NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2020026
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Volume **15**, Number **3**, September **2020** pp. **427–461**

NONLINEAR STABILITY OF STATIONARY SOLUTIONS TO THE KURAMOTO-SAKAGUCHI EQUATION WITH FRUSTRATION

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Abstract. We study measurable stationary solutions for the kinetic Kuramoto-Sakaguchi (in short K-S) equation with frustration and their stability analysis. In the presence of frustration, the total phase is not a conserved quantity anymore, but it is time-varying. Thus, we can not expect the genuinely stationary solutions for the K-S equation. To overcome this lack of conserved quantity, we introduce new variables whose total phase is conserved. In the transformed K-S equation in new variables, we derive all measurable stationary solution representing the incoherent state, complete and partial phase-locked states. We also provide several frameworks in which the complete phase-locked state is stable, whereas partial phase-locked state is semi-stable in the space of Radon measures. In particular, we show that the incoherent state is nonlinearly stable in a large frustration regime, whereas it can exhibit stable behavior or concentration phenomenon in a small frustration regime.

1. **Introduction.** Collective behaviors of oscillatory complex systems are ubiquitous in our nature, i.e., flashing of fireflies, beating of cardiac pacemaker cells, and arrays of Josephson junctions [1, 10, 27, 30, 32] etc. Recently, collective behaviors have received lots of attention from distinct scientific disciplines such as control theory, physics, neuroscience due to its applications in robot system, sensor network, and unmanned aerial vehicle. Among them, we are interested in the synchronization representing adjustment of rhythms of oscillators. The rigorous and systematic study for synchronization goes back to two pioneers Winfree and Kuramoto in a half century ago. In this paper, our focus lies in the Kuramoto model with frustration [29] (sometimes called the Kuramoto-Sakaguchi model). In order to fix the idea, let $\theta_i = \theta_i(t) \in \mathbb{T}$ and α be the phase of the i-th Kuramoto oscillator and the

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2020 *Mathematics Subject Classification.* Primary: 92B25, 35Q92; Secondary: 35B40.

Key words and phrases. Kuramoto-Sakaguchi equation, frustration, stationary solution, stability.

The work of S.-Y. Ha is supported by the National Research Foundation of Korea(NRF) Grant (No. NRF-2020R1A2C3A01003881). The work of Y. Zhang is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. NRF-2017R1A5A1015626 and NRF-2019R1A5A1028324).

uniform frustration (phase shift) between oscillators. Then the phase dynamics of Kuramoto oscillators is governed by the following first-order system [20, 21, 28, 29]:

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{\ell=1}^{N}\sin(\theta_\ell - \theta_i + \alpha), \quad i = 1, \cdots, N, \tag{1.1}$$

where ω_i, κ and N are the natural frequency of the i-th Kuramoto oscillator, coupling strength, and the number of oscillators, respectively.

Note that the K-S model (1.1) can be rewritten as

$$\dot{\theta}_i = \omega_i + \frac{\kappa\cos\alpha}{N}\sum_{\ell=1}^{N}\sin(\theta_\ell - \theta_i) + \frac{\kappa\sin\alpha}{N}\sum_{\ell=1}^{N}\cos(\theta_\ell - \theta_i). \tag{1.2}$$

The terms in the R.H.S. of (1.2) correspond to the natural frequency, synchronization enforcing force and integrable forcing term, respectively. When the system size N is sufficiently large, state of system (1.1) can be approximated by the continuity equation with nonlocal velocity field. More precisely, let $F = F(t, \theta, \omega)$ be a one-particle distribution function, and $f = f(t, \theta, \omega)$ be the conditional probability density function defined by the following relation:

$$F(t, \theta, \omega) = f(t, \theta, \omega)g(\omega), \quad (t, \theta, \omega) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R},$$

where $g(\omega)$ is the probability density for natural frequency. Then, for $\omega \in \mathbb{R}$, we define a measurable map $\omega \longmapsto f_\omega := f(\cdot, \cdot, \omega)$ from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R}_+ \times \mathbb{T})$ (set of all probability measures on $\mathbb{T}$). Then the dynamics of conditional probability density f_ω satisfies

$$\begin{cases} \dfrac{\partial}{\partial t} f_\omega + \partial_\theta(\mathcal{V}[f_\omega]f_\omega) = 0, \quad (t, \theta, \omega) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}, \\ \mathcal{V}[f_\omega](\theta, \omega, t) := \omega + \kappa \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta_* - \theta + \alpha) g(\omega_*) df_{\omega_*} d\omega_*, \\ \displaystyle\int_{\mathbb{T}} df_\omega = 1, \end{cases} \tag{1.3}$$

where differential operator ∂_θ is the derivation on $\mathbb{T}$ interpreted in the sense of distribution: for any $\mathcal{C}^k$ function ϕ on the circle, the action of $\partial_\theta(\mathcal{V}[f_\omega]f_\omega)$ is given by

$$\Big\langle \partial_\theta(\mathcal{V}[f_\omega]f_\omega), \phi \Big\rangle = -\int_{\mathbb{T}} \phi' \mathcal{V}[f_\omega] df_\omega.$$

In the absence of frustration $\alpha = 0$, there have been lots of literatures on the Kuramoto model (1.1), to name a few [4, 5, 8, 9, 14, 15, 17, 19, 31] etc. As discussed in [6, 7, 22, 25, 26, 33], frustration is needed for the realistic modeling of physical and biological oscillators. The mere change (1.1) of the Kuramoto model by adding frustration causes several analytical difficulties in the study of emergent dynamics. For example, the total phases:

$$\sum_{j=1}^{N} \theta_j : \text{ particle system}, \qquad \int_{\mathbb{T}\times\mathbb{R}} \theta g(\omega) df_\omega d\omega : \text{ kinetic K-S equation}.$$

are not conserved quantitites, and gradient flow structure for (1.1) is also destroyed. Thus, we cannot use the useful machineries for the Kuramoto model in the study of emergent dynamics for (1.1) and (1.3). So far, there are only few works on the Kuramoto model with frustration [2, 12, 23]. Recently, the work [16] investigated the emergent property for a finite-N particle model with frustration $|\alpha| < \frac{\pi}{2}$, and Ha and his collaborators studied the stability and instability of incoherent solution in [11, 18], if initial data are sufficient regular. However, As far as the authors know, there are no results concerning all measurable stationary solutions and their stability, if the initial datum is just a Radon measure for system (1.3). Thus, in this paper, we address the following questions:

- (Q1): Are there measurable stationary solutions for the K-S equation (1.3)?
- (Q2): If there exists a stationary solution, are they nonlinearly stable?

The purpose of this paper is to answer the above questions. First, we discuss the first question (Q1). Mirollo and Strogatz [24] presented some special stationary solutions in the absence of frustration. In contrast, when frustration effect is present, many tricks employed in the Kuramoto model do not work mainly due to the non-conservation of the total phase (see Lemma 2.1 for details). Hence, we can not expect genuine stationary solutions. For this, we define a new variable $\widetilde{\theta}(\theta, t)$:

$$\begin{cases} \dfrac{\partial}{\partial t}\widetilde{\theta}(t,\theta) = \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \mathcal{V}[f_\omega]g(\omega)df_\omega d\omega = \kappa R^2(t)\sin\alpha, \quad (t,\theta)\in\mathbb{R}_+\times\mathbb{T}, \\ \widetilde{\theta}(0,\theta) = \theta. \end{cases}$$

By studying a new equation formulated in terms of $\tilde{\theta}$, we present all measurable stationary solutions for the reformulated equation. Second, we study a nonlinear stability of stationary solutions in more general case. Let $\mathcal{M}(\mathbb{T}\times\mathbb{R})$ be the set of non-negative Radon measures on $\mathbb{T}\times\mathbb{R}$. Then, for a Radon measure $\mu\in\mathcal{M}(\mathbb{T}\times\mathbb{R})$, we use the standard duality relation:

$$\langle \mu, h\rangle = \int_{\mathbb{T}\times\mathbb{R}} h(\theta,\omega)\mu(d\theta d\omega), \quad \text{for any } h\in\mathcal{C}_0^\infty(\mathbb{T}\times\mathbb{R}),$$

where $\mathcal{C}_0^\infty$ denotes the set of smooth functions vanishing at infinity.

Consider the stability of stationary solutions arising from the Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\mu_t + \partial_\theta(\mathcal{V}[\mu_t]\mu_t) = 0, \quad (t,\theta,\omega)\in\mathbb{R}_+\times\mathbb{T}\times\mathbb{R}, \\ \mathcal{V}[\mu_t](\theta,\omega,t) := \omega + \kappa\displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta_* - \theta + \alpha)\mu_t(d\theta_* d\omega_*), \\ \mu_t|_{t=0} = \mu_0 \geq 0, \quad \mu_0(\theta+2\pi) = \mu_0(\theta), \quad \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \mu_t(d\theta d\omega) = 1, \end{cases} \tag{1.4}$$

where $\mu_t(d\theta d\omega) := g(\omega)df_\omega d\omega$.

We first study the stability of the complete phase-locked state (stationary solution with $R = 1$, see Definition 2.2 for details), and we show that the complete phase-locked state is nonlinearly stable in space of Radon measures in which the size of initial phase diameter is smaller than $\pi - 2|\alpha|$ (see Theorem 2.3 for more details). On the other hand, we discuss the stability of partial phase-locked state. Let $\mu_t, \nu_t \in \mathcal{C}_w\big([0,T]; \mathcal{M}(\mathbb{T}\times\mathbb{R})\big)$ be measurable solutions to system (1.4) (see Definition 2.4 for details). One of difficulty that we confront with is the lack of exponential stability (see [3]):

$$W_p(\mu_t,\nu_t) \leq e^{-ct}W_p(\mu_0,\nu_0),$$

where $W_p(\mu_t,\nu_t)$ is the p-Wassertein distance between two measures μ_t and ν_t. In fact, let $\phi_i, i = 1,2$ be the pseudo-inverse functions associated to μ_t and ν_t, denote R_i by the order parameters of ϕ_i respectively, then by Lemma 2.1 and proof of Theorem 5.1 in [3], $\frac{d}{dt}W_p(\mu_t,\nu_t)$ is bounded by $|\kappa R_1^2(t)\sin\alpha - \kappa R_2^2(t)\sin\alpha|$, so we have no knowledge whether $W_p(\mu_t, \nu_t)$ will converge to zero. For this, we introduce the characteristic function for the K-S system:

$$\begin{cases} \dfrac{\partial}{\partial t}\Theta(t,\theta,\omega) = \omega + \kappa \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\Theta_* - \Theta + \alpha)\mu(d\theta_* d\omega_*), \\ \Theta(0,\theta,\omega) = \theta, \end{cases}$$

and consider the dynamics of the following quantity:

$$D^1_\theta(\mu_t) := \sup_{\Theta_1,\Theta_2 \in B_\theta(t)} \big(\dot{\Theta}_1(t) - \dot{\Theta}_2(t)\big),$$

where $B_\theta(t)$ is the orthogonal θ-projection of $\mathrm{supp}(\mu_t)$, and then show

$$D^1_\theta(\mu_t) \to 0 \quad \text{exponentially, as } t \to \infty$$

which yields that $\lim_{t\to\infty}(\Theta_1 - \Theta_2)$ exists and finite, that is, phase-locked state emerge asymptotically. Moreover, we obtain the semi-stability of the partially phase-locked state in space of Radon measures under suitable conditions (see Theorem 2.6 for details). Finally, we study the stability of incoherent state to system (1.4) for identical oscillators, i.e., the natural frequency is a constant, and without loss of generality, we set $\omega = 0$ for simplicity. In a large frustration regime, the order parameter R tends to zero, as time tends to infinity, i.e., the incoherent solution is stable, whereas in a small frustration regime, the order parameter will tends to zero if there exists a constant M such that $\mu_t(d\theta) \le M\mu_e(d\theta)$ with μ_e a renormalized measure satisfying $\mu_e(\mathbb{T}) = 1$, otherwise, the probability distribution concentrates and forms a Dirac mass.

The rest of this paper is organized as follows. In Section 2, we briefly present main results of this paper. In Section 3, we discuss measurable stationary solutions. We first verify that the incoherent state is unique, then we find out all phase-locked states. In Sections 4 and 5, we study the stability of phase-locked states and incoherent state by providing suitable frameworks respectively.

2. **Presentation of main results.**

In this section, we present sufficient frameworks and main results on the existence of stationary solutions and their stabilities for the kinetic K-S equation.

2.1. **Existence of stationary solutions.**

In this subsection, we introduce the order parameters and present stationary solutions for the K-S equation (1.3). We first define the order parameters (R, ψ) as follows:

$$Re^{i\psi} := \int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} e^{i\theta} df_\omega\Big) d\omega. \tag{2.1}$$

We divide both sides of relation (2.1) by $e^{i\psi}$, and separate the real and imaginary part to obtain

$$R = \int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} \cos(\theta - \psi) df_\omega\Big) d\omega, \qquad 0 = \int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} \sin(\theta - \psi) df_\omega\Big) d\omega. \tag{2.2}$$

Moreover, we further divide both sides of relation (2.1) by $e^{i(\theta-\alpha)}$, and compare the real and imaginary parts of the resulting relation to get

$$R\sin(\psi - \theta + \alpha) = \int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} \sin(\theta_* - \theta + \alpha) df^*_\omega\Big) d\omega. \tag{2.3}$$

Lemma 2.1. *Suppose the natural frequency distribution $g = g(\omega)$ is integrable. Then, one has*

$$\int_{\mathbb{T}\times\mathbb{R}} \mathcal{V}[f_\omega] g(\omega) df_\omega d\omega = \kappa R^2 \sin\alpha.$$

Proof. We use (2.3) to rewrite the nonlocal velocity $\mathcal{V}$ as

$$\mathcal{V}[f_\omega] = \omega - \kappa R \sin(\theta - \psi - \alpha).$$

Then we use (2.5) and (2.2) to get

$$\begin{aligned}
&\int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} \big(\omega - \kappa R\sin(\theta-\psi-\alpha)\big) df_\omega\Big) d\omega \\
&\quad = \int_{\mathbb{T}} \omega g(\omega) d\omega - \kappa R \int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} \sin(\theta-\psi-\alpha) df_\omega\Big) d\omega \\
&\quad = -\kappa R\cos\alpha \int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} \sin(\theta-\psi) df_\omega\Big) d\omega \\
&\qquad + \kappa R \sin\alpha \int_{\mathbb{R}} g(\omega)\Big(\int_{\mathbb{T}} \cos(\theta-\psi) df_\omega\Big) d\omega \\
&\quad = \kappa R^2 \sin\alpha.
\end{aligned}$$

□

By Lemma 2.1, we can not get equilibria for system (1.3) unless $R \equiv 0$. Thus, we can get a relative equilibrium which is an equilibrium in a rotating frame with a constant velocity. For this, we define

$$\frac{d\tilde{\theta}}{dt} = -\kappa R^2 \sin\alpha, \ t > 0, \qquad \tilde{\theta}(0) = \theta,$$

and consider $f_\omega(t, \tilde{\theta}, \omega)$ instead of $f_\omega(t, \theta, \omega)$. For simplicity, we still write θ for $\tilde{\theta}$. Then our original K-S equation (1.3) can be rewritten as

$$\begin{cases} \dfrac{\partial}{\partial t} f_\omega + \partial_\theta(\mathcal{V}[f_\omega] f_\omega) = 0, \quad (t,\theta,\omega) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}, \\ \mathcal{V}[f_\omega](\theta,\omega,t) := \omega - \kappa R \sin(\theta - \psi - \alpha) - \kappa R^2 \sin\alpha. \end{cases} \tag{2.4}$$

Next, we recall several distinguished states in the following definition.

