

A BGK KINETIC MODEL WITH LOCAL VELOCITY ALIGNMENT FORCES

YOUNG-PIL CHOI*

Department of Mathematics
 Yonsei University
 Seoul 03722, Republic of Korea

SEOK-BAE YUN

Department of Mathematics
 Sungkyunkwan University
 Suwon 440-746, Republic of Korea

ABSTRACT. The global Cauchy problem for a local alignment model with a relaxational inter-particle interaction operator is considered. More precisely, we consider the global-in-time existence of weak solutions of BGK model with local velocity-alignment term when the initial data have finite mass, momentum, energy, and entropy. The analysis involves weak/strong compactness based on the velocity moments estimates and several velocity-weighted L^∞ estimates.

1. Introduction. In this paper, we are interested in the kinetic model which describes the particle system where the alignment effect of the ensemble and the relaxation process through binary collisions compete:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot ((u - v)f) = \mathcal{M}(f) - f, \quad (1.1)$$

subject to initial data

$$f(x, v, 0) =: f_0(x, v).$$

Here $f = f(x, v, t)$ denotes the number density function on the phase point $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ at time $t \in \mathbb{R}_+$. The local Maxwellian $\mathcal{M}(f) = \mathcal{M}(f)(x, v, t)$ is implicitly defined through the moments of f :

$$\rho(x, t) := \int_{\mathbb{R}^d} f(x, v, t) dv, \quad \rho(x, t)u(x, t) := \int_{\mathbb{R}^d} vf(x, v, t) dv,$$

and

$$\rho(x, t)T(x, t) := \frac{1}{d} \int_{\mathbb{R}^d} |v - u(x, t)|^2 f(x, v, t) dv$$

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* Corresponding author: Young-Pil Choi.

by the formula

$$\mathcal{M}(f)(x, v, t) := \frac{\rho(x, t)}{\sqrt{(2\pi T(x, t))^d}} \exp\left(-\frac{|v - u(x, t)|^2}{2T(x, t)}\right).$$

The study of collective dynamics of self-propelled particle systems, in which simple local interaction laws between the particles eventually lead to the emergence of global collective behaviors, has received much attention recently. The most common such rules imposed on the constituent particles are rules of alignment phenomena and collisional interactions. As such, kinetic equations where the local alignment phenomena and collisional interactions are combined as in (1.1) arise in various physical or modelling contexts. For example in [6, 16, 17] a local alignment kinetic model with a Fokker-Planck type inter-particle interaction is considered, which arise as a coarse limit of a variant of Cucker-Smale flocking models [12, 18]. To be more specific, the local particle velocity can be obtained by taking into account the following local averaged particle velocity:

$$u^r(x, t) := \frac{\int_{\mathbb{T}^d \times \mathbb{R}^d} K^r(x - y) w f(y, w) dy dw}{\int_{\mathbb{T}^d \times \mathbb{R}^d} K^r(x - y) f(y, w) dy dw}, \quad (1.2)$$

where K^r is called a communication weight and r represents the radius of particle interactions, i.e., the support of K^r . This type of particle velocity is considered in [18] to describe the flocking behaviors. In [17], the rigorous justification of the limit

$$u^r \rightarrow u \quad \text{as} \quad r \rightarrow 0$$

in the weak sense is investigated. At the formal level, we expect the kernel K^r converges to the Dirac measure δ_0 , thus the limiting function will be the local particle velocity u . We refer to [4, 11] for recent surveys on the Cucker-Smale type flocking models. Similar models also arise in the context of traffic model of Paveri-Fontana type where the evolution of the velocity distribution of the vehicles are explained by the adjustment of velocity with respect to the desired velocity (alignment), and the acceleration of the vehicles (traffic collision operator) [9, 20]. The main motivation of our model (1.1) comes from the collisional alignment model of [16, 17]. Instead of the Fokker-Planck type diffusive approximation of the collision operator as in [16, 17], we are considering relaxational approximation of the collision operator.

Closely related models are kinetic-fluid equations with inter-particle interaction operators, in which a collisional kinetic equation and fluid equations are coupled through drag force terms. More precisely, u in the local alignment force satisfies the fluid equations, for instance, incompressible Navier-Stokes system, see [8, 10] where the existence theories for weak/strong solutions of the Navier-Stokes-BGK system are discussed. When the local velocity alignment force is ignored, our main equation (1.1) becomes the BGK model of Boltzmann equation [3], which is one of the most widely employed model equation of the Boltzmann equation providing reliable results on various problems in rarefied gas dynamics. Some of previous works on the BGK model can be summarized as follows. For the Cauchy problem in the framework of weak solutions, see [21, 28, 29]. The existence of strong solutions with uniqueness can be found in [19, 22, 25]. Behaviors of solutions that stays near global equilibriums are studied in [2, 24, 26, 27]. For results on stationary solutions, see [1, 23].

In the current work, we establish the global-in-time existence of weak solutions to the equation (1.1). The main difficulties are the nonlinear terms fu in the

local alignment force and the local Maxwellian $\mathcal{M}(f)$. In many previous works [5, 7, 14, 15, 18] on kinetic collective behavior models, the method of characteristics are crucially used to estimate the propagation of velocity support of solution f . However, in our case, due to the lack of regularity of u and the BGK operator term, that types of support estimates of f in velocity cannot be applied. Instead of that, we regularize the local particle velocity u , and linearize the equation (1.1). We then do some Cauchy estimates in the velocity-weighted $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ space. Finally, we provide some uniform bound estimates in regularization parameters and pass to the limit which requires some weak and strong compactness arguments.

Let us now introduce a notion of weak solutions to the system (1.1).

Definition 1.1. We say that f is a weak solution to the system (1.1) if the following conditions are satisfied:

- (i) $f \in L^\infty(0, T; (L^1_+ \cap L^\infty)(\mathbb{T}^d \times \mathbb{R}^d))$ ¹,
- (ii) for all $\phi \in \mathcal{C}_c^1(\mathbb{T}^d \times \mathbb{R}^d \times [0, T])$ with $\phi(x, v, T) = 0$,

$$\begin{aligned} & - \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \phi_0 \, dx dv - \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} f (\partial_t \phi + v \cdot \nabla_x \phi + (u - v) \cdot \nabla_v \phi) \, dx dv dt \\ & = \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} (\mathcal{M}(f) - f) \phi \, dx dv dt. \end{aligned}$$

We are now ready to state our main result:

Theorem 1.2. *Let $T > 0$. Suppose that the initial data f_0 satisfy*

$$f_0 \in L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$$

and

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (1 + |v|^2 + |\ln f_0(x, v)|) f_0(x, v) \, dx dv < \infty.$$

Then there exists at least one weak solution to the system (1.1) in the sense of Definition 1.1 satisfying the following estimates:

(i)

$$\|f(\cdot, \cdot, t)\|_{L^\infty} \leq C \|f_0\|_{L^\infty}$$

and

(ii)

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (1 + |v|^2 + |\ln f(x, v, t)|) f(x, v, t) \, dx dv < \infty$$

for almost every $t \in (0, T)$ and $C > 0$.

