COMPRESSIBLE AND VISCOUS TWO-PHASE FLOW IN POROUS MEDIA BASED ON MIXTURE THEORY FORMULATION

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ABSTRACT. The purpose of this work is to carry out investigations of a generalized two-phase model for porous media flow. The momentum balance equations account for fluid-rock resistance forces as well as fluid-fluid drag force effects, in addition, to internal viscosity through a Brinkmann type viscous term. We carry out detailed investigations of a one-dimensional version of the general model. Various a priori estimates are derived that give rise to an existence result. More precisely, we rely on the energy method and use compressibility in combination with the structure of the viscous term to obtain H^1-estimates as well upper and lower uniform bounds of mass variables. These a priori estimates imply existence of solutions in a suitable functional space for a global time $T > 0$. We also derive discrete schemes both for the incompressible and compressible case to explore the role of the viscosity term (Brinkmann type) as well as the incompressible versus the compressible case. We demonstrate similarities and differences between a formulation that is based, respectively, on interstitial velocity and Darcy velocity in the viscous term. The investigations may suggest that interstitial velocity seems more natural to use in the formulation of momentum balance than Darcy velocity.

1. **Introduction.** The importance of multiphase flow in porous media has long been recognized in many fields. Mathematical modelling of multiphase flow is essential in practical applications like enhanced oil recovery and geological CO_2 storage in depleted oil and gas reservoirs [25, 42] as well as biological processes [29, 14, 18, 39, 40, 37]. Traditional formulations of multiphase flow describe macroscopic fluid fluxes with a straightforward extension, first proposed by Muskat [30, 3],

of Darcy's equation for single-phase flow. Unlike in the single-phase case, this extension cannot be rigorously obtained from first principles [22, 23]. The multiphase extension of Darcy's equation may be described as a quasi-linear relation, because the fluid flux depends linearly on the "driving force", which includes viscous, capillary, and gravity forces, and all the nonlinearity is agglutinated in the relative permeability and capillary pressure functions [25]. An instructive overview is given in [32] of how generalizations of the standard Darcy's law for single phase flow can be derived within the context of mixture theory. Starting with more general momentum balance equations and using different sets of assumptions leads to a hierarchy of mathematical models. In particular, it can be shown that popular models due to Brinkman, Biot and many others can be obtained via various approximations. Interesting extensions of the classical multiphase formulation are also discussed by Wu [42].

1.1. **A compressible and viscous two-fluid model for porous media flow.**
The model we are interested in describes flow of two compressible immiscible fluids, e.g., water (w), oil (o), or gas (g), moving in a porous media and takes the following form (we use "w" and "o" in the following as index):

$$
\begin{aligned}
(\phi n)_t + \nabla \cdot (\phi n \mathbf{u}_o) &= Q_o, \qquad n = s_o \rho_o \\
(\phi m)_t + \nabla \cdot (\phi m \mathbf{u}_w) &= Q_w, \qquad m = s_w \rho_w \\
s_o \nabla P_o + n\mathbf{g} &= -\hat{k}_o \mathbf{u}_o + \hat{k}_{ow}(\mathbf{u}_w - \mathbf{u}_o) + \varepsilon_o \nabla \cdot (n\nabla \mathbf{u}_o) \\
s_w \nabla P_w + m\mathbf{g} &= -\hat{k}_w \mathbf{u}_w - \hat{k}_{ow}(\mathbf{u}_w - \mathbf{u}_o) + \varepsilon_w \nabla \cdot (m\nabla \mathbf{u}_w)
\end{aligned}
\tag{1.1}
$$

with capillary pressure P_c defined as the difference between the non-wetting fluid (oil) pressure P_o and wetting fluid (water) pressure P_w

$$
P_c = P_o - P_w = P_c(s_w), \qquad P_c'(s_w) < 0.
\tag{1.2}
$$

Herein, ϕ is the porosity of the medium, ρ_i represents density and s_i the volume fraction (saturation) where $i = w, o$. In addition, we have the fundamental relation that expresses that the water and oil occupy the pore space

$$
s_o + s_w = 1.
\tag{1.3}
$$

Furthermore, $\varepsilon_w, \varepsilon_o$ (assumed to be constant) characterize the magnitude of the viscous terms. The model can be derived (or at least motivated) from general mixture theory [11, 32] where we study a continuum composed of matrix occupying a volume fraction $(1-\phi)$ and a pore space of volume ϕ that is filled with a mixture of water and oil represented, respectively, by ϕs_w and ϕs_o such that $(1-\phi)+\phi s_w+\phi s_o = 1$. The matrix is stagnant whereas the two fluids move with (locally) averaged interstitial velocities $\mathbf{u}_w$ and $\mathbf{u}_o$. We refer to the recent work [31] for more details leading to (1.1). See also [1, 29, 34] and references therein for interesting examples of similar models developed in the mixture theory framework.

Note that the viscous terms $\varepsilon_o \nabla \cdot (n\nabla \mathbf{u}_o)$ and $\varepsilon_w \nabla \cdot (m\nabla \mathbf{u}_w)$ (Brinkman type of term) included in $(1.1)_{3,4}$ involve a mass dependent coefficient whose magnitude is governed by the parameter ε_i. This reflects that we have introduced kinematic viscosity ε that is related to dynamic viscosity μ by $\varepsilon\rho = \mu$ for single-phase flow of a fluid with density ρ [28]. Combined with the two-phase momentum balance for water and oil this gives rise to mass dependent viscosity coefficients of the form $\varepsilon_o n$ and $\varepsilon_w m$. We refer to [14] (and references) therein for more details. More generally, we may think of $\varepsilon_o n$ and $\varepsilon_w m$ as "effective" viscosities since the model (1.1) must

be understood as the result of a volume averaging process where variables have been obtained through averaging over a small representative volume element, implying that detailed information about complex interfaces between the two phases have been lost and are represented only in an averaged sense [11, 32]. Some authors also denote this viscosity as the "Brinkman viscosity" whereas the viscosity associated with the rock-fluid friction term $\hat{k}_i$ $(i = o, w)$ is denoted the "Darcy viscosity" [27]. This issue is also discussed in [38] where it is observed by means of an up-scaling procedure based on volume averaging methods, that the use of a slip boundary condition gives rise to an effective viscosity different from the one corresponding to the fluid phase.

In the following we will focus on nonlinear coupling mechanisms and we therefore assume physical parameters like porosity ϕ, absolute permeability K, Darcy viscosity μ_i (will be introduced later), and Brinkman viscosity ε_i to be constant. Generally, $\hat{k}_o$, $\hat{k}_w$, and $\hat{k}_{ow}$ depend on the fluid composition through s_i and ρ_i.

1.1.1. *Closure relations.* The above model must be endowed with appropriate closure relations for densities $\rho_i = \rho_i(P_i)$. The two phases will be treated as weakly compressible fluids. More precisely, we represent the water and the oil by linear pressure-density relations of the form

$$\rho_w - \tilde{\rho}_{w0} = \frac{P_w}{C_w}, \qquad \rho_o - \tilde{\rho}_{o0} = \frac{P_o}{C_o}, \tag{1.4}$$

where C_w and C_o reflect the compressibility of water and oil, respectively. An essential role is played by the interaction coefficients $\hat{k}_{ow}$, $\hat{k}_w$, and $\hat{k}_o$. We will come back with more details about the choice of these. In addition, a functional form of the capillary pressure $P_c(s_w)$ must also be specified. Combining (1.2), (1.3), and (1.4) it follows that $\rho_w = \rho_w(m, n)$ and $\rho_o = \rho(m, n)$ are well-defined as functions of m and n for $m, n \geq 0$, from which we also can compute $s_w = s_w(m, n)$ and $s_o = s_o(m, n)$, see (2.17)-(2.21) for details.

1.1.2. *Initial and boundary conditions.* Boundary conditions are prescribed as no-flux conditions:

$$\mathbf{u}_i \cdot \nu = 0, \qquad x \in \partial\Omega, \quad t > 0, \qquad i = w, o \tag{1.5}$$

where ν is the outward normal on $\partial\Omega$. The corresponding initial data is

$$n(x, t = 0) = n_0(x), \qquad m(x, t = 0) = m_0(x), \qquad x \in \Omega. \tag{1.6}$$

1.2. **The model (1.1) as a generalization of Darcy's equation based formulation.** We may ignore the effects from the viscous terms in $(1.1)_{3,4}$ by setting $\varepsilon_o = \varepsilon_w = 0$. In addition, we neglect the fluid-fluid interaction effect by setting $\hat{k}_{ow} = 0$, combined with the assumption that fluid-pore resistance force coefficient $\hat{k}_i$ takes the form

$$\hat{k}_i \overset{\text{def}}{:=} s_i^2 \phi \frac{\mu_i}{K k_{ri}}, \qquad i = w, o \tag{1.7}$$

where K is the absolute permeability (assumed here to be a scalar, i.e., we assume a homogeneous media), k_{ri} is relative permeability, and μ_i viscosity. This gives from $(1.1)_{3,4}$ the following reduced momentum equations

$$\mathbf{U}_i \overset{\text{def}}{:=} \phi s_i \mathbf{u}_i = -K \frac{k_{ri}}{\mu_i}(\nabla P_i + \rho_i \mathbf{g}) = -\lambda_i(\nabla P_i + \rho_i \mathbf{g}), \qquad \lambda_i := K \frac{k_{ri}}{\mu_i}, \tag{1.8}$$

for $i = w, o$ which is nothing but the classical Darcy law where $\mathbf{U}_i$ is the Darcy velocity. When combined with $(1.1)_{1,2}$ we arrive at the classical two-phase formulation [42]. The model (1.1) involves two main extensions from classical formulation based on two-phase Darcy's equation, as elaborated upon in the following:

- The interaction forces on the RHS of $(1.1)_{3,4}$ involve a fluid-fluid drag force effect $\hat{k}_{ow}(\mathbf{u}_w - \mathbf{u}_o)$ in addition to fluid-rock drag force $-\hat{k}_o\mathbf{u}_o$ and $-\hat{k}_w\mathbf{u}_w$. The only interaction force in Darcy's equation is the latter one representing friction between fluid and boundaries of the pores [32]. Moreover, while the drag force depends on the velocity, it is by no means necessary that it in general depends linearly on the relative velocity. See for example [42] for extensions that include nonlinear dependence on fluid velocity (Forchheimer). Moreover, it has been observed that inclusion of the fluid-fluid interaction term $\hat{k}_{ow}(\mathbf{u}_w - \mathbf{u}_o)$ can give improvements over standard Darcy's equation based formulation for water-oil flow in porous media. We refer to [35, 31] for a first discussion of this in the context of core scale modelling and generalized permeability functions and [36] for a discussion of this generalized two-phase flow in the context of imbibition (i.e., capillary pressure driven counter-current flow).
- The viscous terms $\varepsilon_o \nabla \cdot (n \nabla \mathbf{u}_o)$ and $\varepsilon_w \nabla \cdot (m \nabla \mathbf{u}_w)$ in $(1.1)_{3,4}$ can account for frictional forces within the fluid due to its viscosity. Ignoring these terms essentially imply that the viscosity of the fluid and the roughness of the solid surface lead to far greater frictional resistance (and hence dissipation) at the porous boundaries of the solid in comparison to the frictional resistance in the fluid [32]. Note that $(1.1)_{3,4}$ can naturally be interpreted as a two-phase version of Brinkman's equation. Brinkman's equation amounts to using a momentum balance equation that takes the following form for a fluid-matrix system [32]:

$$\nabla p + \rho \mathbf{g} = -\alpha_0 \mathbf{u} + \mu \nabla \cdot (\nabla \mathbf{u}), \tag{1.9}$$

where the quantities refer to the fluid phase, i.e., p is fluid pressure and $\mathbf{u}$ is pore velocity, ρ phase density and μ phase viscosity. $\mathbf{g}$ denotes the external gravity force and α_0 the magnitude of the fluid-matrix drag force. The final term, involving second order derivatives of velocity, marks a clear distinction from Darcy's law for single phase flow. This may be relevant for porous media having large permeability and/or being dominated by a network of fractures [27].

Some more precise remarks seem useful in order to set the model (1.1) into a broader context.

Remark 1.1. We may replace the viscous term in $(1.1)_{3,4}$ that accounts for the fluid viscous shear effects that oppose the flow through the porous structure by a more general term

$$\nabla \cdot (\varepsilon_{i,\text{eff}} \nabla \mathbf{u}_i),$$

where the effective viscosity coefficient $\varepsilon_{i,\text{eff}}$ depends on other variables than the mass. We refer to [38] for a discussion of this, both from theoretical and numerical investigations. For example, from physical considerations and experimental investigations it seems clear that it could depend on pressure [32]. It is concluded that it might be reasonable to include dependence on pressure both in modelling of fluid-pore friction force as well as frictional effects within the fluid itself [32]. More generally, one should also account for the possibility that the flow may not be

steady implying that one should add a term $\rho \mathbf{u}_t$ to (1.9) in the single-phase case or $(\phi s_i \rho_i \mathbf{u}_i)_t$ in the more general two-phase model (1.1). Moreover, nonlinear inertial effects in the fluid cannot be ignored if the flow is not sufficiently slow.

Remark 1.2. An interesting study of a two-phase Brinkman type of model is found in [9]. A two-phase formulation based on Darcy's equations of the following form is studied with $s = s_w$

$$s_t + \nabla \cdot (\mathbf{U}_w) = 0, \qquad \mathbf{U}_w = f(s)\mathbf{U}_T = -\lambda_w(s)\nabla P$$
$$\nabla \cdot \mathbf{U}_T = 0 \tag{1.10}$$
$$\mathbf{U}_T = -\lambda_T(s)\nabla P,$$

where $\phi \equiv 1$, $P_w = P_o = P$ (zero capillary pressure), $\lambda_T(s) = \lambda_w(s) + \lambda_o(s)$ is total mobility and $f(s) = \lambda_w(s)/\lambda_T(s)$ is the fractional flow function for water phase. $\mathbf{U}_T$ is the total Darcy velocity $\mathbf{U}_T = \mathbf{U}_w + \mathbf{U}_o$. The following two-phase version of Brinkman's equation based on Darcy's velocity $\mathbf{U}_i$ is proposed as a generalization of Darcy's equation:

$$- \mu\Delta\mathbf{U}_i + \mathbf{U}_i = -\lambda_i\nabla P, \qquad i = w, o. \tag{1.11}$$

Summing the two momentum equations in (1.11) gives rise to a Brinkman type of momentum equation for the mixture of the two phases expressed in terms of the total Darcy velocity $\mathbf{U}_T = \mathbf{U}_w + \mathbf{U}_o$:

$$- \mu\Delta\mathbf{U}_T + \mathbf{U}_T = -\lambda_T(s)\nabla P. \tag{1.12}$$

Taking the divergence of (1.12) gives in light of the total mass balance equation $(1.10)_2$ the following Brinkman based approximation of (1.10)

$$s_t + \nabla \cdot (\mathbf{U}_w) = 0, \qquad \mathbf{U}_w = f(s)\mathbf{U}_T$$
$$-\nabla \cdot (\lambda_T(s)\nabla P) = 0 \tag{1.13}$$
$$-\mu\Delta\mathbf{U}_w + \mathbf{U}_w = -\lambda_w(s)\nabla P, \qquad \lambda_w(s) = f(s)\lambda_T(s).$$

In [9] a notion of weak solutions to the Brinkman model (1.13) is introduced and convergence of an iterative approximation as well as full numerical scheme is demonstrated. Numerical experiments show that the numerical approximation is quite sensitive to the choice of μ and creates oscillatory behavior. Interestingly, numerical experiments also indicate that this solution may not converge to the solution of the model (1.10) corresponding to $\mu = 0$.

Remark 1.3. In the literature there seems to be an ongoing interesting discussion of various formulations of two-phase versions of porous media flow based on Brinkman's equation. See for example the work [41] for a discussion of this. Traditionally, the superficial phase velocity (Darcy velocity) $\mathbf{U}_i$ has been used in two-phase versions of Brinkman's equation similar to (1.11). In [41] it is argued that the most natural choice, at least for the flow system they consider with creeping flow inside moving permeable particles, is to use interstitial (intrinsic) phase velocity $\mathbf{u}_i$ in the macroscopic equations. Their conclusion is based on numerical computations and comparison of the model based on, respectively, phase velocity $\mathbf{U}_i = s_i\mathbf{u}_i$ and interstitial phase velocity $\mathbf{u}_i$.

1.3. **Purpose of this work and brief review of related works.** The aim of this paper is three-fold: (i) Present an example of stability analysis motivated by the study of compressible Navier-Stokes equation (and different from traditional

two-phase porous media stability analysis) which exploits the structure of the viscous term in the momentum balance and accounts for compressibility; (ii) Present an example of a numerical scheme both for the compressible and incompressible version of (1.1) in a one-dimensional setting and demonstrate similarities and differences through some specific simulations; (iii) Gain some insight into the impact the viscosity terms have on the solution.

