) are given by the formulas

$$u_j(t,r,\varphi_j,\theta_j)=\sum_{k=1}^{\infty}\left(b_{j,k}e^{i\omega_{j,k}t}+b_{j,-k}e^{-i\omega_{j,k}t}\right)\sin(\omega_{j,k}r),\quad j=1,\dots,J$$

with suitable complex coefficients $b_{j,\pm k}$, and

$$\sum_{j=1}^{J}u_{j,r}(t,0,\varphi_j,\theta_j)v_j=\sum_{j=1}^{J}\left(\sum_{k=1}^{\infty}\omega_{j,k}\left(b_{j,k}e^{i\omega_{j,k}t}+b_{j,-k}e^{-i\omega_{j,k}t}\right)\right)v_j.$$

The linear map (2) is not always one-to-one. Indeed, if there exists a real number ω such that the set of vectors

$$\left\{v_j\ :\ \text{there exists a } k_j \text{ satisfying } \omega_{j,k_j}=\omega\right\}$$

is linearly dependent, then denoting by J' the set of the corresponding indices j and choosing a nontrivial (finite) linear combination

$$\sum_{j\in J'}\alpha_j v_j=0,$$

the functions

$$u_j(t,r,\varphi_j,\theta_j):=\begin{cases}\alpha_j e^{i\omega t}\sin(\omega r) & \text{if } j\in J',\\ 0 & \text{if } j\in\{1,\dots,J\}\setminus J'\end{cases}$$

define a non-trivial solution of (1) satisfying

$$\sum_{j=1}^{J}u_{j,r}(t,0,\varphi_j,\theta_j)v_j=0\quad\text{for all}\quad t\in\mathbb{R},$$

so that the linear map (2) on H is not one-to-one.

A positive observability result is the following:

Theorem 2.1. *Assume that*

$$\ell_j/\ell_m\quad\textit{is irrational for all}\quad j\neq m. \tag{3}$$

Then there exists a number

$$T_0\in\left[2\max\left\{\ell_1,\dots,\ell_J\right\},2(\ell_1+\dots+\ell_J)\right]$$

such that the restricted linear map

$$(u_{10},u_{11},\dots,u_{J0},u_{J1})\mapsto\sum_{j=1}^{J}u_{j,r}(\cdot,0,\varphi_j,\theta_j)v_j|_I$$

where $u_1,\dots,u_J$ are solutions of (1), is one-to-one for every interval I of length $>T_0$, and for no interval I of length $<T_0$.

Remark 1. The proof of Theorem 2.1 yields a more precise estimation of T_0 in some special cases (see Remark 7 below). We give three examples.

(i) If $J=3$ and v_1,v_2,v_3 are mutually orthogonal, then $T_0=2\max\{\ell_1,\ell_2,\ell_3\}$, see Figure 1-(i).
(ii) If $J=3$ and $v_1\perp v_2=v_3$, then $T_0=2\max\{\ell_1+\ell_2,\ell_3\}$, see Figure 1-(ii).

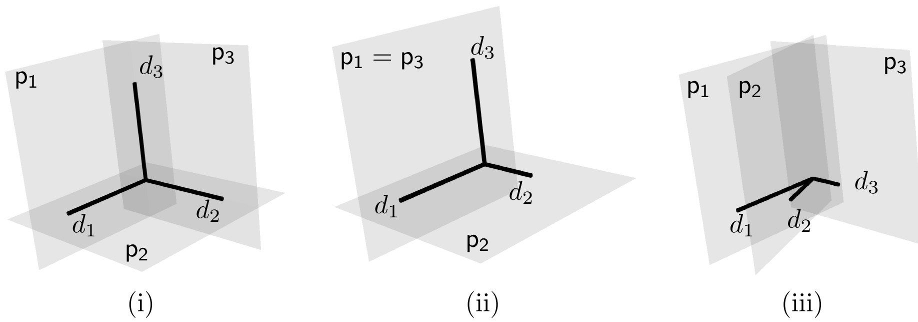

FIGURE 1. A system of three strings with vibration planes $\mathsf{p_j}$ spanned by $d_j := (\ell_j, \phi_j, \theta_j)$ and $v_j \perp d_j$, $j = 1, 2, 3$. In (i) $\ell_1 = \ell_2 = \ell_3 = 1$, the v_j's are pairwise orthogonal, and $v_1 = d_3$, $v_2 = d_1$, $v_3 = d_2$. We have $T_0 = 2\max\{\ell_1, \ell_2, \ell_3\} = 2$. In (ii), we have $\ell_1 = \ell_3 = 1$, $\ell_2 = 2/(2+\sqrt{2})$ and $v_1 = d_3 \perp d_1 = v_2 = v_3$. Then $T_0 = 2\max\{\ell_1 + \ell_2, \ell_3\} \approx 3.1715$. In the planar case (iii) we have $\ell_1 = 1$, $\ell_2 = 2/(2+\sqrt{2})$ and $\ell_3 = 2/(4+\sqrt{2})$, so that $T_0 = 2\max\{\ell_1 + \ell_2 + \ell_3\} \approx 3.9103$.

(iii) If all vectors v_j are equal (we call this the *planar case*), then $T_0 = 2(\ell_1 + \cdots + \ell_J)$– see Figure 1-(iii). Thus we recover an earlier theorem in [5, 6] by using Fourier series. It was also proved in [11] by a method based on d'Alembert's formula. The method of Fourier series enables us to consider more general equations of the type $u_{j,tt} - u_{j,rr} + a_j u_j = 0$ in (1) with arbitrary nonnegative constants (d'Alembert's formulas cannot be generalized to the case where $a_j \neq 0$). Moreover, we may even consider infinitely many strings.

Incidentally note that if the vectors v_j are linearly independent then we do not need the assumption (3). But this may only happen if we have at most three vectors, while the interest of the present paper is to have many, maybe even infinitely many strings or beams, and thus many vectors v_j.

Under some further assumptions on the lengths of the strings we may also get explicit norm estimates. We adopt the following notations. For each fixed $j \geq 1$, let

$$e_{j,k}(x) := \sqrt{2/\ell_j}\sin\left(\frac{k\pi x}{\ell_j}\right), \quad k = 1, 2, \ldots$$

be the usual orthonormal basis of $L^2(0, \ell_j)$. We denote by $D^s(0, \ell_j)$ the Hilbert spaces obtained by completion of the linear hull of $(e_{j,k})$ with respect to the Euclidean norm

$$\left\|\sum_{k=1}^{\infty} c_k e_{j,k}\right\|_{D^s(0,\ell_j)} := \left(\sum_{k=1}^{\infty}\left(\frac{k\pi}{\ell_j}\right)^{2s}|c_k|^2\right)^{1/2}.$$

Note that, identifying $L^2(0,\ell_j)$ with its dual, we have

$$D^0(0,\ell_j)=L^2(0,\ell_j),\quad D^1(0,\ell_j)=H_0^1(0,\ell_j)\quad\text{and}\quad D^{-1}(0,\ell_j)=H^{-1}(0,\ell_j)$$

with equivalent norms.