Remark 1.3. We can easily apply our strategy for the existence result for the equation (1.1) with u^r appeared in (1.2) instead of u under the assumption $K^r \in L^1(\mathbb{T}^d)$.

We introduce several notations used throughout the paper. For functions $f(x, v)$, $g(x)$, $\|f\|_{L^p}$ and $\|g\|_{L^p}$ denote the usual $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ -norm and $L^p(\mathbb{T}^d)$ -norm, respectively. $\|f\|_{L_q^\infty}$ represents a weighted L^∞ -norm:

$$\|f\|_{L_q^\infty} := \operatorname{ess\,sup}_{(x, v) \in \mathbb{T}^d \times \mathbb{R}^d} (1 + |v|^q) f(x, v).$$

For any nonnegative integer s , H^s denotes the s -th order L^2 Sobolev space. $\mathcal{C}^s([0, T]; E)$ is the set of s -times continuously differentiable functions from an interval $[0, T] \subset$

¹ $L^1_+(\mathbb{T}^d \times \mathbb{R}^d)$ denotes the set of nonnegative $L^1(\mathbb{T}^d \times \mathbb{R}^d)$ functions.

\mathbb{R} into a Banach space E , and $L^p(0, T; E)$ is the set of the L^p functions from an interval $(0, T)$ to a Banach space E . We denote by C a generic, not necessarily identical, positive constant, and $C = C_{\alpha, \beta, \dots}$ or $C = C(\alpha, \beta, \dots)$ stands for the positive constant depending on α, β, \dots .

This paper is organized as follows: In the following section 2, we introduce a regularized equation of (1.1) and study the global-in-time existence of weak solutions to the regularized equation. For this, we construct the approximated solutions to the regularized equation, and provide that they are Cauchy sequences in the velocity-weighted $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ space. We also establish uniform bound estimates in the regularizing parameter. In Section 3, we then use this uniform bounds to drive weak solution of (1.1) through the weak limit of the regularized distribution function and the strong limit of macroscopic fields.

2. A regularized equation. In order to show the existence of weak solutions to (1.1), it is required to regularize the equation to remove the singularity in the local alignment force. In this section, we introduce a regularized equation and provide the global-in-time existence of weak solutions of that. Let us regularize the local particle velocity u by using a mollifier $\theta_\varepsilon(x) = \varepsilon^{-d}\theta(x/\varepsilon)$ with $0 \leq \theta \in \mathcal{C}_0^\infty(\mathbb{T}^d)$ satisfying

$$\int_{\mathbb{T}^d} \theta(x) dx = 1.$$

Then our regularized equation is defined as follows:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \nabla_v \cdot ((u_\varepsilon^\varepsilon - v) f_\varepsilon) = \mathcal{M}(f_\varepsilon) - f_\varepsilon, \quad (2.1)$$

subject to regularized initial data:

$$(f_\varepsilon(x, v, 0)) =: (f_{0,\varepsilon}(x, v)), \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d.$$

Here

$$u_\varepsilon^\varepsilon(x, t) := \frac{((\rho_\varepsilon u_\varepsilon) \star \theta_\varepsilon)(x, t)}{(\rho_\varepsilon \star \theta_\varepsilon)(x, t) + \varepsilon(1 + |((\rho_\varepsilon u_\varepsilon) \star \theta_\varepsilon)(x, t)|^2)}$$

with

$$\rho_\varepsilon(x, t) = \int_{\mathbb{R}^d} f_\varepsilon(x, v, t) dv \quad \text{and} \quad (\rho_\varepsilon u_\varepsilon)(x, t) = \int_{\mathbb{R}^d} v f_\varepsilon(x, v, t) dv,$$

and the regularized initial data $f_{0,\varepsilon}$ is defined by

$$f_{0,\varepsilon} = \eta \star \{f_0 1_{f_0 < 1/\varepsilon}\} + \varepsilon e^{-|v|^2},$$

where 1_A denotes the characteristic function on A and η is the standard mollifier. Note that $f_{0,\varepsilon}$ satisfies

- (i) $f_{0,\varepsilon} \rightarrow f_0$ strongly in $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ for all $p < \infty$ and weakly-* in $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$,
- (ii) $\|v\sqrt{f_{0,\varepsilon}}\|_{L^2}^2 \rightarrow \|v\sqrt{f_0}\|_{L^2}^2$ strongly.

In the following two sections, we prove the proposition below on the global-in-time existence of weak solutions and some uniform bound estimates of the regularized system (2.1).

Proposition 2.1. *Let $T > 0$. For any $\varepsilon > 0$, there exists at least one weak solution f_ε of the regularized equation (2.1) on the interval $[0, T]$ in the sense of Definition 1.1. Furthermore we have*

(i) kinetic energy estimate

$$\sup_{0 \leq t \leq T} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_\varepsilon(x, v, t) dx dv \leq C,$$

(ii) third moment & entropy estimates:

$$\sup_{0 \leq t \leq T} \int_{\mathbb{T}^d \times \mathbb{R}^d} (|\ln f_\varepsilon(x, v, t)| + |v|^3) f_\varepsilon(x, v, t) dx dv \leq C,$$

where $C = C(f_0, T) > 0$ is independent of ε .

2.1. A linearized equation. In order to obtain the existence of solutions to the regularized equation (2.1), we first construct the approximated solutions in the following way:

$$\partial_t f_\varepsilon^{n+1} + v \cdot \nabla_x f_\varepsilon^{n+1} + \nabla_v \cdot ((u_\varepsilon^{n,n} - v) f_\varepsilon^{n+1}) = \mathcal{M}(f_\varepsilon^n) - f_\varepsilon^{n+1}, \quad (2.2)$$

with the initial data and first iteration step:

$$f_\varepsilon^n(x, v, t)|_{t=0} = f_{0,\varepsilon}(x, v) \quad \text{for all } n \geq 1$$

and

$$f_\varepsilon^0(x, v, t) = f_{0,\varepsilon}(x, v), \quad (x, v, t) \in \mathbb{T}^d \times \mathbb{R}^d \times (0, T).$$

Here

$$u_\varepsilon^{n,n}(x, t) = \frac{((\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)}{(\rho_\varepsilon^n \star \theta_\varepsilon)(x, t) + \varepsilon(1 + |(\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)|^2)}$$

with

$$\rho_\varepsilon^n(x, t) = \int_{\mathbb{R}^d} f_\varepsilon^n(x, v, t) dv \quad \text{and} \quad (\rho_\varepsilon^n u_\varepsilon^n)(x, t) = \int_{\mathbb{R}^d} v f_\varepsilon^n(x, v, t) dv.$$

In the proposition below, we provide the global-in-time solvability of the regularized and linearized equation (2.2) and uniform-in- n bound estimates of solutions.