Definition 2.2. Let f be a measurable stationary solution to (2.4), and $R = R_f$ is a corresponding order parameter associated with f in (2.1).

1. If $R \equiv 0$, then f is called an incoherent state.
2. If $0 < R < 1$, then f is called a partial phase-locked state.
3. If $R \equiv 1$, then f is called the complete phase-locked state.

Now we are ready to present our first main theorem as follows.

Theorem 2.3. *Suppose that the natural frequency distribution $g = g(\omega) \in L^1(\mathbb{R})$ and*

$$supp\, g = [-L, L], \quad g(-\omega) = g(\omega), \quad \int_{\mathbb{R}} g(\omega) d\omega = 1. \tag{2.5}$$

Then, the following assertions hold.

1. *The incoherent state of* (2.4) *and* (2.5) *is unique, and* $f_\omega = \mu_e$ *is the normalized Lebesgue measure on* $\mathbb{T}$ *with* $\mu_e(\mathbb{T}) = 1$.
2. *If the phase-locked state of* (2.4) *and* (2.5) *exists, then*

$$f_\omega = \begin{cases} \dfrac{C_{\omega-\kappa R^2 \sin\alpha}}{\omega - \kappa R^2 \sin\alpha - \kappa R \sin(\theta-\alpha)}, & \text{for } |\omega - \kappa R^2 \sin\alpha| > \kappa R, \\[2ex] \big(1-\eta(\omega)\big)\delta_{\theta_\omega} + \eta(\omega)\delta_{\theta^*_\omega}, & \text{for } |\omega - \kappa R^2 \sin\alpha| \le \kappa R, \end{cases}$$

and the following relations must hold:

$$\begin{cases} R\sin\alpha = \dfrac{1}{\kappa R}\displaystyle\int_{\kappa R}^{+\infty} \big(2\pi C_\omega - \omega\big)\big(g(\omega+\kappa R^2\sin\alpha) - g(\omega - \kappa R^2 \sin\alpha)\big)d\omega \\[2ex] \qquad\qquad - \dfrac{1}{\kappa R}\displaystyle\int_0^{\kappa R} \omega\big[g(\omega+\kappa R^2\sin\alpha) - g(\omega-\kappa R^2\sin\alpha)\big]d\omega, \\[2ex] R\cos\alpha = \displaystyle\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R} \big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R})^2}d\omega, \end{cases}$$

where $0 \le \eta(\omega) \le 1$ *is a constant,* C_s *is given by*

$$C_s = \frac{1}{2\pi}\sqrt{s^2-(\kappa R)^2} \quad \text{or} \quad -\frac{1}{2\pi}\sqrt{s^2-(\kappa R)^2} \quad \text{for any constant } s,$$

with signs determined by $\int_{\mathbb{T}} \frac{C_s}{s-\kappa R\sin(\theta-\alpha)} = 1$, *phases* $\theta_\omega - \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *and* $\theta^*_\omega - \alpha = \pi - (\theta_\omega - \alpha)$ *are the roots of equation:*

$$\sin x = \frac{\omega - \kappa R^2 \sin\alpha}{\kappa R}.$$

3. *If* $g(\omega)$ *is non-increasing on* $[0, L]$, *then*

$$L \le \kappa R - \kappa R^2 \sin|\alpha|.$$

Remark 1. From the proof of Theorem 2.3, we will see that the same results hold for identical case except the uniqueness of the incoherent state.

2.2. **Stability of phase-locked states.** In this subsection, we present main results on the stability of complete phase-locked state and partial phase-locked state. Denote by $\mathcal{C}_w\big([0,T);\mathcal{M}(\mathbb{T}\times\mathbb{R})\big)$ the space of weakly continuous time-dependent measures. First, we present a definition of measure-valued solution for (1.4) as follows.

Definition 2.4 (Measure-valued solution)**.** For $T \in (0,\infty]$, we say $\mu_t \in \mathcal{C}_w\big([0,T);\mathcal{M}(\mathbb{T}\times\mathbb{R})\big)$ be a measure-valued solution to (1.4) with an initial Radon measure $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ if μ_t satisfies the following conditions:

(i) $\langle \mu_t, h\rangle$ is continuous as a function of time t, for any $h \in \mathcal{C}_0^\infty(\mathbb{T}\times\mathbb{R})$.

(ii) For any $h \in \mathcal{C}_0^\infty([0,T]\times\mathbb{T}\times\mathbb{R})$,

$$\langle \mu_t, h(t,\cdot,\cdot)\rangle - \langle \mu_0, h(0,\cdot,\cdot)\rangle = \int_0^t \langle \mu_s, \partial_s h + \mathcal{V}[\mu]\partial_\theta h\rangle ds, \tag{2.6}$$

where $\mathcal{V}[\mu](s,\theta,\omega)$ is defined by

$$\mathcal{V}[\mu](s,\theta,\omega) := \omega + \kappa\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta_* - \theta + \alpha)\mu(d\theta_* d\omega_*).$$

Remark 2. Below, we provide a brief comment on measure-valued solutions.

1. Recall that $\mathrm{supp}(\mu)$ (the support of a measure μ) is the closure of the set consisting of all points $(x,v) \in \mathbb{R}^{2d}$ such that $\mu(B_r((x,v))) > 0,\ \forall r > 0$. For a finite measure with a compact support, we can use $h \in C^1(\mathbb{R}^{2d})$ as a test function in (2.6). Thus, we choose $h = 1, \omega$ in (2.6) to get
$$\langle \mu_t, 1\rangle = \langle \mu_0, 1\rangle = 1, \quad \langle \mu_t, \omega\rangle = \langle \mu_0, \omega\rangle.$$
2. Let $(\theta_i(t), \omega_i(t))$ be a solution of following ODE system:
$$\begin{aligned}\frac{d\theta_i}{dt} &= \omega_i + \frac{\kappa}{N}\sum_{i=1}^{N} \sin(\theta_j - \theta_i + \alpha),\\ \frac{d\omega_i}{dt} &= 0, \quad t > 0.\end{aligned}$$
Then, the empirical measure
$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\theta_i(t)} \otimes \delta_{\omega_i(t)},$$
is a measure-valued solution to system (1.4).

Now let $\mu_t \in \mathcal{C}_w\big([0,T]; \mathcal{M}(\mathbb{T}\times\mathbb{R})\big)$ be a measurable solution to system (1.4). We define $B_\theta(t)$ and $B_\omega(t)$ be the orthogonal θ and ω-projections of *supp* μ_t, respectively, i.e.,
$$\begin{aligned}&B_\theta(t) := \mathbb{P}_\theta \mathrm{supp}\, \mu_t = \big\{\theta \in \mathbb{T} \mid (\theta,\omega) \in \mathrm{supp}\mu_t\big\},\\ &B_\omega(t) := \mathbb{P}_\omega \mathrm{supp}\, \mu_t = \big\{\omega \in \mathbb{R} \mid (\theta,\omega) \in \mathrm{supp}\mu_t\big\},\\ &D_\theta(\mu_t) := \mathrm{diam} B_\theta(t), \quad D_\omega(\mu_t) := \mathrm{diam} B_\omega(t),\\ &\theta_c(t) := \frac{1}{M(t)}\langle \mu_t, \theta\rangle, \quad \omega_c(t) := \frac{1}{M(t)}\langle \mu_t, \omega\rangle,\end{aligned}$$
where M and $\mathrm{diam}A$ are given as follows.
$$M(t) = \langle \mu_t, 1\rangle \quad \text{and} \quad \mathrm{diam}A := \sup_{x,y\in A} |x-y| \quad \text{for } A \subseteq \mathbb{R}.$$
We use Remark 2 to see
$$M(t) = \langle \mu_t, 1\rangle = \langle \mu_0, 1\rangle = 1, \quad \omega_c(t) = \langle \mu_t, \omega\rangle = \langle \mu_0, \omega\rangle = \omega_c(0).$$
For identical oscillators with $g(\omega) = \delta_{\omega_c}$, we set
$$\mu_\infty(d\theta d\omega) := \delta_{\theta_c(t)} \times \delta_{\omega_c(0)}.$$
Our second main result deals with the stability of complete phase-locked state (i.e. for identical oscillators, $g(\omega) = \delta_{\omega_c(0)}$) as follows.

Theorem 2.5. *Suppose that the initial datum $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ satisfies*
$$D_\theta(\mu_0) \le \pi - 2|\alpha|, \quad D_\omega(\mu_0) = 0. \tag{2.7}$$
Then the measure-valued solution μ_t to system (1.4) satisfies
$$D_\theta(\mu_t) \le D_\theta(\mu_0)e^{-\kappa\Lambda_0 t},\ t \ge 0, \quad \lim_{t\to\infty} d(\mu_t, \mu_\infty) = 0 \ \textit{exponential},$$
where $\Lambda_0 := \frac{2}{\pi}\cos\big(\frac{1}{2}D_\theta(\mu_0) + |\alpha|\big)$ is a positive constant, and $\mu_\infty(d\theta d\omega) = \delta_{\theta_c(t)} \times \delta_{\omega_c(0)}$. In particular, we obtain the stability of the complete phase-locked state in space of Radon measure-valued solution whose initial data satisfy (2.7).

For nonidentical oscillators, a difficulty we need to overcome is that the exponential decay (see [3]):

$$W_p(\mu_t,\nu_t) \le e^{-ct} W_p(\mu_0,\nu_0),$$

where $W_p(\mu_t,\nu_t)$ is the p-Wasserstein distance between two measures μ_t and ν_t. As discussed in the introduction, $\frac{d}{dt}W_p(\mu_t,\nu_t)$ is bounded by $|\kappa R_1^2(t)\sin\alpha - \kappa R_2^2(t)\sin\alpha|$ with $R_i, i=1,2$ the order parameter of the pseudo-inverse functions associated to μ_t and ν_t, so we have no knowledge if $W_p(\mu_t,\nu_t)$ will converge to zero. For this, we study the characteristic function for system (1.4):

$$\begin{cases} \dot\Theta(t,\theta,\omega) = \omega + \kappa \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\Theta_* - \Theta + \alpha)\mu(d\theta_* d\omega_*), \quad t>0, \\ \Theta(t,\theta,\omega)|_{t=0} = \theta, \end{cases}$$

and define

$$D_\theta^1(\mu_t) = \sup_{\Theta_1,\Theta_2\in B_\theta(t)} \big(\dot\Theta_1(t) - \dot\Theta_2(t)\big).$$

Our third main result deals with the exponential stability of D_θ^1.

Theorem 2.6. *Suppose the initial measure $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ satisfies*

$$\begin{aligned} &D_\theta(\mu_0) \le \pi - 2|\alpha|, \quad 0 < D_\omega(\mu_0) < \infty, \\ &\kappa > \kappa_e := \frac{D_\omega(\mu_0)}{\sin\big(D_\theta(\mu_0) + |\alpha|\big) - \sin|\alpha|}, \end{aligned} \tag{2.8}$$

and let μ_t be a measure-valued solution to (1.4). Then, there exists a positive time t_0 such that

$$D_\theta(\mu_t) < D^\infty, \quad D_\theta^1(\mu_t) \le D_\theta^1(\mu_{t_0}) e^{-\frac12 \kappa\cos(D^\infty + |\alpha|)(t-t_0)}, \quad \text{for all } t \ge t_0,$$

where $D^\infty \in (0, \frac{\pi}{2} - |\alpha|)$ is the solution of

$$\sin(x + |\alpha|) = \sin(D_\theta(\mu_0) + |\alpha|).$$

In particular, we obtain the semi-stability of the partial phase-locked state in space of Radon measure-valued solution whose initial data satisfy (2.8).

2.3. **Stability of the incoherent state.** In this subsection, we provide stability and instability estimates for the incoherent state. First, we discuss the small frustration case:

$$\begin{aligned} &\frac{\partial}{\partial t}\mu_t + \partial_\theta(\mathcal{V}[\mu_t]\mu_t) = 0, \quad (\theta,t)\in\mathbb{T}\times\mathbb{R}_+, \\ &\mathcal{V}[\mu_t](\theta,t) = \kappa\int_{\mathbb{T}} \sin(\theta_* - \theta + \alpha)\mu_t(d\theta_*), \end{aligned} \tag{2.9}$$

where $|\alpha| < \frac{\pi}{2}$. Our main results are as follows.

Proposition 2.7. (Small frustration) *Suppose frustration and initial datum satisfy*

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \quad \text{and} \quad \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_0(d\theta)\mu_0(d\theta_*) < \infty,$$

and let μ_t be a measure-valued solution for system (2.9). Then, the following assertions hold.

1. *If there exist a positive constant M such that $\mu_t(d\theta) \le M\mu_e(d\theta)$, then we have*

$$\lim_{t\to\infty} R(t) = 0.$$

2. *If there does not exists a positive constant M such that $\mu_t(d\theta) \le M\mu_e(d\theta)$, then we have*
$$\lim_{t\to\infty}\left\|\frac{\mu_t(d\theta)}{\mu_e(d\theta)}\right\|_{L^\infty} = \infty.$$

Next, we consider a large frustration $|\alpha| \ge \frac{\pi}{2}$ case. For this, we set $\hat{\alpha} := \alpha - \frac{\pi}{2}$. Then, the original identical system can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial t}\mu_t + \partial_\theta(\mathcal{C}[\mu_t]\mu_t) = 0, \quad (t,\theta,\omega) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \\ &\mathcal{C}[\mu_t] = \kappa \int_{\mathbb{T}\times\mathbb{R}} \cos(\theta_* - \theta + \hat{\alpha})\mu_t(d\theta_* d\omega_*). \end{aligned} \tag{2.10}$$

Then, our last result is as follows.

Proposition 2.8. (Large frustration) *Suppose frustration and initial datum satisfy*
$$0 < \hat{\alpha} \le \pi \quad \textit{and} \quad \int_{\mathbb{T}\times\mathbb{T}} \ln\left|\sin\left(\frac{\theta-\theta_*}{2}\right)\right|\mu_0(d\theta)\mu_0(d\theta_*) < \infty,$$
and let μ be a measure-valued solution for system (2.10). Then we have
$$\lim_{t\to\infty} R(t) = 0.$$

3. **Existence of stationary solutions.** In this section, we look for all measurable stationary solutions for the kinetic K-S equation (2.4). Without loss of generality, we may assume that phase order parameter $\psi = 0$.

Note that the stationary solution of (2.4) satisfies the equation:
$$\partial_\theta(\mathcal{V}[f_\omega]f_\omega) = 0.$$
Recall that the distribution ξ on $\mathbb{T}$ satisfying $\partial_\theta\xi = 0$ is equal to a constant $\tilde{C}_\omega$ multiples of normalized Lebesgue measure μ_e on $\mathbb{T}$. In other words, $d\mu_e = \frac{1}{2\pi}d\theta$. Hence, the stationary solution for (2.4) should satisfy
$$\mathcal{V}[f_\omega]f_\omega = \tilde{C}_\omega\mu_e \tag{3.1}$$
In the following proposition, we show that a unique incoherent state is given by μ_e.