Theorem 2.2. *Consider the system* (1). *Assume that all ratios ℓ_j/ℓ_m with $j\neq m$ are quadratic irrational numbers. Then there exists a constant $c>0$ such that the solutions of* (4) *satisfy estimates*

$$\sum_{j=1}^J\left(\|u_{j0}\|^2_{D^{2-J}(0,\ell_j)}+\|u_{j1}\|^2_{D^{1-J}(0,\ell_j)}\right)\le c\int_I\left|\sum_{j=1}^J u_{j,r}(t,0,\varphi_j,\theta_j)v_j\right|^2\,dt$$

for every bounded interval I of length $|I|>2\sum_{j=1}^J\ell_j$.

Next we consider a more general system with given real numbers $a_j\ge 0$, and $\ell_j>0$:

$$\begin{cases}u_{j,tt}-u_{j,rr}+a_ju_j=0 & \text{in}\quad \mathbb{R}\times(0,\ell_j),\\ u_j(t,0,\varphi_j,\theta_j)=u_j(t,\ell_j,\varphi_j,\theta_j)=0 & \text{for}\quad t\in\mathbb{R},\\ u_j(0,r,\varphi_j,\theta_j)=u_{j0}(r)\quad\text{and}\quad u_{j,t}(0,r,\varphi_j,\theta_j)=u_{j1}(r) & \text{for}\quad r\in(0,\ell_j),\\ j=1,2,\dots,J.\end{cases}\tag{4}$$

For any given initial data

$$(u_{10},u_{11},\dots,u_{J0},u_{J1})\in H\tag{5}$$

the system has a unique solution, given by the formula

$$u_j(t,r,\varphi_j,\theta_j)=\sum_{k=1}^\infty\left(b_{j,k}e^{i\omega_{j,k}t}+b_{j,-k}e^{-i\omega_{j,k}t}\right)\sin\left(\frac{k\pi r}{\ell_j}\right),\quad j=1,2,\dots,$$

where now we use the notation

$$\omega_{j,k}:=\sqrt{\left(\frac{k\pi}{\ell_j}\right)^2+a_j}.$$

Theorem 2.3. *Assume that*

$$(j_1,k_1)\neq(j_2,k_2)\Longrightarrow\omega_{j_1,k_1}\neq\omega_{j_2,k_2}.\tag{6}$$

Then the restricted linear map

$$(u_{10},u_{11},\dots,u_{J0},u_{J1})\mapsto\sum_{j=1}^J u_{j,r}(\cdot,0,\varphi_j,\theta_j)v_j|_I\tag{7}$$

where $u_1,\dots,u_J$ are solutions of (4)*, is well defined and continuous from H into $L^2(I;\mathbb{R}^3)$ for every bounded interval I.*

Moreover, there exists a number

$$T_0\in\left[2\max\{\ell_1,\dots,\ell_J\},2(\ell_1+\dots+\ell_J)\right]$$

such that the map (7) *is one-to-one for every interval I of length $>T_0$, and for no interval of length $<T_0$.*

Remark 2.

(i) If $a_j=0$ for all j, then the condition (6) is equivalent to (3).

(ii) We may wonder whether Theorems 2.1, 2.2 and 2.3 remain valid for infinite string systems having a finite total length if the observability time is greater than $2\sum_{j=1}^{\infty}\ell_j$. we will show in Remark 8 below that our proofs cannot be adapted to prove such results.

2.2. **Simultaneous observability of beam systems.** Our approach may be adapted to systems of hinged beams. Moreover, we may even consider systems of infinitely many beams. We consider the following system:

$$\begin{cases} u_{j,tt}+u_{j,rrrr}=0 & \text{in} \quad \mathbb{R}\times(0,\ell_j),\\ u_j(t,0,\varphi_j,\theta_j)=u_j(t,\ell_j,\varphi_j,\theta_j)=0 & \text{for} \quad t\in\mathbb{R},\\ u_{j,rr}(t,0,\varphi_j,\theta_j)=u_{j,rr}(t,\ell_j,\varphi_j,\theta_j)=0 & \text{for} \quad t\in\mathbb{R},\\ u_j(0,r,\varphi_j,\theta_j)=u_{j0}(r) \quad \text{and} \quad u_{j,t}(0,r,\varphi_j,\theta_j)=u_{j1}(r) & \text{for} \quad r\in(0,\ell_j),\\ j=1,2,\dots. \end{cases} \tag{8}$$

For any given initial data

$$(u_{10},u_{11},u_{20},u_{21},\dots)\in\prod_{j=1}^{\infty}(H_0^1(0,\ell_j)\times H^{-1}(0,\ell_j)) \tag{9}$$

the system has a unique solution, given by the formula

$$u_j(t,r,\varphi_j,\theta_j)=\sum_{k=1}^{\infty}\left(b_{j,k}e^{i\omega_{j,k}t}+b_{j,-k}e^{-i\omega_{j,k}t}\right)\sin\left(\frac{k\pi r}{\ell_j}\right),\quad j=1,2,\dots,$$

where now we use the notation

$$\omega_{j,k}:=\left(\frac{k\pi}{\ell_j}\right)^2.$$

Let us denote by H the vector space of those sequences (9) that satisfy the condition

$$\sum_{j=1}^{\infty}\frac{1}{\ell_j}\left(\|u_{j0}\|^2_{H_0^1(0,\ell_j)}+\|u_{j1}\|^2_{H^{-1}(0,\ell_j)}\right)<\infty.$$

The formula

$$\|(u_{10},u_{11},u_{20},u_{21},\dots)\|_H^2:=\sum_{j=1}^{\infty}\frac{1}{\ell_j}\left(\|u_{j0}\|^2_{H_0^1(0,\ell_j)}+\|u_{j1}\|^2_{H^{-1}(0,\ell_j)}\right)$$

defines a Euclidean norm on H for which H becomes a Hilbert space.

Henceforth we consider the solutions of (8) for initial data belonging to H.

Theorem 2.4. *Assume that*

$$(\ell_j/\ell_m)^2 \quad \textit{is irrational for all} \quad j\neq m. \tag{10}$$

Furthermore, assume that there exists a constant $A>0$ such that

$$\operatorname{dist}\left(k\frac{\ell_m}{\ell_j},\mathbb{Z}\right)\geq\frac{A\ell_j\ell_m}{|k|} \quad \textit{for all nonzero integers} \quad k \tag{11}$$

whenever $j\neq m$.