We end this section by giving a brief review of other works on the two-phase porous media model based on Darcy's law. Most works focus on the incompressible, immiscible two-phase flow case. For example, [2] studied the existence of weak solutions for the incompressible two-phase model in fractured porous media based on a dual-porosity formulation. Regularity and stability results were obtained in [8] when analysing a coupled system involving a saturation and a global pressure. In [6, 5] the authors showed existence of a solution for an incompressible two-phase flow within a heterogeneous porous medium made of two rock types. Considering dynamic capillary pressure [24] for the incompressible two-phase flow in porous media, [7] proved the existence of a weak solution to a degenerate elliptic parabolic system whereas in [13, 12] existence conditions for the traveling wave solution were derived. In particular, non-monotone weak solutions for the Buckley-Leverett equation were obtained. Interesting contributions have also been made concerning the compressible immiscible two-phase flow in porous media where phase densities are assumed to depend on their own pressure. Without using the feature of global pressure, Khalil and Saad [26] established an existence result for a three-dimensional model. In addition, the implicit finite volume scheme was studied in [33] to obtain convergence to a weak solution.

2. **Stability analysis and existence of solution in the one-dimensional setting.** The purpose of this section is to derive a priori estimates of the solution of (1.1). The approach is quite different from the approach used for the incompressible model and formulation based on Darcy velocity $\mathbf{U}_i$ where the first step is to derive estimate for pressure [9]. It is also different from mathematical analysis of compressible two-phase models that are based on global pressure [19, 20, 21]. We rely on the energy method where we first derive an energy-type of estimate. In addition, the special structure of the viscous terms allows one to obtain estimate of m, n in H^1 along the lines of the Bresch-Desjardin method [4] for a two-phase Navier-Stokes model. For analysis of related models we refer the interesting reader to [15, 16, 17, 18], and references therein. In the following we consider the 1D version of (1.1) where source terms have been set such that water is injected and possibly produced whereas oil is produced only, i.e., $Q_o = -nQ_p$ whereas $Q_w = -mQ_p + \rho_w Q_I$ for constant Q_p, Q_I. The model takes the following form with (n, m, u_w, u_o) as the main variables

$$
\begin{aligned}
(n)_t + (nu_o)_x &= -nQ_p, & n &= s_o\rho_o \\
(m)_t + (mu_w)_x &= -mQ_p + \rho_w Q_I, & m &= s_w\rho_w \\
s_o(P_o)_x &= -\hat{k}_o u_o + \hat{k}(u_w - u_o) + ng + \varepsilon_o(nu_{ox})_x, & &(2.14) \\
s_w(P_w)_x &= -\hat{k}_w u_w - \hat{k}(u_w - u_o) + mg + \varepsilon_w(mu_{wx})_x, \\
P_c = P_o - P_w &= P_c(s_w),
\end{aligned}
$$

subject to the boundary condition

$$
u_w(x = 0, t) = u_o(x = 0, t) = 0
$$

$$u_w(x = 1, t) = u_o(x = 1, t) = 0, \qquad t > 0 \tag{2.15}$$

and initial condition

$$n(x, t = 0) = n_0(x), \qquad m(x, t = 0) = m_0(x), \qquad x \in [0, 1]. \tag{2.16}$$

Note that the gravity constant g can take both signs depending on the orientation of the x-coordinate axis. Above we assume that positive direction of x-axis points downward and $g > 0$. We also use the notation $\hat{k} = \hat{k}_{ow}$ in (2.14), for simplicity reason only.

Definition of $(\rho_o, \rho_w, s_o, s_w)$**.** Let us first see how we can obtain ρ_w and ρ_o as a function of masses m and n. We focus on the situation where $m, n > 0$. The cases where $m = 0$ or $n = 0$ are treated separately. We rewrite $s_o + s_w = 1$ as

$$m\rho_o + n\rho_w = \rho_w \rho_o \qquad (\text{i.e. } \rho_o = \frac{n\rho_w}{\rho_w - m}). \tag{2.17}$$

On the other hand, from (1.2) we have

$$P_c(s_w) = P_o - P_w = C_o \rho_o - C_w \rho_w + C_w \tilde{\rho}_{w0} - C_o \tilde{\rho}_{o0}. \tag{2.18}$$

Combining (2.18) with (2.17), we get

$$P_o - P_w - P_c(s_w) = C_o \big(\frac{n\rho_w}{\rho_w - m}\big) - C_w \rho_w + C_w \tilde{\rho}_{w0} - C_o \tilde{\rho}_{o0} - P_c\big(\frac{m}{\rho_w}\big) \overset{\text{def}}{:=} F(\rho_w; m, n), \tag{2.19}$$

where we have introduced the function $F(\rho_w; m, n)$ where m, n are thought of as parameters. Clearly, for any choice of $m, n > 0$, we want to verify that $F(u; m, n)$ (where we use u as the main variable) has a unique zero point which we denote as $\rho_w(m, n)$. Let us check some basic properties of $F(u; m, n)$ as a function of u.

By the definition of m, it is natural to look for ρ_w which belongs to $(m, +\infty)$. Moreover, from (2.19) we observe that (i) $F(u \to m^+; m, n) = +\infty$; (ii) $F(u \to +\infty; m, n) = -\infty$. Next, we check monotonicity properties of $F(u; m, n)$ as a function of u.

$$F'_u(u; m, n) = C_o \frac{-mn}{(u - m)^2} - C_w + P'_c\big(\frac{m}{u}\big)\big(\frac{m}{u^2}\big). \tag{2.20}$$

Since $F'_u(u, m, n) < 0$ in $(m, +\infty)$ for any given $m, n > 0$, and $F : (m, +\infty) \mapsto (-\infty, +\infty)$ as observed above, it follows (from the intermediate value theorem) that there is a unique $\rho_w = \rho_w(m, n) \in (m, +\infty)$ such that $F(\rho_w; m, n) = 0$. In addition, since $F'_u(\rho_w; m, n) \neq 0$, it concludes that the function ρ_w is differentiable with respect to m or n (from the implicit function theorem). Furthermore, ρ_o, s_o, and s_w are then obtained as follows:

$$\rho_o(m, n) = \frac{n\rho_w}{\rho_w - m}, \qquad s_w = \frac{m}{\rho_w}, \qquad s_o = 1 - \frac{m}{\rho_w} = \frac{n}{\rho_o}. \tag{2.21}$$

For the limit case when $m = 0$, there are two options: (i) $s_w = 0$, which implies that $\rho_o = n$ and ρ_w is found from (2.18); (ii) $s_w > 0$, which implies that $\rho_w = 0$ and where ρ_o is found from (2.18). Similarly, we can compute ρ_w and ρ_o for the case $n = 0$.

Notation. We first give some notation.

- $L^p = L^p([0, 1])$ for $p \in [1, \infty]$
- We define

$$\tilde{m}(t) = \int_0^1 m(x, t)\, dx; \qquad \tilde{n}(t) = \int_0^1 n(x, t)\, dx. \tag{2.22}$$

Assumptions. The following assumptions are made:

- Capillary pressure $P_c(s_w)$:
  We assume that for $\Phi(s_w)$ such that $\Phi'(s_w) = P_c(s_w)$, the following property holds:

$$\Phi(s_w) \leq P_c(\tilde{s}_w)s_w, \qquad 0 \leq s_w \leq 1 \qquad (2.23)$$

  where $\tilde{s}_w = s_w(\tilde{m}, \tilde{n})$ and $\tilde{m}$ and $\tilde{n}$ refer to the total masses given by (2.22), which are constant due to Remark 2.1. Moreover, we assume that

$$\sup_{s_w \in (0,1)} |P_c(s_w)| < \infty, \qquad \inf_{s_w \in (0,1)} [-P'_c(s_w)] \geq 0 \qquad (2.24)$$

  and that

$$C_w\tilde{\rho}_{w0} - C_o\tilde{\rho}_{o0} - \sup_{s_w \in (0,1)} P_c(s_w) \geq 0. \qquad (2.25)$$

  Note that these constraints on the capillary pressure $P_c(s_w)$ are all mild and physical reasonable conditions.
- Source terms in $(2.14)_{1,2}$ are ignored by setting $Q_p = 0 = Q_I$.
- Interaction term $\hat{k}_w$, $\hat{k}_o$, and $\hat{k}$ are set as follows:

$$\hat{k}_w = I_w\frac{m^2}{m+n}, \qquad \hat{k}_o = I_o\frac{n^2}{m+n}, \qquad \hat{k} = I_{wo}\frac{mn}{m+n}. \qquad (2.26)$$

Remark 2.1. Clearly, in view of $(2.14)_{1,2}$, the condition (2.15), and assumption $Q_p = Q_I = 0$, it follows from (2.22) that

$$\tilde{m}(t) = \int_0^1 m_0(x)\,dx = \tilde{m}_0, \qquad \tilde{n}(t) = \int_0^1 n_0(x)\,dx = \tilde{n}_0 \qquad (2.27)$$

where $\tilde{m}_0, \tilde{n}_0$ are constant.

Remark 2.2. As far as the condition on capillary pressure $P_c(s_w)$ as given by (2.23) is concerned, we may observe that this appears to be a weak structural constraint. Consider for example a capillary pressure curve of the form $P_c(s_w) = -P_c^*\ln(\delta + \frac{s_w}{a})$, for some $\delta, a > 0$ as a typical example of a physical relevant function. Clearly, from the relation $\Phi'(s_w) = P_c(s_w)$ we can introduce two positive constants C_1 and C_2 to be determined such that

$$\begin{aligned}
\Phi(s_w) &= -P_c^*\int_0^{s_w} \ln(\frac{x}{a} + \delta)dx - C_1 - C_2 \\
&= -P_c^*a\int_\delta^{s_w/a+\delta} \ln(u)du - C_1 - C_2 = P_c^*a(u - u\ln(u))\Big|_\delta^{s_w/a+\delta} - C_1 - C_2 \\
&= P_c^*s_w + P_c^*a\Big[\delta\ln(\delta) - (s_w/a + \delta)\ln(s_w/a + \delta)\Big] - C_1 - C_2.
\end{aligned}$$
$$(2.28)$$

Since $x\ln(x)$ is an increasing function for $x \geq e^{-1}$ whereas for $x \in [0, e^{-1})$ decreases from zero for $x = 0$ and takes a minimum $-e^{-1}$, it is clear that we can secure that

$$P_c^*a\Big[\delta\ln(\delta) - (s_w/a + \delta)\ln(s_w/a + \delta)\Big] - C_1 \leq 0, \qquad s_w \in [0,1]$$

for an appropriate choice of C_1 such that we conclude from (2.28) that

$$\Phi(s_w) \leq P_c^*s_w - C_2.$$

What remains to show then is that

$$P_c^*s_w - C_2 \leq P_c(\tilde{s}_w)s_w, \qquad 0 \leq s_w \leq 1.$$

Clearly, $(P_c^* - P_c(\tilde{s}_w))s_w \leq C_2$ for an appropriate choice of the constant $C_2 = C_2(P_c^*, \tilde{s}_w)$ since $s_w \in [0,1]$.

2.1. **Main results.** First, we present a local (in time) existence result whose proof is presented in Appendix A. Then, we state an (almost) global in time existence result which relies on the local existence result combined with certain a priori estimates, see (2.29). Section 2.2 is devoted to these estimates.

Theorem 2.1. *(Local existence) Assume that $m_0 \in H^1$, $n_0 \in H^1$ and $\inf\limits_{x \in [0,1]} n_0 > 0$, $\inf\limits_{x \in [0,1]} m_0 > 0$, and that*

$$\begin{cases} \frac{I_{wo}k_1}{\varepsilon_w k_0} + \frac{I_{wo}k_1}{\varepsilon_o k_0} \leq \frac{1}{4}, \\[2mm] \max\left\{ \frac{I_{wo}k_1}{k_0 \varepsilon_o} + E_{\varepsilon_o,1}, \frac{I_{wo}k_1}{k_0 \varepsilon_w} + E_{\varepsilon_w,1} \right\} \leq \frac{1}{2}, \end{cases}$$

where I_{wo} refers to the coefficient in (2.26), $k_0 = \min\left\{ \frac{\inf n_0}{e}, \frac{\inf m_0}{e} \right\}$ and $k_1 = \max\left\{ e \sup m_0, e \sup n_0 \right\}$, and

$$\begin{cases} E_{\varepsilon_w,1} = \frac{1}{\varepsilon_w}\left[\frac{10C}{(k_0)^2}\left(1 + \frac{10C}{k_0}\right)\frac{I_{wo}k_1}{2k_0} + \frac{10I_w I_{wo}k_1}{\varepsilon_w k_0} + \frac{20I_w I_{wo}k_1}{\varepsilon_w k_0} + \frac{20I_w I_{wo}k_1}{\varepsilon_w k_0} \right], \\[2mm] E_{\varepsilon_o,1} = \frac{1}{\varepsilon_o}\left[\frac{10C}{(k_0)^2}\left(1 + \frac{10C}{k_0}\right)\frac{I_{wo}k_1}{2k_0} + \frac{20(I_{wo})^2 k_1}{\varepsilon_o k_0} + \frac{10I_o I_{wo}k_1}{\varepsilon_o k_0} + \frac{20(I_{wo})^2 k_1}{\varepsilon_o k_0} \right], \end{cases}$$

where I_w, I_o are coefficients given by (2.26) and C is a positive constant depending on k_0, k_1 and some other known data but independent of ε_o and ε_w (see Step 2 in Appendix A for more details). Then there exists a positive constant T_0, such that the system (2.14) with initial-boundary conditions (2.15) and (2.16) has a unique solution (m, n, u_w, u_o) on $[0,1] \times [0, T_0]$ in the sense that

$(m,n) \in C([0,T_0]; H^1) \cap C^1([0,T_0]; L^2), \ (u_w, u_o) \in C([0,T_0]; H^2 \cap H_0^1),$

$\inf m > 0, \inf n > 0.$

Now we are in the position to state our second result on the almost global existence.

Theorem 2.2. *(Almost global existence) In addition to the assumptions of Theorem 2.1, for any given $T > 0$, if $K_1 < \min\left\{ \varepsilon_w \tilde{m}, \varepsilon_o \tilde{n} \right\}$, then the system (2.14) with initial-boundary conditions (2.15) and (2.16) has a unique solution (m, n, u_w, u_o) on $[0,1] \times [0,T]$ in the sense that*

$(m,n) \in C([0,T]; H^1) \cap C^1([0,T]; L^2), \ (u_w, u_o) \in C([0,T]; H^2 \cap H_0^1),$

where K_1 is given by (2.45).

Moreover, we have the following estimates:

$$\int_0^1 \left[(s_w)_x^2 + (s_o)_x^2 + (\rho_w)_x^2 + (\rho_o)_x^2 \right] dx \leq C(T),$$

$$\int_0^1 \left[(s_w)_t^2 + (s_o)_t^2 + (\rho_w)_t^2 + (\rho_o)_t^2 \right] dx \leq C(T),$$

for any $t \in [0,T]$.

Remark 2.3. The constraint $K_1 < \min\left\{\varepsilon_w \tilde{m}, \varepsilon_o \tilde{n}\right\}$ where K_1 is given by (2.45), implies smallness of initial data combined with assumption about sufficiently large viscosity ε_w and ε_o. More precisely, from the definition of K_1 for a fixed $T > 0$ we may choose m_0 and n_0 such that $K_1 \leq 2[g^2(\tilde{m} + \tilde{n})T + K_0]\exp(T)$ where $\varepsilon_w, \varepsilon_o$ are chosen sufficiently large to ensure that
(i) $\bar{a}\max\{\frac{1}{\varepsilon_w}, \frac{1}{\varepsilon_o}\} \leq 1$; (ii) $2[g^2(\tilde{m} + \tilde{n})T + K_0]\exp(T) < \min\left\{\varepsilon_w \tilde{m}, \varepsilon_o \tilde{n}\right\}$. Note that this constraint is only used to get the positive lower bound of m and n. See Corollary 2.3 for more details. Hence, the obtained estimates cannot be used to investigate the limit when $\varepsilon_w, \varepsilon_o \to 0$.