Proposition 3.1. *The incoherent state for (2.4) and (2.5) is unique, and*
$$f_\omega = \mu_e,$$
where μ_e is the normalized Lebesgue measure on $\mathbb{T}$.

Proof. Note that the incoherent solutions for (2.4) satisfies
$$\mathcal{V}[f_\omega]f_\omega = \tilde{C}_\omega\mu_e.$$
Since $R = 0$, we have
$$\mathcal{V}[f_\omega] = \omega.$$
Thus, we have
$$f_\omega = \frac{\tilde{C}_\omega}{\omega}\mu_e.$$
On the other hand, note that
$$1 = \int_{\mathbb{T}} df_\omega = \frac{\tilde{C}_\omega}{\omega}\int_{\mathbb{T}} d\mu_e.$$

Thus,

$$\frac{\tilde{C}_\omega}{\omega} = 1, \quad \text{i.e.,} \quad f_\omega = \mu_e.$$

□

Next, we classify all the phase-locked states.

Proposition 3.2. *The phase-locked state for* (2.4) *and* (2.5) *is*

$$f_\omega = \begin{cases} \dfrac{C_{\omega-\kappa R^2 \sin\alpha}}{\omega - \kappa R^2 \sin\alpha - \kappa R \sin(\theta-\alpha)}, & \textit{for } |\omega - \kappa R^2 \sin\alpha| > \kappa R, \\ \big(1-\eta(\omega)\big)\delta_{\theta_\omega} + \eta(\omega)\delta_{\theta^*_\omega}, & \textit{for } |\omega - \kappa R^2 \sin\alpha| \le \kappa R, \end{cases} \tag{3.2}$$

where $\eta(\omega)$ *is a positive constant, and* $C^{\pm}_{\omega-\kappa R^2\sin\alpha}$ *is given by*

$$C_{\omega-\kappa R^2\sin\alpha} = \frac{1}{2\pi}\sqrt{(\omega-\kappa R^2\sin\alpha)^2 - (\kappa R)^2} \ \textit{or} \ -\frac{1}{2\pi}\sqrt{(\omega-\kappa R^2\sin\alpha)^2 - (\kappa R)^2},$$

with signs determined by

$$\int_{\mathbb{T}} \frac{C_{\omega-\kappa R^2\sin\alpha}}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)}d\theta = 1,$$

phases $\theta_\omega - \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta^*_\omega - \alpha := \pi - (\theta_\omega - \alpha)$ *are the roots of equation:*

$$\sin x = \frac{\omega - \kappa R^2 \sin\alpha}{\kappa R}.$$

Proof. For a proof, we divide the domain of the natural frequency ω into two cases:
$\omega \notin \big[\kappa R(-1+R\sin\alpha), \kappa R(1+R\sin\alpha)\big]$; $\quad \omega \in \big[\kappa R(-1+R\sin\alpha), \kappa R(1+R\sin\alpha)\big]$.

• Case A. Suppose that

$$\omega \notin \big[\kappa R(-1+R\sin\alpha), \kappa R(1+R\sin\alpha)\big], \quad \text{i.e.,} \quad |\omega - \kappa R^2\sin\alpha| > \kappa R.$$

Then we use relation (2.4) and assumption $\psi = 0$ to get

$$\mathcal{V}[f_\omega] \neq 0.$$

Hence, we use relation (3.1) to obtain

$$df_\omega = \frac{\tilde{C}_\omega/2\pi}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)}d\theta.$$

We use formula $\int_{\mathbb{T}} df_\omega = 1$ to get

$$\int_{\mathbb{T}} \frac{\tilde{C}_\omega/2\pi}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)}d\theta = 1.$$

By direct calculation, one has

$$\tilde{C}_\omega = \pm\sqrt{(\omega-\kappa R^2\sin\alpha)^2 - (\kappa R)^2}.$$

We set

$$C_{\omega-\kappa R^2\sin\alpha} = \frac{1}{2\pi}\sqrt{(\omega-\kappa R^2\sin\alpha)^2 - (\kappa R)^2} \quad \text{or} \quad -\frac{1}{2\pi}\sqrt{(\omega-\kappa R^2\sin\alpha)^2 - (\kappa R)^2},$$

with signs of $C_{\omega-\kappa R^2\sin\alpha}$ determined by

$$\int_{\mathbb{T}} \frac{C_{\omega-\kappa R^2\sin\alpha}}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)}d\theta = 1. \tag{3.3}$$

Hence, we have

$$f_\omega = \frac{1}{2\pi}\frac{\sqrt{(\omega-\kappa R^2\sin\alpha)^2-(\kappa R)^2}}{|\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)|} = \frac{C_{\omega-\kappa R^2\sin\alpha}}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)}.$$

• Case B. Suppose that

$$\omega\in\big[\kappa R(-1+R\sin\alpha),\kappa R(1+R\sin\alpha)\big],\quad \text{that is, } |\omega-\kappa R^2\sin\alpha|\le\kappa R.$$

In this case, we claim:

$$\tilde C_\omega = 0. \tag{3.4}$$

Proof of claim (3.4). Suppose not, then for $\theta\notin\big\{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)=0\big\}$

$$df_\omega = \frac{\tilde C_\omega/2\pi}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)}d\theta,$$

Then, there exists $\theta\notin\big\{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)=0\big\}$ such that $df_\omega<0$. This gives a contradiction since $df_\omega\ge 0$, then our claim holds. Hence, we have

$$\mathcal V[f_\omega]f_\omega = 0.$$

We use $\int_{\mathbb T} df_\omega = 1$ to get

$$\mathcal V[f_\omega]=0\quad\text{for } |\omega-\kappa R^2\sin\alpha|\le\kappa R,$$

i.e.,

$$\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)=0\quad \textit{for } |\omega-\kappa R^2\sin\alpha|\le\kappa R.$$

Thus, we can obtain

$$f_\omega = \big(1-\eta(\omega)\big)\delta_{\theta_\omega}+\eta(\omega)\delta_{\theta^*_\omega},$$

where $\theta_\omega-\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}]$ is the solution of equation:

$$\sin(\theta_\omega-\alpha)=\frac{\omega-\kappa R^2\sin\alpha}{\kappa R},$$

and $\theta^*_\omega=\pi-\theta_\omega+2\alpha$. Now we summarize the value of f_ω as follows:

$$f_\omega=\begin{cases}\dfrac{C_{\omega-\kappa R^2\sin\alpha}}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)}, & |\omega-\kappa R^2\sin\alpha|>\kappa R,\\ \big(1-\eta(\omega)\big)\delta_{\theta_\omega}+\eta(\omega)\delta_{\theta^*_\omega}, & |\omega-\kappa R^2\sin\alpha|\le\kappa R,\end{cases}$$

□

In Proposition 3.2, we have determined the ansatz for f_ω in terms of R. It is clear that f_ω should satisfy the following relations with respect to order parameters:

$$R=\int_{\mathbb R} g(\omega)\Big(\int_{\mathbb T}\cos\theta df_\omega\Big)d\omega,\quad 0=\int_{\mathbb R} g(\omega)\Big(\int_{\mathbb T}\sin\theta df_\omega\Big)d\omega. \tag{3.5}$$

Lemma 3.3. *The phase-locked state in* (3.2) *satisfies the following relations:*

$$\begin{aligned}R\sin\alpha &= \frac{1}{\kappa R}\int_{\kappa R}^{+\infty}\big(2\pi C_\omega-\omega\big)\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega\\ &\quad-\frac{1}{\kappa R}\int_0^{\kappa R}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega,\\ R\cos\alpha &= \int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R}\big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}d\omega,\end{aligned}$$

where $C_s = \pm\frac{1}{2\pi}\sqrt{s^2-(\kappa R)^2}$ *for any variable* s, $\eta(\omega)$ *is a constant satisfying* $0 \le \eta(\omega) \le 1$.

Proof. We split the proof into two cases:

$$\text{Either } |\omega - \kappa R^2 \sin\alpha| > \kappa R \quad \text{or} \quad |\omega - \kappa R^2 \sin\alpha| \le \kappa R.$$

• Step A. We first estimate the integral:

$$\begin{aligned}
&\int_{|\omega-\kappa R^2 \sin\alpha|>\kappa R} g(\omega)\Big(\int_{\mathbb{T}} \cos\theta df_\omega\Big) d\omega \\
&= \int_{|\omega-\kappa R^2 \sin\alpha|>\kappa R} g(\omega) C_{\omega-\kappa R^2\sin\alpha}\Big(\int_{\mathbb{T}} \frac{\cos\theta}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)} d\theta\Big) d\omega.
\end{aligned} \tag{3.6}$$

We use the relation (3.3) and

$$\int_{\mathbb{T}} \frac{\cos(\theta-\alpha)}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)} d\theta = 0$$

to get

$$\begin{aligned}
&C_{\omega-\kappa R^2\sin\alpha}\int_{\mathbb{T}} \frac{\cos\theta}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)} d\theta \\
&= C_{\omega-\kappa R^2\sin\alpha}\int_{\mathbb{T}} \frac{\cos(\theta-\alpha)\cos\alpha-\sin(\theta-\alpha)\sin\alpha}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)} d\theta \\
&= -C_{\omega-\kappa R^2\sin\alpha}\sin\alpha\int_{\mathbb{T}} \frac{\sin(\theta-\alpha)}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)} d\theta \\
&= -\frac{C_{\omega-\kappa R^2\sin\alpha}\sin\alpha(\omega-\kappa R^2\sin\alpha)}{\kappa R} \\
&\times \int_{\mathbb{T}} \Big\{\frac{1}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)} - \frac{1}{\omega-\kappa R^2\sin\alpha}\Big\} d\theta \\
&= \frac{2\pi C_{\omega-\kappa R^2\sin\alpha}\sin\alpha}{\kappa R} - \frac{(\omega-\kappa R^2\sin\alpha)\sin\alpha}{\kappa R}\int_{\mathbb{T}} \frac{C_{\omega-\kappa R^2\sin\alpha}}{\omega-\kappa R^2\sin\alpha-\kappa R\sin(\theta-\alpha)} d\theta \\
&= \frac{\sin\alpha}{\kappa R}\big(2\pi C_{\omega-\kappa R^2\sin\alpha} - (\omega-\kappa R^2\sin\alpha)\big),
\end{aligned} \tag{3.7}$$

where

$$C_{\omega-\kappa R^2\sin\alpha} = \frac{1}{2\pi}\sqrt{(\omega-\kappa R^2\sin\alpha)^2-(\kappa R)^2} \quad \text{or} \quad -\frac{1}{2\pi}\sqrt{(\omega-\kappa R^2\sin\alpha)^2-(\kappa R)^2}.$$

In (3.6), we use the estimate (3.7) to see

$$\begin{aligned}
&\int_{|\omega-\kappa R^2 \sin\alpha|>\kappa R} g(\omega)\Big(\int_{\mathbb{T}} \cos\theta df_\omega\Big) d\omega \\
&\qquad = \int_{|\omega-\kappa R^2 \sin\alpha|>\kappa R} \frac{g(\omega)\sin\alpha}{\kappa R}\big[2\pi C_{\omega-\kappa R^2\sin\alpha} - (\omega-\kappa R^2\sin\alpha)\big] d\omega \\
&\qquad = \frac{\sin\alpha}{\kappa R}\int_{|\omega|>\kappa R} g(\omega+\kappa R^2\sin\alpha)\big(2\pi C_\omega - \omega\big) d\omega
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sin\alpha}{\kappa R}\int_{\kappa R}^{+\infty} g(\omega+\kappa R^2\sin\alpha)\big(2\pi C_\omega-\omega\big)d\omega \\
&+ \frac{\sin\alpha}{\kappa R}\int_{-\infty}^{-\kappa R} g(\omega+\kappa R^2\sin\alpha)\big(2\pi C_\omega-\omega\big)d\omega \\
&= \frac{\sin\alpha}{\kappa R}\int_{\kappa R}^{+\infty} \big(2\pi C_\omega-\omega\big)\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega,
\end{aligned} \tag{3.8}$$

where we have used the relations:

$$C_{-\omega}=-C_\omega \quad \text{and} \quad g(-\omega+\kappa R^2\sin\alpha)=g(\omega-\kappa R^2\sin\alpha).$$

Similar to (3.8), we get

$$\begin{aligned}
&\int_{|\omega-\kappa R^2\sin\alpha|>\kappa R} g(\omega)\Big(\int_{\mathbb{T}}\sin\theta df_\omega\Big)d\omega \\
&\quad= \frac{\cos\alpha}{\kappa R}\int_{\kappa R}^{+\infty}\big(-2\pi C_\omega+\omega\big)\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega.
\end{aligned} \tag{3.9}$$

• Step B. Next, we consider the integral:

$$\begin{aligned}
&\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R} g(\omega)\Big(\int_{\mathbb{T}}\cos\theta df_\omega\Big)d\omega \\
&\quad= \int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R} g(\omega)\Big(\big(1-\eta(\omega)\big)\cos\theta_\omega+\eta(\omega)\cos\theta^*_\omega\Big)d\omega.
\end{aligned} \tag{3.10}$$

By direct calculation, one has

$$\begin{aligned}
&\big(1-\eta(\omega)\big)\cos\theta_\omega+\eta(\omega)\cos\theta^*_\omega \\
&\quad= \big(1-\eta(\omega)\big)\big(\cos(\theta_\omega-\alpha)\cos\alpha-\sin(\theta_\omega-\alpha)\sin\alpha\big) \\
&\qquad+\eta(\omega)\big(\cos(\theta^*_\omega-\alpha)\cos\alpha-\sin(\theta^*_\omega-\alpha)\sin\alpha\big) \\
&\quad= \big(1-\eta(\omega)\big)\Big(\cos\alpha\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}-\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\sin\alpha\Big) \\
&\qquad+\eta(\omega)\Big(-\cos\alpha\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}-\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\sin\alpha\Big) \\
&\quad= -\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\sin\alpha+\big(1-2\eta(\omega)\big)\cos\alpha\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}.
\end{aligned} \tag{3.11}$$

Note that

$$\begin{aligned}
&-\frac{\sin\alpha}{\kappa R}\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R}(\omega-\kappa R^2\sin\alpha)g(\omega)d\omega \\
&\quad= -\frac{\sin\alpha}{\kappa R}\int_{|\omega|\le\kappa R}\omega g(\omega+\kappa R^2\sin\alpha)d\omega \\
&\quad= -\frac{\sin\alpha}{\kappa R}\Big\{\int_0^{\kappa R}\omega g(\omega+\kappa R^2\sin\alpha)d\omega+\int_{-\kappa R}^0\omega g(\omega+\kappa R^2\sin\alpha)d\omega\Big\} \\
&\quad= -\frac{\sin\alpha}{\kappa R}\int_0^{\kappa R}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega.
\end{aligned} \tag{3.12}$$

