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SYNCHRONIZATION OF A KURAMOTO-LIKE MODEL FOR POWER GRIDS WITH FRUSTRATION

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Abstract. We discuss the complete synchronization for a Kuramoto-like model for power grids with frustration. For identical oscillators without frustration, it will converge to complete phase and frequency synchronization exponentially fast if the initial phases are distributed in a half circle. For nonidentical oscillators with frustration, we present a framework leading to complete frequency synchronization where the initial phase configurations are located inside the half of a circle. Our estimates are based on the monotonicity arguments of extremal phase and frequency.

1. Introduction.

Synchronization in complex networks has been a focus of interest for researchers from different disciplines[1, 2, 4, 8, 15]. In this paper, we investigate synchronous phenomena in an ensemble of Kuramoto-like oscillators which is regarded as a model for power grid. In [9], a mathematical model for power grid is given by

$$P^i_{source} = I\ddot{\theta}_i\dot{\theta}_i + K_D(\dot{\theta}_i)^2 - \sum_{l=1}^{N} a_{il}\sin(\theta_l - \theta_i), \quad i = 1, 2, \dots, N, \tag{1}$$

where $P^i_{source} > 0$ is power source (the energy feeding rate) of the ith node, $I \geq 0$ is the moment of inertia, $K_D > 0$ is the ratio between the dissipated energy by the turbine and the square of the angular velocity, and $a_{il} > 0$ is the maximum transmitted power between the ith and lth nodes. The phase angle θ_i of ith node is given by $\theta_i(t) = \Omega t + \tilde{\theta}_i(t)$ with a standard frequency Ω (50 or 60 Hz) and small deviations $\tilde{\theta}_i(t)$. Power plants in a big connected network should be synchronized to the same frequency. If loads are too strong and unevenly distributed or if some major fault or a lightening occurs, an oscillator (power plant) may lose synchronization. In that situation the synchronization landscape may change drastically and a blackout may occur. The transient stability of power grids can be regarded as a synchronization problem for nonstationary generator rotor angles aiming to

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restore synchronism subject to local excitations. Therefore, the region of attraction of synchronized states is a central problem for the transient stability.

By denoting $\omega_i = \frac{P^i_{source}}{K_D}$ and assuming $\frac{a_{il}}{K_D} = \frac{K}{N}$ $(\forall i, l \in \{1, 2, \ldots, N\})$ and $I = 0$ in (1), Choi et. al derived a Kuramoto-like model for the power grid as follows [5]:

$$(\dot{\theta}_i)^2 = \omega_i + \frac{K}{N}\sum_{l=1}^{N} \sin(\theta_l - \theta_i), \quad \dot{\theta}_i > 0, \quad i = 1, 2, \ldots, N. \tag{2}$$

Here, the setting $\dot{\theta}_i > 0$ was made in accordance to the observation in system (1) that the grid is operating at a frequency close to the standard frequency with small deviations. In order to ensure that the right-hand side of (2) is positive, it is reasonable to assume that $\omega_i > K,\ i = 1, 2, \ldots, N$. This model can be used to understand the emergence of synchronization in networks of oscillators. As far as the authors know, there are few analytical results. In [5], the authors considered this model with identical natural frequencies and prove that complete phase synchronization occurs if the initial phases are distributed inside an arc with geodesic length less than $\pi/2$.

If $(\dot{\theta}_i)^2$ in (2) is replaced by $\dot{\theta}_i$, the model is the famous Kuramoto model [13]; for its synchronization analysis we refer to [3, 6, 10], etc. Sakaguchi and Kuramoto [16] proposed a variant of the Kuramoto model to describe richer dynamical phenomena by introducing a phase shift (frustration) in the coupling function, i.e., by replacing $\sin(\theta_l - \theta_i)$ with $\sin(\theta_l - \theta_i + \alpha)$. They inferred the need of α from the empirical fact that a pair of oscillators coupled strongly begin to oscillate with a common frequency deviating from the simple average of their natural frequencies. Some physicists also realized via many experiments that the emergence of the phase shift term requires larger coupling strength K and longer relaxation time to exhibit mutual synchronization compared to the zero phase shift case. This is why we call the phase shift a frustration. The effect of frustration has been intensively studied, for example, [11, 12, 14]. In power grid model using Kuramoto oscillators, people use the phase shift to depict the energy loss due to the transfer conductance [7]. In this paper, we will incorporate the phase shift term in (2) and study the Kuramoto-like model with frustration $\alpha \in (-\frac{\pi}{4}, \frac{\pi}{4})$:

$$(\dot{\theta}_i)^2 = \omega_i + \frac{K}{N}\sum_{l=1}^{N} \sin(\theta_l - \theta_i + \alpha), \quad \dot{\theta}_i > 0, \quad i = 1, 2, \ldots, N. \tag{3}$$

We will find a trapping region such that any nonstationary state located in this region will evolve to a synchronous state.