Proposition 2.2. *Let $T > 0$ and $q > d + 2$. For any $n \in \mathbb{N}$, there exists a unique solution f_ε^n of the regularized and linearized system (2.2) such that $f_\varepsilon^n \in L^\infty(0, T; L_q^\infty(\mathbb{T}^d \times \mathbb{R}^d))$. Moreover, we have*

$$\sup_{0 \leq t \leq T} \|f_\varepsilon^n(\cdot, \cdot, t)\|_{L^\infty} \leq C \|f_{0,\varepsilon}\|_{L^\infty}$$

and

$$\sup_{0 \leq t \leq T} \left(\|f_\varepsilon^n(\cdot, \cdot, t)\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n(\cdot, \cdot, t)\|_{L_q^\infty} \right) \leq C_\varepsilon,$$

where $C > 0$ is independent of both n and ε , and $C_\varepsilon > 0$ is independent of n .

Before presenting the details of proof of Proposition 2.2, we list several technical lemmas showing some bound estimates related to the local Maxwellian. First two lemmas below are concerned with the upper bound estimates of local Maxwellian, and the third one is a type of Lipschitz continuity of the local Maxwellian.

Lemma 2.1. [22, p.291] *Suppose $\|f\|_{L_q^\infty} < \infty$ for $q > d + 2$. Then there exists a positive constant C_q , which depends only on q , such that*

$$\|\mathcal{M}(f)\|_{L_q^\infty} \leq C_q \|f\|_{L_q^\infty} \quad (q > d + 2 \text{ or } q = 0).$$

Lemma 2.2. [25, Proposition 4.1] *Assume that f satisfies*

1. $\|f\|_{L_q^\infty} + \|\nabla_{x,v} f\|_{L_q^\infty} < C_1$,
2. $\rho + |u| + T < C_2$,
3. $\rho, T > C_3$,

for some constants $C_i > 0$ ($i = 1, 2, 3$). Then, we have

$$\|\mathcal{M}(f)\|_{L_q^\infty} + \|\nabla_{x,v} \mathcal{M}(f)\|_{L_q^\infty} \leq C_T \left\{ \|f\|_{L_q^\infty} + \|\nabla_{x,v} f\|_{L_q^\infty} \right\},$$

where $C_T > 0$ depends only on C_1, C_2, C_3 and the final time T .

Lemma 2.3. [25, Proposition 6.1] *Assume f, g satisfy (h denotes either f or g)*

1. $\|h\|_{L_q^\infty} < C_1$,
2. $\rho_h + |u_h| + T_h < C_2$,
3. $\rho_h, T_h > C_3$,

for some constants $C_i > 0$ ($i = 1, 2, 3$). Then, we have

$$\|\mathcal{M}(f) - \mathcal{M}(g)\|_{L_q^\infty} \leq C_T \|f - g\|_{L_q^\infty},$$

where $C_T > 0$ depends only on C_1, C_2, C_3 and the final time T .

Proof of Proposition 2.2. For the proof, we first introduce the following backward characteristics:

$$Z_\varepsilon^{n+1}(s) := (X_\varepsilon^{n+1}(s), V_\varepsilon^{n+1}(s)) := (X_\varepsilon^{n+1}(s; t, x, v), V_\varepsilon^{n+1}(s; t, x, v))$$

defined by

$$\begin{aligned} \frac{d}{ds} X_\varepsilon^{n+1}(s) &= V_\varepsilon^{n+1}(s), \quad 0 \leq s \leq T, \\ \frac{d}{ds} V_\varepsilon^{n+1}(s) &= u_\varepsilon^{\varepsilon, n}(X_\varepsilon^{n+1}(s), s) - V_\varepsilon^{n+1}(s), \end{aligned} \tag{2.3}$$

with the terminal datum

$$Z_\varepsilon^{n+1}(t) = (x, v).$$

Due to the regularization, the characteristics (2.3) is well-defined, thus global-in-time existence of solutions to (2.2) can be obtained by the Cauchy-Lipschitz existence theory. It follows from (2.3) that $V_\varepsilon^{n+1}(s)$ satisfies

$$V_\varepsilon^{n+1}(s) = v e^{t-s} - e^{-s} \int_s^t u_\varepsilon^{\varepsilon, n}(X_\varepsilon^{n+1}(\tau), \tau) e^\tau d\tau. \tag{2.4}$$

On the other hand, we get

$$\begin{aligned} |\rho_\varepsilon^n(x)| &= \left| \int_{\mathbb{R}^d} f_\varepsilon^n(x, v) dv \right| \\ &= \int_{\mathbb{R}^d} (1 + |v|^q) (1 + |v|^q)^{-1} f_\varepsilon^n(x, v) dv \\ &\leq C \|f_\varepsilon^n\|_{L_q^\infty} \end{aligned}$$

for $q > d$ and

$$\begin{aligned} |(\rho_\varepsilon^n u_\varepsilon^n)(x)| &= \left| \int_{\mathbb{R}^d} v f_\varepsilon^n(x, v) dv \right| \\ &= \int_{\mathbb{R}^d} |v| (1 + |v|^q) (1 + |v|^q)^{-1} f_\varepsilon^n(x, v) dv \\ &\leq C \|f_\varepsilon^n\|_{L_q^\infty} \end{aligned}$$

for $q > d + 1$. Subsequently, these imply

$$\begin{aligned} |u_\varepsilon^{\varepsilon, n}| &= \left| \frac{((\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)}{(\rho_\varepsilon^n \star \theta_\varepsilon)(x, t) + \varepsilon (1 + |(\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)|^2} \right| \\ &\leq \frac{1}{\varepsilon} |((\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)| \\ &\leq \frac{C}{\varepsilon} \|\rho_\varepsilon^n u_\varepsilon^n\|_{L^\infty} \\ &\leq C_\varepsilon \|f_\varepsilon^n\|_{L_q^\infty} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
& |\nabla_x u_\varepsilon^{\varepsilon,n}| \\
& \leq \frac{|((\rho_\varepsilon^n u_\varepsilon^n) \star \nabla_x \theta_\varepsilon)(x, t)|}{(\rho_\varepsilon^n \star \theta_\varepsilon)(x, t) + \varepsilon(1 + |(\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)|^2)} \\
& \quad + \frac{|((\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)| (|\rho_\varepsilon^n \star \nabla_x \theta_\varepsilon| + 2\varepsilon |\rho_\varepsilon^n u_\varepsilon^n \star \theta_\varepsilon| |\rho_\varepsilon^n u_\varepsilon^n \star \nabla_x \theta_\varepsilon|)}{((\rho_\varepsilon^n \star \theta_\varepsilon)(x, t) + \varepsilon(1 + |(\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x, t)|^2)^2} \quad (2.6) \\
& \leq \frac{1}{\varepsilon} |\rho_\varepsilon^n u_\varepsilon^n \star \nabla_x \theta_\varepsilon| + \frac{1}{\varepsilon^2} |\rho_\varepsilon^n \star \nabla_x \theta_\varepsilon| + \frac{2}{\varepsilon} |\rho_\varepsilon^n u_\varepsilon^n \star \nabla_x \theta_\varepsilon| \\
& \leq C_\varepsilon \|f_\varepsilon^n\|_{L_q^\infty},
\end{aligned}$$

where $C_\varepsilon > 0$ is independent of n . This together with (2.4) yields

$$|v| \leq |V_\varepsilon^{n+1}(s)| + \int_s^t |u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(\tau), \tau)| d\tau \leq |V_\varepsilon^{n+1}(s)| + C_{T,\varepsilon}. \quad (2.7)$$