In (3.10), we combine estimates (3.11) and (3.12) to obtain

$$\begin{aligned}&\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R} g(\omega)\Big(\int_{\mathbb{T}}\cos\theta df_\omega\Big)d\omega\\&\quad=-\frac{\sin\alpha}{\kappa R}\int_0^{\kappa R}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega\\&\qquad+\cos\alpha\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R}\big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}d\omega.\end{aligned}\tag{3.13}$$

Similarly, we get

$$\begin{aligned}&\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R} g(\omega)\Big(\int_{\mathbb{T}}\sin\theta df_\omega\Big)d\omega\\&\quad=\frac{\cos\alpha}{\kappa R}\int_0^{\kappa R}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega\\&\qquad+\sin\alpha\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R}\big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}d\omega.\end{aligned}\tag{3.14}$$

Now we substitute estimates (3.8), (3.9), (3.13) and (3.14) into (3.5) to obtain

$$\begin{aligned}R&=\frac{\sin\alpha}{\kappa R}\int_{\kappa R}^{+\infty}\big(2\pi C_\omega-\omega\big)\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega\\&\quad-\frac{\sin\alpha}{\kappa R}\int_0^{\kappa R}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega\\&\quad+\cos\alpha\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R}\big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}d\omega,\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}0&=\frac{\cos\alpha}{\kappa R}\int_{\kappa R}^{+\infty}\big(-2\pi C_\omega+\omega\big)\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega\\&\quad+\frac{\cos\alpha}{\kappa R}\int_0^{\kappa R}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega\\&\quad+\sin\alpha\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R}\big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}d\omega.\end{aligned}\tag{3.16}$$

We multiply relation (3.15) by $\sin\alpha$, and multiply relation (3.16) by $\cos\alpha$, and then take the difference of the two resulting relations to derive

$$\begin{aligned}R\sin\alpha&=\frac{1}{\kappa R}\int_{\kappa R}^{+\infty}\big(2\pi C_\omega-\omega\big)\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega\\&\quad-\frac{1}{\kappa R}\int_0^{\kappa R}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega.\end{aligned}\tag{3.17}$$

Next, we substitute relation (3.17) into (3.15) to get

$$R=R\sin^2\alpha+\cos\alpha\int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R}\big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}d\omega.$$

Since $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we know $\cos\alpha \neq 0$. Thus, we have

$$R\cos\alpha = \int_{|\omega-\kappa R^2\sin\alpha|\le\kappa R} \big(1-2\eta(\omega)\big)g(\omega)\sqrt{1-\Big(\frac{\omega-\kappa R^2\sin\alpha}{\kappa R}\Big)^2}\,d\omega.$$

$\square$

Lemma 3.4. *If the phase-locked state for* (2.4) *and* (2.5) *exists, and $g(\omega)$ is non-increasing on $[0, L]$, then the upper bound L of natural frequency should satisfy*

$$L \le \kappa R - \kappa R^2 \sin|\alpha|.$$

Proof. Recall that Lemma 3.3 yields

$$\begin{aligned} R\sin\alpha = &\frac{2\pi}{\kappa R}\int_{\kappa R}^{+\infty} C_\omega\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega \\ &-\frac{1}{\kappa R}\int_0^{\infty}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} &\int_0^{+\infty}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega \\ &\quad=\frac{1}{2}\int_{-\infty}^{+\infty}\omega\big[g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big]d\omega \\ &\quad=-\kappa R^2\sin\alpha\int_{-\infty}^{+\infty}g(\omega)d\omega=-\kappa R^2\sin\alpha. \end{aligned}$$

Thus, we have

$$0=\frac{2\pi}{\kappa R}\int_{\kappa R}^{+\infty} C_\omega\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega. \tag{3.18}$$

Next, we will prove only $\alpha \ge 0$ case. The other case $\alpha < 0$ can be treated similarly. Suppose that

$$L > \kappa R - \kappa R^2\sin\alpha.$$

Then, we will derive a contradiction by ruling out the following two cases:
• Case A ($\ell \ge \kappa R + \kappa R^2 \sin\alpha$). In this case, we analyze R.H.S term of relation (3.18) by using the non-increasing property of g on $[0, +\infty)$ to get

$$\begin{aligned} &\int_{\kappa R}^{+\infty} C_\omega\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega \\ &\quad=\int_{\kappa R}^{\ell-\kappa R^2\sin\alpha} C_\omega\big(g(\omega+\kappa R^2\sin\alpha)-g(\omega-\kappa R^2\sin\alpha)\big)d\omega \\ &\qquad-\int_{\ell-\kappa R^2\sin\alpha}^{\ell+\kappa R^2\sin\alpha} C_\omega g(\omega-\kappa R^2\sin\alpha)d\omega \\ &\quad\le-\int_{\ell-\kappa R^2\sin\alpha}^{\ell+\kappa R^2\sin\alpha} C_\omega g(\omega-\kappa R^2\sin\alpha)d\omega<0. \end{aligned}$$

This contradicts to relation (3.18).

• Case B ($\kappa R - \kappa R^2 \sin\alpha < \ell < \kappa R + \kappa R^2 \sin\alpha$). Note that

$$\begin{aligned}&\int_{\kappa R}^{+\infty} C_\omega \big(g(\omega + \kappa R^2 \sin\alpha) - g(\omega - \kappa R^2 \sin\alpha)\big) d\omega \\ &\quad = -\int_{\kappa R}^{\ell + \kappa R^2 \sin\alpha} C_\omega g(\omega - \kappa R^2 \sin\alpha) d\omega < 0.\end{aligned}$$

This also contradicts to relation (3.18). Hence, we derived the desired estimate. □

Proof of Theorem 2.3. First, Proposition 3.1 gives (1) of Theorem 2.3 directly. Next, Proposition 3.2 shows if the phase-locked state exists, then

$$f_\omega = \begin{cases} \dfrac{C_{\omega - \kappa R^2 \sin\alpha}}{\omega - \kappa R^2 \sin\alpha - \kappa R \sin(\theta - \alpha)}, & \text{for } |\omega - \kappa R^2 \sin\alpha| > \kappa R, \\ \big(1 - \eta(\omega)\big)\delta_{\theta_\omega} + \eta(\omega)\delta_{\theta^*_\omega}, & \text{for } |\omega - \kappa R^2 \sin\alpha| \le \kappa R, \end{cases}$$

where $0 \le \eta(\omega) \le$ is a constant, $C^{\pm}_{\omega - \kappa R^2 \sin\alpha}$ is given by

$$C_{\omega - \kappa R^2 \sin\alpha} = \frac{1}{2\pi}\sqrt{(\omega - \kappa R^2 \sin\alpha)^2 - (\kappa R)^2} \text{ or } -\frac{1}{2\pi}\sqrt{(\omega - \kappa R^2 \sin\alpha)^2 - (\kappa R)^2},$$

with signs determined by

$$\int_{\mathbb{T}} \frac{C_{\omega - \kappa R^2 \sin\alpha}}{\omega - \kappa R^2 \sin\alpha - \kappa R \sin(\theta - \alpha)} d\theta = 1,$$

and phases $\theta_\omega - \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta^*_\omega - \alpha := \pi - (\theta_\omega - \alpha)$ are the roots of equation:

$$\sin x = \frac{\omega - \kappa R^2 \sin\alpha}{\kappa R}.$$

Furthermore, Lemmas 3.3 and 3.4 show that the following relations must hold:

$$\begin{cases} R\sin\alpha = \dfrac{1}{\kappa R}\displaystyle\int_{\kappa R}^{+\infty} \big(2\pi C_\omega - \omega\big)\big(g(\omega + \kappa R^2 \sin\alpha) - g(\omega - \kappa R^2 \sin\alpha)\big) d\omega, \\ \qquad\qquad -\dfrac{1}{\kappa R}\displaystyle\int_0^{\kappa R} \omega\big[g(\omega + \kappa R^2 \sin\alpha) - g(\omega - \kappa R^2 \sin\alpha)\big] d\omega, \\ R\cos\alpha = \displaystyle\int_{|\omega - \kappa R^2 \sin\alpha| \le \kappa R} \big(1 - 2\eta(\omega)\big) g(\omega) \sqrt{1 - \Big(\frac{\omega - \kappa R^2 \sin\alpha}{\kappa R}\Big)^2} d\omega, \\ L \le \kappa R - \kappa R^2 \sin|\alpha|. \end{cases}$$

□

4. **Stability of phase-locked states.** In this section, we study stability estimates of the phase-locked states for equation (1.4), i.e., stability of the complete phase-locked state and partial phase-locked state, respectively.

4.1. **Existence of measure-valued solutions.** We first derive a global existence of measure-valued solution for the corresponding mean-field model.

Recall that the order parameters R and ψ can be redefined as follows.

$$Re^{\mathrm{i}\psi} := \int_{\mathbb{T}\times\mathbb{R}} e^{\mathrm{i}\theta} \mu(d\theta d\omega). \tag{4.1}$$

Then as in Section 2, one has

$$
\begin{aligned}
R &= \langle \mu_t, \cos(\theta - \psi)\rangle = \int_{\mathbb{T}\times\mathbb{R}} \cos(\theta-\psi)\mu(d\theta d\omega),\\
0 &= \langle \mu_t, \sin(\theta - \psi)\rangle = \int_{\mathbb{T}\times\mathbb{R}} \sin(\theta-\psi)\mu(d\theta d\omega),
\end{aligned}
\tag{4.2}
$$

and

$$
\mathcal{V}[\mu](\theta,\omega,t) = \omega - \kappa R\sin(\theta-\psi-\alpha).
$$

Thus, equation (1.4) can be rewritten as

$$
\begin{aligned}
&\frac{d}{dt}\mu_t + \partial_\theta(\mathcal{V}[\mu_t]\mu_t) = 0,\\
&\mathcal{V}[\mu_t](\theta,\omega,t) := \omega - \kappa R\sin(\theta-\psi-\alpha).
\end{aligned}
\tag{4.3}
$$

4.1.1. *Emergence of phase-locked states.* In this subsection, we list the emergent estimates for the Kuramoto-Sakaguchi model for later use. Since the methodology for proofs is similar to the arguments given in [12, 13, 23], we leave detailed proofs in Appendix A.

Consider the following N-particle Kuramoto model with frustration:

$$
\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{j=1}^N \sin(\theta_j - \theta_i + \alpha), \quad t>0, \quad |\alpha| < \frac{\pi}{2}, \tag{4.4}
$$

and for a given phase vector $\Theta = (\theta_1, \cdots, \theta_N)$, we define

$$
\theta_M := \max_{1\le i\le N}\theta_i, \quad \theta_m := \min_{1\le i\le N}\theta_i, \quad D(\Theta) := \theta_M - \theta_m, \quad D(\omega) := \max_{1\le i,j\le N}|\omega_i - \omega_j|. \tag{4.5}
$$

Below, we state two asymptotic phase-locking for identical and non-identical ensembles.

Proposition 4.1. *The following assertions hold.*

1. *Suppos natural frequencies, coupling strength and initial data satisfy*
$$
\omega_i = 0, \quad 1\le i\le N, \quad \kappa>0, \quad D(\Theta^{in}) < \pi - 2|\alpha|,
$$
and let θ_i be a solution to (A.1). Then, we have exponential synchronization:
$$
D(\Theta(t)) \le D(\Theta^{in})\exp\Big[-\frac{2\kappa}{\pi}\cos\Big(\frac{1}{2}D(\Theta^{in}) + |\alpha|\Big)t\Big], \quad \textit{for } t\ge 0.
$$
2. *Suppose natural frequencies, coupling strength and initial data satisfy*
$$
\begin{aligned}
&0 < D(\Theta^{in}) < \pi - 2|\alpha|, \quad 0 < D(\Omega) < \infty,\\
&\kappa > \kappa_e := \frac{D(\Omega)}{\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|}.
\end{aligned}
$$
Then, we have
$$
D(\Theta(t)) < D^\infty, \quad \textit{for any } t > t_0 := \frac{D(\Theta^{in}) - D^\infty}{(1-\frac{\kappa}{\kappa_e})D(\Omega)},
$$
where $D^\infty \in (0, \frac{\pi}{2} - |\alpha|)$ is the root of the following trigonometric equation:
$$
\sin(x + |\alpha|) = \sin(D(\Theta^{in}) + |\alpha|).
$$

Proof. We leave its proof in Appendix A. ☐

4.1.2. *Measure-valued solutions.* Next, we briefly study a global existence of measure-valued solution for system (1.4) and its property.

Theorem 4.2. *For any $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$, let μ_t be a unique measure-valued solution to system (1.4) with the initial data μ_0. Then μ_t can be approximated by a sequence of empirical measures:*

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\theta_i(t)} \otimes \delta_{\omega_i(t)}.$$

Furthermore, one has

$$d\left(\mu_t, \mu_t^N\right) \to 0, \quad \text{as } N \to \infty.$$

Proof. Since the proof is nearly the same as in [3] using the N-particle theory in Section 4.1.1, we omit its proof. □

Lemma 4.3. *Suppose the initial measure satisfies*

$$\langle \mu_0, \omega \rangle = 0,$$

and let μ_t be a measure-valued solution of system (1.4). Then, for $t \geq 0$, one has

$$\langle \mu_t, \theta\rangle = \langle \mu_0, \theta\rangle + \int_0^t \kappa R^2 \sin\alpha ds.$$

Proof. We take $h = \theta$ in relation (2.6) and use system (4.3) to get

$$\langle \mu_t, \theta\rangle = \langle \mu_0, \theta\rangle + \int_0^t \langle \mu_s, \mathcal{V}[\mu]\rangle ds = \langle \mu_0, \theta\rangle + \int_0^t \langle \mu_s, \omega - \kappa R\sin(\theta-\psi-\alpha)\rangle ds.$$

Note that assumption on initial datum and Remark 2 (1) yield

$$\langle \mu_t, \omega\rangle = \langle \mu_0, \omega\rangle = 0.$$

Thus, we use relation (4.2) to obtain

$$\begin{aligned}
\langle \mu_t, \theta\rangle &= \langle \mu_0, \theta\rangle + \int_0^t \kappa R\langle \mu_s, -\sin(\theta-\psi-\alpha)\rangle ds \\
&= \langle \mu_0, \theta\rangle - \int_0^t \kappa R\cos\alpha\langle \mu_s, \sin(\theta-\psi)\rangle ds + \int_0^t \kappa R\sin\alpha\langle \mu_s, \cos(\theta-\psi)\rangle ds \\
&= \langle \mu_0, \theta\rangle + \int_0^t \kappa R^2 \sin\alpha ds.
\end{aligned}$$

□

4.2. **Complete phase-locked state.** In this subsection, we study the stability estimate of the complete phase-locked state. Let $\mu_t \in \mathcal{C}\big([0,T];\mathcal{M}(\mathbb{T}\times\mathbb{R})\big)$ be a measurable weak solution to system (1.4).