The contributions of this paper are twofold: First, for identical oscillators without frustration, we show that the initial phase configurations located in the half circle will converge to complete phase and frequency synchronization. This extends the analytical results in [5] in which the initial phase configuration for synchronization needs to be confined in a quarter of circle. Second, we consider the nonidentical oscillators with frustration and present a framework leading to the boundness of the phase diameter and complete frequency synchronization. To the best of our knowledge, this is the first result for the synchronization of (3) with nonidentical oscillators and frustration.

The rest of this paper is organized as follows. In Section 2, we recall the definitions for synchronization and summarize our main results. In Section 3, we give synchronization analysis and prove the main results. Finally, Section 4 is devoted to a concluding summary.

Notations. We use the following simplified notations throughout this paper:

$$\begin{aligned}
&\nu_i := \dot{\theta}_i,\ i = 1,2,\dots,N,\ \omega := (\omega_1,\omega_2,\dots,\omega_N),\\
&\bar{\omega} := \max_{1\le i\le N}\omega_i,\ \underline{\omega} := \min_{1\le i\le N}\omega_i,\ D(\omega) := \bar{\omega} - \underline{\omega},\\
&\theta_M := \max_{1\le i\le N}\theta_i,\ \theta_m := \min_{1\le i\le N}\theta_i,\ D(\theta) := \theta_M - \theta_m,\\
&\nu_M := \max_{1\le i\le N}\nu_i,\ \nu_m := \min_{1\le i\le N}\nu_i,\ D(\nu) := \nu_M - \nu_m,\\
&\theta_M^\nu \in \{\theta_j | \nu_j = \nu_M\},\ \theta_m^\nu \in \{\theta_j | \nu_j = \nu_m\}.
\end{aligned}$$

2. **Preliminaries.** In this paper, we consider the system

$$\begin{aligned}
&(\dot{\theta}_i)^2 = \omega_i + \frac{K}{N}\sum_{l=1}^{N}\sin(\theta_l - \theta_i + \alpha), \quad \dot{\theta}_i > 0, \quad \alpha \in (-\frac{\pi}{4}, \frac{\pi}{4}),\\
&\theta_i(0) = \theta_i^0, \qquad i = 1,2,\dots,N.
\end{aligned} \tag{4}$$

Next we introduce the concepts of complete synchronization and conclude this introductory section with the main result of this paper.

Definition 2.1. Let $\theta := (\theta_1,\theta_2,\dots,\theta_N)$ be a dynamical solution to the system (4). We say

1. it exhibits asymptotically complete phase synchronization if
$$\lim_{t\to\infty}(\theta_i(t) - \theta_j(t)) = 0,\ \forall i \neq j.$$
2. it exhibits asymptotically complete frequency synchronization if
$$\lim_{t\to\infty}(\dot{\theta}_i(t) - \dot{\theta}_j(t)) = 0,\ \forall i \neq j.$$

For identical oscillators without frustration, we have the following result.

Theorem 2.2. *Let $\theta := (\theta_1,\theta_2,\dots,\theta_N)$ be a solution to the coupled system (4) with $\alpha = 0, \omega_i = \omega_0, i = 1,2,\dots,N$ and $\omega_0 > K$. If the initial configuration*

$$\theta^0 \in \mathcal{A} := \{\theta \in [0,2\pi)^N :\ D(\theta) < \pi\},$$

then there exits $\lambda_1, \lambda_2 > 0$ and $t_0 > 0$ such that

$$D(\theta(t)) \le D(\theta^0)e^{-\lambda_1 t}, \quad t \ge 0. \tag{5}$$

and

$$D(\nu(t)) \le D(\nu(t_0))e^{-\lambda_2(t-t_0)}, \quad t \ge t_0. \tag{6}$$

Next we introduce the main result for nonidentical oscillators with frustration. For $\bar{\omega} < \frac{1}{2\sin^2|\alpha|}$, we set

$$K_c := \frac{D(\omega)\sqrt{2\bar{\omega}}}{1 - \sqrt{2\bar{\omega}}\sin|\alpha|} > 0.$$