2.2. **Proof of Theorem 2.2.** Equipped with Theorem 2.1, we are going to prove Theorem 2.2. More precisely, let T^* denote the maximum time for the existence of solutions as in Theorem 2.1[1]. Then Theorem 2.1 implies that $T^* > 0$. To prove the almost global existence, it suffices to show that T^* is larger than the given T which can be taken as large as possible. For otherwise, i.e., $T^* \leq T$, it will lead to a contradiction based on the following estimates uniformly for t, i.e.,

$$\begin{cases} \|(m, n, s_w, s_o, \rho_w, \rho_o)(t)\|_{H^1} + \|(u_w, u_o)(t)\|_{H^2} \leq C(T), \\ \|(m_t, n_t, (s_w)_t, (s_o)_t, (\rho_w)_t, (\rho_o)_t)(t)\|_{L^2} \leq C(T), \\ \inf_{(x,t)\in Q_{T^*}} m(x,t) > 0, \quad \inf_{(x,t)\in Q_{T^*}} n(x,t) > 0, \end{cases} \tag{2.29}$$

for any $t \in [0, T^*)$, where $Q_{T^*} = [0,1] \times [0, T^*]$. In fact, (2.29) implies that T^* is not the maximum time for the existence which is the desired contradiction.

To get (2.29), we need the following lemmas. To simplify the proof, we let $C(T)$ denote a generic positive constant depending on the initial data and T. Moreover, for any given $T > 0$, $C(T) < \infty$. We let $t < T^* \leq T$ throughout the rest of this section, i.e., in Lemma 2.2–Corollary 2.5. Note that $C(T) \geq C(T^*)$ and $K_1 = K_1(T) \geq K_1(T^*)$ in Lemma 2.2.

(a) Energy estimate.

Lemma 2.1. *For any $t \in [0, T^*)$, it holds that*

$$E(t) + \int_0^t \int_0^1 (\varepsilon_w m u_{wx}^2 + \varepsilon_o n u_{ox}^2)\, dxdt$$
$$+ \int_0^t \int_0^1 \hat{k}(u_w - u_o)^2\, dxdt + \int_0^t \int_0^1 \hat{k}_w u_w^2\, dxdt + \int_0^t \int_0^1 \hat{k}_o u_o^2\, dxdt = E(0), \tag{2.30}$$

where $E(t)$ is given by

$$E(t) = C_w \int_0^1 m \int_{\tilde{\rho}_w}^{\rho_w} \frac{s - \tilde{\rho}_w}{s^2}\, ds\, dx + C_o \int_0^1 n \int_{\tilde{\rho}_o}^{\rho_o} \frac{s - \tilde{\rho}_o}{s^2}\, ds\, dx$$
$$+ \int_0^1 \left[P_c(\tilde{s}_w)s_w - \Phi(s_w)\right] dx + \int_0^1 \int_0^x g(n + m)\, dy\, dx, \tag{2.31}$$

where $\tilde{\rho}_w = \rho_w(\tilde{m}, \tilde{n}), \tilde{\rho}_o = \rho_o(\tilde{m}, \tilde{n})$, and $\tilde{s}_w = s_w(\tilde{m}, \tilde{n})$.

---

[1]It means that the solution exists on $[0, T^*)$ but not on $[0, T^*]$.

Proof. From the two momentum equations of $(2.14)_{3,4}$ we get after a multiplication, respectively, by u_o and u_w, followed by integration over $[0, 1]$, integration by parts and use of (2.15)

$$\int_0^1 (\varepsilon_w m u_{wx}^2 + \varepsilon_o n u_{ox}^2) \, dx + \int_0^1 \hat{k}(u_w - u_o)^2 \, dx + \int_0^1 \hat{k}_w u_w^2 \, dx + \int_0^1 \hat{k}_o u_o^2 \, dx$$

$$= (\int_0^1 n g u_o \, dx + \int_0^1 m g u_w \, dx) - \int_0^1 (s_o P_{ox} u_o + s_w P_{wx} u_w) \, dx := I_0 + I_1.$$

$$(2.32)$$

For I_0, integrating the two mass equations $(2.14)_{1,2}$ on $(0, x)$ for any given $x \in (0, 1)$, and using the boundary condition, we have

$$\begin{cases} \dfrac{d}{dt} \displaystyle\int_0^x n(y, t) \, dy = -n u_o(x, t), \\[3mm] \dfrac{d}{dt} \displaystyle\int_0^x m(y, t) \, dy = -m u_w(x, t). \end{cases}$$

Thus we have

$$I_0 = -\frac{d}{dt} \int_0^1 \int_0^x g(n + m) \, dy \, dx. \tag{2.33}$$

As to I_1, we observe that by using (1.4) and $(2.14)_{1,2}$ we can compute as follows

$$\int_0^1 s_o u_o P_{ox} \, dx = C_o \int_0^1 s_o u_o (\rho_o)_x \, dx = C_o \int_0^1 n u_o (\ln(\rho_o))_x \, dx$$

$$= -C_o \int_0^1 (n u_o)_x \ln(\rho_o) \, dx = C_o \int_0^1 n_t \ln(\rho_o) \, dx$$

$$= C_o \frac{d}{dt} \int_0^1 n \ln(\rho_o) \, dx - C_o \int_0^1 s_o (\rho_o)_t \, dx \tag{2.34}$$

$$= C_o \frac{d}{dt} \int_0^1 n \ln(\rho_o) \, dx - C_o \frac{d}{dt} \int_0^1 n \, dx + C_o \int_0^1 (s_o)_t \rho_o \, dx$$

$$= C_o \frac{d}{dt} \int_0^1 n \ln(\rho_o) \, dx + \int_0^1 (s_o)_t P_o \, dx + C_o \tilde{\rho}_{o0} \int_0^1 (s_o)_t \, dx,$$

and, by similar calculations

$$\int_0^1 s_w u_w P_{wx} \, dx = C_w \int_0^1 s_w u_w (\rho_w)_x \, dx = C_w \int_0^1 m u_w (\ln(\rho_w))_x \, dx$$

$$= -C_w \int_0^1 (m u_w)_x \ln(\rho_w) \, dx = C_w \int_0^1 m_t \ln(\rho_w) \, dx$$

$$= C_w \frac{d}{dt} \int_0^1 m \ln(\rho_w) \, dx - C_w \int_0^1 s_w (\rho_w)_t \, dx$$

$$= C_w \frac{d}{dt} \int_0^1 m \ln(\rho_w) \, dx - C_w \frac{d}{dt} \int_0^1 m \, dx + C_w \int_0^1 (s_w)_t \rho_w \, dx$$

$$= C_w \frac{d}{dt} \int_0^1 m \ln(\rho_w) \, dx + \int_0^1 (s_w)_t P_w \, dx + C_w \tilde{\rho}_{w0} \int_0^1 (s_w)_t \, dx.$$

$$(2.35)$$

Note that here we have also used (2.27). Consequently, using that $P_w = P_o - P_c$ and (1.3), we find from summing (2.34) and (2.35)

$$-I_1 = C_o \frac{d}{dt} \int_0^1 n \ln(\rho_o) \, dx + C_w \frac{d}{dt} \int_0^1 m \ln(\rho_w) \, dx - \int_0^1 s_{wt} P_c(s_w) \, dx$$
$$+ C_o \tilde{\rho}_{o0} \frac{d}{dt} \int_0^1 s_o \, dx + C_w \tilde{\rho}_{w0} \frac{d}{dt} \int_0^1 s_w \, dx.$$

That is,

$$-I_1 = C_o \frac{d}{dt} \int_0^1 n \ln(\rho_o) \, dx + C_w \frac{d}{dt} \int_0^1 m \ln(\rho_w) \, dx - \int_0^1 \Phi(s_w)_t \, dx$$
$$+ C_o \tilde{\rho}_{o0} \frac{d}{dt} \int_0^1 s_o \, dx + C_w \tilde{\rho}_{w0} \frac{d}{dt} \int_0^1 s_w \, dx, \qquad \Phi'(s_w) = P_c(s_w).$$

Moreover, we see that

$$\int_0^1 m \int_{\tilde{\rho}_w}^{\rho_w} \frac{s - \tilde{\rho}_w}{s^2} \, ds \, dx = \int_0^1 m \left[\ln(s) + \frac{\tilde{\rho}_w}{s} \right] \Big|_{\tilde{\rho}_w}^{\rho_w} dx$$
$$= \int_0^1 m \left[\ln(\rho_w) - \ln(\tilde{\rho}_w) + \frac{\tilde{\rho}_w}{\rho_w} - \frac{\tilde{\rho}_w}{\tilde{\rho}_w} \right] dx$$
$$= \int_0^1 m \ln(\rho_w) \, dx - \ln(\tilde{\rho}_w) \int_0^1 m \, dx + \tilde{\rho}_w \int_0^1 s_w \, dx - \int_0^1 m \, dx,$$
$$(2.36)$$

for some reference density $\tilde{\rho}_w > 0$. Hence, again by using (2.27)

$$C_w \frac{d}{dt} \int_0^1 m \ln(\rho_w) \, dx = C_w \frac{d}{dt} \int_0^1 m \int_{\tilde{\rho}_w}^{\rho_w} \frac{s - \tilde{\rho}_w}{s^2} \, ds \, dx - C_w \tilde{\rho}_w \frac{d}{dt} \int_0^1 s_w \, dx \quad (2.37)$$

and

$$C_o \frac{d}{dt} \int_0^1 n \ln(\rho_o) \, dx = C_o \frac{d}{dt} \int_0^1 n \int_{\tilde{\rho}_o}^{\rho_o} \frac{s - \tilde{\rho}_o}{s^2} \, ds \, dx - C_o \tilde{\rho}_o \frac{d}{dt} \int_0^1 s_o \, dx. \quad (2.38)$$

Thus, it follows that

$$- I_1 + \frac{d}{dt} \int_0^1 \Phi(s_w) \, dx$$
$$= C_o \frac{d}{dt} \int_0^1 n \ln(\rho_o) \, dx + C_w \frac{d}{dt} \int_0^1 m \ln(\rho_w) \, dx$$
$$+ C_o \tilde{\rho}_{o0} \frac{d}{dt} \int_0^1 s_o \, dx + C_w \tilde{\rho}_{w0} \frac{d}{dt} \int_0^1 s_w \, dx \qquad (2.39)$$
$$= C_w \frac{d}{dt} \int_0^1 m \int_{\tilde{\rho}_w}^{\rho_w} \frac{s - \tilde{\rho}_w}{s^2} \, ds \, dx + C_o \frac{d}{dt} \int_0^1 n \int_{\tilde{\rho}_o}^{\rho_o} \frac{s - \tilde{\rho}_o}{s^2} \, ds \, dx$$
$$+ C_o [\tilde{\rho}_{o0} - \tilde{\rho}_o] \frac{d}{dt} \int_0^1 s_o \, dx + C_w [\tilde{\rho}_{w0} - \tilde{\rho}_w] \frac{d}{dt} \int_0^1 s_w \, dx.$$

Note that in view of (1.4), the last line of (2.39) gives us

$$C_o [\tilde{\rho}_{o0} - \tilde{\rho}_o] \frac{d}{dt} \int_0^1 s_o \, dx + C_w [\tilde{\rho}_{w0} - \tilde{\rho}_w] \frac{d}{dt} \int_0^1 s_w \, dx$$
$$= -P_o(\tilde{\rho}_o) \frac{d}{dt} \int_0^1 s_o \, dx - P_w(\tilde{\rho}_w) \frac{d}{dt} \int_0^1 s_w \, dx$$

$$= -P_o(\tilde{\rho}_o)\frac{d}{dt}\int_0^1 s_o\,dx - P_o(\tilde{\rho}_o)\frac{d}{dt}\int_0^1 s_w\,dx + P_c(\tilde{s}_w)\frac{d}{dt}\int_0^1 s_w\,dx$$

$$= P_c(\tilde{s}_w)\frac{d}{dt}\int_0^1 s_w\,dx,$$

where $\tilde{\rho}_o$, $\tilde{\rho}_w$, and $\tilde{s}_w$ are related to each other by common masses $\tilde{m}, \tilde{n}$, i.e., we have that

(i) $\tilde{\rho}_o = \rho_o(\tilde{m}, \tilde{n})$, $\tilde{\rho}_w = \rho_w(\tilde{m}, \tilde{n})$, and $\tilde{s}_w = s_w(\tilde{m}, \tilde{n})$;

(ii) $P_o(\tilde{\rho}_o) = P_w(\tilde{\rho}_w) + P_c(\tilde{s}_w)$.

Hence, it follows from (2.39) that

$$-I_1 = C_w\frac{d}{dt}\int_0^1 m\int_{\tilde{\rho}_w}^{\rho_w}\frac{s - \tilde{\rho}_w}{s^2}\,ds\,dx + C_o\frac{d}{dt}\int_0^1 n\int_{\tilde{\rho}_o}^{\rho_o}\frac{s - \tilde{\rho}_o}{s^2}\,ds\,dx \tag{2.40}$$
$$+ \frac{d}{dt}\int_0^1 [P_c(\tilde{s}_w)s_w - \Phi(s_w)]\,dx.$$

Inserting (2.40) and (2.33) in (2.32) we get

$$C_w\frac{d}{dt}\int_0^1 m\int_{\tilde{\rho}_w}^{\rho_w}\frac{s - \tilde{\rho}_w}{s^2}\,ds\,dx + C_o\frac{d}{dt}\int_0^1 n\int_{\tilde{\rho}_o}^{\rho_o}\frac{s - \tilde{\rho}_o}{s^2}\,ds\,dx$$
$$+ \frac{d}{dt}\int_0^1 [P_c(\tilde{s}_w)s_w - \Phi(s_w)]\,dx + \frac{d}{dt}\int_0^1\int_0^x g(n + m)\,dy\,dx$$
$$+ \int_0^1 (\varepsilon_w mu_{wx}^2 + \varepsilon_o nu_{ox}^2)\,dx + \int_0^1 \hat{k}(u_w - u_o)^2\,dx + \int_0^1 \hat{k}_w u_w^2\,dx + \int_0^1 \hat{k}_o u_o^2\,dx$$
$$= 0. \tag{2.41}$$

We can rewrite (2.41) to be

$$\frac{d}{dt}E(t) + \int_0^1 (\varepsilon_w mu_{wx}^2 + \varepsilon_o nu_{ox}^2)\,dx + \int_0^1 \hat{k}(u_w - u_o)^2\,dx \tag{2.42}$$
$$+ \int_0^1 \hat{k}_w u_w^2\,dx + \int_0^1 \hat{k}_o u_o^2\,dx = 0$$

with $E(t)$ as given by (2.31). Hence, we conclude that (2.30) holds and where it is also clear from (2.23) that $\int_0^1 [P_c(\tilde{s}_w)s_w - \Phi(s_w)]\,dx \geq 0$. $\square$

Lemma 2.1 implies the following corollary.

Corollary 2.1. *For any $t \in [0, T^*)$, it holds that*

$$\int_0^t\int_0^1 (\varepsilon_w mu_{wx}^2 + \varepsilon_o nu_{ox}^2)\,dx\,ds + \int_0^t\int_0^1 \left[\hat{k}(u_w - u_o)^2 + \hat{k}_w u_w^2 + \hat{k}_o u_o^2\right]\,dx\,ds \leq K_0$$

where

$$K_0 = g(\tilde{n}_0 + \tilde{m}_0) + C_w\int_0^1 m_0\int_{\tilde{\rho}_w}^{\rho_{w0}}\frac{s - \tilde{\rho}_w}{s^2}\,ds\,dx + C_o\int_0^1 n_0\int_{\tilde{\rho}_o}^{\rho_{o0}}\frac{s - \tilde{\rho}_o}{s^2}\,ds\,dx$$
$$+ \int_0^1 \left[P_c(\tilde{s}_w)s_{w0} - \Phi(s_{w0})\right]\,dx,$$

where ρ_{i0} and s_{i0} refer to initial states.

Remark 2.4. The energy equality as expressed by (2.30) contains several dissipation terms on its left-hand-side. The three last terms reflect that there is a loss of energy (i.e., an energy transformation) through the three different friction terms, respectively, with coefficients $\hat{k}$, $\hat{k}_w$, and $\hat{k}_o$ and is a consequence of the viscous property of the involved fluids leading to drag force effects. Similarly, the internal viscosity of each fluid also creates resistance to move and is reflected by the two viscous terms, respectively, with coefficients ε_w and ε_o.

Remark 2.5. The energy $E(t)$ defined in (2.31) contains different terms each having a specific physical meaning. The use of compressible fluids implies that energy can be stored and released in the fluid due to pressure variations. An intuitive example of this is when there is influx of gas (oil) in a low reservoir layer where pressure is high. As this gas migrates towards a higher zone where pressure is lower, the gas (oil) will expand. That is, ρ_o decreases according to (1.4) and s_o increases since mass m_o is conserved and $m_o = s_o\rho_o$ and therfore typically will displace the surrounding water phase represented by s_w. This energy exchange is accounted for through the two first terms of $E(t)$. Capillary pressure $P_c = P_o - P_w$, accounts for the difference between the water and oil pressure P_w and P_o, and also acts as a driver (an additional pressure effect) for fluid motion. It naturally occurs in the energy functional $E(t)$ similar to the gravitational energy, see the last line of (2.31).