Then there exist a number $T_0 \geq 0$ and a constant $c > 0$ such that the solutions of (8) *satisfy the relation*

$$\sum_{j=1}^{\infty} \frac{1}{\ell_j} \left(\|u_{j0}\|^2_{H^1_0(0,\ell_j)} + \|u_{j1}\|^2_{H^{-1}(0,\ell_j)} \right) \leq c \int_I \left\| \sum_{j=1}^{\infty} u_{j,r}(t,0,\varphi_j,\theta_j) v_j \right\|^2 \, dt$$

for every bounded interval I of length $> T_0$, where u_j with $j = 1, 2, \ldots$ are the solutions of (8).

It is not obvious that there exist infinite sequences (ℓ_j) satisfying (10) and (11). We give three different examples.

We recall that a *Perron number* is a real algebraic integer q of degree ≥ 2 whose conjugates are all smaller than q in absolute value. For example, the Golden Ratio $q \approx 1.618$ and more generally the Pisot and Salem numbers are Perron numbers. In what follows we use the symbol $[x]$ to denote the lower integer part of x.

Proposition 1. *The sequence $(\ell_j)_{j=1}^{\infty}$ satisfies* (10) *and* (11) *in the following three cases:*

(i) *q is a quadratic Perron number and $\ell_j = q^{-j}$;*
(ii) $\ell_j = \frac{1}{2^j + \sqrt{2}}$;
(iii) $\ell_j = \frac{\pi}{j + 1 + \sqrt{2}}$.

Remark 3.

(i) The conclusion of Theorem 2.4 remains valid for systems of Schrdinger equations of the form

$$\begin{cases} u_{j,t} + i u_{j,rr} = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t,0,\varphi_j,\theta_j) = u_j(t,\ell_j,\varphi_j,\theta_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0,r,\varphi_j,\theta_j) = u_{j0}(r) & \text{for} \quad r \in (0,\ell_j), \\ j = 1, 2, \ldots \end{cases} \tag{12}$$

(we may also change some $u_{j,t} + i u_{j,rr} = 0$ to $u_{j,t} - i u_{j,rr} = 0$ for some or all indices j) because every solution of (12) also satisfies (8): see, e.g., [18, Section 6.4, p. 82].

(ii) The beams in Proposition 1 (iii) have an infinite total length. Since we have an infinite propagation speed for beams (see [18, Theorem 6.7]), this does not exclude the observability of the system.

(iii) We conjecture that $T_0 = 0$ in all cases covered by Proposition 1.

3. **Review of some Ingham type theorems.** We recall some tools we need in this paper. We refer to [19] for more details and proofs. Every increasing sequence $(\omega_k)_{k \in \mathbb{Z}}$ of real numbers has an upper density

$$D^+ = D^+(\{\omega_k \ : \ k \in \mathbb{Z}\}) := \lim_{r \to \infty} \frac{n^+(r)}{r} \in [0, \infty],$$

where $n^+(r)$ denotes the largest number of terms of the sequence $(\omega_k)_{k \in \mathbb{Z}}$ contained in an interval of length r. It is shown in [6, Proposition 1.4] that the upper density is finite if and only if the following *weakened gap condition* is satisfied: there exists an integer $M \geq 1$ and a real number $\gamma > 0$ such that

$$\omega_{k+M} - \omega_k \geq M\gamma \quad \text{for all} \quad k \in \mathbb{Z}. \tag{13}$$

(This proposition is crucial for enabling us to consider *infinite* string and beam systems.)

If the sequence is *uniformly separated*, i.e., if (13) is satisfied with $M = 1$ *(uniform gap condition)*, then $D^+ \le \frac{1}{\gamma}$.

First we state a vectorial generalization of Parseval's formula. Given two expressions $A(u)$ and $B(u)$ depending on the solutions henceforth we write $A \lesssim B$ or $B \gtrsim A$ if $A \le cB$ with some constant $c > 0$, independent of the choice of the particular solution. Furthermore, we write $A \asymp B$ if $A \lesssim B$ and $A \gtrsim B$.

Theorem 3.1. *Let $(\omega_k)_{k\in\mathbb{Z}}$ be a* uniformly separated *increasing sequence with upper density D^+, $(U_k)_{k\in\mathbb{Z}}$ a sequence of unit vectors in some complex Hilbert space G, and I a bounded interval of length $|I|$.*

(i) *The functions*
$$x(t) = \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}$$
are well defined in $L^2(I;G)$ for all square summable sequences (x_k) of complex numbers, and
$$\int_I \|x(t)\|_G^2 \ dt \lesssim \sum_{k\in\mathbb{Z}} |x_k|^2 .$$

(ii) *If $|I| > 2\pi D^+$, then the inverse inequality also holds:*
$$\int_I \|x(t)\|_G^2 \ dt \gtrsim \sum_{k\in\mathbb{Z}} |x_k|^2 .$$

Proof. The scalar case $G = \mathbb{C}$ is classical, due to Ingham [14] and Beurling [9]; see also [17] for higher-dimensional generalizations. In the general case we choose an orthonormal basis $(E_n)_{n\in N}$ of the linear hull of (U_k) in G, and we write $U_k = \sum_{n\in N} u_{k,n} E_n$ with suitable complex numbers $u_{k,n}$. Then we have
$$\int_I \|x(t)\|_G^2 \ dt = \int_I \left\| \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t} \right\|_G^2 \ dt = \sum_{n\in N} \int_I \left| \sum_{k\in\mathbb{Z}} x_k u_{k,n} e^{i\omega_k t} \right|^2 \ dt$$
Applying the scalar case of the theorem to each integral on rifght hand side, and using the Bessel equality
$$\sum_{n\in N} |x_k u_{k,n}|^2 = |x_k|^2 \|U_k\|^2 = |x_k|^2$$
for every k, the theorem follows. □

Remark 4. In the scalar case Beurling proved that the value $2\pi D^+$ is the best possible. In the vectorial case the best possible value may be smaller: see [8].

Next we recall from [6] (see also [19, Theorem 9.4]) a generalization of the scalar case of Theorem 3.1 for arbitrary increasing sequences $(\omega_k)_{k\in\mathbb{Z}}$ having a finite upper density:

Theorem 3.2. *Let $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence with a finite upper density D^+, and I a bounded interval of length $|I|$.*

(i) *The functions*
$$x(t) = \sum_{k\in\mathbb{Z}} x_k e^{i\omega_k t}$$

are well defined in $L^2(I)$ for all square summable sequences (x_k) of complex numbers, and

$$\int_I |x(t)|^2 \ dt \lesssim \sum_{k\in\mathbb{Z}} |x_k|^2 .$$

(ii) *There exists another basis $(f_k(t))$ of the linear span of the functions $(e^{i\omega_k t})$ such that if we rewrite the functions $x(t)$ in this basis:*

$$x(t) = \sum_{k\in\mathbb{Z}} x_k e^{i\omega_k t} = \sum_{k\in\mathbb{Z}} y_k f_k(t),$$

then

$$y_k = 0 \quad \text{for all} \quad k \Longleftrightarrow x_k = 0 \quad \text{for all} \quad k,$$

and

$$\int_I |x(t)|^2 \ dt \gtrsim \sum_{k\in\mathbb{Z}} |y_k|^2$$

whenever $|I| > 2\pi D^+$.