For suitable parameters, we denote by D_1^∞ and D_*^∞ the two angles as follows:

$$\sin D_1^\infty = \sin D_*^\infty := \frac{\sqrt{\bar{\omega} + K}(D(\omega) + K\sin|\alpha|)}{K\sqrt{\underline{\omega} - K}}, \quad 0 < D_1^\infty < \frac{\pi}{2} < D_*^\infty < \pi.$$

Theorem 2.3. *Let* $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ *be a solution to the coupled system* (4) *with* $K > K_c$ *and* $[\underline{\omega}, \bar{\omega}] \subset \left[K+1, \frac{1}{2\sin^2|\alpha|}\right)$*. If*

$$\theta^0 \in \mathcal{B} := \{\theta \in [0, 2\pi)^N \mid D(\theta) < D_*^\infty - |\alpha|\},$$

then for any small $\varepsilon > 0$ *with* $D_1^\infty + \varepsilon < \frac{\pi}{2}$*, there exists* $\lambda_3 > 0$ *and* $T > 0$ *such that*

$$D(\nu(t)) \leq D(\nu(T))e^{-\lambda_3(t-T)}, \ t \geq T. \tag{7}$$

Remark 1. If the parametric conditions in Theorem 2.3 are fulfilled, the reference angles D_1^∞ and D_*^∞ are well-defined. Indeed, because $K > K_c$ and $\bar{\omega} < \frac{1}{2\sin^2|\alpha|}$, we have

$$\frac{D(\omega)\sqrt{2\bar{\omega}}}{1 - \sqrt{2\bar{\omega}}\sin|\alpha|} < K, \qquad 1 - \sqrt{2\bar{\omega}}\sin|\alpha| > 0.$$

This implies

$$\frac{\sqrt{2\bar{\omega}}(D(\omega) + K\sin|\alpha|)}{K} < 1.$$

Then, by $\underline{\omega} \geq K+1$ and $K \leq \bar{\omega}$ we obtain

$$\sin D_1^\infty = \sin D_*^\infty := \frac{\sqrt{\bar{\omega}+K}(D(\omega) + K\sin|\alpha|)}{K\sqrt{\underline{\omega}-K}} \leq \frac{\sqrt{2\bar{\omega}}(D(\omega) + K\sin|\alpha|)}{K} < 1.$$

Remark 2. In order to make $1 < K+1 < \frac{1}{2\sin^2|\alpha|}$, it is necessary to assume $\alpha \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. This is the reason for the setting $\alpha \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

3. Synchronization analysis.

3.1. Synchronization estimates: Identical oscillators without frustration.

In this subsection we consider the system (4) with identical natural frequencies and zero frustration:

$$(\dot{\theta}_i)^2 = \omega_0 + \frac{K}{N}\sum_{l=1}^{N}\sin(\theta_l - \theta_i), \quad \dot{\theta}_i > 0, \quad i = 1, 2, \dots, N. \tag{8}$$

To obtain the complete synchronization, we need to derive a trapping region. We start with two elementary estimates for the transient frequencies.

Lemma 3.1. *Suppose* $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ *be a solution to the coupled system* (8)*, then for any* $i, j \in \{1, 2, \dots, N\}$*, we have*

$$(\dot{\theta}_i - \dot{\theta}_j)(\dot{\theta}_i + \dot{\theta}_j) = \frac{2K}{N}\sum_{l=1}^{N}\cos(\theta_l - \frac{\theta_i + \theta_j}{2})\sin\frac{\theta_j - \theta_i}{2}.$$

Proof. It is immediately obtained by (8). □

Lemma 3.2. *Suppose* $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ *be a solution to the coupled system* (8) *and* $\Omega > K$*, then we have*

$$\dot{\theta}_i \leq \sqrt{\omega_0 + K}.$$

Proof. It follows from (8) and $\dot{\theta}_i > 0$ that we have

$$(\dot{\theta}_i)^2 = \omega_0 + \frac{K}{N}\sum_{l=1}^{N}\sin(\theta_l - \theta_i) \leq \omega_0 + K.$$

□

Next we give an estimate for trapping region and prove Theorem 2.2. For this aim, we will use the time derivative of $D(\theta(t))$ and $D(\nu(t))$. Note that $D(\theta(t))$ is Lipschitz continuous and differentiable except at times of collision between the extremal phases and their neighboring phases. Therefore, for the collision time t at which $D(\theta(t))$ is not differentiable, we can use the so-called Dini derivative to replace the classic derivative. In this manner, we can proceed the analysis with differential inequality for $D(\theta(t))$. For $D(\nu(t))$ we can proceed in the similar way. In the following context, we will always use the notation of classic derivative for $D(\theta(t))$ and $D(\nu(t))$.