(b) More regularity estimates.

Lemma 2.2. *For any $t \in [0, T^*)$, it holds that*

$$\int_0^1 [\varepsilon_w \frac{m_x^2}{m} + \varepsilon_o \frac{n_x^2}{n}] \, dx \le K_1, \tag{2.43}$$

and

$$\int_0^t \int_0^1 s_o[(\rho_o^{1/2})_x]^2 \, dx \, ds + \int_0^t \int_0^1 s_w[(\rho_w^{1/2})_x]^2 \, dx \, ds$$
$$- \int_0^t \int_0^1 P_c'(s_w)s_{wx}^2 \, dx \, ds \le C(T), \tag{2.44}$$

where

$$K_1 =$$
$$\left[\int_0^1 [\varepsilon_w \frac{(m_0)_x^2}{m_0} + \varepsilon_o \frac{(n_0)_x^2}{n_0}] \, dx + g^2(\tilde{m} + \tilde{n})T + K_0 \right] \exp\left\{ \overline{a} \max\{\frac{1}{\varepsilon_w}, \frac{1}{\varepsilon_o}\}T \right\}, \tag{2.45}$$

and $\overline{a} = \max\{1 + I_{wo} + I_w, 1 + I_{wo} + I_o\}$.

Proof. Note that from (2.14)$_2$ we get the following reformulated equation after expanding the advective term and taking a derivative in space:

$$(m_x)_t + (m_x u_w)_x = -(m u_{wx})_x. \tag{2.46}$$

Note the appearance of the viscosity term on the RHS of (2.46). Combining (2.46) with (2.14)$_4$ we arrive at

$$(\varepsilon_w m_x)_t + (\varepsilon_w m_x u_w)_x = -\varepsilon_w(m u_{wx})_x = -s_w P_{wx} - \hat{k}_w u_w - \hat{k}(u_w - u_c) + mg.$$

This is the same as

$$\left[m(\varepsilon_w \frac{m_x}{m}) \right]_t + \left[m(\varepsilon_w \frac{m_x}{m})u_w \right]_x = -s_w P_{wx} - \hat{k}_w u_w - \hat{k}(u_w - u_c) + mg$$

or

$$[\varepsilon_w m w]_t + [\varepsilon_w m w u_w]_x = -s_w P_{wx} - \hat{k}_w u_w - \hat{k}(u_w - u_c) + mg$$

for

$$w = \frac{m_x}{m}$$

which clearly, by using $(2.14)_2$, is the same as

$$\varepsilon_w m w_t + \varepsilon_w m u_w w_x = -s_w P_{wx} - \hat{k}_w u_w - \hat{k}(u_w - u_c) + mg. \qquad (2.47)$$

Now, we test (2.47) with w and combine it with $(2.14)_2$ and (2.15) which leads us to

$$\frac{\varepsilon_w}{2} \frac{d}{dt} \int_0^1 m w^2 \, dx = -\int_0^1 s_w P_{wx} w \, dx - \int_0^1 \hat{k}_w u_w w \, dx$$
$$- \int_0^1 \hat{k}(u_w - u_c) w \, dx + \int_0^1 mgw \, dx \qquad (2.48)$$

Similarly, for the oil phase we obtain

$$\frac{\varepsilon_o}{2} \frac{d}{dt} \int_0^1 n v^2 \, dx = -\int_0^1 s_o P_{ox} v \, dx - \int_0^1 \hat{k}_o u_o v \, dx$$
$$+ \int_0^1 \hat{k}(u_w - u_o) v \, dx + \int_0^1 ngv \, dx \qquad (2.49)$$

with

$$v = \frac{n_x}{n}.$$

Next, we focus on the terms appearing on the RHS of (2.48):

$$-\int_0^1 s_w P_{wx} w \, dx = -\int_0^1 s_w P_{wx}\left(\frac{m_x}{m}\right) dx := J_{w,1}.$$

We note that

$$J_{w,1} = -\int_0^1 s_w P_{wx} \frac{m_x}{m} \, dx = -\int_0^1 s_{wx} P_{wx} \, dx - \int_0^1 s_w P_{wx} \frac{s_w \rho_{wx}}{m} \, dx$$
$$= -\int_0^1 s_{wx} P_{wx} \, dx - 4C_w \int_0^1 s_w [(\rho_w^{1/2})_x]^2 \, dx. \qquad (2.50)$$

Similarly, for $J_{o,1}$ associated with (2.49)

$$J_{o,1} = -\int_0^1 s_o P_{ox} \frac{n_x}{n} \, dx = -\int_0^1 s_{ox} P_{ox} \, dx - \int_0^1 s_o P_{ox} \frac{\alpha_o \rho_{ox}}{n} \, dx$$
$$= -\int_0^1 s_{ox} P_{ox} \, dx - 4C_o \int_0^1 s_o [(\rho_o^{1/2})_x]^2 \, dx. \qquad (2.51)$$

To conclude, we see that by summing (2.48) and (2.49), using (2.50), and (2.51), we get

$$\frac{1}{2} \frac{d}{dt} \int_0^1 [\varepsilon_w m w^2 + \varepsilon_o n v^2] \, dx + 4C_o \int_0^1 s_o [(\rho_o^{1/2})_x]^2 \, dx + 4C_w \int_0^1 s_w [(\rho_w^{1/2})_x]^2 \, dx$$
$$= -\int_0^1 s_{wx} P_{wx} \, dx - \int_0^1 s_{ox} P_{ox} \, dx + \int_0^1 mgw \, dx + \int_0^1 ngv \, dx$$

$$-\int_0^1 \hat{k}(u_w - u_o)w\,dx + \int_0^1 \hat{k}(u_w - u_o)v\,dx - \int_0^1 \hat{k}_w u_w w\,dx - \int_0^1 \hat{k}_o u_o v\,dx$$

$$= \int_0^1 s_{wx}^2 P_c'(s_w)\,dx + \int_0^1 mgw\,dx + \int_0^1 ngv\,dx$$

$$-\int_0^1 \hat{k}(u_w - u_o)w\,dx + \int_0^1 \hat{k}(u_w - u_o)v\,dx - \int_0^1 \hat{k}_w u_w w\,dx - \int_0^1 \hat{k}_o u_o v\,dx$$

$$(2.52)$$

where we again have used $P_c(s_w) = P_o - P_w$ and $s_w + s_o = 1$. That is,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [\varepsilon_w m w^2 + \varepsilon_o n v^2]\,dx$$

$$+ 4C_o \int_0^1 s_o[(\rho_o^{1/2})_x]^2\,dx + 4C_w \int_0^1 s_w[(\rho_w^{1/2})_x]^2\,dx - \int_0^1 s_{wx}^2 P_c'(s_w)\,dx$$

$$= \int_0^1 g(m_x + n_x)\,dx - \int_0^1 \hat{k}(u_w - u_o)(\frac{m_x}{m})\,dx + \int_0^1 \hat{k}(u_w - u_o)(\frac{n_x}{n})\,dx$$

$$- \int_0^1 \hat{k}_w u_w(\frac{m_x}{m})\,dx - \int_0^1 \hat{k}_o u_o(\frac{n_x}{n})\,dx$$

$$:= K_{ow0} + K_{w1} + K_{o1} + K_{w2} + K_{o2}.$$

$$(2.53)$$

For K_{ow0}, we use Cauchy inequality and the mass equations and have

$$K_{ow0} = g\int_0^1 \frac{m_x}{\sqrt{m}}\sqrt{m}\,dx + g\int_0^1 \frac{n_x}{\sqrt{n}}\sqrt{n}\,dx$$

$$\leq \int_0^1 (\frac{m_x^2}{m} + \frac{n_x^2}{n})\,dx + \frac{1}{4}g^2\int_0^1 (m+n)\,dx = \int_0^1 (mw^2 + nv^2)\,dx + \frac{1}{4}g^2(\tilde{m} + \tilde{n}).$$

$$(2.54)$$

In the following we make use of the functional form of the interaction coefficients $\hat{k}_w$, $\hat{k}_o$, and $\hat{k}$ as expressed in (2.26).

$$K_{w1} = -\int_0^1 \hat{k}(u_w - u_o)(\frac{m_x}{m})\,dx$$

$$\leq \frac{1}{4}\int_0^1 \hat{k}(u_w - u_o)^2\,dx + \int_0^1 \hat{k}\frac{m_x^2}{m^2}\,dx$$

$$\leq \frac{1}{4}\int_0^1 \hat{k}(u_w - u_o)^2\,dx + I_{wo}\int_0^1 \frac{m_x^2}{m}\,dx = \frac{1}{4}\int_0^1 \hat{k}(u_w - u_o)^2\,dx + I_{wo}\int_0^1 mw^2\,dx$$

$$(2.55)$$

and

$$K_{o1} = \int_0^1 \hat{k}(u_w - u_o)(\frac{n_x}{n})\,dx$$

$$\leq \frac{1}{4}\int_0^1 \hat{k}(u_w - u_o)^2\,dx + \int_0^1 \hat{k}\frac{n_x^2}{n^2}\,dx$$

$$\leq \frac{1}{4}\int_0^1 \hat{k}(u_w - u_o)^2\,dx + I_{wo}\int_0^1 \frac{n_x^2}{n}\,dx = \frac{1}{4}\int_0^1 \hat{k}(u_w - u_o)^2\,dx + I_{wo}\int_0^1 nv^2\,dx.$$

$$(2.56)$$

Furthermore,

$$
\begin{aligned}
K_{w2} &= -\int_0^1 \hat{k}_w u_w \left(\frac{m_x}{m}\right) dx \\
&\leq \frac{1}{4} \int_0^1 \hat{k}_w u_w^2 dx + \int_0^1 \hat{k}_w \frac{m_x^2}{m^2} dx \\
&= \frac{1}{4} \int_0^1 \hat{k}_w u_w^2 dx + I_w \int_0^1 \frac{m_x^2}{m} dx = \frac{1}{4} \int_0^1 \hat{k}_w u_w^2 dx + I_w \int_0^1 m w^2 dx
\end{aligned}
\tag{2.57}
$$

and

$$
\begin{aligned}
K_{o2} &= -\int_0^1 \hat{k}_o u_o \left(\frac{n_x}{n}\right) dx \\
&\leq \frac{1}{4} \int_0^1 \hat{k}_o u_o^2 dx + \int_0^1 \hat{k}_o \frac{n_x^2}{n^2} dx \\
&= \frac{1}{4} \int_0^1 \hat{k}_o u_o^2 dx + I_o \int_0^1 \frac{n_x^2}{n} dx = \frac{1}{4} \int_0^1 \hat{k}_o u_o^2 dx + I_o \int_0^1 n v^2 dx.
\end{aligned}
\tag{2.58}
$$

Putting the estimates (2.54)–(2.58) into (2.53), integrating the result over $(0, t)$, and using Corollary 2.1, we have

$$
\begin{aligned}
&\int_0^1 [\varepsilon_w m w^2 + \varepsilon_o n v^2]\, dx + 4 C_o \int_0^t \int_0^1 s_o [(\rho_o^{1/2})_x]^2\, dx\, ds \\
&+ 4 C_w \int_0^t \int_0^1 s_w [(\rho_w^{1/2})_x]^2\, dx\, ds - \int_0^t \int_0^1 P_c'(s_w) s_{wx}^2\, dx\, ds \\
&\leq \int_0^1 [\varepsilon_w m_0 w_0^2 + \varepsilon_o n_0 v_0^2]\, dx + \bar{a} \int_0^t \int_0^1 (m w^2 + n v^2) dx\, ds + \frac{1}{4} g^2 (\tilde{m} + \tilde{n}) t \\
&+ \frac{1}{2} \int_0^t \int_0^1 \hat{k}(u_w - u_o)^2 dx\, ds + \frac{1}{4} \int_0^t \int_0^1 \hat{k}_w u_w^2 dx\, ds + \frac{1}{4} \int_0^t \int_0^1 \hat{k}_o u_o^2 dx\, ds \\
&\leq \int_0^1 [\varepsilon_w m_0 w_0^2 + \varepsilon_o n_0 v_0^2]\, dx + \bar{a} \max\left\{\frac{1}{\varepsilon_w}, \frac{1}{\varepsilon_o}\right\} \int_0^t \int_0^1 (\varepsilon_w m w^2 + \varepsilon_o n v^2) dx\, ds \\
&+ g^2 (\tilde{m} + \tilde{n}) T + K_0,
\end{aligned}
\tag{2.59}
$$

where $\bar{a} = \max\{1 + I_{wo} + I_w, 1 + I_{wo} + I_o\}$.

Using Gronwall's inequality and (2.59), we get (2.43). (2.44) is given by (2.43) and (2.59). □

In view of Lemma 2.2 it follows that $\sqrt{m}, \sqrt{n} \in H^1(0,1)$. Combined with the Sobolev inequality $H^1(0,1) \hookrightarrow C([0,1])$ we have the following corollary.

Corollary 2.2. *It holds that*

$$
\begin{cases}
m(x,t) + n(x,t) \leq C(T), \\
\int_0^1 (m_x^2 + n_x^2)\, dx \leq C(T),
\end{cases}
\tag{2.60}
$$

for any $(x,t) \in [0,1] \times [0, T^*]$.

(c) Upper and lower bounds of density and related quantities.

Lemma 2.3. *It holds that*

$$\begin{cases} n \le \rho_o \le C(T), \\ m \le \rho_w \le C(T), \end{cases} \tag{2.61}$$

for any $(x,t) \in [0,1] \times [0,T^*)$.

Proof. Since

$$\rho_o = \frac{C_w}{C_o}\rho_w - \frac{C_w}{C_o}\tilde{\rho}_{w0} + \tilde{\rho}_{o0} + \frac{1}{C_o}P_c(s_w), \tag{2.62}$$

we have

$$\rho_o = (s_o + s_w)\rho_o = n + s_w\left[\frac{C_w}{C_o}\rho_w - \frac{C_w}{C_o}\tilde{\rho}_{w0} + \tilde{\rho}_{o0} + \frac{1}{C_o}P_c(s_w)\right], \tag{2.63}$$

$$\rho_w = (s_w + s_o)\rho_w = m + s_o\left[\frac{C_o}{C_w}\rho_o + \tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0} - \frac{1}{C_w}P_c(s_w)\right]. \tag{2.64}$$

Armed with the upper bounds of m and n from Corollary 2.2, we get the upper bounds of ρ_o and ρ_w from (2.63) and (2.64). Note that the term ρ_i appearing on the right-hand-side is grouped with the corresponding s_i which gives either m or n. In addition, we make use of the uniform bound on $P_c(s_w)$ given by (2.24). The lower bounds can be derived from the definitions $n = s_o\rho_o$ and $m = s_w\rho_w$ combined with (1.3). $\qquad\square$

Corollary 2.3. *The following uniform lower bound holds*

$$m, n \ge \frac{1}{C}, \qquad \text{for any } (x,t) \in [0,1] \times [0,T^*), \tag{2.65}$$

subject to the constraint that $K_1 < \min\{\varepsilon_w\tilde{m}, \varepsilon_o\tilde{n}\}$ where K_1 is given by (2.45).

Proof.

$$|m(x,t) - \tilde{m}| = \left|\int_0^1 [m(x,t) - m(y,t)]\,dy\right| = \left|\int_0^1\int_y^x m_\xi(\xi,t)\,d\xi\,dy\right|$$

$$\le \int_0^1 |m_x(x,t)|\,dx = \int_0^1 \sqrt{m(x,t)}\frac{|m_x(x,t)|}{\sqrt{m(x,t)}}\,dx$$

$$\le \left(\int_0^1 m(x,t)\,dx\right)^{\frac{1}{2}}\left(\int_0^1 \frac{|m_x(x,t)|^2}{m}\,dx\right)^{\frac{1}{2}} \le \left(\frac{\tilde{m}K_1}{\varepsilon_w}\right)^{\frac{1}{2}}.$$

Letting $\left(\frac{\tilde{m}K_1}{\varepsilon_w}\right)^{\frac{1}{2}} < \tilde{m}$, i.e., $K_1 < \varepsilon_w\tilde{m}$, we have

$$m \ge \frac{1}{C}$$

on $[0,1] \times [0,T^*)$ for some positive constant C. Similarly, letting $\left(\frac{\tilde{n}K_1}{\varepsilon_o}\right)^{\frac{1}{2}} < \tilde{n}$, i.e., $K_1 < \varepsilon_o\tilde{n}$, we get the positive lower bound of n. $\qquad\square$

(d) Higher-order estimates.