Recall that

$$\begin{aligned}
&B_\theta(t) := \mathbb{P}_\theta \mathrm{supp}\mu_t = \big\{\theta\in\mathbb{T} \mid (\theta,\omega)\in \mathrm{supp}\mu_t\big\},\\
&B_\omega(t) := \mathbb{P}_\omega \mathrm{supp}\mu_t = \big\{\omega\in\mathbb{R} \mid (\theta,\omega)\in \mathrm{supp}\mu_t\big\},\\
&D_\theta(\mu_t) := \mathrm{diam} B_\theta(t),\ \ D_\omega(\mu_t) := \mathrm{diam} B_\omega(t),\\
&\theta_c(t) := \frac{1}{M(t)}\langle \mu_t, \theta\rangle, \quad \omega_c(t) := \frac{1}{M(t)}\langle \mu_t, \omega\rangle,
\end{aligned}$$

where

$$M(t) = \langle \mu_t, 1\rangle = \langle \mu_0, 1\rangle = 1, \quad \text{and} \quad \mathrm{diam} A := \sup_{x,y\in A}|x-y|.$$

Since Remark 2 (1) gives that $\langle \mu_t, \omega\rangle = \langle \mu_0, \omega\rangle$, one has

$$B_\omega(t) = B_\omega(0), \quad t \geq 0.$$

Note that for identical oscillators, without loss of generality, we may assume $\omega_c(0) = 0$. Thus,

$$\omega_c(t) = 0, \quad t \geq 0.$$

Now we use Theorem 4.2 and Proposition 4.1 to get an exponential decay of $D_\theta(\mu_t)$.

Lemma 4.4. *Suppose the initial datum $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ satisfies*

$$D_\theta(\mu_0) \leq \pi - 2|\alpha|, \quad D_\omega(\mu_0) = 0,$$

and let μ_t be a measure-valued solution to system (1.4). Then, there exists a positive constant $\Lambda_0 := \frac{2}{\pi}\cos\left(\frac{1}{2}D_\theta(\mu_0) + |\alpha|\right)$ such that

$$D_\theta(\mu_t) \leq D_\theta(\mu_0)e^{-\kappa\Lambda_0 t}, \quad t \geq 0.$$

Proof. We define μ_0^N as in reference [3]. Note that Proposition 4.1 gives that the approximate measure valued solution $\mu_t^N \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ satisfies

$$D_\theta(\mu_t^N) \leq D_\theta(\mu_0^N)e^{-\kappa\Lambda_0 t}, \quad t \geq 0.$$

Now we use Theorem 4.2:

$$d(\mu, \mu_t^N) \to 0 \quad \text{as } N \to \infty$$

to see

$$D_\theta(\mu_t^N) \to D_\theta(\mu_t) \quad \text{as } N \to \infty.$$

Hence, our desired stability estimate is obtained. □

Now, we set

$$\mu_\infty(d\theta d\omega) := \delta_{\theta_c(t)} \times \delta_{\omega_c(0)}.$$

Theorem 4.5. *Suppose the initial datum $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ satisfies*

$$D_\theta(\mu_0) \leq \pi - 2|\alpha|, \quad D_\omega(\mu_0) = 0,$$

and let μ_t be a measure-valued solution to system (1.4). Then, one has

$$\lim_{t\to\infty} d(\mu_t, \mu_\infty) = 0 \quad \textit{exponentially.}$$

Proof. Let $h \in \mathcal{C}(\mathbb{T})$ be an arbitrary test function satisfying

$$\|h\|_\infty \leq 1 \quad \text{and} \quad \|h\|_{Lip} \leq 1.$$

Then we have

$$\begin{aligned}\Big|\int_{\mathbb{T}\times\mathbb{R}} h(\theta)\mu_t(d\theta, d\omega) - \int_{\mathbb{T}\times\mathbb{R}} h(\theta)\mu_\infty(d\theta, d\omega)\Big| &= \Big|\int_{\mathbb{T}} h(\theta)\bar{\mu}_t(d\theta) - h(\theta_c)\Big| \\ &\leq \int_{\mathbb{T}} |\theta - \theta_c|\bar{\mu}_t(d\theta) \leq D_\theta(\mu_0)e^{-c_0\kappa t},\end{aligned}$$

where

$$\bar{\mu}_t(d\theta) := \int_{\mathbb{R}} \mu_t(d\theta, d\omega).$$

We use Theorem 4.2 to conclude that

$$d(\mu_t, \mu_\infty) \to 0 \text{ exponentially}, \quad \text{as } t \to \infty.$$

□

As a corollary of Theorem 4.5, we obtain the exponential stability of the complete phase-locked state in the space of Radon measure-valued solutions.

Corollary 4.6. *Suppose that the initial datum $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ satisfies*

$$D_\theta(\mu_0) \le \pi - 2|\alpha|, \quad D_\omega(\mu_0) = 0, \tag{4.6}$$

and let μ_t be a measure-valued solution to system (1.4). Then, μ_t is asymptotically phase-locked. In particular, we obtain the stability of the complete phase-locked state in the space of Radon measure solution with initial datum satisfying (4.6).

Proof. Let Θ_1^0 and Θ_2^0 be initial measures satisfying

$$|\Theta_1^0 - \Theta_2^0| \le \pi - 2|\alpha|.$$

Then, it follows from Theorem 4.5 that

$$\Theta_1 - \Theta_2 \to 0 \quad \text{exponentially},$$

where $\dot\Theta(t,\theta,\omega) = \kappa\int_{\mathbb{T}\times\mathbb{R}} \sin(\Theta_* - \Theta + \alpha)\mu(d\theta_* d\omega_*)$ is the characteristic function. □

4.3. **Partial phase-locked state.** In this subsection, we study the nonlinear stability of partial phase-locked state to the K-S equation (1.4).

Consider the characteristic function defined by the following system:

$$\begin{cases} \dot\Theta(t,\theta,\omega) = \omega + \kappa \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\Theta_* - \Theta + \alpha)\mu(d\theta_* d\omega_*), \\ \Theta(t,\theta,\omega)|_{t=0} = \theta. \end{cases}$$

In this case, we set

$$\Phi(t,\theta,\omega) := \dot\Theta(t,\theta,\omega).$$

Then, the new variable $\Phi(t)$ satisfies

$$\dot\Phi(t,\theta,\omega) = \kappa\int_{\mathbb{T}\times\mathbb{R}} \cos(\Theta_* - \Theta + \alpha)(\Phi_* - \Phi)\mu(d\theta_* d\omega_*). \tag{4.7}$$

We also define

$$D_\theta^1(\mu_t) = \sup_{\Theta_1,\Theta_2\in B_\theta(t)} \big(\dot\Theta_1(t) - \dot\Theta_2(t)\big) := \sup_{\Theta_1,\Theta_2\in B_\theta(t)} \big(\Phi_1(t) - \Phi_2(t)\big).$$

Note that $0 \le D_\theta^1(\mu_t) < \infty$. In fact, for any $\Theta(t) \in B_\theta(t)$, we use the compactness of g in Lemma 3.4 to have

$$|\Phi| = |\dot\Theta| \le L + \kappa < \infty.$$

In the sequel, we will prove that the quantity $D_\theta^1(\mu_t)$ tends to zero exponentially fast as $t \to \infty$.

Lemma 4.7. *Let $\mu_0 \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ be a given initial measure such that*

$$0 < D_\theta(\mu_0) \le \pi - 2|\alpha|, \quad 0 < D_\omega(\mu_0) < \infty,$$
$$\kappa > \kappa_e := \frac{D_\omega(\mu_0)}{\sin\big(D_\theta(\mu_0) + |\alpha|\big) - \sin|\alpha|},$$

and let μ_t be a measure-valued solution to equation (1.4) with an initial datum μ_0. Then, there exists a time t_0 such that

$$D_\theta(\mu_t) < D^\infty, \quad \textit{for all } t > t_0,$$

where $D^\infty \in (0, \frac{\pi}{2} - |\alpha|)$ *is the solution of*

$$\sin(x + |\alpha|) = \sin(D_\theta(\mu_0) + |\alpha|).$$

Proof. For given $N > 0$, we consider the approximation μ_0^N for μ_0:

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\theta_0} \otimes \delta_{\omega_0}.$$

We solve the Cauchy problem for the following N-particle system:

$$\begin{cases} \dfrac{d\theta_i}{dt} = \omega_i + \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N} \sin(\theta_j - \theta_i + \alpha), \\ \dfrac{d\omega_i}{dt} = 0, \end{cases}$$

with the initial data (θ_i^0, ω_i^0). We use Theorem 4.2 to obtain

$$d\left(\mu_t, \mu_t^N\right) \to 0, \quad \text{as } N \to \infty.$$

Furthermore, Proposition 4.1 shows that there exists time t_0:

$$t_0 := \frac{D_\theta(\mu_0^N) - D^{\infty,N}}{D_\omega(\mu_0^N) - \kappa\Big(\sin\big(D_\theta(\mu_0^N) + |\alpha|\big) - \sin|\alpha|\Big)},$$

with $D^{\infty,N} + |\alpha| = \arcsin(D_\theta(\mu_0^N) + |\alpha|) \in (0, \frac{\pi}{2})$ such that

$$D_\theta(\mu_t^N) < D^{\infty,N}, \quad \text{for all } t > t_0^N,$$

for N large enough. Now let N tends to infinity to obtain the desired results. $\square$

Theorem 4.8. *Let* $\mu_0 \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$ *be a given initial measure such that*

$$0 < D_\theta(\mu_0) \le \pi - 2|\alpha|, \quad 0 < D_\omega(\mu_0) < \infty,$$
$$\kappa > \kappa_e := \frac{D_\omega(\mu_0)}{\sin(D_\theta(\mu_0) + |\alpha|) - \sin|\alpha|},$$

and let μ_t *be a measure valued solution to equation* (1.4)*. Then there exists a positive time* t_0 *such that*

$$D_\theta^1(\mu_t) \le D_\theta^1(\mu_{t_0}) e^{-\frac{1}{2}\kappa\cos(D^\infty + |\alpha|)(t - t_0)}, \quad \textit{for all } t \ge t_0.$$

Proof. For $t > t_0$ and any $\varepsilon \in (0, \frac{1}{8})$, we take any $\Theta_{M,\varepsilon} \in B_{M,\varepsilon}$ and $\Theta_{m,\varepsilon} \in B_{m,\varepsilon}$ such that

$$\int_{\Phi \ge \Phi_{M,\varepsilon}} \mu(d\theta_* d\omega_*) \le \varepsilon \quad \text{and} \quad \int_{\Phi \le \Phi_{m,\varepsilon}} \mu(d\theta_* d\omega_*) \le \varepsilon.$$

Then, we have

$$\frac{d}{dt}(\Phi_{M,\varepsilon} - \Phi_{m,\varepsilon}) = \int_{\Phi_* \ge \Phi_{M,\varepsilon}} + \int_{\Phi_* \le \Phi_{m,\varepsilon}} + \int_{\Phi_{m,\varepsilon} \le \Phi_* \le \Phi_{M,\varepsilon}} =: \mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{13}.$$

Below, we estimate $\mathcal{J}_{1i},\ i = 1, 2, 3$ one by one.

• Case A (Estimate on $\mathcal{J}_{11}$). It follows from relation (4.7) and Lemma 4.7 that

$$\begin{aligned}\mathcal{J}_{11} = \kappa \int_{\Phi_* \geq \Phi_{M,\varepsilon}} &\big[\cos(\Theta_* - \Theta_{M,\varepsilon} + \alpha)(\Phi_* - \Phi_{M,\varepsilon}) \\ &- \cos(\Theta_* - \Theta_{m,\varepsilon} + \alpha)(\Phi_* - \Phi_{m,\varepsilon})\big]\mu(d\theta_* d\omega_*) \\ &\leq \kappa \int_{\Phi_* \geq \Phi_{M,\varepsilon}} \cos(\Theta_* - \Theta_{M,\varepsilon} + \alpha)(\Phi_* - \Phi_{M,\varepsilon})\mu(d\theta_* d\omega_*) \\ &\leq \kappa\varepsilon D^1_\theta(\mu_t).\end{aligned}$$

• Case B (Estimate on $\mathcal{J}_{12}$). Similar to Case A, we get

$$\begin{aligned}\mathcal{J}_{12} = \kappa \iint_{\Phi_* \leq \Phi_{m,\varepsilon}} &\big[\cos(\Theta_* - \Theta_{M,\varepsilon} + \alpha)(\Phi_* - \Phi_{M,\varepsilon}) \\ &- \cos(\Theta_* - \Theta_{m,\varepsilon} + \alpha)(\Phi_* - \Phi_{m,\varepsilon})\big]\mu(d\theta_* d\omega_*) \\ &\leq \kappa \iint_{\Phi_* \geq \Phi_{M,\varepsilon}} -\cos(\Theta_* - \Theta_{m,\varepsilon} + \alpha)(\Phi_* - \Phi_{m,\varepsilon})\mu(d\theta_* d\omega_*) \\ &\leq \kappa\varepsilon D^1_\theta(\mu_t).\end{aligned}$$

• Case C (Estimate on $\mathcal{J}_{13}$). We use Lemma 4.7 to obtain

$$\begin{aligned}\mathcal{J}_{13} = \kappa \int_{\Phi_{m,\varepsilon} \leq \Phi_* \leq \Phi_{M,\varepsilon}} &\big[\cos(\Theta_* - \Theta_{M,\varepsilon} + \alpha)(\Phi_* - \Phi_{M,\varepsilon}) \\ &- \cos(\Theta_* - \Theta_{m,\varepsilon} + \alpha)(\Phi_* - \Phi_{m,\varepsilon})\big]\mu(d\theta_* d\omega_*) \\ &\leq \kappa \cos(D^\infty + |\alpha|) \int_{\Phi_{m,\varepsilon} \leq \Phi_* \leq \Phi_{M,\varepsilon}} \big[(\Phi_* - \Phi_{M,\varepsilon}) - (\Phi_* - \Phi_{m,\varepsilon})\big]\mu(d\theta_* d\omega_*) \\ &= -\kappa \cos(D^\infty + |\alpha|) \int_{\Phi_{m,\varepsilon} \leq \Phi_* \leq \Phi_{M,\varepsilon}} (\Phi_{M,\varepsilon} - \Phi_{m,\varepsilon})\mu(d\theta_* d\omega_*) \\ &\leq -\kappa(1 - 2\varepsilon)(1 - \varepsilon)\cos(D^\infty + |\alpha|) D^1_\theta(\mu_t).\end{aligned}$$

Now, we combine all estimates to derive

$$\frac{d}{dt}\big(\Phi_{M,\varepsilon} - \Phi_{m,\varepsilon}\big) \leq -\kappa \cos D^\infty + |\alpha|\big[1 - (3 + \frac{2}{\cos(D^\infty + |\alpha|)} - 2\varepsilon)\varepsilon\big] D^1_\theta(\mu_t).$$

Thus, we can derive

$$\frac{d}{dt} D^1_\theta(\mu_t) \leq \sup_{\Phi_{M,\varepsilon}, \Phi_{m,\varepsilon}} \frac{d}{dt}\big(\Phi_{M,\varepsilon} - \Phi_{m,\varepsilon}\big) \leq -\frac{1}{2}\kappa \cos(D^\infty + |\alpha|) D^1_\theta(\mu_t).$$

Finally, we use Gronwall's lemma to get

$$D^1_\theta(\mu_t) \leq D^1_\theta(\mu_{t_0}) e^{-\frac{1}{2}\kappa(\cos D^\infty)(t - t_0)}, \quad \text{for all } t \geq t_0.$$

□

Corollary 4.9. *Let $\mu_0 \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$ be an initial measure satisfying*

$$0 < D_\theta(\mu_0) \leq \pi - 2|\alpha|, \quad 0 < D_\omega(\mu_0) < \infty,$$
$$\kappa > \kappa_e := \frac{D_\omega(\mu_0)}{\sin(D_\theta(\mu_0) + |\alpha|) - \sin|\alpha|}.$$

Then, the measure-valued solution μ_t to equation (1.4) is asymptotically phase-locked. In particular, we obtain the semi-stability of the partially phase-locked state in space of Radon measure solution whose initial data satisfy (2.8).