Furthermore, we have

$$\begin{aligned}
|\nabla_{x,v} X_\varepsilon^{n+1}(s)| & \leq C + \int_s^t |\nabla_{x,v} V_\varepsilon^{n+1}(\tau)| d\tau, \\
|\nabla_{x,v} V_\varepsilon^{n+1}(s)| & \leq C + \int_s^t \|\nabla_x u_\varepsilon^{\varepsilon,n}(\cdot, \tau)\|_{L^\infty} |\nabla_{x,v} X_\varepsilon^{n+1}(\tau)| d\tau \quad (2.8) \\
& \leq C + C_\varepsilon \int_s^t \|f_\varepsilon^n\|_{L_q^\infty} |\nabla_{x,v} X_\varepsilon^{n+1}(\tau)| d\tau.
\end{aligned}$$

Along that characteristics we obtain from (2.2) that

$$\begin{aligned}
f_\varepsilon^{n+1}(x, v, t) & = e^{(d-1)t} f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0)) \\
& \quad + \int_0^t e^{(d-1)(t-s)} \mathcal{M}(f_\varepsilon^n(Z_\varepsilon^{n+1}(s), s)) ds. \quad (2.9)
\end{aligned}$$

Then it is easy to check from (2.9) that

$$\begin{aligned}
\|f_\varepsilon^{n+1}(\cdot, \cdot, t)\|_{L^\infty} & \leq C_T \|f_{0,\varepsilon}\|_{L^\infty} + C_T \int_0^t \|\mathcal{M}(f_\varepsilon^n(Z_\varepsilon^{n+1}(s; t, \cdot, \cdot), s))\|_{L^\infty} ds \\
& \leq C_T \|f_{0,\varepsilon}\|_{L^\infty} + C_T \int_0^T \|f_\varepsilon^n(\cdot, \cdot, s)\|_{L^\infty} ds
\end{aligned}$$

due to Lemma 2.1. Applying the Gronwall's inequality, we conclude the first assertion. In order to prove the second estimate, we use (2.7) to obtain

$$1 + |v| \leq 1 + |V_\varepsilon^{n+1}(s)| + C_{T,\varepsilon}.$$

Using the above estimate, we find

$$\begin{aligned}
f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0)) & = f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0))(1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(0)|)^q (1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(0)|)^{-q} \\
& \leq C_{T,\varepsilon,q} \|f_{0,\varepsilon}\|_{L_q^\infty} (1 + |v|)^{-q}
\end{aligned}$$

for $0 < q < \infty$. Similarly, with the aid of Lemma 2.1, we estimate

$$\begin{aligned}
& \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s) \\
& \leq \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s)(1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(s)|)^q (1 + C_{T,\varepsilon} + |V_\varepsilon^{n+1}(s)|)^{-q} \\
& \leq C_{T,\varepsilon,q} \|\mathcal{M}(f_\varepsilon^n)\|_{L_q^\infty} (1 + |v|)^{-q} \\
& \leq C_{T,\varepsilon,q} \|f_\varepsilon^n\|_{L_q^\infty} (1 + |v|)^{-q}.
\end{aligned}$$

This together with (2.9) gives

$$\|f_\varepsilon^{n+1}(\cdot, \cdot, t)\|_{L_q^\infty} \leq C_{T,\varepsilon,q} \|f_{0,\varepsilon}\|_{L_q^\infty} + C_{T,\varepsilon,q} \int_0^t \|f_\varepsilon^n(\cdot, \cdot, s)\|_{L_q^\infty} ds. \quad (2.10)$$

This yields

$$\sup_{0 \leq t \leq T} \sup_{n \in \mathbb{N}} \|f_\varepsilon^n(\cdot, \cdot, t)\|_{L_q^\infty} \leq C_{T,\varepsilon,q} \|f_{0,\varepsilon}\|_{L_q^\infty},$$

and subsequently, we find

$$|\nabla_{x,v} Z_\varepsilon^{n+1}(s)| \leq C_{T,\varepsilon}$$

for all $n \geq 1$ due to (2.8). Then we also use similar argument as above to get

$$\begin{aligned} & |\nabla_{x,v} f_\varepsilon^{n+1}(x, v, t)| \\ & \leq C_T |\nabla_{x,v} f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0))| |\nabla_{x,v} Z_\varepsilon^{n+1}(0)| \\ & \quad + C_T \int_0^t |\nabla_{x,v} \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s)| |\nabla_{x,v} Z_\varepsilon^{n+1}(s)| ds \\ & \leq C_{T,\varepsilon} \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} (1 + |v|)^{-q} \\ & \quad + C_{T,\varepsilon} \int_0^t \|\nabla_{x,v} \mathcal{M}(f_\varepsilon^n)\|_{L_q^\infty} (1 + |v|)^{-q} ds \\ & \leq C_{T,\varepsilon} \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} (1 + |v|)^{-q} \\ & \quad + C_{T,\varepsilon} \int_0^t \left(\|f_\varepsilon^n\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n\|_{L_q^\infty} \right) (1 + |v|)^{-q} ds \end{aligned}$$

due to Lemma 2.2. Hence we obtain

$$\begin{aligned} & \|\nabla_{x,v} f_\varepsilon^{n+1}\|_{L_q^\infty} \leq C_{T,\varepsilon} \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} \\ & \quad + C_{T,\varepsilon} \int_0^t \left(\|f_\varepsilon^n\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n\|_{L_q^\infty} \right) ds. \end{aligned} \quad (2.11)$$

Combining (2.10) and (2.11), we have

$$\begin{aligned} & \|f_\varepsilon^{n+1}(t)\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^{n+1}(t)\|_{L_q^\infty} \\ & \leq C_{T,\varepsilon} \left(\|f_{0,\varepsilon}\|_{L_q^\infty} + \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} \right) + C_{T,\varepsilon} \int_0^t \left(\|f_\varepsilon^n\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n\|_{L_q^\infty} \right) ds, \end{aligned}$$

which concludes the desired result. \square

Corollary 2.1. *Let $q > d + 1$. For any $T > 0$ and $n \in \mathbb{N}$, there exists a unique solution f_ε^n of the regularized and linearized system (2.2) such that $f_\varepsilon^n \in L^\infty(0, T; L_q^\infty(\mathbb{T}^d \times \mathbb{R}^d))$. Moreover, we have*

(i) *lower bound estimates for the local particle density and temperature:*

$$\rho_\varepsilon^n > C_\varepsilon \quad \text{and} \quad T_\varepsilon^n > C_\varepsilon,$$

(ii) *upper bound estimates for the macroscopic fields:*

$$\rho_\varepsilon^n + |u_\varepsilon^n| + T_\varepsilon^n < C_\varepsilon$$

for all $0 \leq t \leq T$, where C_ε is independent of n .