Remark 5. The value $2\pi D^+$ is still the best possible: see [25].

Remark 6. In fact, the theorem in [6] is more precise because the new basis is explicitly defined by Newton's formula of divided differences. Hence there is an estimate between the coefficients x_k and y_k. To explain this, let $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence satisfying the weakened gap condition (13), and fix $\gamma' \in (0,\gamma]$ arbitrarily. Then we may partition the sequence (ω_n) into disjoint finite subsequences $\omega_{j+1},\dots,\omega_{j+m}$ with $1 \le m \le M$, such that

$$\omega_i - \omega_{i-1} < \gamma' \quad \text{for} \quad i = j+2,\dots,j+m,$$

but

$$\omega_{j+1} - \omega_j \ge \gamma' \quad \text{and} \quad \omega_{j+m+1} - \omega_{j+m} \ge \gamma'.$$

For each such group we define the divided differences $e_{j+1},\dots,e_{j+m}$ of the exponential functions $e^{i\omega_{j+1}t},\dots,e^{i\omega_{j+m}t}$ by the formula

$$e_k(t) := \prod_{p=j}^{k} \left(\prod_{q=j,q\neq p}^{k} (\omega_p - \omega_q) \right)^{-1} e^{i\omega_p t} \quad \text{for all} \quad k = j+1,\dots,j+m.$$

Then we have

$$\sum_{k=-\infty}^{\infty} x_k e^{i\omega_k t} = \sum_{k=-\infty}^{\infty} y_k e_k(t)$$

with an invertible linear transformation

$$(x_{j+1},\dots,x_{j+m}) \mapsto (y_{j+1},\dots,y_{j+m}).$$

Furthermore, we may infer from the structure of the divided differences that $y_{j+1} = x_{j+1}$ whenever $m = 1$, and there exists a constant $c > 0$ such that

$$\min\left\{|\omega_p - \omega_q| \ : \ j+1 \le p < q \le j+m\right\}^{2M-2} \sum_{k=j+1}^{j+m} |x_n|^2 \le c \sum_{k=j+1}^{j+m} |y_n|^2$$

for all j with $m \ge 2$.

We end this section by stating a consequence of Theorem 3.2 for vector valued functions.

Corollary 1. *Let $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence with a finite upper density D^+, $(U_k)_{k\in\mathbb{Z}}$ a sequence of unit vectors in some complex Hilbert space G, and I a bounded interval of length $|I|$.*

(i) *The functions*

$$x(t)=\sum_{k\in\mathbb{Z}} x_kU_ke^{i\omega_k t}$$

are well defined in $L^2(I;G)$ for all square summable sequences (x_k) of complex numbers, and

$$\int_I \|x(t)\|_G^2\ dt \lesssim \sum_{k\in\mathbb{Z}} |x_k|^2 .$$

(ii) *If $x(t)=0$ for all $t\in I$ for some interval I of length $>2\pi D^+$, then all coefficients x_k vanish.*

Proof. Choosing an orthonormal basis $(E_n)_{n\in N}$ of the linear hull of (U_k), and writing $U_k=\sum_{n\in N}u_{k,n}E_n$ with suitable complex numbers $u_{k,n}$ as in the proof of Theorem 3.1, we have

$$\int_I \|x(t)\|_G^2\ dt=\int_I \left\|\sum_{k\in\mathbb{Z}} x_kU_ke^{i\omega_k t}\right\|_G^2\ dt=\sum_{n\in N}\int_I \left|\sum_{k\in\mathbb{Z}} x_ku_{k,n}e^{i\omega_k t}\right|^2\ dt \tag{14}$$

and

$$\sum_{n\in N}|x_ku_{k,n}|^2=|x_k|^2\|U_k\|^2=|x_k|^2 \tag{15}$$

for every k.

(i) Applying Theorem 3.2 (i) to each integral on the right hand side of (14), and using (15), we have

$$\int_I \|x(t)\|_G^2\ dt\lesssim \sum_{n\in N}\sum_{k\in\mathbb{Z}}|x_ku_{k,n}|^2=\sum_{k\in\mathbb{Z}}|x_k|^2 .$$

We have used here the fact that the hidden constants in the relations $\lesssim$ do not depend on $n\in N$ because we apply Theorem 3.2 for the same exponent sequence (ω_k).

(ii) Applying Theorem 3.2 (ii) to each integral on the right hand side of (14) we obtain that $x_ku_{k,n}=0$ for all k and n. Since $\sum_{n\in N}|u_{k,n}|^2=\|U_k\|^2=1$ for all k, for each k there exists n such that $u_{k,n}\neq 0$, and therefore $x_k=0$. $\square$

Remark 7. Let us introduce the sets

$$\Omega_n:=\{\omega_k\ :\ u_{k,n}\neq 0\},\quad n\in N.$$

The proof of Corollary 1 (see (14)) shows that we may replace D^+ by

$$\min\left\{D^+(\Omega_n)\ :\ n\in N\right\}.$$

4. **Proof of Theorems 2.1 and 2.3.** We only prove Theorem 2.3 because Theorem 2.1 is similar and simpler. It follows by an elementary consideration using translation invariance that there exists a value $T_0 \in [0, \infty]$ such that the uniqueness property in Theorem 2.3 holds on every interval of length $> T_0$, and it fails on every interval of length $< T_0$.

We recall that the solution of (4) have the form

$$u_j(t, r, \varphi_j, \theta_j) = \sum_{k=1}^{\infty} \left(b_{j,k} e^{i\omega_{j,k} t} + b_{j,-k} e^{-i\omega_{j,k} t}\right) \sin\left(\frac{k\pi r}{\ell_j}\right), \quad j = 1, 2, \dots, J$$

with

$$\omega_{j,k} := \sqrt{\left(\frac{k\pi}{\ell_j}\right)^2 + a_j}.$$

Hence

$$\sum_{j=1}^{J} u_{j,r}(\cdot, 0, \varphi_j, \theta_j) v_j = \sum_{j=1}^{J} \sum_{k=1}^{\infty} \left(\frac{k\pi}{\ell_j}\right) \left(b_{j,k} e^{i\omega_{j,k} t} + b_{j,-k} e^{-i\omega_{j,k} t}\right). \tag{16}$$