Proof. Note that Theorem 4.8 yields

$$\lim_{t\to\infty}\int_0^t |\partial_s\Theta_1 - \partial_s\Theta_2| ds < \infty.$$

Then we have

$$\lim_{t\to\infty}\big(\Theta_1(t) - \Theta_2(t)\big) = \big(\Theta_1^0 - \Theta_2^0\big) + \lim_{t\to\infty}\int_0^t (\partial_s\Theta_1 - \partial_s\Theta_2) ds < \infty.$$

This means that for any Θ_1 and Θ_2,

$$\lim_{t\to\infty}\big(\Theta_1(t) - \Theta_2(t)\big) \quad \text{exists and finite.}$$

Now we use Theorem 4.8 and definition of order parameters to get

$$\omega - \kappa R\sin(\Theta - \psi - \alpha) - \kappa R^2 \sin\alpha \to 0 \quad \text{exponentially fast.}$$

We set

$$R^\infty := \lim_{t\to\infty} R \quad \text{and} \quad \psi^\infty := \lim_{t\to\infty}\psi.$$

Then, we can get

$$\lim_{t\to\infty}\sin(\Theta^\infty - \psi^\infty - \alpha) = \frac{\omega - \kappa(R^\infty)^2\sin\alpha}{\kappa R^\infty}.$$

Hence, for any $\Theta(0)$ satisfy assumption (2.8), $\Theta(t)$ approaches to an equilibrium described in (2) of Theorem 2.3. In particular, the partially phase-locked state we obtained in Section 3 is semi-stable. □

5. **Stability of the incoherent state.** In this section, we study the stability of incoherent solution to the K-S equation with a frustration for identical oscillators. For this, we define a new quantity:

$$\mathcal{I}(t) := \Big\langle \mu_t, \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\Big\rangle = \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\mu_t(d\theta_*).$$

Lemma 5.1. *Let $\mu_t := \frac{\mu_e}{2\pi}$ be a uniform distribution on $\mathbb{S}^1$ such that*

$$\mu_e(\mathbb{T}) = 1 \quad \text{or} \quad d\mu_e = \frac{1}{2\pi}d\theta.$$

Then we can have

$$\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\mu_t(d\theta_*) = -\ln 2.$$

Proof. By direct estimate, we have

$$\begin{aligned}
&\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\mu_t(d\theta_*) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_e(d\theta)\mu_e(d\theta_*)\\
&\quad = \frac{1}{(2\pi)^2}\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta}{2}\Big)\Big|\mu_e(d\theta)\mu_e(d\theta_*) = \frac{1}{2\pi}\int_{\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta}{2}\Big)\Big|\mu_e(d\theta)\\
&\quad = \frac{1}{\pi}\int_0^\pi \ln(\sin\theta)\mu_e(d\theta) = -\ln 2.
\end{aligned}$$

□

By Lemma 5.1, we can see that $\mathcal{I}(t)$ works well (at least for the uniform distribution on S^1). As a corollary of Lemma 5.1, we have upper and lower bound estimates for $\mathcal{I}$.

Lemma 5.2. *Suppose there exists positive constants m and M such that*

$$m\mu_e(d\theta) \le \mu_t(d\theta) \le M\mu_e(d\theta) \quad \textit{for all } \theta \in \mathbb{T}. \tag{5.1}$$

Then, the quantity $\mathcal{I}(t)$ satisfies

$$-4\pi^2 M^2 \ln 2 \le \mathcal{I}(t) \le -4\pi^2 m^2 \ln 2, \quad t \ge 0.$$

Proof. We use the assumptions (5.1) to get that for all $t \ge 0$,

$$\begin{aligned} m^2 \int_{\mathbb{T}\times\mathbb{T}} \ln \Big| \sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big| \mu_e(d\theta)\mu_e(d\theta_*) &\le \mathcal{I}(t) \\ &\le M^2 \int_{\mathbb{T}\times\mathbb{T}} \ln \Big| \sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big| \mu_e(d\theta)\mu_e(d\theta_*). \end{aligned}$$

Note that

$$\int_{\mathbb{T}\times\mathbb{T}} \ln \Big| \sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big| \mu_e(d\theta)\mu_e(d\theta_*) = -4\pi^2 \ln 2.$$

Thus, we have

$$-4\pi^2 M^2 \ln 2 \le \mathcal{I}(t) \le -4\pi^2 m^2 \ln 2, \quad t \ge 0.$$

□

5.1. **Small frustrations.** In this subsection, we consider small frustration case with $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Similar to system (4.3), we use order parameters to rewrite equation (2.9) as

$$\frac{d}{dt}\mu_t + \partial_\theta\big\{\mathcal{V}[\mu_t]\mu_t(d\theta)\big\} = 0, \quad \mathcal{V}[\mu_t] = -\kappa R\sin(\theta-\psi-\alpha). \tag{5.2}$$

We differentiate both sides of (4.1) with respect to t and use equation (5.2) to get

$$\begin{aligned} &e^{\mathrm{i}\psi}\big[\dot R(t) + \mathrm{i}R(t)\dot\psi(t)\big] = \int_{\mathbb{T}} e^{\mathrm{i}\theta}\frac{\partial}{\partial t}\mu_t(d\theta) \\ &= \int_{\mathbb{T}} e^{\mathrm{i}\theta}\partial_\theta\big\{\kappa R\sin(\theta-\psi-\alpha)\mu_t(d\theta)\big\} = -\mathrm{i}\int_{\mathbb{T}} e^{\mathrm{i}\theta}\kappa R\cos(\theta-\psi-\alpha)\mu(d\theta). \end{aligned} \tag{5.3}$$

Now we divide both sides of relation (5.3) by $e^{\mathrm{i}\psi}$, and compare the real and imaginary parts to obtain

$$\begin{aligned} \dot R &= \kappa R\int_{\mathbb{T}} \sin(\theta-\psi)\sin(\theta-\psi-\alpha)\mu_t(d\theta), \\ R\dot\psi &= -\kappa R\int_{\mathbb{T}\times\mathbb{R}} \cos(\theta-\psi)\sin(\theta-\psi-\alpha)\mu_t(d\theta). \end{aligned} \tag{5.4}$$

Lemma 5.3. *Let $\mu_t \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ be a measure-valued solution for equation (2.9). Then, for all $t \ge 0$,*

$$\frac{d}{dt}\mathcal{I}(t) = -\kappa R^2\cos\alpha \quad \textit{and} \quad \frac{d^2}{dt^2}\mathcal{I}(t) = -2\kappa^2R^2\cos\alpha\int_{\mathbb{T}}\sin(\theta-\psi)\sin(\theta-\psi-\alpha)\mu_t(d\theta).$$

Proof. (i) We use equation (5.2) to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) &= \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta_*)\frac{d}{dt}\mu_t(d\theta)\\ &+ \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\frac{d}{dt}\mu_t(d\theta_*)\\ &= \kappa R\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\partial_\theta\big(\sin(\theta-\psi-\alpha)\mu_t(d\theta)\big)\mu_t(d\theta_*)\\ &+ \kappa R\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\partial_{\theta_*}\big(\sin(\theta_*-\psi-\alpha)\mu_t(d\theta_*)\big)\mu_t(d\theta).\end{aligned}$$

Integrations by parts yield

$$\begin{aligned}&\frac{d}{dt}\mathcal{I}(t) = -\kappa R\int_{\mathbb{T}\times\mathbb{T}}\Big(\sin(\theta-\psi-\alpha)\partial_\theta\ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\\ &+\sin(\theta_*-\psi-\alpha)\partial_{\theta_*}\ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &= -\kappa R\int_{\mathbb{T}\times\mathbb{T}}\big(\sin(\theta-\psi-\alpha)-\sin(\theta_*-\psi-\alpha)\big)\partial_\theta\ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\mu_t(d\theta_*).\end{aligned}$$

By direct calculation, one has

$$\sin(\theta-\psi-\alpha)-\sin(\theta_*-\psi-\alpha) = 2\cos\Big(\frac{\theta+\theta_*-2\psi}{2}-\alpha\Big)\sin\Big(\frac{\theta_*-\theta}{2}\Big).$$

Now we use the above estimates and relation:

$$\partial_\theta\ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big| = \frac{\cos(\frac{\theta-\theta_*}{2})}{2\sin(\frac{\theta-\theta_*}{2})}$$

to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) &= -\kappa R\int_{\mathbb{T}\times\mathbb{T}}\cos\Big(\frac{\theta+\theta_*-2\psi}{2}-\alpha\Big)\sin\Big(\frac{\theta_*-\theta}{2}\Big)\frac{\cos(\frac{\theta-\theta_*}{2})}{\sin(\frac{\theta-\theta_*}{2})}\mu_t(d\theta)\mu_t(d\theta_*)\\ &= -\kappa R\cos\alpha\int_{\mathbb{T}\times\mathbb{T}}\cos\Big(\frac{\theta+\theta_*-2\psi}{2}\Big)\cos\Big(\frac{\theta-\theta_*}{2}\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &\quad-\kappa R\sin\alpha\int_{\mathbb{T}\times\mathbb{T}}\sin\Big(\frac{\theta+\theta_*-2\psi}{2}\Big)\cos\Big(\frac{\theta-\theta_*}{2}\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &=: \mathcal{J}_{21}+\mathcal{J}_{22}.\end{aligned}$$

We use relation (4.2) to derive

$$\begin{aligned}&\int_{\mathbb{T}\times\mathbb{T}}\cos\Big(\frac{\theta+\theta_*-2\psi}{2}\Big)\cos\Big(\frac{\theta-\theta_*}{2}\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &\qquad=\int_{\mathbb{T}\times\mathbb{T}}\big(\cos(\theta-\psi)+\cos(\theta_*-\psi)\big)\mu_t(d\theta)\mu_t(d\theta_*) = R,\\ &\int_{\mathbb{T}\times\mathbb{T}}\sin\Big(\frac{\theta+\theta_*-2\psi}{2}\Big)\cos\Big(\frac{\theta-\theta_*}{2}\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &\qquad=\frac{1}{2}\int_{\mathbb{T}\times\mathbb{T}}\big(\sin(\theta-\psi)+\sin(\theta_*-\psi)\big)\mu_t(d\theta)\mu_t(d\theta_*) = 0.\end{aligned}$$

Thus, we have

$$\mathcal{J}_{21} = -\kappa R^2\cos\alpha,\quad \mathcal{J}_{22} = 0. \tag{5.5}$$

This yields

$$\frac{d}{dt}\mathcal{I}(t) = -\kappa R^2 \cos\alpha, \quad t > 0. \tag{5.6}$$

(ii) Now we differentiate relation (5.6) with respect to time t to get

$$\frac{d^2}{dt^2}\mathcal{I}(t) = -2\kappa \cos\alpha R\dot{R}, \quad t > 0.$$

Hence, we use relation (5.4) to obtain

$$\frac{d^2}{dt^2}\mathcal{I}(t) = -2\kappa^2 R^2 \cos\alpha \int_{\mathbb{T}} \sin(\theta - \psi)\sin(\theta - \psi - \alpha)\mu_t(d\theta), \quad t > 0.$$

□

Proposition 5.4. *Suppose the frustration and initial datum satisfy*

$$\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \textit{and} \quad \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_0(d\theta)\mu_0(d\theta_*) < \infty,$$

and let μ be a measure-valued solution for equation (2.9).

1. *If there exist constant M such that $\mu(d\theta) \le M\mu_e(d\theta)$, then we have*
$$\lim_{t\to\infty} R(t) = 0.$$
2. *If not, we have*
$$\lim_{t\to\infty} \Big\|\frac{\mu_t(d\theta)}{\mu_e(d\theta)}\Big\|_{L^\infty} = \infty.$$

Proof. (1) We use Lemma 5.3 to see that $\frac{d}{dt}\mathcal{I}(t)$ is uniformly continuous and $\mathcal{I}(t)$ is decreasing. Now if there exist a constant M such that $\mu(d\theta) \le M\mu_e(d\theta)$, we can use Lemma 5.2 to get

$$\mathcal{I}(t) \ge -4\pi^2 M^2 \ln 2.$$

Hence, we deduce

$$\lim_{t\to\infty} \mathcal{I}(t) \text{ exists}.$$

Together with initial assumption

$$\mathcal{I}(0) := \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_0(d\theta)\mu_0(d\theta_*) < \infty,$$

we obtain

$$\lim_{t\to\infty}\int_0^t \frac{d}{ds}\mathcal{I}(s)ds = \lim_{t\to\infty}\mathcal{I}(t) - \mathcal{I}(0) \text{ exists}.$$

Thus, we can use Barbalat's Lemma to conclude

$$\frac{d}{dt}\mathcal{I}(t) \to 0, \quad \text{as } t \to \infty.$$

Now we use

$$\frac{d}{dt}\mathcal{I}(t) = -\kappa R^2\cos\alpha \quad \text{and} \quad |\alpha| < \frac{\pi}{2},$$

to see

$$\lim_{t\to\infty} R(t) = 0.$$

(2) If we can not find a positive constant M such that $\mu(d\theta) \le M\mu_e(d\theta)$, then we obtain

$$\lim_{t\to\infty}\mathcal{I}(t) = -\infty.$$

Furthermore, we can see

$$
\begin{aligned}
\mathcal{I}(t) &= \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\mu_t(d\theta_*)\\
&\geq \Big\|\frac{\mu_t(d\theta)}{\mu_e(d\theta)}\Big\|^2_{L^\infty}\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_e(d\theta)\mu_e(d\theta_*) = -\ln 2\Big\|\frac{\mu_t(d\theta)}{\mu_e(d\theta)}\Big\|^2_{L^\infty}.
\end{aligned}
$$

Thus, we can deduce

$$
\Big\|\frac{\mu_t(d\theta)}{\mu_e(d\theta)}\Big\|_{L^\infty} \to \infty, \quad \text{as } t\to\infty.
$$

□

5.2. **Large frustrations.** In this subsection, we consider a large frustration case ($|\alpha| \geq \frac{\pi}{2}$). For this, we define $\hat{\alpha} = \alpha - \frac{\pi}{2}$. Then, the original K-S equation becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}\mu_t + \partial_\theta(\mathcal{C}[\mu_t]\mu_t) = 0, \quad (t,\theta,\omega)\in\mathbb{R}_+\times\mathbb{T}\times\mathbb{R},\\
&\mathcal{C}[\mu_t] = \kappa\int_{\mathbb{T}\times\mathbb{R}}\cos(\theta_*-\theta+\hat{\alpha})\mu_t(d\theta_* d\omega_*).
\end{aligned}
\tag{5.7}
$$