Proof. (i) We take into account the integration of (2.9) and recall how we regularized f_0 to see

$$\begin{aligned} \int_{\mathbb{R}^d} f_\varepsilon^n(x, v, t) dv &\geq e^{2t} \int_{\mathbb{R}^d} f_{0,\varepsilon}(Z_\varepsilon^n(0)) dv \\ &\geq \int_{\mathbb{R}^d} \varepsilon e^{-|V_\varepsilon^n(0)|^2} dv \\ &\geq \int_{\mathbb{R}^d} \varepsilon e^{-C_\varepsilon(1+|v|)^2} dv \geq C_\varepsilon, \end{aligned} \quad (2.12)$$

where we used

$$|V_\varepsilon^{n+1}(s)| \leq C|v| + C \int_s^t |u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(\tau), \tau)| d\tau \leq C(|v| + C_\varepsilon) \leq C_\varepsilon(1 + |v|)$$

for all $0 \leq s \leq T$. This gives the lower bound for ρ_ε^n . On the other hand, it follows from [21, Proposition 2.1] that

$$\rho_\varepsilon^n(x, t) \leq C_q \|f_\varepsilon^n\|_{L_q^\infty} (T_\varepsilon^n)^{d/2}$$

for $q > d$ or $q = 0$. This combined with the above lower bound estimate for ρ_ε^n asserts $T_\varepsilon^n > C_\varepsilon$ for some C_ε , which is independent of n . Note that this is essential for the local Maxwellian to be well-defined.

(ii) Straightforward computations give

$$\begin{aligned} \rho_\varepsilon^n &= \int_{\mathbb{R}^d} f_\varepsilon^n dv \leq C \|f_\varepsilon^n\|_{L_q^\infty} \leq C_\varepsilon, \\ |u_\varepsilon^n| &= \frac{1}{\rho_\varepsilon^n} \left| \int_{\mathbb{R}^d} f_\varepsilon^n v dv \right| \leq \frac{C}{\rho_\varepsilon^n} \|f_\varepsilon^n\|_{L_q^\infty} \leq C_\varepsilon, \quad \text{and} \\ T_\varepsilon^n &= \frac{1}{\rho_\varepsilon^n} \int_{\mathbb{R}^d} f_\varepsilon^n |v|^2 dv - \frac{1}{\rho_\varepsilon^n} \left| \int_{\mathbb{R}^d} f_\varepsilon^n v dv \right|^2 \leq \frac{C}{\rho_\varepsilon^n} \|f_\varepsilon^n\|_{L_q^\infty} + \frac{C}{\rho_\varepsilon^n} \|f_\varepsilon^n\|_{L_q^\infty}^2 \leq C_\varepsilon. \end{aligned}$$

□

2.2. Cauchy estimate for f_ε^n . In this part, we show that f_ε^n is a Cauchy sequence in $L^\infty(0, T; L_q^\infty(\mathbb{T}^d \times \mathbb{R}^d))$. It follows from (2.9) that

$$\begin{aligned} &f_\varepsilon^{n+1}(x, v, t) - f_\varepsilon^n(x, v, t) \\ &= e^{(d-1)t} (f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0)) - f_{0,\varepsilon}(Z_\varepsilon^n(0))) \\ &\quad + \int_0^t e^{(d-1)(t-s)} (\mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s) - \mathcal{M}(f_\varepsilon^{n-1})(Z_\varepsilon^n(s), s)) ds \\ &= e^{(d-1)t} (f_{0,\varepsilon}(Z_\varepsilon^{n+1}(0)) - f_{0,\varepsilon}(Z_\varepsilon^n(0))) \\ &\quad + \int_0^t e^{(d-1)(t-s)} (\mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^{n+1}(s), s) - \mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^n(s), s)) ds \\ &\quad + \int_0^t e^{(d-1)(t-s)} (\mathcal{M}(f_\varepsilon^n)(Z_\varepsilon^n(s), s) - \mathcal{M}(f_\varepsilon^{n-1})(Z_\varepsilon^n(s), s)) ds \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (2.13)$$

Here I_1 is readily estimated as

$$I_1 \leq C(1 + |v|)^{-q} \|\nabla_{x,v} f_{0,\varepsilon}\|_{L_q^\infty} |Z_\varepsilon^{n+1}(0) - Z_\varepsilon^n(0)|,$$

where $C > 0$ is independent of n and ε . For the estimate of I_2 , we use Lemma 2.2 to find

$$\begin{aligned} I_2 &= \int_0^t e^{(d-1)(t-s)} \nabla_{x,v} \mathcal{M}(f_\varepsilon^n) (\alpha Z_\varepsilon^{n+1}(s) + (1-\alpha) Z_\varepsilon^n(s), s) \\ &\quad \cdot (Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)) ds \\ &\leq C_\varepsilon (1+|v|)^{-q} \int_0^t \|\nabla_{x,v} \mathcal{M}(f_\varepsilon^n)\|_{L_q^\infty} |Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| ds \\ &\leq C_\varepsilon (1+|v|)^{-q} \int_0^t \left(\|f_\varepsilon^n(\cdot, \cdot, s)\|_{L_q^\infty} + \|\nabla_{x,v} f_\varepsilon^n(\cdot, \cdot, s)\|_{L_q^\infty} \right) \\ &\quad \times |Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| ds. \end{aligned}$$

It follows from Lemma 2.3 that

$$I_3 \leq C_\varepsilon (1+|v|)^{-q} \int_0^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(\cdot, \cdot, s)\|_{L_q^\infty} ds.$$

Putting these estimates into (2.13) gives

$$\begin{aligned} &\|(f_\varepsilon^{n+1} - f_\varepsilon^n)(\cdot, \cdot, t)\|_{L_q^\infty} \\ &\leq C_\varepsilon |Z_\varepsilon^{n+1}(0) - Z_\varepsilon^n(0)| + C_\varepsilon \int_0^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(\cdot, \cdot, s)\|_{L_q^\infty} ds \\ &\quad + C_\varepsilon \int_0^t |Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| ds. \end{aligned} \tag{2.14}$$