Lemma 2.4. *The following estimate holds*

$$\int_0^1 \left[(s_w)_x^2 + (s_o)_x^2 + (\rho_w)_x^2 + (\rho_o)_x^2 \right] dx \leq C(T) \tag{2.66}$$

for any $t \in [0, T^*)$.

Proof. Differentiating (2.64) with respect to x, we have

$$\begin{aligned}
(\rho_w)_x =& m_x + \frac{C_o}{C_w}(s_o \rho_o)_x + \tilde{\rho}_{w0}(s_o)_x - \frac{C_o}{C_w}\tilde{\rho}_{o0}(s_o)_x \\
& - \frac{1}{C_w}(s_o)_x P_c(s_w) - \frac{1}{C_w}s_o P_c'(s_w)(s_w)_x \\
=& m_x + \frac{C_o}{C_w}n_x - (\tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0})(s_w)_x \\
& + \left[\frac{1}{C_w}P_c(s_w) - \frac{1}{C_w}s_o P_c'(s_w) \right](s_w)_x,
\end{aligned} \tag{2.67}$$

where we have used $s_o + s_w = 1$ such that $(s_o)_x = -(s_w)_x$.

With (2.67), we proceed to estimate $(s_w)_x$.

$$\begin{aligned}
(s_w)_x =& (\frac{m}{\rho_w})_x = \frac{m_x}{\rho_w} - \frac{s_w(\rho_w)_x}{\rho_w} \\
=& \frac{m_x}{\rho_w} - \frac{s_w}{\rho_w}\left[m_x + \frac{C_o}{C_w}n_x \right] + \frac{s_w}{\rho_w}(\tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0})(s_w)_x \\
& - \frac{s_w}{\rho_w}\left[\frac{1}{C_w}P_c(s_w) - \frac{1}{C_w}s_o P_c'(s_w) \right](s_w)_x,
\end{aligned}$$

which implies that

$$\begin{aligned}
(s_w)_x &\left[\rho_w - s_w(\tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0}) + s_w\left[\frac{1}{C_w}P_c(s_w) - \frac{1}{C_w}s_o P_c'(s_w) \right] \right] \\
=& m_x - s_w(m_x + \frac{C_o}{C_w}n_x).
\end{aligned} \tag{2.68}$$

By (2.62), we have

$$\rho_w = \frac{C_o}{C_w}\rho_o + \tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0} - \frac{1}{C_w}P_c(s_w).$$

Substituting this identity and that $1 - s_w = s_o$ into (2.68), we have

$$\begin{aligned}
(s_w)_x =& (s_o m_x - s_w \frac{C_o}{C_w}n_x)\left[\rho_w - s_w(\tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0}) \right. \\
& \left. + s_w\left[\frac{1}{C_w}P_c(s_w) - \frac{1}{C_w}s_o P_c'(s_w) \right] \right]^{-1}.
\end{aligned} \tag{2.69}$$

Since

$$\rho_w = \frac{C_o}{C_w}\rho_o + \tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0} - \frac{1}{C_w}P_c(s_w), \tag{2.70}$$

we have

$$\begin{aligned}
\rho_w &- s_w(\tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0}) + s_w\left[\frac{1}{C_w}P_c(s_w) - \frac{1}{C_w}s_o P_c'(s_w) \right] \\
&= \frac{C_o}{C_w}\rho_o + s_o\left[\tilde{\rho}_{w0} - \frac{C_o}{C_w}\tilde{\rho}_{o0} - \frac{1}{C_w}P_c(s_w) \right] - \frac{1}{C_w}s_w s_o P_c'(s_w).
\end{aligned} \tag{2.71}$$

Combining (2.71) with (2.69), (2.61), (2.65), and the assumptions (2.24) and (2.25), allow us to conclude that

$$
\begin{aligned}
\int_0^1 (s_w)_x^2\, dx &\leq C \int_0^1 \frac{1}{\rho_o^2}(m_x^2 + n_x^2)\, dx \\
&\leq C \int_0^1 \frac{s_o^2}{n^2}(m_x^2 + n_x^2)\, dx \leq C(T).
\end{aligned}
\tag{2.72}
$$

This combined with (2.67) and (2.60) gives

$$
\int_0^1 (\rho_w)_x^2\, dx \leq C(T).
\tag{2.73}
$$

By virtue of (2.70), the fact that $(s_w)_x = -(s_o)_x$, (2.72), and (2.73), we get the estimates of $(s_o)_x$ and $(\rho_o)_x$. $\qquad\square$

Lemma 2.5. *For any* $t \in [0, T^*)$*, it holds that*

$$
\int_0^1 \left[(u_w)_x^2 + (u_o)_x^2\right] dx \leq C(T).
\tag{2.74}
$$

Proof. By virtue of (2.32), we have

$$
\begin{aligned}
&\int_0^1 (\varepsilon_w m u_{wx}^2 + \varepsilon_o n u_{ox}^2)\, dx + \int_0^1 \left[\hat{k}(u_w - u_o)^2 + \hat{k}_w u_w^2 + \hat{k}_o u_o^2\right] dx \\
&= -\left(\int_0^1 n g u_o\, dx + \int_0^1 m g u_w\, dx\right) - \int_0^1 (s_o P_{ox} u_o + s_w P_{wx} u_w)\, dx \\
&\leq C(\|u_o\|_{L^\infty} + \|u_w\|_{L^\infty}) + C\|(\rho_o)_x\|_{L^2}\|u_o\|_{L^2} + C\|(\rho_w)_x\|_{L^2}\|u_w\|_{L^2} \\
&\leq C(T)\left(\|(u_o)_x\|_{L^2} + \|(u_w)_x\|_{L^2}\right) \leq C(T)\left(\|\sqrt{n}(u_o)_x\|_{L^2} + \|\sqrt{m}(u_w)_x\|_{L^2}\right) \\
&\leq \frac{1}{2}\int_0^1 (\varepsilon_w m u_{wx}^2 + \varepsilon_o n u_{ox}^2)\, dx + C(T),
\end{aligned}
\tag{2.75}
$$

where we use $\int_0^1 m\, dx = \tilde{m}$, $\int_0^1 n\, dx = \tilde{n}$ and Hölder inequality in the first inequality, and use the inequality $\|u_i\|_\infty \leq C\|u_{ix}\|_{L^2}$, (2.66), (2.65), and Cauchy inequality in the rest.

Using (2.65) again, and (2.75), we get

$$
\int_0^1 (u_{wx}^2 + u_{ox}^2)\, dx \leq C(T).
\tag{2.76}
$$

$$\qquad\square$$

Corollary 2.4. *For any* $t \in [0, T^*)$*, it holds that*

$$
\int_0^1 \left[(u_w)_{xx}^2 + (u_o)_{xx}^2\right] dx \leq C(T).
\tag{2.77}
$$

Proof. From the equation of u_w, we have

$$
\varepsilon_w m (u_w)_{xx} = -\varepsilon_w m_x u_{wx} + s_w P_{wx} + \hat{k}_w u_w + \hat{k}(u_w - u_o) + mg,
$$

which together with (2.65), (2.60), (2.66), and (2.74) gives

$$
\int_0^1 (u_w)_{xx}^2 \, dx \leq C(T) \int_0^1 m_x^2 u_{wx}^2 \, dx + C(T)
$$

$$
\leq C(T) \|u_{wx}^2\|_{L^\infty} \int_0^1 m_x^2 \, dx + C(T)
$$

$$
\leq C(T) \left(\|(u_w)_x\|_{L^2}^2 + \int_0^1 |(u_w)_x (u_w)_{xx}| \, dx \right) + C(T)
$$

$$
\leq \frac{1}{2} \int_0^1 (u_w)_{xx}^2 \, dx + C(T),
$$

where we use $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$ in the third inequality, and use $\int ab \, dx \leq C \int a^2 \, dx + \varepsilon \int b^2 \, dx$ with appropriate choice of ε in the last one. This implies

$$
\int_0^1 (u_w)_{xx}^2 \, dx \leq C(T).
$$

Similarly, we get the estimate of $(u_o)_{xx}$. $\qquad\qquad\qquad\qquad\square$

Corollary 2.5. *For any $t \in [0, T^*)$, it holds that*

$$
\int_0^1 \left[m_t^2 + n_t^2 + (s_w)_t^2 + (s_o)_t^2 + (\rho_w)_t^2 + (\rho_o)_t^2 \right] dx \leq C(T). \tag{2.78}
$$

Proof. By using the equation of m, Cauchy inequality, (2.60), and (2.74), we have

$$
\int_0^1 m_t^2 \, dx \leq 2 \int_0^1 \left[m^2 (u_w)_x^2 + m_x^2 u_w^2 \right] dx
$$

$$
\leq 2\|m\|_{L^\infty}^2 \int_0^1 (u_w)_x^2 \, dx + 2\|u_w\|_{L^\infty}^2 \int_0^1 m_x^2 \, dx \tag{2.79}
$$

$$
\leq C(T).
$$

Similarly we get

$$
\int_0^1 n_t^2 \, dx \leq C(T). \tag{2.80}
$$

We consider (2.69) with ∂_x replaced by ∂_t, and use similar analysis as (2.72). Then

$$
\int_0^1 (s_w)_t^2 \, dx \leq C \int_0^1 \frac{1}{\rho_o^2} (m_t^2 + n_t^2) \, dx
$$

$$
\leq C \int_0^1 \frac{s_o^2}{n^2} (m_t^2 + n_t^2) \, dx \leq C(T), \tag{2.81}
$$

where we use (2.65), (2.79), and (2.80). Since $(s_o)_t = -(s_w)_t$, we get

$$
\int_0^1 (s_o)_t^2 \, dx \leq C(T).
$$

By virtue of (2.67) where ∂_x is replaced by ∂_t, (2.79), (2.80), and (2.81), we have

$$
\int_0^1 (\rho_w)_t^2 \, dx \leq C(T).
$$

This combined with (2.70) and (2.81) gives

$$\int_0^1 (\rho_o)_t^2 \, dx \le C(T).$$

□

With the above estimates, we get (2.29). Thus the proof of Theorem 2.2 is complete.

3. **Numerical results.** The main objective of this section is to carry out some testing of the numerical schemes presented in Appendix D, respectively, for the compressible and incompressible version of (1.1). First, we want to test general stability properties. Second, we seek some insight into the role played by using Darcy velocity $U_i = s_i u_i$ versus interstitial fluid velocity u_i ($i = w, o$) in the viscous terms. In other words, we modify (4.145)$_{3,4}$ and use the following momentum equations for the compressible case

$$
\begin{aligned}
s_o(P_w + P_c)_x &= -(\hat{k}_o + \hat{k})u_o + \hat{k}u_w + ng + \varepsilon_o(\rho_o U_{ox})_x \\
s_w(P_w)_x &= -(\hat{k}_w + \hat{k})u_w + \hat{k}u_o + mg + \varepsilon_w(\rho_w U_{wx})_x,
\end{aligned}
\tag{3.82}
$$

and modify (4.165)$_{3,4}$ as follows for the incompressible case

$$
\begin{aligned}
s_o(P_w + P_c)_x &= -(\hat{k}_o + \hat{k})u_o + \hat{k}u_w + ng + (\varepsilon_o\rho_o)U_{oxx} \\
s_w(P_w)_x &= -(\hat{k}_w + \hat{k})u_w + \hat{k}u_o + mg + (\varepsilon_w\rho_w)U_{wxx}.
\end{aligned}
\tag{3.83}
$$

Third, we also test the behavior of the scheme as the coefficients $\varepsilon_w, \varepsilon_o$ are varied to see what kind of impact this term will have on the solution. This also allows us to get some understanding of whether the viscous model seems to converge to the inviscid version obtained by letting $\varepsilon_w, \varepsilon_o \to 0$. We conduct numerical tests similar to those reported in [9]. We refer to Remark 1.2 for more details regarding the model they solve. Most importantly, the viscous term in their model depends on the Darcy velocity U_w, U_o. We apply the scheme for the incompressible model for these investigations, see Section 3.1. However, in Section 3.2 we also include examples where we use the scheme derived for the compressible model (see Appendix D) and do some comparison with the results from the incompressible model. The following input data are chosen for the numerical examples.

We choose parameters as specified in Table 1. In particular, when combined with the relations (4.132) it gives rise to a fractional flow function given by

$$f_w = \frac{k_w/\mu_w}{k_w/\mu_w + k_o/\mu_o} = \frac{s_w^2}{s_w^2 + (1 - s_w)^2} = f_w(s_w).$$

The function is illustrated in Fig. 1 (left figure). The initial data of water saturation is set to be as in [9]:

$$s_w(x, t = 0) = \begin{cases} 0.8 & 0 \le x \le 2, \\ 0.8 \exp(-150(x-2)^2) & 2 < x \le 100. \end{cases} \tag{3.84}$$

A horizontal reservoir layer is considered in this case and porosity is also assumed to be 1. The whole test is a 10-day flooding process with a constant interstitial water injection rate at left boundary, $Q = 8.004 m^3/d$. The relevant boundary values are: $s_w(x = 0, t) = 0.8$ and $u_w(x = 0, t) = 8.004 m/d$. In addition, 2001 grids are used to simulate this displacement. Water and oil have the same properties such as density and viscosity. Fluid-fluid interaction effect is ignored here by setting $I = 0$ in the

Parameter	Dimensional Value	Parameter	Dimensional Value
L	100 m	I_w	1.5
ϕ	1	I_o	1.5
A	1 m^2	I	0 (Pa·s)$^{-1}$
$\tilde{\rho}_{w0}$	1 g/cm^3	α	0
$\tilde{\rho}_{o0}$	1 g/cm^3	β	0
C_w	10^6 m^2/s^2	ε_w	$10^7, 10^6, 10^5, 10^4, 10^3, 10^2$ cP
C_o	$5 \cdot 10^5$ m^2/s^2	ε_o	$10^7, 10^6, 10^5, 10^4, 10^3, 10^2$ cP
μ_w	1 cP	K	1000 mD
μ_o	1 cP	$k_{rw}^{max} = 1/I_w$	0.667
Q	8.004 m^3/day	$k_{ro}^{max} = 1/I_o$	0.667
P_{wL}	10^6 Pa	T	10 days
N_x	2001	$\triangle t$	8640 s

TABLE 1. Input parameters of reservoir and fluid properties used for for the below simulations. Note that P_{wL} is the boundary pressure at left for the incompressible model whereas for the compressible model it represents the initial pressure distribution.
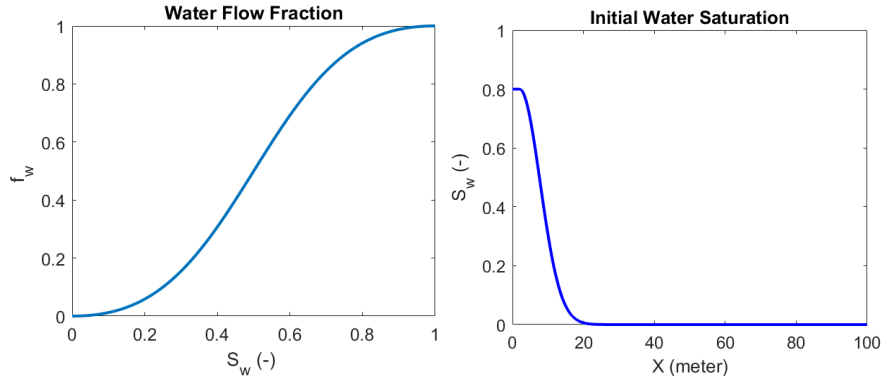

FIGURE 1. Water fractional flow function $\hat{f}_w(s_w)$ as given by (4.127) for the incompressible model obtained by using the parameters specified in Table 1 (left figure) and initial water saturation (3.84) profile (right), both similar to that used in [9].

correlations (4.132). The corresponding initial water saturation profile is shown in Fig. 1 (right figure).