Furthermore,we obtain by a direct computation that

$$\|u_{j0}\|^2_{H_0^1(0,\ell_j)} = \frac{\ell_j}{2} \sum_{k=1}^{\infty} \left(\frac{k\pi}{\ell_j}\right)^2 |b_{j,k} + b_{j,-k}|^2$$

and

$$\|u_{j1}\|^2_{L^2(0,\ell_j)} = \frac{\ell_j}{2} \sum_{k=1}^{\infty} \omega_{j,k}^2 |b_{j,k} - b_{j,-k}|^2$$

whence

$$\sum_{j=1}^{J} \frac{1}{\ell_j} \left(\|u_{j0}\|^2_{H_0^1(0,\ell_j)} + \|u_{j1}\|^2_{L^2(0,\ell_j)}\right) \asymp \sum_{j=1}^{J} \sum_{k=1}^{\infty} \left(\frac{k\pi}{\ell_j}\right)^2 \left(|b_{j,k}|^2 + |b_{j,-k}|^2\right). \tag{17}$$

(We have an equality if $a_j = 0$ for all j.) Thanks to assumption (6) we may combine all exponents $\omega_{j,k'}$ into a unique increasing sequence (ω_k). Setting $U_k := v_j$ if $\omega_k = \pm\omega_{j,k'}$, the theorem will follow from (16) and (17) with some $T_0 \le 2\sum_{j=1}^{J} \ell_j$ by applying Corollary 1 and the following

Lemma 4.1. *The sequence (ω_k) has a finite upper density $D^+ = \frac{1}{\pi}\sum_{j=1}^{J} \ell_j$.*

Proof. Since $\omega_{j,k} - \frac{k\pi}{\ell_j} \to 0$ as $k \to \infty$, each set

$$A_j := \{\omega_{j,k} \ : \ k = 1, 2, \dots\}$$

has the upper density ℓ_j/π. Moreover, every interval of length r contains

$$\frac{r\ell_j}{\pi} + O(1) \quad (r \to \infty)$$

elements of A_j. Therefore

$$D^+ = \limsup_{r\to\infty} \frac{n^+(r)}{r} = \limsup_{r\to\infty} \left(\frac{\ell_1 + \dots + \ell_J}{\pi} + O(1/r)\right) = \frac{\ell_1 + \dots + \ell_J}{\pi}. \quad \square$$

It remains to show that $T_0 \ge 2\ell_m$ for every m. For any fixed m, given an arbitrary interval I of length $< 2\ell_m$, by a well-known classical result (see for instance [18, Ch. 3]) on the one-dimensional wave equation there exists a non-trivial solution of the problem

$$\begin{cases} u_{m,tt} - u_{m,rr} + a_m u_m = 0 & \text{in } \ \mathbb{R} \times (0,\ell_m), \\ u_m(t,0,\varphi_m,\theta_m) = u_m(t,\ell_m,\varphi_m,\theta_m) = 0 & \text{for } \ t \in \mathbb{R}, \\ u_m(0,r,\varphi_m,\theta_m) = u_{m0}(r) \quad \text{and} \quad u_{m,t}(0,r,\varphi_m,\theta_m) = u_{m1}(r) & \text{for } \ r \in (0,\ell_m) \end{cases}$$

such that

$$u_{m,r}(t,0,\varphi_m,\theta_m) = 0 \quad \text{for all} \quad t \in I.$$

Choosing u_j to be identically zero for all $j \ne m$, we obtain a non-trivial solution of the system (4) for which the right hand side of (7) vanishes on I.

5. **Proof of Theorem 2.2.** Instead of (17) now we have for every real number s the following equality by a direct computation:

$$\sum_{j=1}^J \frac{1}{\ell_j} \left(\|u_{j0}\|^2_{D^s(0,\ell_j)} + \|u_{j1}\|^2_{D^{s-1}(0,\ell_j)} \right) = \sum_{j=1}^J \sum_{k=1}^\infty \left(\frac{k\pi}{\ell_j} \right)^{2s} \left(|b_{j,k}|^2 + |b_{j,-k}|^2 \right).$$

We used the assumption $a_j = 0$ implying that $\omega_{j,k} = k\pi/\ell_j$. In view of (16) we have to find a real number s such that

$$\sum_{j=1}^J \sum_{k=1}^\infty \left(\frac{k\pi}{\ell_j} \right)^{2s} \left(|b_{j,k}|^2 + |b_{j,-k}|^2 \right) \lesssim \int_I \left\| \sum_{j=1}^J \left(\sum_{k=1}^\infty \left(\frac{k\pi}{\ell_j} \right) (b_{j,k} e^{i\omega_{j,k} t} + b_{j,-k} e^{-i\omega_{j,k} t}) \right) v_j \right\|^2 \, dt$$

whenever $|I| > 2\sum_{j=1}^J \ell_j$. Setting $x_{j,\pm k} := \frac{k\pi}{\ell_j} b_{j,\pm k}$ we may rewrite it in the form

$$\sum_{j=1}^J \sum_{k=1}^\infty \omega_{j,k}^{2s-2} \left(|x_{j,k}|^2 + |x_{j,-k}|^2 \right) \lesssim \int_I \left\| \sum_{j=1}^J \left(\sum_{k=1}^\infty (x_{j,k} e^{i\omega_{j,k} t} + x_{j,-k} e^{-i\omega_{j,k} t}) \right) v_j \right\|^2 \, dt.$$

Choosing an orthonormal basis $(E_n)_{n\in N}$ of the linear hull of (U_k) as in the proof of Theorem 3.1 and writing $v_j = \sum_{n\in N} v_{j,n} E_n$, this is equivalent to

$$\sum_{j=1}^J \sum_{k=1}^\infty \omega_{j,k}^{2s-2} \left(|x_{j,k}|^2 + |x_{j,-k}|^2 \right) \lesssim \sum_{n\in N} \int_I \left| \sum_{j=1}^J \left(\sum_{k=1}^\infty x_{j,k} e^{i\omega_{j,k} t} + x_{j,-k} e^{-i\omega_{j,k} t} \right) v_{j,n} \right|^2 \, dt. \tag{18}$$

Now we need a lemma.

Lemma 5.1. *Assume (11), and let $0 < \gamma'' < \min_j \frac{\pi}{\ell_j}$. Then*

$$0 < |\omega_{j,m} - \omega_{k,n}| \le \gamma'' \Longrightarrow j \ne k,$$

and there exists a positive constant B such that

$$|\omega_{j,m} - \omega_{k,n}| \ge \frac{B}{|\omega_{j,m}|} \tag{19}$$

whenever $(j,m)\neq(k,n)$.

Proof. If $|\omega_{j,m}-\omega_{j,n}|\le\gamma''$ for some j,m,n, then, since the sequence $(\omega_{j,m})_{m\in\mathbb{Z}^*}=(\pi m/\ell_j)_{m\in\mathbb{Z}^*}$ has a uniform gap $\pi/\ell_j>\gamma''$, we have $m=n$, and therefore $|\omega_{j,m}-\omega_{j,n}|=0$. This proves the first implication.