We next present Cauchy estimate for the characteristic Z_ε^{n+1} . For this, we first notice that

$$|\rho_\varepsilon^n - \rho_\varepsilon^{n-1}| = \left| \int_{\mathbb{R}^d} (f_\varepsilon^n - f_\varepsilon^{n-1}) dv \right| \leq C \|f_\varepsilon^n - f_\varepsilon^{n-1}\|_{L_q^\infty}$$

for $q > d$ and

$$|\rho_\varepsilon^n u_\varepsilon^n - \rho_\varepsilon^{n-1} u_\varepsilon^{n-1}| = \left| \int_{\mathbb{R}^d} v (f_\varepsilon^n - f_\varepsilon^{n-1}) dv \right| \leq C \|f_\varepsilon^n - f_\varepsilon^{n-1}\|_{L_q^\infty}$$

for $q > d+1$. This yields

$$\begin{aligned} &|u_\varepsilon^{\varepsilon,n} - u_\varepsilon^{\varepsilon,n-1}| \\ &= \left| \frac{(\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon}{\rho_\varepsilon^n \star \theta_\varepsilon + \varepsilon(1 + |\rho_\varepsilon^n u_\varepsilon^n \star \theta_\varepsilon|^2)} - \frac{(\rho_\varepsilon^{n-1} u_\varepsilon^{n-1}) \star \theta_\varepsilon}{\rho_\varepsilon^{n-1} \star \theta_\varepsilon + \varepsilon(1 + |\rho_\varepsilon^{n-1} u_\varepsilon^{n-1} \star \theta_\varepsilon|^2)} \right| \\ &\leq \left| \frac{(\rho_\varepsilon^n u_\varepsilon^n - \rho_\varepsilon^{n-1} u_\varepsilon^{n-1}) \star \theta_\varepsilon}{\rho_\varepsilon^n \star \theta_\varepsilon + \varepsilon(1 + |\rho_\varepsilon^n u_\varepsilon^n \star \theta_\varepsilon|^2)} \right| \\ &\quad + \left| \frac{((\rho_\varepsilon^{n-1} u_\varepsilon^{n-1}) \star \theta_\varepsilon) ((\rho_\varepsilon^{n-1} - \rho_\varepsilon^n) \star \theta_\varepsilon + \varepsilon (|\rho_\varepsilon^{n-1} u_\varepsilon^{n-1} \star \theta_\varepsilon|^2 - |\rho_\varepsilon^n u_\varepsilon^n \star \theta_\varepsilon|^2))}{(\rho_\varepsilon^n \star \theta_\varepsilon + \varepsilon(1 + |\rho_\varepsilon^n u_\varepsilon^n \star \theta_\varepsilon|^2)) (\rho_\varepsilon^{n-1} \star \theta_\varepsilon + \varepsilon(1 + |\rho_\varepsilon^{n-1} u_\varepsilon^{n-1} \star \theta_\varepsilon|^2))} \right| \\ &\leq \frac{1}{\varepsilon} |(\rho_\varepsilon^n u_\varepsilon^n - \rho_\varepsilon^{n-1} u_\varepsilon^{n-1}) \star \theta_\varepsilon| + \frac{1}{\varepsilon^2} |(\rho_\varepsilon^{n-1} u_\varepsilon^{n-1}) \star \theta_\varepsilon| |(\rho_\varepsilon^{n-1} - \rho_\varepsilon^n) \star \theta_\varepsilon| \\ &\quad + \frac{2}{\varepsilon} |(\rho_\varepsilon^{n-1} u_\varepsilon^{n-1}) \star \theta_\varepsilon| |(\rho_\varepsilon^{n-1} u_\varepsilon^{n-1} - \rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon| \\ &\leq C_\varepsilon \|f_\varepsilon^n - f_\varepsilon^{n-1}\|_{L_q^\infty}, \end{aligned}$$

where $C_\varepsilon > 0$ is independent of n . Then, by using the above combined with (2.6), we obtain from (2.3) that

$$\begin{aligned} |V_\varepsilon^{n+1}(s) - V_\varepsilon^n(s)| &= \left| e^{-s} \int_s^t (u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(\tau), \tau) - u_\varepsilon^{\varepsilon,n-1}(X_\varepsilon^n(\tau), \tau)) e^\tau d\tau \right| \\ &\leq C \int_s^t |u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(\tau), \tau) - u_\varepsilon^{\varepsilon,n-1}(X_\varepsilon^{n+1}(\tau), \tau)| d\tau \\ &\quad + C \int_s^t |u_\varepsilon^{\varepsilon,n-1}(X_\varepsilon^{n+1}(\tau), \tau) - u_\varepsilon^{\varepsilon,n-1}(X_\varepsilon^n(\tau), \tau)| d\tau \\ &\leq C_\varepsilon \int_s^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(\cdot, \cdot, \tau)\|_{L_q^\infty} d\tau \\ &\quad + C_\varepsilon \int_s^t |X_\varepsilon^{n+1}(\tau) - X_\varepsilon^n(\tau)| d\tau. \end{aligned}$$

Thus we get

$$\begin{aligned} |Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| &\leq C_\varepsilon \int_0^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(\cdot, \cdot, \tau)\|_{L_q^\infty} d\tau \\ &\quad + C_\varepsilon \int_s^t |Z_\varepsilon^{n+1}(\tau) - Z_\varepsilon^n(\tau)| d\tau. \end{aligned}$$

Applying Gronwall inequality to the above, we have

$$|Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)| \leq C_\varepsilon \int_0^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(\cdot, \cdot, \tau)\|_{L_q^\infty} d\tau$$

for all $0 \leq s \leq t \leq T$. We then combine this with (2.14) to conclude

$$\begin{aligned} &\|Z_\varepsilon^{n+1}(t) - Z_\varepsilon^n(t)\|_{L^\infty} + \|(f_\varepsilon^{n+1} - f_\varepsilon^n)(\cdot, \cdot, t)\|_{L_q^\infty} \\ &\leq C_\varepsilon \int_0^t \|Z_\varepsilon^{n+1}(s) - Z_\varepsilon^n(s)\|_{L^\infty} + \|(f_\varepsilon^n - f_\varepsilon^{n-1})(\cdot, \cdot, s)\|_{L_q^\infty} ds. \end{aligned} \tag{2.15}$$

2.3. Proof of Proposition 2.1. We now provide the details of proof of the global-in-time existence of weak solutions to the regularized equation (2.1).