Lemma 3.3. *Let $\theta := (\theta_1, \theta_2, \ldots, \theta_N)$ be a solution to the coupled system (8) and $\Omega > K$. If the initial configuration $\theta(0) \in \mathcal{A}$, then $\theta(t) \in \mathcal{A}$, $t \geq 0$.*

Proof. For any $\theta^0 \in \mathcal{A}$, there exists $D^\infty \in (0, \pi)$ such that $D(\theta^0) < D^\infty$. Let

$$\mathcal{T} := \left\{ T \in [0, +\infty) \,\middle|\, D(\theta(t)) < D^\infty, \ \forall\, t \in [0, T) \right\}.$$

Since $D(\theta^0) < D^\infty$, and $D(\theta(t))$ is a continuous function of t, there exists $\eta > 0$ such that

$$D(\theta(t)) < D^\infty, \ t \in [0, \eta).$$

Therefore, the set $\mathcal{T}$ is not empty. Let $T^* := \sup \mathcal{T}$. We claim that

$$T^* = \infty. \tag{9}$$

Suppose to the contrary that $T^* < \infty$. Then from the continuity of $D(\theta(t))$, we have

$$D(\theta(t)) < D^\infty, \ t \in [0, T^*), \quad D(\theta(T^*)) = D^\infty.$$

We use Lemma 3.1 and Lemma 3.2 to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} D(\theta(t))^2 &= D(\theta(t)) \frac{d}{dt} D(\theta(t)) = (\theta_M - \theta_m)\left(\dot{\theta}_M - \dot{\theta}_m\right) \\
&= (\theta_M - \theta_m) \frac{1}{\dot{\theta}_M + \dot{\theta}_m} \frac{2K}{N} \sum_{l=1}^N \cos\left(\theta_l - \frac{\theta_M + \theta_m}{2}\right) \sin\left(\frac{\theta_m - \theta_M}{2}\right) \\
&\leq (\theta_M - \theta_m) \frac{1}{\dot{\theta}_M + \dot{\theta}_m} \frac{2K}{N} \sum_{l=1}^N \cos\frac{D^\infty}{2} \sin\left(\frac{\theta_m - \theta_M}{2}\right) \\
&\leq (\theta_M - \theta_m) \frac{1}{\sqrt{\omega_0 + K}} \frac{K}{N} \sum_{l=1}^N \cos\frac{D^\infty}{2} \sin\left(\frac{\theta_m - \theta_M}{2}\right) \\
&= -\frac{2K \cos\frac{D^\infty}{2}}{\sqrt{\omega_0 + K}} \frac{D(\theta)}{2} \sin\frac{D(\theta)}{2} \\
&\leq -\frac{K \cos\frac{D^\infty}{2}}{\pi\sqrt{\omega_0 + K}} D(\theta)^2, \ t \in [0, T^*).
\end{aligned}$$

Here we used the relations

$$-\frac{D^\infty}{2} < -\frac{D(\theta)}{2} \leq \frac{\theta_l - \theta_M}{2} \leq 0 \leq \frac{\theta_l - \theta_m}{2} \leq \frac{D(\theta)}{2} < \frac{D^\infty}{2}$$

and

$$x \sin x \geq \frac{2}{\pi} x^2, \ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Therefore, we have

$$\frac{d}{dt}D(\theta) \le -\frac{K\cos\frac{D^\infty}{2}}{\pi\sqrt{\omega_0+K}}D(\theta),\ t\in[0,T^*), \tag{10}$$

which implies that

$$D(\theta(T^*)) \le D(\theta^0)e^{-\frac{K\cos\frac{D^\infty}{2}}{\pi\sqrt{\omega_0+K}}T^*} < D(\theta^0) < D^\infty.$$

This is contradictory to $D(\theta(T^*)) = D^\infty$. The claim (9) is proved, which yields the desired result. □

Now we can give a proof for Theorem 2.2.

Proof of Theorem 2.2. According to Lemma 3.3, we substitute $T^* = \infty$ into (10), then (5) is proved with $\lambda_1 = \frac{K\cos\frac{D^\infty}{2}}{\pi\sqrt{\omega_0+K}}$.