Next we have

$$|\omega_{j,m}-\omega_{k,n}|=\pi\left|\frac{m}{\ell_j}-\frac{n}{\ell_k}\right|=\frac{\pi}{\ell_k}\left|m\frac{\ell_k}{\ell_j}-n\right|\ge\frac{\pi}{\ell_k}\operatorname{dist}\left(m\frac{\ell_k}{\ell_j},\mathbb{Z}\right).$$

Thanks to the quadratic irrationality assumption and the corresponding Diophantine approximation property, with suitable constants $A_{k,j}$ we have

$$\frac{\pi}{\ell_k}\operatorname{dist}\left(m\frac{\ell_k}{\ell_j},\mathbb{Z}\right)\ge\frac{\pi A_{k,j}}{\ell_k\,|m|}=\frac{\pi^2A_{k,j}}{\ell_k\ell_j|\omega_{j,m}|},$$

and the lemma follows with

$$B:=\min_{j\neq k}\frac{\pi^2A_{k,j}}{\ell_k\ell_j}.$$

□

Proof of (18). Introducing the sequence (ω_k) as in the preceding section, it suffices to show that

$$\sum_{k=-\infty}^{\infty}|\omega_k|^{2s-2}|x_k|^2\lesssim\int_I\left|\sum_{k=-\infty}^{\infty}x_ke^{i\omega_kt}\right|^2\,dt.\tag{20}$$

Indeed, applying for each fixed $n\in N$ this estimate with $x_k:=x_{j,\pm k'}v_{j,n}$ where $\omega_k=\pm\omega_{j,k'}$, (18) will follow because $\sum_{n\in N}|v_{j,n}|^2=1$.

Applying Theorem 3.2 we obtain for every bounded interval of length $>2\sum_{j=-\infty}^{J}\ell_j$ the relation

$$\int_I\left|\sum_{k=-\infty}^{\infty}x_ke^{i\omega_kt}\right|^2\,dt\gtrsim\sum_{k=-\infty}^{\infty}|y_k|^2.$$

Using Remark 6 hence we infer that

$$\int_I\left|\sum_{k=-\infty}^{\infty}x_ke^{i\omega_kt}\right|^2\,dt\gtrsim\sum_{k=-\infty}^{\infty}\delta_k\,|x_k|^2$$

with

$$\delta_k:=\min\{|\omega_p-\omega_q|\;:\;j+1\le p<q\le j+m\}^{2M-2}$$

if $j+1\le k\le j+m$ for some finite sequence $j+1,\dots,j+m$ of length $m\ge2$ in the partition of Remark 6, and $\delta_k:=1$ otherwise.

If we choose $0<\gamma'<\min_j\frac{\pi}{M\ell_j}$ in Remark 6, and then we apply Lemma 5.1 with $\gamma'':=M\gamma'$, then we get

$$\delta_k\gtrsim|\omega_k|^{2-2M}$$

whenever $j+1\le k\le j+m$ for some finite sequence $j+1,\dots,j+m$ of length $m\ge2$. Since $M\ge1$ and $|\omega_k|\to\infty$ as $|k|\to\infty$, this remains valid also by including the values of k for which $\delta_k=1$, so that

$$\sum_{k=-\infty}^{\infty}|\omega_k|^{2-2M}|x_k|^2\lesssim\int_I\left|\sum_{k=-\infty}^{\infty}x_ke^{i\omega_kt}\right|^2\,dt.$$

This proves (20) with $s=2-M$.

We complete the proof by observing that, since (ω_k) is the union of J uniformly separated sequences, we may choose $M=J$ in the application of Remark 6: if γ' is chosen small enough, then no chain of close exponents can contain more than one exponent corresponding to each string, so that it contains at most J elements. $\square$

6. **Proof of Theorem 2.4.** In this section we set $\omega_{j,k}=\left(\frac{k\pi}{\ell_j}\right)^2$ for brevity, and we denote by $(\omega_k)_{k=-\infty}^{\infty}$ the increasing enumeration of the elements of the set $\{\pm\omega_{j,k}\ :\ j,k\ge 1\}$.

We need a variant of Lemma 5.1.

Lemma 6.1. *Under the conditions of Theorem 2.4 we have*

$$(j,m)\ne(k,n)\Longrightarrow \omega_{j,m}\ne\omega_{k,n}, \tag{21}$$

and the combined sequence $(\omega_k)_{k=-\infty}^{\infty}$ is uniformly separated.

Proof. The condition (10) implies (21). Since $\omega_{j,-m}=-\omega_{j,m}$, and

$$\inf\{\omega_{j,m}\ :\ j,m\ge 1\}=\inf\left\{\left(\frac{\pi}{\ell_j}\right)^2\ :\ j\ge 1\right\}>0$$

because $\pi^2/\ell_j^2\to+\infty$, it suffices to show that

$$\left(\frac{m\pi}{\ell_j}\right)^2-\left(\frac{n\pi}{\ell_k}\right)^2\ge\min\left\{2B,\min_j\frac{3\pi^2}{\ell_j^2}\right\}\quad\text{whenever}\quad\frac{m\pi}{\ell_j}>\frac{n\pi}{\ell_k}>0,$$

where B is the positive constant in Lemma 5.1. If $j=k$, then this follows from the inequality

$$m^2-n^2=(m+n)(m-n)\ge m+n\ge 3.$$

If $j\ne k$, then applying Lemma 5.1 we have

$$\left(\frac{m\pi}{\ell_j}\right)^2-\left(\frac{n\pi}{\ell_k}\right)^2=\left(\frac{m\pi}{\ell_j}+\frac{n\pi}{\ell_k}\right)\cdot\left(\frac{m\pi}{\ell_j}-\frac{n\pi}{\ell_k}\right)\ge\frac{2m\pi}{\ell_j}\cdot\frac{B\ell_j}{m\pi}=2B. \qquad \square$$

Proof of Theorem 2.4. We proceed as in the proof of Theorem 2.2, by taking

$$\omega_{j,k}=\left(\frac{k\pi}{\ell_j}\right)^2\quad\text{instead of}\quad\omega_{j,k}=\frac{k\pi}{\ell_j}.$$

Then (16) remains valid by replacing the sums $\sum_{j=1}^{J}$ by $\sum_{j=1}^{\infty}$, while (17) is replaced by the equality

$$\sum_{j=1}^{\infty}\frac{1}{\ell_j}\left(\|u_{j0}\|^2_{H_0^1(0,\ell_j)}+\|u_{j1}\|^2_{H^{-1}(0,\ell_j)}\right)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{k\pi}{\ell_j}\right)^2\left(|b_{j,k}|^2+|b_{j,-k}|^2\right).$$

Since the combined sequence $(\omega_k)_{k=-\infty}^{\infty}$ is uniformly separated by Lemma 6.1, it has a finite upper density D^+. Therefore the theorem follows by applying Corollary 1 with $T_0:=2\pi D^+$. $\square$

Remark 8. We show that the crucial Lemma 6.1 and Theorem 2.4 have no counterparts for infinite string systems. For this we show that if (ℓ_j) is an infinite sequence of positive numbers such that ℓ_j/ℓ_m is irrational for all $j\ne m$, then the sequence formed by the numbers $k\pi/\ell_j$ with $k\in\mathbb{Z}^*$ and $j=1,2,\dots$ is not uniformly separated, and it even has an infinite upper density. We need the following theorem of Minkowski [10, Corollary 3, p. 151] concerning a system of inequalities

$$\left|\sum_{j=1}^{n}a_{ij}k_j\right|<c_i,\quad i=1,\dots,m \tag{22}$$

for given positive integers $m \le n$, and real numbers a_{ij}, c_i with $c_i > 0$: if $m < n$, or $m = n$ and $\det(a_{ij}) = 0$, then there are integers $k_1, \dots, k_n$, not all zero, such that (22) is satisfied.