2.3.1. Existence of weak solutions. It follows from (2.15) that f_ε^n is a Cauchy sequence in $L^\infty(0, T; L_q^\infty(\mathbb{T}^d \times \mathbb{R}^d))$ from which, for a fixed $\varepsilon > 0$, there exists a limiting function f_ε such that

$$\sup_{0 \leq t \leq T} \|f_\varepsilon^n(\cdot, \cdot, t) - f_\varepsilon(\cdot, \cdot, t)\|_{L_q^\infty} \rightarrow 0 \tag{2.16}$$

as $n \rightarrow \infty$. Then it remains to show that the limiting function f_ε solves the regularized equation (2.1). Since there is no singularity in the local alignment force, it suffices to have

$$\|\mathcal{M}(f_\varepsilon^n) - \mathcal{M}(f_\varepsilon)\|_{L_q^\infty} \rightarrow 0$$

as $n \rightarrow \infty$. Since $q > d + 2$, for $\phi(v) = 1, v$, and $|v|^2$, we estimate

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} f_\varepsilon^n \phi(v) dv - \int_{\mathbb{R}^d} f_\varepsilon \phi(v) dv \right| \\ &\leq \int_{\mathbb{R}^d} |f_\varepsilon^n - f_\varepsilon| \phi(v) dv \leq \|f_\varepsilon^n - f_\varepsilon\|_{L_q^\infty} \int_{\mathbb{R}^d} \frac{\phi(v)}{(1 + |v|)^q} dv \leq C \|f_\varepsilon^n - f_\varepsilon\|_{L_q^\infty}. \end{aligned}$$

This together with (2.16) yields

$$\rho_\varepsilon^n \rightarrow \rho_\varepsilon, \quad \rho_\varepsilon^n u_\varepsilon^n \rightarrow \rho_\varepsilon u_\varepsilon,$$

and

$$\rho_\varepsilon^n |u_\varepsilon^n|^2 + d\rho_\varepsilon^n T_\varepsilon^n \rightarrow \rho_\varepsilon |u_\varepsilon|^2 + d\rho_\varepsilon T_\varepsilon$$

in $L^\infty(\mathbb{T}^d \times (0, T))$ as $n \rightarrow \infty$. Here $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ denote the macroscopic fields constructed from f_ε . On the other hand, the lower bound estimate of ρ_ε^n is obtained in (2.12), and this gives

$$\rho_\varepsilon^n \rightarrow \rho_\varepsilon, \quad u_\varepsilon^n \rightarrow u_\varepsilon, \quad \text{and} \quad T_\varepsilon^n \rightarrow T_\varepsilon$$

as $n \rightarrow \infty$, uniformly in x and t . The convergence of f_ε^n in $\|\cdot\|_{L^\infty(0, T; L_q^\infty)}$ and the uniform convergence of $(\rho_\varepsilon^n, u_\varepsilon^n, T_\varepsilon^n)$ to $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ imply that f_ε and $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ also satisfy the assumptions of Lemma 2.3. This concludes from Lemma 2.3 and (2.16) that

$$\|\mathcal{M}(f_\varepsilon^n) - \mathcal{M}(f_\varepsilon)\|_{L_q^\infty} \leq C_\varepsilon \|f_\varepsilon^n - f_\varepsilon\|_{L_q^\infty} \rightarrow 0$$

as $n \rightarrow \infty$. This completes the proof of the existence part.

2.3.2. Uniform-in- ε bound estimates. In this part, we show the uniform-in- ε bound estimates of solutions obtained in the above. Since the third moment and entropy estimates can be obtained by using almost the same arguments as in [10, Section 5], we only provide the kinetic energy estimate (i). We first notice from [16, Lemma 2.5] that

$$\sup_{y \in \mathbb{T}^d} \int_{\mathbb{T}^d} \theta_\varepsilon(x - y) \frac{\rho_\varepsilon(x)}{\theta_\varepsilon \star \rho_\varepsilon(x)} dx \leq C, \quad (2.17)$$

where $C > 0$ is independent of ε . On the other hand, a straightforward computation yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_\varepsilon dx dv &= \int_{\mathbb{T}^d \times \mathbb{R}^d} v \cdot (u_\varepsilon^\varepsilon - v) f_\varepsilon dx dv \\ &\leq \frac{1}{2} \int_{\mathbb{T}^d} |u_\varepsilon^\varepsilon|^2 \rho_\varepsilon dx - \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_\varepsilon dx dv \\ &\leq \frac{1}{2} \int_{\mathbb{T}^d} |u_\varepsilon^\varepsilon|^2 \rho_\varepsilon dx. \end{aligned}$$

Note that

$$\begin{aligned} |u_\varepsilon^\varepsilon(x, t)|^2 &= \left| \frac{((\rho_\varepsilon u_\varepsilon) \star \theta_\varepsilon)(x, t)}{(\rho_\varepsilon \star \theta_\varepsilon)(x, t) + \varepsilon(1 + |((\rho_\varepsilon u_\varepsilon) \star \theta_\varepsilon)(x, t)|^2)} \right|^2 \\ &\leq \left| \frac{\int_{\mathbb{T}^d \times \mathbb{R}^d} \theta_\varepsilon(x - y) w f_\varepsilon(y, w, t) dy dw}{\theta_\varepsilon \star \rho_\varepsilon(x, t)} \right|^2 \\ &\leq \frac{\int_{\mathbb{T}^d \times \mathbb{R}^d} \theta_\varepsilon(x - y) |w|^2 f_\varepsilon(y, w, t) dy dw}{\theta_\varepsilon \star \rho_\varepsilon(x, t)}, \end{aligned}$$

and this together with (2.17) gives

$$\begin{aligned} \int_{\mathbb{T}^d} \rho_\varepsilon |u_\varepsilon^\varepsilon|^2 dx &\leq \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\int_{\mathbb{T}^d} \theta_\varepsilon(x - y) \frac{\rho_\varepsilon(x)}{\theta_\varepsilon \star \rho_\varepsilon(x)} dx \right) |w|^2 f_\varepsilon(y, w) dy dw \\ &\leq C \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_\varepsilon(x, v) dx dv, \end{aligned}$$

where $C > 0$ is independent of ε . Hence we have

$$\frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_\varepsilon(x, v, t) dx dv \leq C \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_\varepsilon(x, v, t) dx dv,$$

i.e.,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_\varepsilon(x, v, t) dx dv \leq C \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_{0,\varepsilon}(x, v) dx dv,$$

where $C > 0$ is independent of ε . This concludes the desired result.

3. Proof of Theorem 1.2. We are now ready to send the regularization parameter $\varepsilon \rightarrow \infty$ in the regularized equation (2.1). We will use several compactness arguments based on the Dunford-Pettis theorem and the velocity averaging lemma. Since the compactness of the BGK relaxation operator and the macroscopic fields are already discussed in [10, Section 6], we sketch the idea of proofs here. In the previous section, we obtained

$$\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} (1 + |v|^3 + |\ln f_\varepsilon(x, v, t)|) f_\varepsilon(x, v, t) dx dv dt \leq C, \quad (3.1)$$

where $C > 0$ is independent of ε . Then by Dunford-Pettis theorem, we find that f_ε , $f_\varepsilon v$ and $f_\varepsilon |v|^2$ are weakly compact in $L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$, and thus there exists $f \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$ such that f_ε , $f_\varepsilon v$, $f_\varepsilon |v|^2$ converge to f , $f v$, $f |v|^2$ weakly in $L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$ respectively, which also implies