On the other hand, by (5) there exist t_0 and $\delta(0<\delta<\frac{\pi}{2})$ such that $D(\theta(t))\le\delta$ for $t\ge t_0$. Now we differentiate (8) to find

$$\dot{\nu}_i = \frac{K}{2N\nu_i}\sum_{l=1}^{N}\cos(\theta_l-\theta_i)(\nu_l-\nu_i).$$

Using Lemma 3.2, we now consider the temporal evolution of $D(\nu(t))$:

$$\begin{aligned}
&\frac{d}{dt}D(\nu) = \dot{\nu}_M - \dot{\nu}_m\\
&= \frac{K}{2N\nu_M}\sum_{l=1}^{N}\cos(\theta_l-\theta_M^\nu)(\nu_l-\nu_M) - \frac{K}{2N\nu_m}\sum_{l=1}^{N}\cos(\theta_l-\theta_m^\nu)(\nu_l-\nu_m)\\
&\le \frac{K\cos\delta}{2N\nu_M}\sum_{l=1}^{N}(\nu_l-\nu_M) - \frac{K\cos\delta}{2N\nu_m}\sum_{l=1}^{N}(\nu_l-\nu_m)\\
&\le \frac{K}{2N}\frac{\cos\delta}{\sqrt{\omega_0+K}}\sum_{l=1}^{N}(\nu_l-\nu_M) - \frac{K}{2N}\frac{\cos\delta}{\sqrt{\omega_0+K}}\sum_{l=1}^{N}(\nu_l-\nu_m)\\
&= \frac{K\cos\delta}{2N\sqrt{\omega_0+K}}\sum_{l=1}^{N}(\nu_l-\nu_M-\nu_l+\nu_m)\\
&= -\frac{K\cos\delta}{2\sqrt{\omega_0+K}}D(\nu),\ t\ge t_0.
\end{aligned}$$

This implies that

$$D(\nu(t)) \le D(\nu(t_0))e^{-\frac{K\cos\delta}{2\sqrt{\omega_0+K}}(t-t_0)},\quad t\ge t_0,$$

and proves (6) with $\lambda_2 = \frac{K\cos\delta}{2\sqrt{\omega_0+K}}$. □

Remark 3. Theorem 2.2 shows, as long as the initial phases are confined inside an arc with geodesic length strictly less than π, complete phase synchronization occurs exponentially fast. This extends the main result in [5] where the initial phases for synchronization need to be confined in an arc with geodesic length less than $\pi/2$.

3.2. **Synchronization estimates: Nonidentical oscillators with frustration.** In this subsection, we prove the main result for nonidentical oscillators with frustration.

Lemma 3.4. *Let* $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ *be a solution to the coupled system* (4), *then for any* $i, j \in \{1, 2, \dots, N\}$, *we have*

$$(\dot{\theta}_i - \dot{\theta}_j)(\dot{\theta}_i + \dot{\theta}_j) \le D(\omega) + \frac{K}{N}\sum_{l=1}^{N}\left[\sin(\theta_l - \theta_i + \alpha) - \sin(\theta_l - \theta_j + \alpha)\right].$$

Proof. By (4) and for any $i, j \in \{1, 2, \dots, N\}$

$$(\dot{\theta}_i - \dot{\theta}_j)(\dot{\theta}_i + \dot{\theta}_j) = (\dot{\theta}_i)^2 - (\dot{\theta}_j)^2,$$

the result is immediately obtained. □

Lemma 3.5. *Let* $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ *be a solution to the coupled system* (4) *and* $\underline{\omega} \ge K + 1$, *then we have*

$$\dot{\theta}_i \in \left[\sqrt{\underline{\omega} - K}, \sqrt{\bar{\omega} + K}\right], \quad \forall i = 1, 2, \dots, N.$$

Proof. From (4), we have

$$\underline{\omega} - K \le (\dot{\theta}_i)^2 \le \bar{\omega} + K, \quad \forall i = 1, 2, \dots, N,$$

and also because $\dot{\theta}_i > 0$. □

The following lemma gives a trapping region for nonidentical oscillators.

Lemma 3.6. *Let* $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ *be a solution to the coupled system* (4) *with* $K > K_c$ *and* $[\underline{\omega}, \bar{\omega}] \subset [K + 1, \frac{1}{2\sin^2|\alpha|})$. *If the initial configuration* $\theta^0 \in \mathcal{B}$, *then* $\theta(t) \in \mathcal{B}$ *for* $t \ge 0$.