3.1. The incompressible model.

Case 1. First, we want to compare the numerical results obtained by using the scheme from Appendix D (incompressible variant) and compare with similar results presented in [9], which are based on the model (1.13). We also mimic their scheme by slightly modifying the scheme prescribed in Appendix D (incompressible variant) where $s_i u_{ix}$ is replaced by U_{ix} in the viscous term, as described by (3.83). Results are illustrated in Fig. 2. We show water saturation profiles after 10 days flooding with, respectively, $\varepsilon_w = \varepsilon_o = 10^7$ and $\varepsilon_w = \varepsilon_o = 10^6$ (upper row) and compare with the corresponding results from Coclite et al. [9] (lower row). Main observations are:
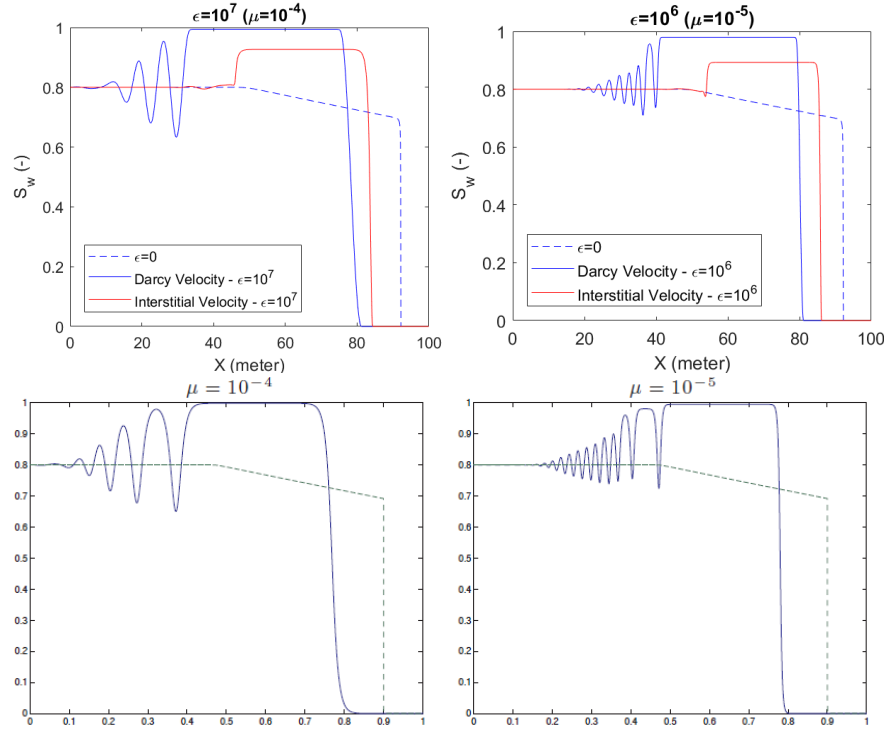
FIGURE 2.   **Upper row**: Results produced by the discrete scheme
described in Appendix D (incompressible model). Three kinds of
curves are plotted including the case without viscous effect, i.e.,
$\varepsilon_w = \varepsilon_o = 0$, the one based on using Darcy velocity U_i $(i = w, o)$ in
the viscous term, and the one with interstitial velocity u_i $(i = w, o)$
in the viscous term. The left figure shows results with $\varepsilon = \varepsilon_w = \varepsilon_o = 10^7$ whereas the right figure shows results with $\varepsilon = \varepsilon_w = \varepsilon_o = 10^6$. **Lower row**: The results of two corresponding cases
with $\varepsilon_w = \varepsilon_o = 10^7$ and $\varepsilon_w = \varepsilon_o = 10^6$ after a dimensionless
time, 0.65, produced by the numerical scheme described in [9] to
solve the model (1.13). From these computations we see that the
solution is sensitive to whether the interstitial velocity u_i or the
Darcy velocity U_i appear in the viscous term. In particular, the use
of Darcy velocity seems to generate considerably more oscillatory
behavior behind the "water bank" formed at the front.

(i) A new "water bank" is formed behind the front of the water as a result of the
viscous terms. This is a local effect restricted to the region right behind the water
front where large gradients in velocity are present;

(ii) Internal viscous forces slow down the transport effect, especially at the satura-
tion front where velocity involves dramatic changes. Right behind the water bank,
the model with Darcy velocity involved in the viscous term tends to develop oscil-
latory behavior;

(iii) The solution shows a clear sensitivity to the magnitude of $\varepsilon_o, \varepsilon_w$ (i.e., 10^7
versus 10^6) for the scheme based on Darcy velocity U_i in the viscous terms. The
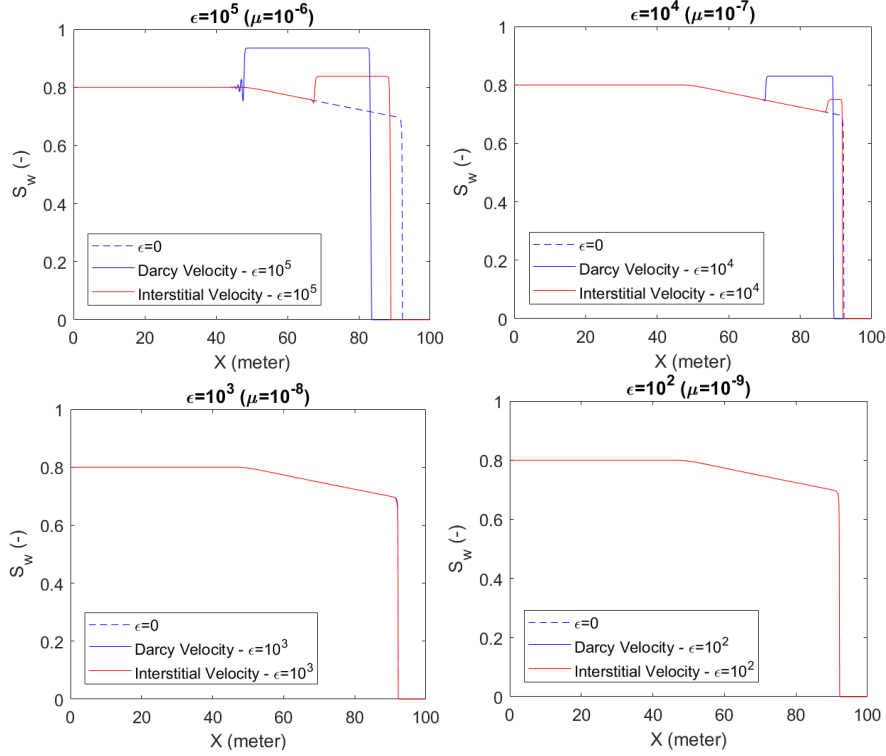
FIGURE 3.   Simulation results with smaller viscous parameters after 10 days of water flooding. Three kinds of curves are compared: zero viscous effect, Darcy velocity U_i in viscous term and interstitial velocity u_i in viscous term. It shows that the viscous constant water level gradually vanishes when ε is as low as 10^3 and 10^2.

scheme with viscous term based on interstitial velocity u_i shows less sensitivity to this change in $\varepsilon_o, \varepsilon_w$.

The classical Buckley-Leverett model solution (i.e., $\varepsilon_w = \varepsilon_o = 0$) is composed of a sharp water front followed by a rare-faction wave which is due to a viscous effect associated with resistance forces between fluid (water and oil) and walls of the pore space. The new water bank is a consequence of internal viscosity within the fluid felt at the region behind the water front. The difference between solution when viscous term is based on Darcy velocity U_i versus solution when viscous term is based on interstitial velocity u_i, can be naturally understood in light of the expansion

$$\partial_x(\partial_x(s_i u_i)) = \partial_x(s_i \partial_x(u_i)) + \partial_x(u_i \partial_x(s_i)).$$

Clearly, the viscous term based on $U_i = s_i u_i$ gives rise to an additional term that naturally can be linked to the observed difference between the two schemes used to produce solutions in Fig. 2. It should be noted that Brinkman equation was developed empirically for single phase flow and afterwards has been extended to the multiphase setting in a heuristic manner. As noted in Remark 1.3 there seems to be an ongoing discussion in the literature whether to base the viscous term on Darcy velocity or interstitial velocity. Finally, in Fig. 3 is shown the result for the two schemes as $\varepsilon_w, \varepsilon_o$ get smaller. Both schemes seem to reflect convergence toward
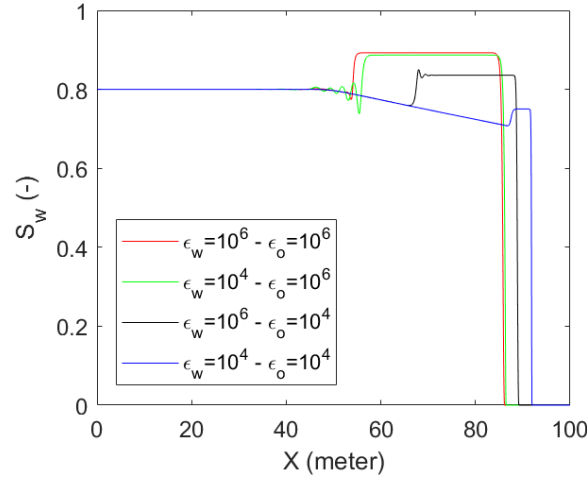
FIGURE 4.   The results after 10 days with initial data are shown in Fig. 1 with interstitial velocity in viscous term. Four curves are compared: the one with large values of ε_w and ε_o, 10^6; the second one with large ε_o, 10^6 and small ε_w, 10^4; the third one with large ε_w, 10^6 and small ε_o, 10^4 and the last one with small values of ε_w and ε_o, 10^4. It shows that the displaced oil influences significantly on the constant level of water which displaces oil.

the solution of the inviscid model (i.e., $\varepsilon_w = \varepsilon_o = 0$) with a considerably faster convergence produced by the scheme with viscous term based on interstitial fluid u_i.

Case 2. In Fig. 4 we show simulation results when we vary the internal relation between ε_w and ε_o. It is intuitively understandable that oil viscous effects can have a strong impact on the constant water level right behind the water front. Apparently, the same change of magnitude of water viscous parameter ε_w from 10^4 to 10^6 with a constant ε_o has a dramatic effect when ε_o is small (i.e., 10^4) whereas the effect is rather small when ε_o is large (i.e., 10^6). We refer to Fig. 4 for simulation results.

Case 3. Now, we move to another case which has a different initial condition but still with the same injection rate of water, $8.004 m^3/d$, as interstitial velocity at left boundary. The initial water saturation is illustrated in Fig. 5. Numerical results are shown in Fig. 6 where we compare the scheme based on interstitial viscosity (right figure) with the simulation result reported in [9] (left figure). In particular, it seems that the numerical solution based on using Darcy velocity produces an unphysical water saturation value near the left boundary. The numerical solution illustrated in Fig. 6 (right) does not contain this "defect". In addition, apparently the solution converges to the classical Buckley-Leverett type result with a small ε such as 10^3. This behavior seems different from the conclusion in [9].

3.2. **The compressible model.** Next, we use the numerical scheme described in Appendix D (compressible variant) to compute and illustrate the numerical behavior for the compressible model. Comparison is made with the cases shown in Fig. 2 for the incompressible model. The compressible fluids are assumed to be given by
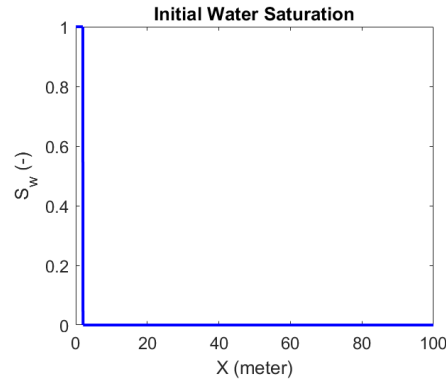
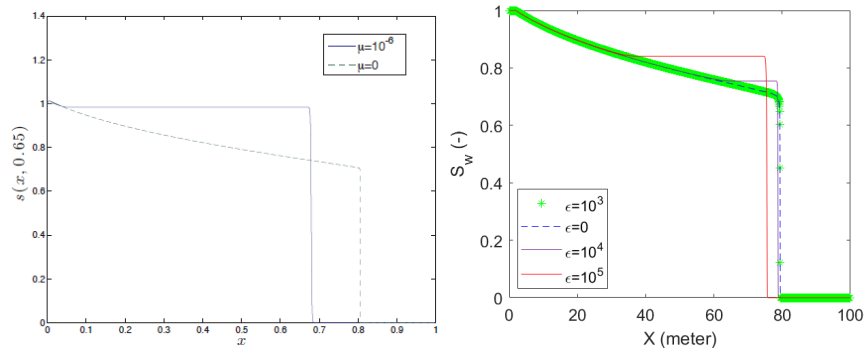FIGURE 5. Initial water saturation profile from Coclite et al. [9].

FIGURE 6. Left: The results from Coclite et al. [9] based on Darcy velocity in viscous term. Right: Numerical scheme (after 8 days) which uses interstitial velocity in viscous term with different viscous values $\varepsilon = 0, 10^3, 10^4, 10^5$ and $\varepsilon_w = \varepsilon_o = \varepsilon$.

the pressure-density relation (1.4). The initial water saturation is also the same as shown in Fig. 1. The water saturation s_w at left boundary is constrained with 0.8 and the initial water pressure at left boundary is 10^6 Pa. The numerical behavior is shown in Fig. 7. The essential difference is a delay in the solution of the compressible model.

In order to shed light on the difference observed in Fig. 7, we explore the pressure profiles at various times (shown in Fig. 8). It is clear from these plots that water pressure keeps increasing, especially for the water displacing part. Water can be compressed in the compressible model therefore the water density will also increase, which leads to a larger viscous effect since density is included in the coefficient of the viscous term. Water will feel more resistance forces and it is more difficult to displace oil resulting in a delay effect.

Injection of water versus gas. Finally, a numerical example is shown with gas injection to displace oil in the compressible model instead of water. The parameters of gas are the same as for water, as described in Table 1, except using the pressure-density relation: $\rho_g = P_g/C_g$ where $C_g = 10^5$. The left figure of Fig. 9 compares the results of gas injection and water injection. As expected, it is a much slower
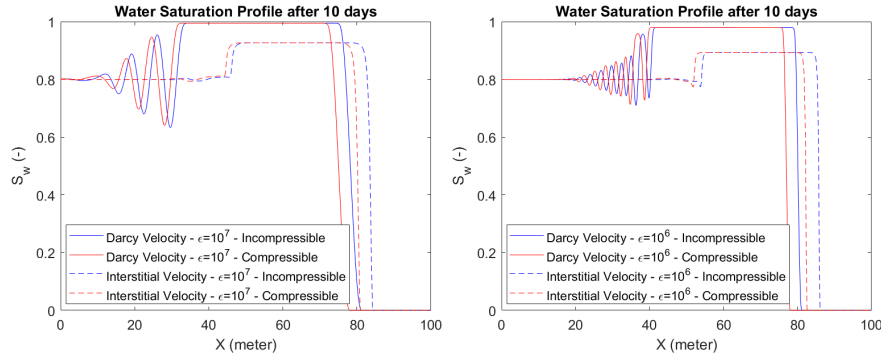
FIGURE 7. Comparison between the compressible model and the incompressible model for water-oil flow with $\varepsilon_w = \varepsilon_o = \varepsilon = 10^7, 10^6$. After the same period of 10 days, water flow in the compressible model is delayed compared with water profiles in the incompressible model, for both situations with interstitial velocity and Darcy velocity in viscous terms.
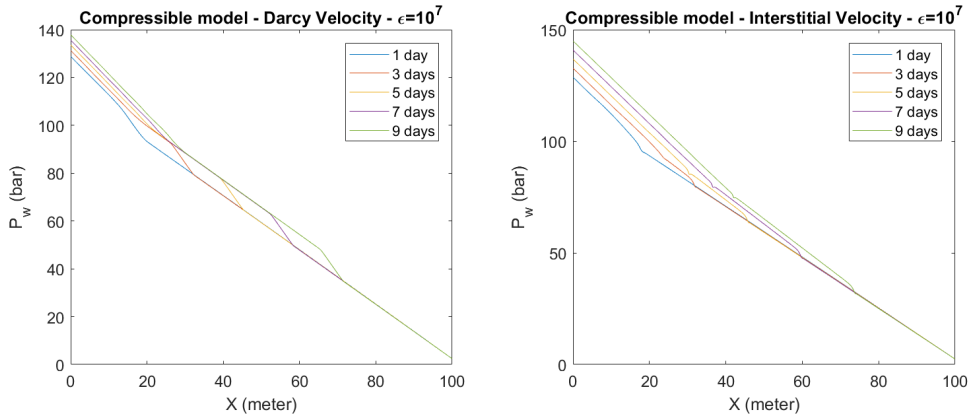

FIGURE 8. The water pressure evolution in the compressible model for the case with Darcy velocity in viscous term (left figure) and the case with interstitial velocity in viscous term (right figure). Water pressure increases with time in the water displacing part of the reservoir layer which leads to a compression effect where the magnitude of the viscous terms increase and thereby slows down the displacement of the water front.

process for gas to displace oil. This is a natural consequence of the fact that gas is much more compressible than water. The high gas pressure which results from the increased viscous effect will generate a strong compression of gas. It is also interesting to see that the fluctuation in the gas saturation profile becomes stronger as time elapses, which is not observed in the case of water injection. However, the elevated constant water level is almost the same in both cases (compare the left and the right subfigures in Fig. 9).
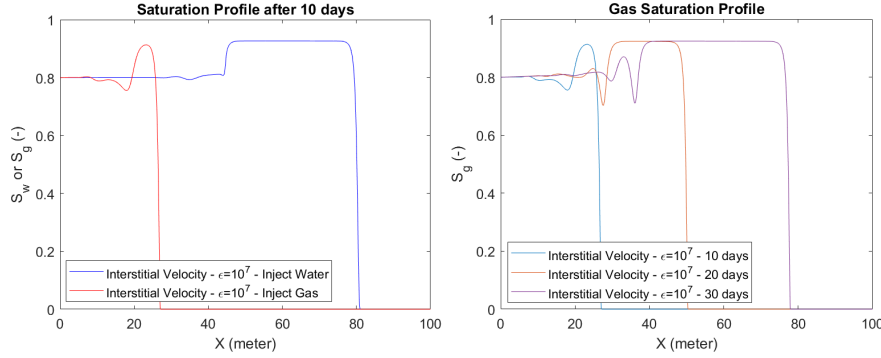
FIGURE 9. Left: Comparison of saturation profiles for water injection and gas injection, respectively, after the same time period (10 days) in the compressible model using interstitial velocity in viscous term ($\varepsilon_w = \varepsilon_o = \varepsilon = 10^7$). Right: The gas saturation profile shown at different times.