Note that the assumption on α and definition of $\hat{\alpha}$, we have $\hat{\alpha}\in[0,\pi]$, and since $\alpha\in[\frac{\pi}{2},\pi]$, it is easy to see that $\hat{\alpha}\in[0,\frac{\pi}{2}]$. As $\alpha\in(-\pi,-\frac{\pi}{2}]$, we have

$$
\hat{\alpha}\in\Big(-\frac{3\pi}{2},-\pi\Big] = \Big(\frac{\pi}{2},\pi\Big].
$$

We divide both sides of relation (4.1) by $e^{\mathrm{i}(\theta-\hat{\alpha})}$ and take the real part of the resulting relation to have

$$
R\cos(\theta-\psi-\hat{\alpha}) = \int_{\mathbb{T}\times\mathbb{R}}\cos(\theta_*-\theta+\hat{\alpha})\mu_t(d\theta_* d\omega_*). \tag{5.8}
$$

We use relation (5.8) to rewrite the equation (5.7) to rewrite as

$$
\frac{\partial}{\partial t}\mu_t + \partial_\theta\big(\mathcal{C}[\mu_t]\mu_t\big) = 0, \quad \mathcal{C}[\mu_t] = \kappa R\cos(\theta-\psi-\hat{\alpha}). \tag{5.9}
$$

We differentiate both sides of (4.1) with respect to t and use system (5.9) to get

$$
\begin{aligned}
e^{\mathrm{i}\psi}\big[\dot{R}(t) + \mathrm{i}R(t)\dot{\psi}(t)\big] &= \int_{\mathbb{T}\times\mathbb{R}} e^{\mathrm{i}\theta}\frac{d}{dt}\mu_t(d\theta d\omega)\\
&= -\int_{\mathbb{T}\times\mathbb{R}} e^{i\theta}\partial_\theta\big\{\kappa R\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta d\omega)\big\}\\
&= \mathrm{i}\int_{\mathbb{T}\times\mathbb{R}} e^{i\theta}\kappa R\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta d\omega).
\end{aligned}
$$

Now, we divide both sides of above relation by $e^{i\psi}$ and separate the real and imaginary parts to obtain

$$
\begin{aligned}
\dot{R}(t) &= -\kappa R\int_{\mathbb{T}\times\mathbb{R}}\sin(\theta-\psi)\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta d\omega),\\
R(t)\dot{\psi}(t) &= \kappa R\int_{\mathbb{T}\times\mathbb{R}}\cos(\theta-\psi)\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta d\omega).
\end{aligned}
\tag{5.10}
$$

Now, we define the first phase moment $m_1(t)$ as follows.

$$
m_1(t) := \int_{\mathbb{T}}\theta\mu_t(d\theta).
$$

Lemma 5.5. *Let $\mu_t \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ be a measure-valued solution for equation (2.10). Then one has*

$$(i)\ \frac{d}{dt}\mathcal{I}(t) = \kappa R^2 \sin\hat{\alpha}, \quad \frac{d}{dt}m_1(t) = \kappa R^2\cos\hat{\alpha}.$$

$$(ii)\ \frac{d^2}{dt^2}\mathcal{I}(t) = -2\kappa^2 R^2 \sin\hat{\alpha}\int_{\mathbb{T}} \sin(\theta-\psi)\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta).$$

Proof. (i) We use relation (5.9) to obtain that

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) &= \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta_*)\frac{d}{dt}\mu_t(d\theta)\\ &\quad + \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\frac{d}{dt}\mu_t(d\theta_*)\\ &= -\kappa R\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\partial_\theta\big(\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta)\big)\mu_t(d\theta_*)\\ &\quad -\kappa R\int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\partial_{\theta_*}\big(\cos(\theta_*-\psi-\hat{\alpha})\mu_t(d\theta_*)\big)\mu_t(d\theta).\end{aligned}$$

We use integration by parts to get

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) &= \kappa R\int_{\mathbb{T}\times\mathbb{T}} \Big(\cos(\theta-\psi-\hat{\alpha})\partial_\theta \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\\ &\qquad + \cos(\theta_*-\psi-\hat{\alpha})\partial_{\theta_*}\ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &= \kappa R\int_{\mathbb{T}\times\mathbb{T}}\big(\cos(\theta-\psi-\hat{\alpha})-\cos(\theta_*-\psi-\hat{\alpha})\big)\partial_\theta\ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_t(d\theta)\mu_t(d\theta_*).\end{aligned} \tag{5.11}$$

By direct calculation, one has

$$\cos(\theta-\psi-\hat{\alpha})-\cos(\theta_*-\psi-\hat{\alpha}) = -2\sin\Big(\frac{\theta+\theta_*-2\psi}{2}-\hat{\alpha}\Big)\sin\Big(\frac{\theta_*-\theta}{2}\Big). \tag{5.12}$$

Now, we use (5.11), (5.12) and relation

$$\partial_\theta \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big| = \frac{\cos(\frac{\theta-\theta_*}{2})}{2\sin(\frac{\theta-\theta_*}{2})}$$

to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) &= -\kappa R\int_{\mathbb{T}\times\mathbb{T}}\sin\Big(\frac{\theta+\theta_*-2\psi}{2}-\hat{\alpha}\Big)\sin\Big(\frac{\theta_*-\theta}{2}\Big)\frac{\cos(\frac{\theta-\theta_*}{2})}{\sin(\frac{\theta-\theta_*}{2})}\mu_t(d\theta)\mu_t(d\theta_*)\\ &= -\kappa R\cos\hat{\alpha}\int_{\mathbb{T}\times\mathbb{T}}\sin\Big(\frac{\theta+\theta_*-2\psi}{2}\Big)\cos\Big(\frac{\theta-\theta_*}{2}\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &\quad +\kappa R\sin\hat{\alpha}\int_{\mathbb{T}\times\mathbb{T}}\cos\Big(\frac{\theta+\theta_*-2\psi}{2}\Big)\cos\Big(\frac{\theta-\theta_*}{2}\Big)\mu_t(d\theta)\mu_t(d\theta_*)\\ &=: \mathcal{J}_{31}+\mathcal{J}_{32}.\end{aligned}$$

Similar to the derivation of (5.5) in Lemma 4.7, we have

$$\mathcal{J}_{31} = 0, \quad \mathcal{J}_{32} = \kappa R^2\sin\hat{\alpha}.$$

Hence, we conclude

$$\frac{d}{dt}\mathcal{I}(t) = \kappa R^2\sin\hat{\alpha}, \quad t>0. \tag{5.13}$$

We use relation (5.9) to obtain

$$\begin{aligned}\frac{d}{dt}m_1(t) &= \int_{\mathbb{T}} \theta \frac{d}{dt}\mu(d\theta) = -\kappa R \int_{\mathbb{T}} \theta\partial_\theta\big(\cos(\theta-\psi-\hat{\alpha})\mu_t\big) \\ &= \kappa R \int_{\mathbb{T}} \cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta) \\ &= \kappa R\cos\hat{\alpha}\int_{\mathbb{T}}\cos(\theta-\psi)\mu_t(d\theta) + \kappa R\sin\hat{\alpha}\int_{\mathbb{T}}\sin(\theta-\psi)\mu_t(d\theta) = \kappa R^2\cos\hat{\alpha}.\end{aligned}$$

(ii) Now we differentiate relation (5.13) with respect to time t to get

$$\frac{d^2}{dt^2}\mathcal{I}(t) = 2\kappa R\dot{R}\sin\hat{\alpha}, \quad t>0.$$

We use relation (5.10) to deduce

$$\frac{d^2}{dt^2}\mathcal{I}(t) = -2\kappa^2R^2\sin\hat{\alpha}\int_{\mathbb{T}}\sin(\theta-\psi)\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta), \quad t>0.$$

□

Corollary 5.6. *Let $\mu_t \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$ be a measure-valued solution for equation (2.10). Then, the quantity $\cos\hat{\alpha}\mathcal{I}(t) - \sin\hat{\alpha}m_1(t)$ is invariant with respect to time t:*

$$\cos\hat{\alpha}\mathcal{I}(t) - \sin\hat{\alpha}m_1(t) = \cos\hat{\alpha}\mathcal{I}(0) - \sin\hat{\alpha}m_1(0), \quad \textit{for all } t\geq 0.$$

Proof. We combine estimate (i) of Lemma 5.5 (i) to get

$$\cos\hat{\alpha}\frac{d}{dt}m_1(t)\mathcal{I}(t) - \sin\hat{\alpha}\frac{d}{dt}m_1(t) = 0, \quad t>0.$$

This yields our desired estimate. □

Remark 3. For all $t\geq 0$, for $\hat{\alpha}=0$, Lemma 5.5 gives

$$\mathcal{I}(t) = \mathcal{I}(0),$$

and for $\hat{\alpha} = \frac{\pi}{2}$, Lemma 5.5 gives $m_1(t) = m_1(0)$.

Proposition 5.7. *Suppose frustration and initial datum satisfy*

$$\hat{\alpha}\in(0,\pi] \quad \textit{and} \quad \int_{\mathbb{T}\times\mathbb{T}} \ln\Big|\sin\Big(\frac{\theta-\theta_*}{2}\Big)\Big|\mu_0(d\theta)d\mu_0(d\theta_*) < \infty,$$

and let $\mu_t\in\mathcal{M}(\mathbb{T}\times\mathbb{R})$ be a measure-valued solution for equation (2.10). Then we have

$$\lim_{t\to\infty} R(t) = 0.$$

Proof. It follows from Lemma 5.5 that

$$\frac{d}{dt}\mathcal{I}(t) = \kappa R^2\sin\hat{\alpha}, \quad \frac{d^2}{dt^2}\mathcal{I}(t) = -2\kappa^2R^2\sin\hat{\alpha}\int_{\mathbb{T}}\sin(\theta-\psi)\cos(\theta-\psi-\hat{\alpha})\mu_t(d\theta).$$

This yields that $\frac{d}{dt}\mathcal{I}(t)$ is uniformly continuous. Furthermore, we use the assumption on μ_0 to get

$$\lim_{t\to\infty}\int_0^t \frac{d}{ds}\mathcal{I}(s)ds = \lim_{t\to\infty}\mathcal{I}(t) - \mathcal{I}(0) \leq -\mathcal{I}(0) < \infty.$$

Thus, we apply Barbalat's Lemma to conclude

$$\lim_{t\to\infty}\frac{d}{dt}\mathcal{I}(t) = 0. \tag{5.14}$$

Now we use the relation $\frac{d}{dt}\mathcal{I}(t) = \kappa R^2 \sin\hat{\alpha}$, $\hat{\alpha} \in (0, \pi)$ and (5.14) to obtain

$$\lim_{t\to\infty} R(t) = 0.$$

□

Appendix A. **Proof of Proposition 4.1.** In this appendix, we provide proofs for Proposition 4.1 on the emergent dynamics of the Kuramoto-Sakaguchi equation.

A.1. **Proof of the first part.** Consider an ensemble of identical oscillators. Without loss of generality, we may assume

$$\omega_i = 0, \quad i = 1, \cdots, N.$$

In this case, the Kuramoto model becomes

$$\dot{\theta}_i = \frac{\kappa}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i + \alpha), \quad t > 0, \quad |\alpha| < \frac{\pi}{2}. \tag{A.1}$$

Lemma A.1. (Phase coherence) *Suppose the coupling strength and initial data satisfy*

$$\kappa > 0, \quad D(\Theta^{in}) < \pi - 2|\alpha|,$$

and let $\{\theta_i\}$ be a solution to (A.1). Then, the following relations hold:

$$\sup_{0\le t<\infty} D\big(\Theta(t)\big) \le \pi - 2|\alpha|, \qquad \sup_{0\le t<\infty} D\big(\Theta(t)\big) \le D(\Theta^{in}).$$

Proof. (i) The first relation can be obtained from Lemma 3.1 in [12].
(ii) Note that the phase diameter $D(\Theta)$ satisfies

$$\begin{aligned}\frac{d}{dt}D(\Theta) &= \frac{d}{dt}(\theta_M - \theta_m)\\ &= \frac{\kappa}{N}\Big\{\sum_{j=1}^{N}\sin(\theta_j - \theta_M + \alpha) - \sum_{j=1}^{N}\sin(\theta_j - \theta_m + \alpha)\Big\} \le 0,\end{aligned}$$

where we have used the following relations:

$$\begin{aligned}|\theta_j - \theta_M + \alpha| \le D(\Theta) + |\alpha| \le \pi - |\alpha|,\\ |\theta_j - \theta_m + \alpha| \le D(\Theta) + |\alpha| \le \pi - |\alpha|.\end{aligned}$$

□

Now, we are ready to provide the first estimate in Proposition 4.1.
Let Θ be a solution to system (A.1). By definition of $D(\Theta)$, one has

$$\begin{aligned}\frac{d}{dt}D(\theta) &= \frac{\kappa}{N}\sum_{j=1}^{N}\big\{\sin(\theta_j - \theta_M + \alpha) - \sin(\theta_j - \theta_m + \alpha)\big\}\\ &= -\frac{2\kappa}{N}\sum_{j=1}^{N}\cos\Big(\frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} + \alpha\Big)\sin\Big(\frac{\theta_M - \theta_m}{2}\Big).\end{aligned} \tag{A.2}$$

Since

$$\frac{\theta_j - \theta_M}{2} + \alpha \le \frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} + \alpha \le \frac{\theta_j - \theta_m}{2} + \alpha,$$

one has

$$\begin{aligned}&\frac{\theta_j-\theta_M}{2}+\alpha\in\Big(-\frac{1}{2}D(\Theta^{in})-|\alpha|,|\alpha|\Big)\subseteq\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big),\\&\frac{\theta_j-\theta_m}{2}+\alpha\in\Big(|\alpha|,\frac{1}{2}D(\Theta^{in})+|\alpha|\Big)\subseteq\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big).\end{aligned}\tag{A.3}$$

Then, we use (A.3) to get

$$\frac{\theta_j-\theta_M}{2}+\frac{\theta_j-\theta_m}{2}+\alpha\in\Big(-\frac{1}{2}D(\Theta^{in})-|\alpha|,\frac{1}{2}D(\Theta^{in})+|\alpha|\Big)\subseteq(-\frac{\pi}{2},\frac{\pi}{2}).$$

This yields

$$\cos\Big(\frac{\theta_j-\theta_M}{2}+\frac{\theta_j-\theta_m}{2}+\alpha\Big)\ge\cos\Big(\frac{1}{2}D(\Theta^{in})+|\alpha|\Big).\tag{A.4}$$

Now we substitute (A.4) into (A.2) to derive a differential inequality:

$$\frac{dD(\Theta)}{dt}\le-\frac{2\kappa}{N}\cos\Big(\frac{1}{2}D(\Theta^{in})+|\alpha|\Big)\sum_{j=1}^{N}\sin\big(\frac{\theta_M-\theta_m}{2}\big).\tag{A.5}$$

Since $\frac{\theta_M-\theta_m}{2}\in(0,\pi)$, one has

$$\sin\Big(\frac{\theta_M-\theta_m}{2}\Big)\ge\frac{2}{\pi}\frac{D(\Theta)}{2}=\frac{D(\Theta)}{\pi}.\tag{A.6}$$

Now we combine (A.5) and (A.6) to obtain

$$\frac{dD(\Theta)}{dt}\le-\frac{2\kappa}{\pi}\cos\Big(\frac{1}{2}D(\Theta^{in})+|\alpha|\Big)D(\Theta).\tag{A.7}$$

We integrate the differential inequality (A.7) with respect to time t to get

$$D\big(\Theta(t)\big)\le D(\Theta^{in})\exp\Big\{-\frac{2\kappa}{\pi}\cos\Big(\frac{1}{2}D(\Theta^{in})+|\alpha|\Big)t\Big\},\quad\text{for } t\ge 0.$$

A.2. **Proof of the second part.** In this subsection, we consider the non-identical Kuramoto-Sakaguchi equation (4.4). We find the exact time t_0 such that all the particles trapped in half circle after time t_0, which is important for asymptotic phase-locking of non-identical oscillators in Section 4.3.