Proof. We define the set $\mathcal{T}$ and its supremum:

$$\mathcal{T} := \left\{T \in [0, +\infty) \,\middle|\, D(\theta(t)) < D_*^\infty - |\alpha|,\ \forall\, t \in [0, T)\right\}, \quad T^* := \sup \mathcal{T}.$$

Since $\theta^0 \in \mathcal{B}$, and note that $D(\theta(t))$ is a continuous function of t, we see that the set $\mathcal{T}$ is not empty and T^* is well-defined. We now claim that

$$T^* = \infty.$$

Suppose to the contrary that $T^* < \infty$. Then from the continuity of $D(\theta(t))$, we have

$$D(\theta(t)) < D_*^\infty - |\alpha|,\ t \in [0, T^*), \quad D(\theta(T^*)) = D_*^\infty - |\alpha|.$$

We use Lemma 3.4 to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}D(\theta)^2 &= D(\theta)\frac{d}{dt}D(\theta) = D(\theta)\left(\dot{\theta}_M - \dot{\theta}_m\right)\\ &\le D(\theta)\frac{1}{\dot{\theta}_M + \dot{\theta}_m}\underbrace{\left[D(\omega) + \frac{K}{N}\sum_{l=1}^{N}(\sin(\theta_l - \theta_M + \alpha) - \sin(\theta_l - \theta_m + \alpha))\right]}_{I}.\end{aligned}$$

For I, we have

$$I = D(\omega) + \frac{K\cos\alpha}{N}\sum_{l=1}^{N}\left[\sin(\theta_l-\theta_M)-\sin(\theta_l-\theta_m)\right]$$
$$+\frac{K\sin\alpha}{N}\sum_{l=1}^{N}\left[\cos(\theta_l-\theta_M)-\cos(\theta_l-\theta_m)\right].$$

We now consider two cases according to the sign of α.
(1) $\alpha \in [0, \frac{\pi}{4})$. In this case, we have

$$\begin{aligned}
I &\le D(\omega) + \frac{K\cos\alpha\sin D(\theta)}{ND(\theta)}\sum_{l=1}^{N}\left[(\theta_l-\theta_M)-(\theta_l-\theta_m)\right]\\
&\quad+\frac{K\sin\alpha}{N}\sum_{l=1}^{N}\left[1-\cos D(\theta)\right]\\
&= D(\omega) - K\left[\sin(D(\theta)+\alpha)-\sin\alpha\right]\\
&= D(\omega) - K\left[\sin(D(\theta)+|\alpha|)-\sin|\alpha|\right].
\end{aligned}$$

(2) $\alpha \in (-\frac{\pi}{4}, 0)$. In this case, we have

$$\begin{aligned}
I &\le D(\omega) + \frac{K\cos\alpha\sin D(\theta)}{ND(\theta)}\sum_{l=1}^{N}\left[(\theta_l-\theta_M)-(\theta_l-\theta_m)\right]\\
&\quad+\frac{K\sin\alpha}{N}\sum_{l=1}^{N}\left[\cos D(\theta)-1\right]\\
&= D(\omega) - K\left[\sin(D(\theta)-\alpha)+\sin\alpha\right]\\
&= D(\omega) - K\left[\sin(D(\theta)+|\alpha|)-\sin|\alpha|\right].
\end{aligned}$$

Here we used the relations

$$\frac{\sin(\theta_l-\theta_M)}{\theta_l-\theta_M},\ \ \frac{\sin(\theta_l-\theta_m)}{\theta_l-\theta_m} \ge \frac{\sin D(\theta)}{D(\theta)},$$

and

$$\cos D(\theta) \le \cos(\theta_l-\theta_M),\ \cos(\theta_l-\theta_m) \le 1, \quad l = 1, 2, \dots, N.$$

Since $D(\theta(t)) + |\alpha| < D_*^\infty < \pi$ for $t \in [0, T^*)$, we obtain

$$\begin{aligned}
I &\le D(\omega) - K\left[\sin(D(\theta)+|\alpha|)-\sin|\alpha|\right] &(11)\\
&\le D(\omega) + K\sin|\alpha| - K\frac{\sin D_*^\infty}{D_*^\infty}(D(\theta)+|\alpha|). &(12)
\end{aligned}$$

By (12) and Lemma 3.5 we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}D(\theta)^2\\
&\le D(\theta)\frac{1}{\dot\theta_M+\dot\theta_m}\left(D(\omega)+K\sin|\alpha|-K\frac{\sin D_*^\infty}{D_*^\infty}(D(\theta)+|\alpha|)\right)\\
&= \frac{D(\omega)+K\sin|\alpha|}{\dot\theta_M+\dot\theta_m}D(\theta) - \frac{K\sin D_*^\infty}{D_*^\infty(\dot\theta_M+\dot\theta_m)}D(\theta)(D(\theta)+|\alpha|)\\
&\le \frac{D(\omega)+K\sin|\alpha|}{2\sqrt{\underline{\omega}-K}}D(\theta) - \frac{K\sin D_*^\infty}{D_*^\infty 2\sqrt{\bar\omega+K}}D(\theta)(D(\theta)+|\alpha|), \quad t \in [0, T^*).
\end{aligned}$$