4. **Concluding remarks.** Some main observations from the investigations of the present manuscript are:

- We have found that exploiting the fact that the viscous term depends on the interstitial fluid flow velocity u_w, u_o, we can derive stability estimates (energy-type estimate and BD-estimate) that also naturally deal with the capillary pressure term $P_c(s_w)$. This approach seems strongly linked to the special structure of the viscous coefficients.
- We formulated finite difference schemes, both for the incompressible and compressible version of the model. These schemes allow us to systematically gain some insight into the effect of compressibility as well as the effect from the viscous terms that account for the frictional resistance within the fluid. We also observe that by using Darcy velocity in the viscous term, the resulting scheme tends to give more oscillatory behavior similar to that reported in [9].
- In particular, when the viscous coefficients $\varepsilon_w, \varepsilon_o$ become small enough, the numerical experiments carried out in a one-dimensional setting indicate that the approximate solution converges to the solution of the inviscid model. The stability estimates for the model based on interstitial fluid velocity, however, do depend on $\varepsilon_w, \varepsilon_o$ and cannot be used to ensure convergence to the inviscid model. Hence, this remains an interesting open problem.

Appendix A: Proof of Theorem 2.1. We apply a method similar to the one used in our previous work [16] to prove the local existence and uniqueness. Hence, in order to make the proof more compact we will heavily refer to that paper for details and highlight what is different.

First, we consider the solution space

$$S := S_{T_0, A_1} = \left\{ v \in C([0, T_0]; H_0^1 \cap H^2) \Big| \|v\|_{C([0, T_0]; H^2)} \leq A A_1 \right\}$$

where $A = \max\{\frac{1}{\epsilon_o}, \frac{1}{\epsilon_w}\}$, A_1 and T_0 satisfy (4.101) and (4.87), (4.88), and (4.117), respectively.

Step 1. Construct an iteration sequence.

We define an iteration sequence to approximate (2.14) which takes the following form

$$
\begin{aligned}
(n^k)_t + (n^k u_o^{k-1})_x &= 0 \\
(m^k)_t + (m^k u_w^{k-1})_x &= 0 \\
s_o^k P_{ox}^k &= -\hat{k}_o^k u_o^k + \hat{k}^k(u_w^{k-1} - u_o^k) + \varepsilon_o(n^k u_{ox}^k)_x - n^k g \\
s_w^k P_{wx}^k &= -\hat{k}_w^k u_w^k - \hat{k}^k(u_w^k - u_o^k) + \varepsilon_w(m^k u_{wx}^k)_x - m^k g
\end{aligned}
\tag{4.85}
$$

with the initial-boundary value conditions:

$$
(u_w^k, u_o^k)(0, t) = (u_w^k, u_o^k)(1, t) = 0, \ t \geq 0,
$$

and

$$
(m^k, n^k)(x, 0) = (m_0, n_0)(x), \ x \in [0, 1],
$$

for $k = 1, 2, 3, \cdots$, where $(u_w^0, u_o^0) = (0, 0)$, $s_w^k = s_w(m^k, n^k)$, $s_o^k = s_o(m^k, n^k)$, $P_w^k = P_w(m^k, n^k)$, $P_o^k = P_o(m^k, n^k)$, $\hat{k}_w^k = \hat{k}_w(m^k, n^k)$, $\hat{k}_o^k = \hat{k}_o(m^k, n^k)$, $\hat{k}^k = \hat{k}(m^k, n^k)$, and

$$
(m^k, n^k) \in C([0, T_0]; H^1) \cap C^1([0, T_0]; L^2), \ (u_w^k, u_o^k) \in C([0, T_0]; H_0^1 \cap H^2).
$$

Step 2. Boundedness of the sequence.

Assume that $u_w^{k-1}, u_o^{k-1} \in S$. To prove $u_w^i, u_o^i \in S$ for all $i = 0, 1, 2, 3, \ldots,$ it suffices to prove $u_w^k, u_o^k \in S$.

In fact, as a consequence of that $u_w^{k-1}, u_o^{k-1} \in S$, we have

$$
\begin{cases}
\|u_{w,x}^{k-1}(\cdot, t)\|_{L^\infty} \leq \|u_{w,x}^{k-1}(\cdot, t)\|_{W^{1,1}} \leq AA_1 \\
\|u_{o,x}^{k-1}(\cdot, t)\|_{L^\infty} \leq \|u_{o,x}^{k-1}(\cdot, t)\|_{W^{1,1}} \leq AA_1
\end{cases}
\tag{4.86}
$$

for $t \in [0, T_0]$.

By virtue of $(4.85)_1$, $(4.85)_2$, and (4.86), we can find positive constants C, k_0 and k_1 independent of T_0, A, and A_1 and where $T_0 \leq T_1$ for some $T_1 > 0$ which reflects smallness on time, such that

$$
\begin{cases}
k_0 \leq m^k \leq k_1, \text{ and } k_0 \leq n^k \leq k_1 \text{ on } [0, 1] \times [0, T_0], \\
\|\frac{\partial P_w(m^k, n^k)}{\partial n^k}\|_{L^\infty([0,1] \times [0,T_0])} \leq C, \ \|\frac{\partial P_w(m^k, n^k)}{\partial m^k}\|_{L^\infty([0,1] \times [0,T_0])} \leq C, \\
\|\frac{\partial P_o(m^k, n^k)}{\partial n^k}\|_{L^\infty([0,1] \times [0,T_0])} \leq C, \ \|\frac{\partial P_o(m^k, n^k)}{\partial m^k}\|_{L^\infty([0,1] \times [0,T_0])} \leq C,
\end{cases}
\tag{4.87}
$$

and

$$
\begin{cases}
\int_0^1 |n_x^k(x, t)|^2 dx \leq C, \\
\int_0^1 |m_x^k(x, t)|^2 dx \leq C.
\end{cases}
\tag{4.88}
$$

In fact, $(4.87)_1$ as well as (4.88) are straightforward consequences of the fact that m^k and n^k are transported by the smooth vector fields u_w^{k-1} and u_o^{k-1}. Moreover, $(4.87)_{2,3}$ can then be deduced from the regularity of $(m, n) \mapsto \rho_s(m, n)$ and of $\rho_s \mapsto P_s(\rho_s)$ for $s = w, o$. See Step 2, Section 2.2 in [16] for more details.

Multiplying $(4.85)_3$ by u_o^k and u_{oxx}^k respectively, and integrating the results over $(0,1)$, we have

$$
\begin{aligned}
\varepsilon_o \int_0^1 n^k |u_{ox}^k|^2 \, dx &= -\int_0^1 s_o^k P_{ox}^k u_o^k \, dx - \int_0^1 \hat{k}_o^k |u_o^k|^2 \, dx \\
&\quad + \int_0^1 \hat{k}^k (u_w^{k-1} - u_o^k) u_o^k \, dx - \int_0^1 n^k g u_o^k \, dx \\
&\leq -\frac{1}{2} \int_0^1 \hat{k}_o^k |u_o^k|^2 \, dx + \frac{1}{2} \int_0^1 \hat{k}^k |u_w^{k-1}|^2 \, dx - \frac{1}{2} \int_0^1 \hat{k}^k |u_o^k|^2 \, dx + C \\
&\leq -\frac{1}{2} \int_0^1 \hat{k}_o^k |u_o^k|^2 \, dx + \frac{I_{wo} k_1}{2} (AA_1)^2 - \frac{1}{2} \int_0^1 \hat{k}^k |u_o^k|^2 \, dx + C,
\end{aligned}
\tag{4.89}
$$

and

$$
\begin{aligned}
\varepsilon_o \int_0^1 n^k |u_{oxx}^k|^2 \, dx &= -\varepsilon_o \int_0^1 (n^k)_x u_{ox}^k u_{oxx}^k \, dx + \int_0^1 s_o^k P_{ox}^k u_{oxx}^k \, dx \\
&\quad + \int_0^1 \hat{k}_o^k u_o^k u_{oxx}^k \, dx - \int_0^1 \hat{k}^k (u_w^{k-1} - u_o^k) u_{oxx}^k \, dx + \int_0^1 n^k g u_{oxx}^k \, dx \\
&\leq \frac{\varepsilon_o}{2} \int_0^1 n^k |u_{oxx}^k|^2 \, dx + \frac{5\varepsilon_o}{2} \int_0^1 \frac{1}{n^k} |(n^k)_x|^2 |u_{ox}^k|^2 \, dx + \frac{C}{\varepsilon_o} \\
&\quad + \frac{5}{2\varepsilon_o} \int_0^1 \frac{1}{n^k} |\hat{k}_o^k|^2 |u_o^k|^2 \, dx + \frac{5}{2\varepsilon_o} \int_0^1 \frac{1}{n^k} |\hat{k}^k|^2 |u_w^{k-1} - u_o^k|^2 \, dx \\
&\leq \frac{\varepsilon_o}{2} \int_0^1 n^k |u_{oxx}^k|^2 \, dx + \frac{5\varepsilon_o}{2k_0} \int_0^1 |(n^k)_x|^2 |u_{ox}^k|^2 \, dx + \frac{C}{\varepsilon_o} + \frac{5I_o}{2\varepsilon_o} \int_0^1 \hat{k}_o^k |u_o^k|^2 \, dx \\
&\quad + \frac{5(I_{wo})^2 k_1 (AA_1)^2}{\varepsilon_o} + \frac{5I_{wo}}{\varepsilon_o} \int_0^1 \hat{k}^k |u_o^k|^2 \, dx \\
&\leq \frac{\varepsilon_o}{2} \int_0^1 n^k |u_{oxx}^k|^2 \, dx + \frac{5C\varepsilon_o}{2k_0} \left\| |u_{ox}^k|^2 \right\|_{L^\infty} + \frac{C}{\varepsilon_o} + \frac{5I_o}{2\varepsilon_o} \int_0^1 \hat{k}_o^k |u_o^k|^2 \, dx \\
&\quad + \frac{5(I_{wo})^2 k_1 (AA_1)^2}{\varepsilon_o} + \frac{5I_{wo}}{\varepsilon_o} \int_0^1 \hat{k}^k |u_o^k|^2 \, dx,
\end{aligned}
\tag{4.90}
$$

where we have used Cauchy inequality, (4.87), (4.88). Note that we obtain

$$
\begin{aligned}
\left\| |u_{ox}^k|^2 \right\|_{L^\infty} &\leq \int_0^1 |u_{ox}^k|^2 \, dx + 2\|u_{ox}^k\|_{L^2} \|u_{oxx}^k\|_{L^2} \\
&\leq (1 + \frac{1}{\delta}) \int_0^1 |u_{ox}^k|^2 \, dx + \delta \int_0^1 |u_{oxx}^k|^2 \, dx,
\end{aligned}
\tag{4.91}
$$

for any small $\delta > 0$, by using Sobolev inequality, Hölder inequality and Cauchy inequality.

By virtue of (4.87), (4.89), (4.90), and (4.91), we have

$$
\begin{cases}
\int_0^1 |u_{ox}^k|^2 \, dx \leq \dfrac{1}{\varepsilon_o} \Big[\dfrac{I_{wo} k_1}{2k_0} (AA_1)^2 + \dfrac{C}{k_0} \Big], \\[3mm]
\int_0^1 \hat{k}_o^k |u_o^k|^2 \, dx + \int_0^1 \hat{k}^k |u_o^k|^2 \, dx \leq I_{wo} k_1 (AA_1)^2 + 2C,
\end{cases}
\tag{4.92}
$$

and

$$\int_0^1 |u_{oxx}^k|^2 \, dx \le \frac{10C}{(k_0)^2}(1 + \frac{10C}{k_0}) \int_0^1 |u_{ox}^k|^2 \, dx + \frac{4C}{(\varepsilon_o)^2 k_0}$$
$$+ \frac{10I_o}{(\varepsilon_o)^2 k_0} \int_0^1 \hat{k}_o^k |u_o^k|^2 \, dx + \frac{20(I_{wo})^2 k_1 (AA_1)^2}{(\varepsilon_o)^2 k_0} + \frac{20I_{wo}}{(\varepsilon_o)^2 k_0} \int_0^1 \hat{k}^k |u_o^k|^2 \, dx, \tag{4.93}$$

where we take $\delta = \frac{k_0}{10C}$.

Putting (4.92) into (4.93), we have

$$\int_0^1 |u_{oxx}^k|^2 \, dx \le \frac{10C}{(k_0)^2}(1 + \frac{10C}{k_0}) \Big(\frac{I_{wo}k_1}{2\varepsilon_o k_0}(AA_1)^2 + \frac{C}{\varepsilon_o k_0} \Big) + \frac{4C}{(\varepsilon_o)^2 k_0}$$
$$+ \frac{20(I_{wo})^2 k_1 (AA_1)^2}{(\varepsilon_o)^2 k_0} + \Big[\frac{10I_o}{(\varepsilon_o)^2 k_0} + \frac{20I_{wo}}{(\varepsilon_o)^2 k_0} \Big] (I_{wo}k_1(AA_1)^2 + 2C) \tag{4.94}$$
$$= E_{\varepsilon_o, 1}(AA_1)^2 + E_{\varepsilon_o, 2},$$

where

$$\begin{cases} E_{\varepsilon_o, 1} = \frac{1}{\varepsilon_o} \Big[\frac{10C}{(k_0)^2}(1 + \frac{10C}{k_0}) \frac{I_{wo}k_1}{2k_0} + \frac{20(I_{wo})^2 k_1}{\varepsilon_o k_0} + \frac{10I_o I_{wo}k_1}{\varepsilon_o k_0} + \frac{20(I_{wo})^2 k_1}{\varepsilon_o k_0} \Big], \\ E_{\varepsilon_o, 2} = \frac{1}{\varepsilon_o} \Big[\frac{10C}{(k_0)^2}(1 + \frac{10C}{k_0}) \frac{C}{k_0} + \frac{4C}{\varepsilon_o k_0} + \frac{20CI_o}{\varepsilon_o k_0} + \frac{40CI_{wo}}{\varepsilon_o k_0} \Big]. \end{cases}$$

In view of (4.92) and (4.94), we have

$$\|u_o^k(t)\|_{H^2}^2 \le \Big[\frac{I_{wo}k_1}{k_0 \varepsilon_o} + E_{\varepsilon_o, 1} \Big](AA_1)^2 + \frac{2C}{\varepsilon_o k_0} + E_{\varepsilon_o, 2}. \tag{4.95}$$