$$\rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon dv \rightharpoonup \int_{\mathbb{R}^d} f dv = \rho, \quad \rho_\varepsilon u_\varepsilon = \int_{\mathbb{R}^d} v f_\varepsilon dv \rightharpoonup \int_{\mathbb{R}^d} v f dv = \rho u,$$

and

$$d\rho_\varepsilon T_\varepsilon + \rho_\varepsilon |u_\varepsilon|^2 = \int_{\mathbb{R}^d} f_\varepsilon |v|^2 dv \rightharpoonup \int_{\mathbb{R}^d} f |v|^2 dv = d\rho T + \rho |u|^2$$

in $L^1(\mathbb{T}^d \times (0, T))$. Then with the aid of the velocity averaging lemma [13], the above convergence actually is strong, which gives the almost everywhere convergence of the macroscopic fields:

$$\rho_\varepsilon \rightarrow \rho \quad \text{a.e on } \mathbb{T}^d \times [0, T], \quad u_\varepsilon \rightarrow u \quad \text{a.e on } E, \quad \text{and} \quad T_\varepsilon \rightarrow T \quad \text{a.e on } E, \quad (3.2)$$

where

$$E = \{(x, t) \in \mathbb{T}^d \times (0, T) \mid \rho(x, t) \neq 0\}. \quad (3.3)$$

Concerning the compactness of the local Maxwellian, we use the entropy bound estimate in (3.1) together with Dunford-Pettis theorem to have the weak compactness of $\mathcal{M}(f)$ in $L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$. In fact, we can also show that $\mathcal{M}(f_\varepsilon)$ converges weakly to $\mathcal{M}(f)$ in $L^1(\mathbb{T}^d \times \mathbb{R}^d \times (0, T))$, see [10, Section 6.3] for details.

Employing the previous convergence estimates, we can pass to the limit $\varepsilon \rightarrow 0$ in the regularized equation (2.1) except the term $f_\varepsilon u_\varepsilon^\varepsilon$ in the local alignment force. Thus in the rest of this section, we will show that

$$f_\varepsilon u_\varepsilon^\varepsilon \rightharpoonup f u \quad \text{in} \quad L^\infty(0, T; L^p(\mathbb{T}^d \times \mathbb{R}^d)) \quad \text{for} \quad p \in \left(1, \frac{d+2}{d+1}\right).$$

For this, we use a similar argument as in [16]. By the strong convergence (3.2), we get

$$(\rho_\varepsilon u_\varepsilon) \star \theta_\varepsilon \rightarrow \rho u, \quad \rho_\varepsilon \star \theta_\varepsilon \rightarrow \rho \quad \text{a.e.} \quad \text{and} \quad L^p(\mathbb{T}^d \times (0, T))\text{-strong},$$

up to a subsequence, for all $p \in (1, (d+2)/(d+1))$. Let

$$\rho_\varepsilon^\varphi(x, t) = \int_{\mathbb{R}^d} f_\varepsilon(x, v, t) \varphi(v) dv$$

for a given test function $\varphi(v)$. Consider a test function $\psi(x, v, t) := \phi(x, t) \varphi(v)$ with $\phi \in \mathcal{C}_c^\infty(\mathbb{T}^d \times (0, T))$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then we obtain

$$\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon u_\varepsilon^\varepsilon \psi dx dv dt = \int_0^T \int_{\mathbb{T}^d} u_\varepsilon^\varepsilon \rho_\varepsilon^\varphi \phi dx dt.$$

Note that if $p \in (1, (d+2)/(d+1))$, then $p/(2-p) \in (1, (d+2)/d)$, and we use Hölder's inequality and Proposition 2.1 (i) to deduce

$$\|u_\varepsilon^\varepsilon \rho_\varepsilon^\varphi\|_{L^p} \leq \|\varphi\|_{L^\infty} \|\rho_\varepsilon\|_{L^{p/(2-p)}}^{1/2} \|\sqrt{\rho_\varepsilon} u_\varepsilon^\varepsilon\|_{L^2} < \infty.$$

Thus there exists a function $m \in L^\infty(0, T; L^p(\mathbb{T}^d))$ such that

$$u_\varepsilon^\varepsilon \rho_\varepsilon^\varphi \rightharpoonup m \quad \text{in} \quad L^\infty(0, T; L^p(\mathbb{T}^d)) \quad \text{for all} \quad p \in \left(1, \frac{d+2}{d+1}\right),$$

up to a subsequence. We then claim that

$$m = u \rho^\varphi, \quad \text{where} \quad \rho^\varphi = \int_{\mathbb{R}^d} f \varphi dv \quad \text{and} \quad \rho u = \int_{\mathbb{R}^d} v f dv.$$

By using the set E appeared in (3.3), we first easily estimate

$$\|u_\varepsilon^\varepsilon \rho_\varepsilon^\varphi\|_{L^p(E)} \leq C \|\rho_\varepsilon\|_{L^{p/(2-p)}(E)}^{1/2} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$

This implies that it suffices to check

$$m = u \rho^\varphi \quad \text{whenever} \quad \rho > 0.$$

For this, we introduce a set

$$E^\delta := \{(x, t) \in \mathbb{T}^d \times (0, T) : \rho(x, t) > \delta\}.$$

By the compactness of ρ_ε and $\rho_\varepsilon \star \theta_\varepsilon$ and Egorov's theorem, for any $\eta > 0$, there exists a set $C_\eta \subset E^\delta$ with $|E^\delta \setminus C_\eta| < \eta$ on which both ρ_ε and $\rho_\varepsilon \star \theta_\varepsilon$ uniformly converge to ρ as $\varepsilon \rightarrow 0$. This yields that $\rho_\varepsilon \star \theta_\varepsilon > \delta/2$ in C_η for sufficiently small $\varepsilon > 0$. This yields

$$u_\varepsilon^\varepsilon \rho_\varepsilon^\varphi = \frac{(\rho_\varepsilon u_\varepsilon) \star \theta_\varepsilon}{\varepsilon + \rho_\varepsilon \star \theta_\varepsilon} \rho_\varepsilon^\varphi \rightarrow m = u \rho^\varphi \quad \text{in} \quad C_\eta,$$

and subsequently, this asserts

$$m = u \rho^\varphi \quad \text{on} \quad E,$$

since $\eta > 0$ and $\delta > 0$ were arbitrary. Hence we have

$$\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon u_\varepsilon^\varepsilon \psi dx dv dt \rightarrow \int_0^T \int_{\mathbb{T}^d} u \rho^\varphi \phi dx dt = \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} f u \psi dx dv dt$$

for all test functions of the form $\psi(x, v, t) = \phi(x, t) \varphi(v)$.

Incorporating all of the above observations allows us to send $\varepsilon \rightarrow 0$ in (2.1) to conclude that the limiting function f is the weak solution to our main equation (1.1) in the sense of Definition 1.1. L^∞ bound and kinetic energy estimates in Theorem 1.2 can be easily obtained.

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E-mail address: ypchoi@yonsei.ac.kr

E-mail address: sbyun01@skku.edu