Then we obtain

$$\frac{d}{dt}D(\theta) \le \frac{D(\omega)+K\sin|\alpha|}{2\sqrt{\underline{\omega}-K}} - \frac{K\sin D_*^\infty}{2D_*^\infty\sqrt{\bar\omega+K}}(D(\theta)+|\alpha|),\ t\in[0,T^*),$$

i.e.,

$$\begin{aligned}\frac{d}{dt}(D(\theta)+|\alpha|) &\le \frac{D(\omega)+K\sin|\alpha|}{2\sqrt{\underline{\omega}-K}} - \frac{K\sin D_*^\infty}{2D_*^\infty\sqrt{\bar\omega+K}}(D(\theta)+|\alpha|)\\ &= \frac{K\sin D_*^\infty}{2\sqrt{\bar\omega+K}} - \frac{K\sin D_*^\infty}{2D_*^\infty\sqrt{\bar\omega+K}}(D(\theta)+|\alpha|),\ t\in[0,T^*).\end{aligned}$$

Here we used the definition of $\sin D_*^\infty$. By Gronwall's inequality, we obtain

$$D(\theta(t))+|\alpha| \le D_*^\infty + (D(\theta^0)+|\alpha|-D_*^\infty)e^{-\frac{K\sin D_*^\infty}{2D_*^\infty\sqrt{\bar\omega+K}}t},\ t\in[0,T^*),$$

Thus

$$D(\theta(t)) \le (D(\theta^0)+|\alpha|-D_*^\infty)e^{-\frac{K\sin D_*^\infty}{2D_*^\infty\sqrt{\bar\omega+K}}t} + D_*^\infty - |\alpha|,\ t\in[0,T^*).$$

Let $t\to T^{*-}$ and we have

$$D(\theta(T^*)) \le (D(\theta^0)+|\alpha|-D_*^\infty)e^{-\frac{K\sin D_*^\infty}{2D_*^\infty\sqrt{\bar\omega+K}}T^*} + D_*^\infty - |\alpha| < D_*^\infty - |\alpha|,$$

which is contradictory to $D(\theta(T^*)) = D_*^\infty - |\alpha|$. Therefore, we have

$$T^* = \infty.$$

That is,

$$D(\theta(t)) \le D_*^\infty - |\alpha|, \quad \forall\, t\ge 0.$$

□

Lemma 3.7. *Let $\theta := (\theta_1,\theta_2,\ldots,\theta_N)$ be a solution to the coupled system (4) with $K > K_c$ and $[\underline{\omega},\bar\omega] \subset [K+1, \frac{1}{2\sin^2|\alpha|})$. If the initial configuration $\theta(0)\in\mathcal{B}$, then*

$$\frac{d}{dt}D(\theta(t)) \le \frac{D(\omega)+K\sin|\alpha|}{2\sqrt{\underline{\omega}-K}} - \frac{K}{2\sqrt{\bar\omega+K}}\sin(D(\theta)+|\alpha|), \quad t\ge 0.$$

Proof. It follows from (11) and Lemma 3.5, Lemma 3.6 and that we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}D(\theta)^2 &= D(\theta)\frac{d}{dt}D(\theta)\\ &\le D(\theta)\frac{1}{\dot\theta_M+\dot\theta_m}\left[D(\omega) - K\left(\sin(D(\theta)+|\alpha|)-\sin|\alpha|\right)\right]\\ &= \frac{D(\omega)+K\sin|\alpha|}{\dot\theta_M+\dot\theta_m}D(\theta) - \frac{K\sin(D(\theta)+|\alpha|)}{\dot\theta_M+\dot\theta_m}D(\theta)\\ &\le \frac{D(\omega)+K\sin|\alpha|}{2\sqrt{\underline{\omega}-K}}D(\theta) - \frac{K\sin(D(\theta)+|\alpha|)}{2\sqrt{\bar\omega+K}}D(\theta), \quad t\ge 0.\end{aligned}$$