Fix an arbitrarily large positive integer n, and a positive number

$$\varepsilon < \min \left\{ \frac{\pi}{\ell_1}, \dots, \frac{\pi}{\ell_n} \right\}.$$

Applying Minkowski's theorem with $m = n-1$, $c_1 = \dots = c_{n-1} = \varepsilon$ and

$$a_{ij} := \begin{cases} \frac{\pi}{\ell_1} & \text{if } j = 1, \\ -\frac{\pi}{\ell_{i+1}} & \text{if } j = i+1, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \dots, n-1$, we obtain that there exist integers $k_1, \dots, k_m$, not all zero, such that

$$\left| \frac{k_1 \pi}{\ell_1} - \frac{k_{i+1} \pi}{\ell_{i+1}} \right| < \varepsilon \quad \text{for} \quad i = 1, \dots, n-1. \tag{23}$$

Now it follows from the choice of ε that *all* integers $k_1, \dots, k_m$ are different from zero. Indeed, if $k_1 = 0$, then

$$\left| \frac{k_{i+1} \pi}{\ell_{i+1}} \right| = \left| \frac{k_1 \pi}{\ell_1} - \frac{k_{i+1} \pi}{\ell_{i+1}} \right| < \varepsilon < \frac{\pi}{\ell_{i+1}}$$

for every $i = 1, \dots, n-1$, so that all integers $k_2, \dots, k_n$ also vanish, contradicting our choice.

Next, if $k_{i+1} = 0$ for some $i = 1, \dots, n-1$, then

$$\left| \frac{k_1 \pi}{\ell_1} \right| = \left| \frac{k_1 \pi}{\ell_1} - \frac{k_{i+1} \pi}{\ell_{i+1}} \right| < \varepsilon < \frac{\pi}{\ell_1},$$

and therefore $k_1 = 0$. This leads to the contradiction $k_1 = \ldots = k_n = 0$ as before.

Since $k_1, \dots, k_n \in \mathbb{Z}^*$, we infer from (23) that $n^+(r) \ge n$ for all $r > 2\varepsilon$, and a fortiori for all $r > 2\pi/\ell_1$. Since n was arbitrary, hence $n^+(r) = +\infty$ for all $r > 2\pi/\ell_1$, and therefore $D^+ = +\infty$.

7. **Proof of Proposition 1.**

A classical result of Liouville (1844) on Diophantine approximation states that if p is an algebraic number of degree $n \ge 2$, then there exists a constant $A(p) > 0$ such that

$$\operatorname{dist}(kp, \mathbb{Z}) \ge \frac{A(p)}{|k|^{n-1}}$$

for all nonzero integers k. We shall need an explicit value of $A(p)$ in case $n = 2$.

Lemma 7.1. *Let p be a non purely quadratic irrational number, i.e., having as minimal polynomial $f(x) = ax^2 - bx + c$, with $a, b, c \in \mathbb{Z}$, $a, c \ne 0$. Set $\Delta := b^2 - 4ac > 0$ and*

$$A(p) := \frac{1}{4|a|} \left(\sqrt{\Delta + 8|a|} - \sqrt{\Delta} \right) = \frac{2}{\sqrt{\Delta + 8|a|} + \sqrt{\Delta}}.$$

Then

$$\operatorname{dist}(kp, \mathbb{Z}) \ge \frac{A(p)}{|k|}$$

for all nonzero integers k.

Proof. We have to prove the inequality

$$\left|p - \frac{n}{k}\right| \ge \frac{A(p)}{k^2}$$

for all $n, k \in \mathbb{Z}$, $k \ne 0$. The inequality is obvious if the left hand side is $\ge A(p)$.

Henceforth we assume that $\left|p - \frac{n}{k}\right| < A(p)$. First we observe that, since f has integer coefficients, $|f(n/k)| \ge 1/k^2$. Next, by the mean value theorem there exists a real number ξ between p and n/k such that

$$\left|p - \frac{n}{k}\right| = \frac{|f(n/k)|}{|f'(\xi)|},$$

and therefore

$$\left|p - \frac{n}{k}\right| \ge \frac{1}{k^2|f'(\xi)|}.$$

We conclude the proof by showing that $\frac{1}{|f'(\xi)|} \ge A(p)$. We have

$$\begin{aligned}|f'(\xi)| = 2|a|\left|\xi - \frac{b}{2a}\right| &\le 2|a|\left(|\xi - p| + \left|p - \frac{b}{2a}\right|\right)\\ &\le 2|a|\left(\left|\frac{n}{k} - p\right| + \left|p - \frac{b}{2a}\right|\right)\\ &< 2|a|\left(A(p) + \frac{\sqrt{\Delta}}{2|a|}\right),\end{aligned}$$

and therefore

$$\frac{1}{|f'(\xi)|} > \frac{1}{2|a|A(p) + \sqrt{\Delta}} = \frac{2}{\sqrt{\Delta + 8|a|} + \sqrt{\Delta}} = A(p). \qquad \square$$

Proof of Proposition 1 (i). Since the set of Perron numbers is closed for multiplications [27], q^n is irrational for every positive integer n, and hence also for every nonzero integer n. Therefore, fixing $j \ne m$ arbitrarily, $p := \ell_m/\ell_j = q^{j-m}$ is a non purely quadratic irrational number as well, thus it satisfies the assumptions of Lemma 7.1. Since

$$\frac{A(p)}{\ell_j \ell_m} = A(q^{j-m})q^{j+m},$$

its suffices to show that

$$\inf_{j \ne m} A(q^{j-m})q^{j+m} > 0.$$

Since $q^{j+m} \ge q^{j-m}$ and $q^{j+m} \ge q^{m-j}$, setting $n := |j - m|$ is suffices to show that

$$\inf_{n \ge 1} A(q^n)q^n > 0 \quad \text{and} \quad \inf_{n \ge 1} A(q^{-n})q^n > 0. \tag{24}$$