Lemma A.2. *Suppose initial data, natural frequencies and coupling strength satisfy*

$$0<D(\Theta^{in})<\pi-2|\alpha|,\quad 0<D(\Omega)<\infty,\quad \kappa>\kappa_e:=\frac{D(\Omega)}{\sin\big(D(\Theta^{in})+|\alpha|\big)-\sin|\alpha|}.$$

Then we have

$$\sup_{0\le t<\infty}D(\Theta(t))<D(\Theta^{in}).$$

Proof. We will use the continuity argument. For this, we define

$$T_*:=\sup\Big\{T\mid D(\Theta(t))<D(\Theta^{in}),\quad\text{for all } t\in(0,T]\Big\}.$$

We claim:

$$T_*=+\infty.\tag{A.8}$$

Suppose not, i.e., $T_*<+\infty$. Then there exists t_0 such that

$$D(\Theta(t))<D(\Theta^{in}),\ \text{for all } t\in(0,t_0)\quad\text{and}\quad D(\Theta)(t_0)=D(\Theta^{in}).\tag{A.9}$$

Then we use continuity of $D(\Theta)(t)$ to see

$$\frac{d}{dt}\Big|_{t=t_0} D(\Theta) > 0. \tag{A.10}$$

On the other hand, note that the system (4.4) can be rewritten as

$$\dot{\theta}_i = \omega_i + \frac{\kappa \cos\alpha}{N}\sum_{j=1}^N \sin(\theta_j - \theta_i) + \frac{\kappa \sin\alpha}{N}\sum_{j=1}^N \cos(\theta_j - \theta_i), \quad t > 0, \quad |\alpha| < \frac{\pi}{2}.$$

Hence, we use definition of $D(\Theta)$ in (4.5) to obtain

$$\begin{aligned}\frac{d}{dt}D(\Theta) \le D(\Omega) &+ \frac{\kappa\cos\alpha}{N}\sum_{j=1}^N \big(\sin(\theta_j - \theta_M) - \sin(\theta_j - \theta_m)\big) \\ &+ \frac{\kappa\sin\alpha}{N}\sum_{j=1}^N \big(\cos(\theta_j - \theta_M) - \cos(\theta_j - \theta_m)\big).\end{aligned} \tag{A.11}$$

We use (A.9) to see that for $t \in (0, t_0)$

$$\cos(\theta_j - \theta_M) - \cos(\theta_j - \theta_m) \le 1 - \cos D(\Theta), \tag{A.12}$$

and

$$\frac{\sin(\theta_M - \theta_j)}{\theta_M - \theta_j} > \frac{\sin D(\Theta)}{D(\Theta)} \quad \text{and} \quad \frac{\sin(\theta_j - \theta_m)}{\theta_j - \theta_m} > \frac{\sin D(\Theta)}{D(\Theta)}, \quad t \in (0, t_0).$$

Hence, for $t \in (0, t_0)$, one has

$$\begin{aligned}&\sin(\theta_j - \theta_M) - \sin(\theta_j - \theta_m) \\ &\quad < \frac{\sin D(\Theta)}{D(\Theta)}\big((\theta_j - \theta_M) - (\theta_j - \theta_m)\big) = -\sin D(\Theta).\end{aligned} \tag{A.13}$$

Now we substitute (A.12) and (A.13) into (A.11) to obtain

$$\begin{aligned}\frac{dD(\Theta)}{dt} &\le D(\Omega) - \kappa\cos\alpha\sin D(\Theta) + \kappa\sin|\alpha|\big(1 - \cos D(\Theta)\big) \\ &= D(\Omega) - \kappa\Big(\sin\big(D(\Theta) + |\alpha|\big) - \sin|\alpha|\Big), \quad t \in (0, t_0).\end{aligned} \tag{A.14}$$

Thus, we use (A.14) to get

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=t_0} D(\Theta) \le D(\Omega) - \kappa\Big(\sin\big(D(\Theta)(t_0) + |\alpha|\big) - \sin|\alpha|\Big) \\ &< D(\Omega) - \frac{D(\Omega)}{\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|}\cdot\Big(\sin\big(D(\Theta)(t_0) + |\alpha|\big) - \sin|\alpha|\Big) = 0.\end{aligned}$$

This contradicts (A.10). Thus, we deduce that $T_* = +\infty$, and our assertion holds. □

Now, we are ready to provide a proof of the second part.

Suppose initial data, natural frequencies and coupling strength satisfy

$$0 < D(\Theta^{in}) < \pi - 2|\alpha|, \quad 0 < D(\Omega) < \infty, \quad \kappa > \kappa_e := \frac{D(\Omega)}{\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|}.$$

Then we claim: for all $t > t_0 = \frac{D(\Theta^{in}) - D^\infty}{(1 - \frac{\kappa}{\kappa_e})D(\Omega)}$,

$$D(\Theta(t)) < D^\infty, \tag{A.15}$$

where $D^\infty \in (0, \frac{\pi}{2} - |\alpha|)$ is a root of the equation:

$$\sin(x + |\alpha|) = \sin(D(\Theta^{in}) + |\alpha|).$$

We split the proof of claim (A.15) into two steps.

• Step A. We need to find time t_0 such that

$$D(\Theta(t_0)) < D^\infty.$$

For this, we consider two cases.

⋄ Case I. Suppose

$$0 < D(\Theta^{in}) \le \frac{\pi}{2}.$$

Then, one has

$$D^\infty = D(\Theta^{in}).$$

Thus, it follows from Lemma A.2 that we have the desired estimate (A.15).

⋄ Case II. Suppose

$$D(\Theta^{in}) \in (\frac{\pi}{2}, \pi).$$

Then, it follows from Lemma A.2 that

$$D(\Theta(t)) < D(\Theta^{in}) \quad \text{for all } t > 0.$$

Now we claim:

$$\frac{d}{dt}D(\Theta) < 0, \quad \text{for a.e. } t \text{ such that } D(\Theta(t)) \in (D^\infty, D(\Theta^{in})). \tag{A.16}$$

Proof of Claim (A.16). Since $(D^\infty + |\alpha|, D(\theta_0) + |\alpha|) \subseteq (|\alpha|, \pi - |\alpha|)$, we have

$$\sin\big(D(\Theta) + |\alpha|\big) \ge \sin(D^\infty + \alpha) = \sin(D(\Theta^{in}) + |\alpha|). \tag{A.17}$$

We use relation (A.11), relation (A.17) and assumption of κ to get

$$\begin{aligned}
\frac{d}{dt}D(\Theta) &\le D(\Omega) - \kappa\Big(\sin\big(D(\Theta) + |\alpha|\big) - \sin|\alpha|\Big)\\
&\le D(\Omega) - \kappa\Big(\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|\Big)\\
&< D(\Omega) - \frac{D(\Omega)}{\sin\big(D(\Theta_0) + |\alpha|\big) - \sin|\alpha|}\cdot\Big(\sin(D\big(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|\Big) = 0.
\end{aligned}$$

In fact, we can get

$$\begin{aligned}
\frac{d}{dt}D(\Theta) &\le D(\Omega) - \kappa\Big(\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|\Big)\\
&= D(\Omega) - \frac{\kappa}{\kappa_e}D(\Omega) = \big(1 - \frac{\kappa}{\kappa_e}\big)D(\Omega).
\end{aligned}$$

Hence, we have

$$D(\Theta(t)) < D(\Theta^{in}) + \big(1 - \frac{\kappa}{\kappa_e}\big)D(\Omega)t, \quad t > 0.$$

Thus, in order to have $D(\Theta(t)) < D^\infty$, we need

$$t > t_e := \frac{D(\Theta^{in}) - D^\infty}{(1 - \frac{\kappa}{\kappa_e})D(\Omega)}.$$

• Step B. We need to verify

$$D(\Theta(t)) < D^\infty, \quad \text{for all } t > t_0.$$

Suppose that there exists a time $t_1 > t_0$ such that

$$D(\Theta(t)) < D^\infty \text{ for } t \in (t_0, t_1) \quad D(\Theta(t_1)) = D^\infty.$$

This yields

$$\frac{d}{dt}\Big|_{t=t_1} D(\Theta) > 0. \tag{A.18}$$

However, we use relation (A.11) again to obtain that

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=t_1} D(\Theta) \le D(\Omega) - \kappa\Big(\sin\big(D^\infty + |\alpha|\big) - \sin|\alpha|\Big)\\
&\quad = D(\Omega) - \kappa\Big(\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|\Big)\\
&\quad < D(\Omega) - \frac{D(\Omega)}{\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|} \cdot \Big(\sin\big(D(\Theta^{in}) + |\alpha|\big) - \sin|\alpha|\Big) = 0.
\end{aligned}$$

This contradicts to the relation (A.18). Thus, our assertion holds.

REFERENCES

[1] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, *Rev. Mod. Phys.*, **77** (2005), 137–185.

[2] M. Brede and A. C. Kalloniatis, Frustration tuning and perfect phase synchronization in the Kuramoto-Sakaguchi model, *Phys. Rev. E*, **93** (2016), 13pp.

[3] J. A. Carrillo, Y.-P. Choi, S.-Y. Ha, M.-J. Kang and Y. Kim, Contractivity of transport distances for the kinetic Kuramoto equation, *J. Stat. Phys.*, **156** (2014), 395–415.

[4] H. Chiba, A proof of the Kuramoto conjecture for a bifurcation structure of the infinite-dimensional Kuramoto model, *Ergodic Theory Dynam. Systems*, **35** (2015), 762–834.

[5] N. Chopra and M. W. Spong, On exponential synchronization of Kuramoto oscillators, *IEEE Trans. Automat. Control*, **54** (2009), 353–357.

[6] H. Daido, Quasientrainment and slow relaxation in a population of oscillators with random and frustrated interactions, *Phys. Rev. Lett.*, **68** (1992), 1073–1076.

[7] F. De Smet and D. Aeyels, Partial entrainment in the finite Kuramoto-Sakaguchi model, *Phys. D*, **234** (2007), 81–89.

[8] J.-G. Dong and X. Xue, Synchronization analysis of Kuramoto oscillators, *Commun. Math. Sci.*, **11** (2013), 465–480.

[9] F. Dörfler and F. Bullo, Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators, *SIAM J. Control Optim.*, **50** (2012), 1616–1642.

[10] F. Dörfler and F. Bullo, Synchronization in complex networks of phase oscillators: A survey, *Automatica J. IFAC*, **50** (2014), 1539–1564.

[11] S.-Y. Ha, D. Kim, J. Lee and Y. Zhang, Remarks on the stability properties of the Kuramoto-Sakaguchi-Fokker-Planck equation with frustration, *Z. Angew. Math. Phys.*, **69** (2018), 25pp.

[12] S.-Y. Ha, Y. Kim and Z. Li, Asymptotic synchronous behavior of Kuramoto type models with frustrations, *Netw. Heterog. Media*, **9** (2014), 33–64.

[13] S.-Y. Ha, Y. Kim and Z. Li, Large-time dynamics of Kuramoto oscillators under the effects of inertia and frustration, *SIAM J. Appl. Dyn. Syst.*, **13** (2014), 466–492.

[14] S.-Y. Ha, H. K. Kim and J. Park, Remarks on the complete synchronization of Kuramoto oscillators, *Nonlinearity*, **28** (2015), 1441–1462.

[15] S.-Y. Ha, H. K. Kim and S. W. Ryoo, Emergence of phase-locked states for the Kuramoto model in a large coupling regime, *Commun. Math. Sci.*, **14** (2016), 1073–1091.

[16] S.-Y. Ha, D. Ko and Y. Zhang, Emergence of phase-locking in the Kuramoto model for identical oscillators with frustration, *SIAM J. Appl. Dyn. Syst.*, **17** (2018), 581–625.

[17] S.-Y. Ha, Z. Li and X. Xue, Formation of phase-locked states in a population of locally interacting Kuramoto oscillators, *J. Differential Equations*, **255** (2013), 3053–3070.

[18] S.-Y. Ha, J. Li and Y. Zhang, Robustness in the instability of the incoherent state for the Kuramoto-Sakaguchi-Fokker-Planck equation with frustration, *Quart. Appl. Math.*, **77** (2019), 631–654.

[19] S.-Y. Ha and Q. Xiao, Remarks on the nonlinear stability of the Kuramoto-Sakaguchi equation, *J. Differential Equations*, **259** (2015), 2430–2457.

[20] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*, Springer Series in Synergetics, 19, Springer-Verlag, Berlin, 1984.

[21] Y. Kuramoto, *International Symposium on Mathematical Problems in Mathematical Physics*, Lecture Notes in Physics, 39, Springer-Verlag, Berlin, 1975.

[22] Z. Levnajić, Emergent multistability and frustration in phase-repulsive networks of oscillators, *Phys. Rev. E*, **84** (2011).

[23] Z. Li and S.-Y. Ha, Uniqueness and well-ordering of emergent phase-locked states for the Kuramoto model with frustration and inertia, *Math. Models Methods Appl. Sci.*, **26** (2016), 357–382.

[24] R. Mirollo and S. H. Strogatz, The spectrum of the partially locked state for the Kuramoto model, *J. Nonlinear Sci.*, **17** (2017), 309–347.

[25] E. Oh, C. Choi, B. Kahng and D. Kim, Modular synchronization in complex networks with a gauge Kuramoto model, *EPL*, **83** (2008).

[26] K. Park, S. W. Rhee and M. Y. Choi, Glass synchronization in the network of oscillators with random phase shifts, *Phys. Rev. E*, **57** (1998), 5030–5035.

[27] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences*, Cambridge Nonlinear Science Series, 12, Cambridge University Press, Cambridge, 2001.

[28] H. Sakaguchi, Cooperative phenomena in coupled oscillator systems sunder external fields, *Prog. Theoret. Phys.*, **79** (1988), 39–46.

[29] H. Sakaguchi and Y. Kuramoto, A soluble active rotator model showing phase transitions via mutual entraintment, *Progr. Theoret. Phys.*, **76** (1986), 576–581.

[30] S. H. Strogatz, From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, *Phys. D*, **143** (2000), 1–20.

[31] S. H. Strogatz and R. E. Mirollo, Stability of incoherence in a population of coupled oscillators, *J. Statist. Phys.*, **63** (1991), 613–635.

[32] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, *J. Theor. Biol.*, **16** (1967), 15–42.

[33] Z.-G. Zheng, Frustration effect on synchronization and chaos in coupled oscillators, *Chin. Phys. Soc.*, **10** (2001), 703–707.

Received August 2019; revised May 2020.

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