The proof is completed. □

Lemma 3.8. *Let $\theta := (\theta_1,\theta_2,\ldots,\theta_N)$ be a solution to the coupled system (4) with $K > K_c$ and $[\underline{\omega},\bar\omega] \subset [K+1, \frac{1}{2\sin^2|\alpha|})$. If the initial configuration $\theta(0)\in\mathcal{B}$, then for any small $\varepsilon>0$ with $D_1^\infty+\varepsilon<\frac{\pi}{2}$, there exists some time $T>0$ such that*

$$D(\theta(t)) < D_1^\infty - |\alpha| + \varepsilon, \qquad t\ge T.$$

Proof. Consider the ordinary differential equation:

$$\dot{y} = \frac{D(\omega) + K\sin|\alpha|}{2\sqrt{\underline{\omega} - K}} - \frac{K}{2\sqrt{\bar{\omega} + K}}\sin y, \qquad y(0) = y^0 \in [0, D_*^\infty). \tag{13}$$

It is easy to find that $y_* = D_1^\infty$ is a locally stable equilibrium of (13), while $y_{**} = D_*^\infty$ is unstable. Therefore, for any initial data y^0 with $0 < y^0 < D_*^\infty$, the trajectory $y(t)$ monotonically approaches y_*. Then, for any $\varepsilon > 0$ with $D_1^\infty + \varepsilon < \frac{\pi}{2}$, there exists some time $T > 0$ such that

$$|y(t) - y_*| < \varepsilon, \quad t \geq T.$$

In particular, $y(t) < y_* + \varepsilon$ for $t \geq T$. By Lemma 3.7 and the comparison principle, we have

$$D(\theta(t)) + |\alpha| < D_1^\infty + \varepsilon, \quad t \geq T,$$

which is the desired result. □

Remark 4. Since

$$\sin D_1^\infty \geq \frac{D(\omega)}{K} + \sin|\alpha| > \sin|\alpha|,$$

we have $D_1^\infty > |\alpha|$.

Proof of Theorem 2.3. It follows from Lemma 3.8 that for any small $\varepsilon > 0$, there exists some time $T > 0$ such that

$$\sup_{t\geq T} D(\theta(t)) < D_1^\infty - |\alpha| + \varepsilon < \frac{\pi}{2}.$$

We differentiate the equation (4) to find

$$\dot{\nu}_i = \frac{K}{2N\nu_i}\sum_{l=1}^{N}\cos(\theta_l - \theta_i + \alpha)(\nu_l - \nu_i), \quad \nu_i > 0.$$

We now consider the temporal evolution of $D(\nu(t))$:

$$\begin{aligned}
&\frac{d}{dt}D(\nu) = \dot{\nu}_M - \dot{\nu}_m \\
=&\frac{K}{2N\nu_M}\sum_{l=1}^{N}\cos(\theta_l - \theta_M^\nu + \alpha)(\nu_l - \nu_M) - \frac{K}{2N\nu_m}\sum_{l=1}^{N}\cos(\theta_l - \theta_m^\nu + \alpha)(\nu_l - \nu_m) \\
\leq&\frac{K}{2N\nu_M}\sum_{l=1}^{N}\cos(D_1^\infty + \varepsilon)(\nu_l - \nu_M) - \frac{K}{2N\nu_m}\sum_{l=1}^{N}\cos(D_1^\infty + \varepsilon)(\nu_l - \nu_m) \\
\leq&\frac{K\cos(D_1^\infty + \varepsilon)}{2N\sqrt{\bar{\omega} + K}}\sum_{l=1}^{N}(\nu_l - \nu_M - \nu_l + \nu_m) \\
=&-\frac{K\cos(D_1^\infty + \varepsilon)}{2\sqrt{\bar{\omega} + K}}D(\nu), \quad t \geq T,
\end{aligned}$$

where we used

$$\cos(\theta_l - \theta_M^\nu + \alpha), \cos(\theta_l - \theta_m^\nu + \alpha) \geq \cos(D_1^\infty + \varepsilon), \quad \text{and} \quad \nu_M, \nu_m \leq \sqrt{\bar{\omega} + K}.$$

Thus we obtain

$$D(\nu(t)) \leq D(\nu(T))e^{-\frac{K\cos(D_1^\infty + \varepsilon)}{2\sqrt{\bar{\omega}+K}}(t-T)}, \ t \geq T,$$

and proves (7) with $\lambda_3 = \frac{K\cos(D_1^\infty + \varepsilon)}{2\sqrt{\bar{\omega}+K}}$. □

4. **Conclusions.** In this paper, we presented synchronization estimates for the Kuramoto-like model. We show that for identical oscillators with zero frustration, complete phase synchronization occurs exponentially fast if the initial phases are confined inside an arc with geodesic length strictly less than π. For nonidentical oscillators with frustration, we present a framework to guarantee the emergence of frequency synchronization.

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