Let us recall that Perron numbers are closed under multiplication, see for instance [27]. Therefore q^n is a quadratic Perron number and, consequently, its minimal polynomial has the form $x^2 - bx + c$ with integers $b \ge 1$ and $c \ne 0$ satisfying $\Delta := b^2 - 4c > 0$. Therefore we have $q^n = (b + \sqrt{\Delta})/2$ and

$$A(q^n)q^n = \frac{b + \sqrt{\Delta}}{\sqrt{\Delta + 8} + \sqrt{\Delta}}.$$

Next we remark that the minimal polynomial of q^{-n} is cx^2-bx+1 and it has the same discriminant $\Delta>0$. Therefore

$$A(q^{-n})q^n=\frac{b+\sqrt{\Delta}}{\sqrt{\Delta+8|c|}+\sqrt{\Delta}}.$$

Setting $t:=\Delta/b^2$ we have $8|c|/b^2=2\,|t-1|$, so that

$$A(q^{-n})q^n=\frac{1+\sqrt{t}}{\sqrt{t+2\,|t-1|}+\sqrt{t}}$$

The relations (24) follow by observing that the functions

$$\Delta\mapsto\frac{b+\sqrt{\Delta}}{\sqrt{\Delta+8}+\sqrt{\Delta}}\quad\text{and}\quad t\mapsto\frac{1+\sqrt{t}}{\sqrt{t+2\,|t-1|}+\sqrt{t}}$$

are continuous in $[0,\infty)$, positive, and have positive limits at ∞, and hence they are bounded from below by some positive constants. $\square$

Proof of Proposition 1 (ii). First we show that $x:=(\ell_j/\ell_m)^2$ is irrational for all $j\neq m$. We may asume that $m>j$. A straightforward computation yields the equality

$$\left(2^{m+1}-2^{j+1}x\right)\sqrt{2}=\left(2^{2j}+2\right)x-\left(2^{2m}+2\right)$$

If x were rational, then the irrationality of $\sqrt{2}$ would imply the equalities

$$2^{m+1}-2^{j+1}x=0\quad\text{and}\quad\left(2^{2j}+2\right)x-\left(2^{2m}+2\right)=0.$$

Eliminating x hence we would infer that

$$2^m+2^{1-m}=2^j+2^{1-j}.$$

However, since $m>j\geq 1$, we have

$$2^m+2^{1-m}>2^m\geq 2^{j+1}\geq 2^j+2>2^j+2^{1-j}.$$

Now we prove (11). If $j\neq m$, then $p:=\ell_j/\ell_m$ satisfies the relations

$$\begin{aligned}(2^j+\sqrt{2})p=2^m+\sqrt{2}&\iff 2^jp-2^m=\sqrt{2}(1-p)\\&\iff 2^{2j}p^2-2^{j+m+1}p+2^{2m}=2p^2-4p+2\\&\iff(2^{2j}-2)p^2-(2^{j+m+1}-4)p+(2^{2m}-2)=0,\end{aligned}$$

so that the minimal polynomial of p is

$$f_{j,m}(x)=(2^{2j}-2)x^2-(2^{j+m+1}-4)x+(2^{2m}-2).$$

Hence

$$A(p)=\frac{2}{\sqrt{\Delta+8(2^{2j}-2)}+\sqrt{\Delta}}>\frac{1}{\sqrt{\Delta+8(2^{2j}-2)}}$$

with

$$\Delta=(2^{j+m+1}-4)^2-4(2^{2j}-2)(2^{2m}-2).$$

This implies the inequality

$$\Delta+8(2^{2j}-2)<2^{2j+4}+2^{2m+4},$$

and therefore

$$\frac{A(\ell_j/\ell_m)}{\ell_j\ell_m}>\frac{(2^j+\sqrt{2})(2^m+\sqrt{2})}{\sqrt{2^{2j+4}+2^{2m+4}}}>\frac{2^{j+m}}{2^{j+2}+2^{m+2}}.$$

We conclude by observing that the last expression is $> 1/4$ for all $j \neq m$. Indeed, assuming by symmetry that $j > m$, we have

$$\frac{2^{j+m}}{2^{j+2}+2^{m+2}} > \frac{2^{j+1}}{2^{j+2}+2^{j+2}} = 1/4. \qquad \square$$

Proof of Proposition 1 (iii). Let $p := \ell_{j-1}/\ell_{m-1}$ for some $j \neq m$, $j, m \geq 2$. First we prove (10). We have

$$\begin{aligned} p^2 = \left(\frac{m+\sqrt{2}}{j+\sqrt{2}}\right)^2 &= \frac{m^2+2+2\sqrt{2}m}{j^2+2+2\sqrt{2}j} \\ &= \frac{(m^2+2+2\sqrt{2}m)(j^2+2-2\sqrt{2}j)}{(j^2+2)^2-8j^2} \\ &= \frac{(m^2+2)(j^2+2)-8mj+2\sqrt{2}(mj-2)(j-m)}{(j^2+2)^2-8j^2}. \end{aligned}$$

Since $(mj-2)(j-m) \neq 0$ by our assumptions on j and m, we conclude that p^2 is irrational.

Next we prove (11). First we observe the following imoplications:

$$\begin{aligned} (j+\sqrt{2})p = m+\sqrt{2} &\Longrightarrow jp - m = \sqrt{2}(1-p) \\ &\Longrightarrow j^2p^2 - 2jmp + m^2 = 2p^2 - 4p + 2 \\ &\Longrightarrow (j^2-2)p^2 - 2(jm-2)p + (m^2-2) = 0. \end{aligned}$$

Since p is irrational by the first part of the proof, it follows that the minimal polynomial of p is

$$f_{j,m}(x) = (j^2-2)x^2 - 2(jm-2)x + (m^2-2).$$

Hence

$$A(p) = \frac{2}{\sqrt{\Delta + 8(j^2-2)} + \sqrt{\Delta}} > \frac{1}{\sqrt{\Delta + 8(j^2-2)}}$$

with

$$\Delta = 4(jm-2)^2 - 4(j^2-2)(m^2-2) = 8(j-m)^2 < 8j^2 + 8m^2.$$

Hence,

$$\Delta + 8(j^2-2) < 16j^2 + 16m^2$$

and therefore (we recall that $j, m \geq 2$)

$$\frac{A(\ell_{j-1}/\ell_{m-1})}{\ell_{j-1}\ell_{m-1}} > \frac{(j+\sqrt{2})(m+\sqrt{2})}{\sqrt{16j^2+16m^2}} > \frac{\sqrt{2}(j+m)}{4(j+m)} = \frac{1}{2\sqrt{2}}.$$

Since the last positive lower bound is independent of the choice of j and m, the condition (11) follows by Lemma 7.1.

Applying Theorem 2.4 we conclude the observability relations for some $T_0 < \infty$. $\square$

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