Similar to (4.89) and (4.90), we have

$$\varepsilon_w \int_0^1 m^k |u_{wx}^k|^2 \, dx + \frac{1}{2} \int_0^1 \hat{k}_w^k |u_w^k|^2 \, dx + \frac{1}{2} \int_0^1 \hat{k}^k |u_w^k|^2 \, dx \le \frac{1}{2} \int_0^1 \hat{k}^k |u_o^k|^2 \, dx + C,$$

and

$$\frac{\varepsilon_w}{2} \int_0^1 m^k |u_{wxx}^k|^2 \, dx \le \frac{5C\varepsilon_w}{k_0}(1 + \frac{10C}{k_0}) \int_0^1 |u_{wx}^k|^2 \, dx + \frac{2C}{\varepsilon_w} + \frac{5I_w}{\varepsilon_w} \int_0^1 \hat{k}_w^k |u_w^k|^2 \, dx$$
$$+ \frac{10I_w}{\varepsilon_w} \int_0^1 \hat{k}^k |u_w^k|^2 \, dx + \frac{10I_w}{\varepsilon_w} \int_0^1 \hat{k}^k |u_o^k|^2 \, dx,$$

which yield

$$\begin{cases} \int_0^1 |u_{wx}^k|^2 \, dx \le \frac{1}{\varepsilon_w} \Big[\frac{I_{wo}k_1}{2k_0}(AA_1)^2 + \frac{2C}{k_0} \Big], \\ \int_0^1 \hat{k}_w^k |u_w^k|^2 \, dx + \int_0^1 \hat{k}^k |u_w^k|^2 \, dx \le I_{wo}k_1(AA_1)^2 + 4C, \end{cases} \tag{4.96}$$

and

$$\int_0^1 |u_{wxx}^k|^2 \, dx \le \frac{10C}{(k_0)^2}(1 + \frac{10C}{k_0}) \frac{1}{\varepsilon_w} \Big[\frac{I_{wo}k_1}{2k_0}(AA_1)^2 + \frac{2C}{k_0} \Big] + \frac{4C}{(\varepsilon_w)^2 k_0}$$
$$+ \Big[\frac{10I_w}{(\varepsilon_w)^2 k_0} + \frac{20I_w}{(\varepsilon_w)^2 k_0} + \frac{20I_w}{(\varepsilon_w)^2 k_0} \Big] (I_{wo}k_1(AA_1)^2 + 4C) \tag{4.97}$$
$$= E_{\varepsilon_w, 1}(AA_1)^2 + E_{\varepsilon_w, 2}.$$

where

$$\begin{cases} E_{\varepsilon_w,1} = \frac{1}{\varepsilon_w}\Big[\frac{10C}{(k_0)^2}\big(1 + \frac{10C}{k_0}\big)\frac{I_{wo}k_1}{2k_0} + \frac{10I_wI_{wo}k_1}{\varepsilon_wk_0} + \frac{20I_wI_{wo}k_1}{\varepsilon_wk_0} + \frac{20I_wI_{wo}k_1}{\varepsilon_wk_0} \Big], \\[2mm] E_{\varepsilon_w,2} = \frac{1}{\varepsilon_w}\Big[\frac{10C}{(k_0)^2}\big(1 + \frac{10C}{k_0}\big)\frac{2C}{k_0} + \frac{4C}{\varepsilon_wk_0} + \frac{40CI_w}{\varepsilon_wk_0} + \frac{80CI_w}{\varepsilon_wk_0} + \frac{80CI_w}{\varepsilon_wk_0} \Big]. \end{cases}$$

In view of (4.96) and (4.97), we have

$$\|u_w^k(t)\|_{H^2}^2 \le \Big[\frac{I_{wo}k_1}{k_0\varepsilon_w} + E_{\varepsilon_w,1}\Big](AA_1)^2 + \frac{4C}{\varepsilon_wk_0} + E_{\varepsilon_w,2}. \tag{4.98}$$

Letting

$$\max\Big\{ \frac{I_{wo}k_1}{k_0\varepsilon_o} + E_{\varepsilon_o,1}, \frac{I_{wo}k_1}{k_0\varepsilon_w} + E_{\varepsilon_w,1} \Big\} \le \frac{1}{2} \tag{4.99}$$

which can be done by assuming for instance that ε_o and ε_w are large enough, we obtain from (4.95) and (4.98)

$$\|u_o^k(t)\|_{H^2} \le AA_1, \ \|u_w^k(t)\|_{H^2} \le AA_1, \tag{4.100}$$

where we choose that

$$A_1 \ge \frac{1}{A}\max\Big\{ \Big(\frac{4C}{\varepsilon_ok_0} + 2E_{\varepsilon_o,2}\Big)^{\frac{1}{2}}, \Big(\frac{8C}{\varepsilon_wk_0} + 2E_{\varepsilon_w,2}\Big)^{\frac{1}{2}} \Big\} \tag{4.101}$$

Consequently, we have $u_w^k, u_o^k \in S$ for all $k = 1, 2, 3, \cdots$, if we assume that (4.99) is satisfied for A_1 given by (4.101) and for $T_0 \le T_1$.

Step 3. Compactness arguments.

By virtue of the estimates uniformly for k in Step 2, there exist a subsequence k_i $(i = 1, 2, 3, ...)$ and a (u_w, u_o, m, n), such that

$$\begin{aligned} &(u_w^{k_i}, u_o^{k_i}) \rightharpoonup (u_w, u_o) \text{ weak-* in } L^\infty([0,T_0]; H_0^1 \cap H^2), \\ &n^{k_i} \rightharpoonup n \text{ weak-* in } L^\infty([0,T_0]; H^1), \\ &m^{k_i} \rightharpoonup m \text{ weak-* in } L^\infty([0,T_0]; H^1), \\ &(n_t^{k_i}, m_t^{k_i}) \rightharpoonup (n_t, m_t) \text{ weak-* in } L^\infty([0,T_0]; L^2) \end{aligned} \tag{4.102}$$

as $k_i \to \infty$, where

$$(u_w, u_o) \in L^\infty([0,T_0]; H^2 \cap H_0^1), \text{ and } (m, n) \in L^\infty([0,T_0]; H^1),$$

and $n_t, m_t \in L^\infty([0,T_0]; L^2)$. Using the Aubin-Lions' compactness theorem, we can obtain some strong convergence. More precisely, there exists a subsequence still denoted by k_i without loss of generality $(i = 1, 2, 3, ...)$, such that

$$\begin{aligned} &n^{k_i} \to n \text{ in } C([0,1] \times [0,T_0]), \\ &m^{k_i} \to m \text{ in } C([0,1] \times [0,T_0]), \end{aligned} \tag{4.103}$$

as $k_i \to \infty$. It follows from (4.103) and (4.87) that

$$k_0 \le m \le k_1, \text{ and } k_0 \le n \le k_1 \text{ on } [0,1] \times [0,T_0].$$

Step 4. Convergence of $(u_w^{k_i-1}, u_o^{k_i-1})$.

We are going to investigate the convergence of the neighbor sequence of $(u_w^{k_i}, u_o^{k_i})$, i.e., $(u_w^{k_i-1}, u_o^{k_i-1})$, in order to make sure that their limits are the same, since they both appear in the approximate system (4.85).

To begin with, we need the estimates of the difference between m^{k+1} (n^{k+1}) and m^k (n^k), since there is a connection between velocity and mass due to the

momentum equation. Denote $\bar{m}^{k+1} = m^{k+1} - m^k$ and $\bar{n}^{k+1} = n^{k+1} - n^k$. Then, from $(4.85)_{1,2}$ it follows

$$\begin{cases} \bar{m}_t^{k+1} + \bar{m}_x^{k+1} u_w^k + m_x^k(u_w^k - u_w^{k-1}) + \bar{m}^{k+1}[u_w^k]_x + m^k(u_w^k - u_w^{k-1})_x = 0, \\ \bar{m}^{k+1}(x,0) = 0 \end{cases} \quad (4.104)$$

and

$$\begin{cases} \bar{n}_t^{k+1} + \bar{n}_x^{k+1} u_o^k + n_x^k(u_o^k - u_o^{k-1}) + \bar{n}^{k+1}[u_o^k]_x + n^k(u_o^k - u_o^{k-1})_x = 0, \\ \bar{n}^{k+1}(x,0) = 0 \end{cases} \quad (4.105)$$

for $(x,t) \in [0,1] \times [0,T_0]$.

Using (4.104), we have

$$\frac{d}{dt} \int_0^1 |\bar{m}^{k+1}|^2 \, dx \le \bar{C} \|(u_w^k - u_w^{k-1})_x\|_{L^2} \|\bar{m}^{k+1}\|_{L^2} + \bar{C}A \|\bar{m}^{k+1}\|_{L^2}^2, \quad (4.106)$$

where $\bar{C}$ is a generic positive constant depending only on the initial data, the upper bound of T_0 and other known constants but independent of k, A and T_0. Here we have used the facts that m_x^k is bounded in L^2 and that $u_w^k \in S$, and the Poincaré inequality. Similarly, we have

$$\frac{d}{dt} \int_0^1 |\bar{n}^{k+1}|^2 \, dx \le \bar{C} \|(u_o^k - u_o^{k-1})_x\|_{L^2} \|\bar{n}^{k+1}\|_{L^2} + \bar{C}A \|\bar{n}^{k+1}\|_{L^2}^2. \quad (4.107)$$

Using $(4.85)_3$, we have

$$\begin{aligned} \varepsilon_o[n^k(u_o^k - u_o^{k-1})_x]_x = &-\varepsilon_o[(n^k - n^{k-1})(u_o^{k-1})_x]_x + (s_o^k - s_o^{k-1})P_{ox}^k \\ &+ s_o^{k-1}[P_o^k - P_o^{k-1}]_x + g(n^k - n^{k-1}) + [\hat{k}_o(m^k, n^k) - \hat{k}_o(m^{k-1}, n^{k-1})]u_o^{k-1} \\ &+ \hat{k}_o^k(u_o^k - u_o^{k-1}) - \hat{k}^k[u_w^{k-1} - u_w^{k-2} - (u_o^k - u_o^{k-1})] \\ &- [\hat{k}(m^k, n^k) - \hat{k}(m^{k-1}, n^{k-1})](u_w^{k-2} - u_o^{k-1}), \end{aligned} \quad (4.108)$$

where $k = 2, 3, 4, \cdots$.

Denote $\bar{u}_o^{k+1} = u_o^{k+1} - u_o^k$ and $\bar{u}_w^{k+1} = u_w^{k+1} - u_w^k$. Then similar to the estimate of u_o^k (see (4.89)), the $\|(u_o^k - u_o^{k-1})_x\|_{L^2}$ can be evaluated as follows:

$$\begin{aligned} \|[\bar{u}_o^k]_x\|_{L^2}^2 \le &\bar{C} \|\bar{n}^k(u_o^{k-1})_x\|_{L^2}^2 + \frac{\bar{C}}{(\varepsilon_o)^2} \|(s_o^k - s_o^{k-1})P_{ox}^k\|_{L^1}^2 \\ &+ \frac{\bar{C}}{(\varepsilon_o)^2} \|[s_o^{k-1}]_x(P_o^k - P_o^{k-1})\|_{L^1}^2 + \frac{\bar{C}}{(\varepsilon_o)^2} \|s_o^{k-1}(P_o^k - P_o^{k-1})\|_{L^2}^2 + \frac{\bar{C}}{(\varepsilon_o)^2} \|\bar{m}^k\|_{L^2}^2 \\ &- \frac{1}{\varepsilon_o k_0} \|\sqrt{\hat{k}_o^k} \bar{u}_o^k\|_{L^2}^2 + \frac{\bar{C}}{(\varepsilon_o)^2} \|\hat{k}_o(m^k, n^k) - \hat{k}_o(m^{k-1}, n^{k-1})\|_{L^2}^2 \\ &- \frac{1}{2\varepsilon_o k_0} \|\sqrt{\hat{k}^k} \bar{u}_o^k\|_{L^2}^2 + \frac{1}{2\varepsilon_o k_0} \|\sqrt{\hat{k}^k} \bar{u}_w^{k-1}\|_{L^2}^2 \\ &+ \frac{\bar{C}}{(\varepsilon_o)^2} \|\hat{k}(m^k, n^k) - \hat{k}(m^{k-1}, n^{k-1})\|_{L^2}^2, \end{aligned} \quad (4.109)$$

where $k = 2, 3, 4, \cdots$. Note that

$$\begin{cases} |s_o^k - s_o^{k-1}| \le \bar{C}(|\bar{m}^k| + |\bar{n}^k|) \text{ and } |P_o^k - P_o^{k-1}| \le \bar{C}(|\bar{m}^k| + |\bar{n}^k|), \\ |\hat{k}_o(m^k, n^k) - \hat{k}_o(m^{k-1}, n^{k-1})| \le \bar{C}(|\bar{m}^k| + |\bar{n}^k|) \\ \text{and } |\hat{k}(m^k, n^k) - \hat{k}(m^{k-1}, n^{k-1})| \le \bar{C}(|\bar{m}^k| + |\bar{n}^k|). \end{cases}$$

This, together with Hölder inequality, (4.109) and the boundedness of P_{ox}^k and $[s_o^{k-1}]_x$ in L^2, implies that

$$\|[\bar{u_o}^k]_x\|_{L^2}^2 \leq \bar{C}(1+A^2)(\|\bar{m}^k\|_{L^2}^2 + \|\bar{n}^k\|_{L^2}^2) - \frac{1}{\varepsilon_o k_0}\|\sqrt{\hat{k}_o^k}\bar{u_o}^k\|_{L^2}^2$$
$$- \frac{1}{2\varepsilon_o k_0}\|\sqrt{\hat{k}^k}\bar{u_o}^k\|_{L^2}^2 + \frac{1}{2\varepsilon_o k_0}\|\sqrt{\hat{k}^k}\bar{u_w}^{k-1}\|_{L^2}^2, \tag{4.110}$$

where $k = 2, 3, 4, \cdots$.

Similarly, by virtue of $(4.85)_4$, we have

$$\varepsilon_w[m^k(u_w^k - u_w^{k-1})_x]_x$$
$$= -\varepsilon_w[(m^k - m^{k-1})(u_w^{k-1})_x]_x + (s_w^k - s_w^{k-1})P_{wx}^k + s_w^{k-1}[P_w^k - P_w^{k-1}]_x$$
$$+ g(m^k - m^{k-1}) + \hat{k}_w^k(u_w^k - u_w^{k-1}) + [\hat{k}_w(m^k, n^k) - \hat{k}_w(m^{k-1}, n^{k-1})]u_w^{k-1}$$
$$+ \hat{k}^k[u_w^k - u_w^{k-1} - (u_o^k - u_o^{k-1})] + [\hat{k}(m^k, n^k) - \hat{k}(m^{k-1}, n^{k-1})](u_w^{k-1} - u_o^{k-1}), \tag{4.111}$$

and

$$\|[\bar{u_w}^k]_x\|_{L^2}^2 \leq \bar{C}(1+A^2)(\|\bar{m}^k\|_{L^2}^2 + \|\bar{n}^k\|_{L^2}^2) - \frac{1}{\varepsilon_w k_0}\|\sqrt{\hat{k}_w^k}\bar{u_w}^k\|_{L^2}^2$$
$$- \frac{1}{2\varepsilon_w k_0}\|\sqrt{\hat{k}^k}\bar{u_w}^k\|_{L^2}^2 + \frac{1}{2\varepsilon_w k_0}\|\sqrt{\hat{k}^k}\bar{u_o}^k\|_{L^2}^2, \tag{4.112}$$

where $k = 2, 3, 4, \cdots$.

Putting (4.110) into (4.112), we have

$$\|[\bar{u_w}^k]_x\|_{L^2}^2 \leq \bar{C}_1(\|\bar{m}^k\|_{L^2}^2 + \|\bar{n}^k\|_{L^2}^2) + \frac{1}{2\varepsilon_w k_0}\|\sqrt{\hat{k}^k}\bar{u_w}^{k-1}\|_{L^2}^2$$
$$\leq \bar{C}_1(\|\bar{m}^k\|_{L^2}^2 + \|\bar{n}^k\|_{L^2}^2) + \frac{I_{wo}k_1}{2\varepsilon_w k_0}\|[\bar{u_w}^{k-1}]_x\|_{L^2}^2, \tag{4.113}$$

for $k = 2, 3, 4, \cdots$, where we have used (4.87) and Sobolev inequality.

Combining (4.106), (4.107), (4.110) and (4.113) with Cauchy inequality, we have

$$\frac{d}{dt}\int_0^1 (|\bar{m}^{k+1}|^2 + |\bar{n}^{k+1}|^2)\,dx + \|(\bar{u_w}^{k+1})_x\|_{L^2}^2 + \|(\bar{u_o}^{k+1})_x\|_{L^2}^2$$
$$\leq \bar{C}_2(\|\bar{m}^{k+1}\|_{L^2}^2 + \|\bar{n}^{k+1}\|_{L^2}^2) + \left(\frac{I_{wo}k_1}{\varepsilon_w k_0} + \frac{I_{wo}k_1}{\varepsilon_o k_0}\right)(\|(\bar{u_w}^k)_x\|_{L^2}^2 + \|(\bar{u_o}^k)_x\|_{L^2}^2),$$

which yields

$$\frac{d}{dt}f^{k+1}(t) + g^{k+1}(t) \leq \bar{C}_2 f^{k+1}(t) + \left(\frac{I_{wo}k_1}{\varepsilon_w k_0} + \frac{I_{wo}k_1}{\varepsilon_o k_0}\right)g^k(t), \tag{4.114}$$

where

$$f^k = \int_0^1 (|\bar{m}^k|^2 + |\bar{n}^k|^2)\,dx, \ g^k = \|(\bar{u_w}^k)_x\|_{L^2}^2 + \|(\bar{u_o}^k)_x\|_{L^2}^2.$$

Then, (4.114) together with Gronwall inequality yields

$$f^{k+1}(t) \leq \left(\frac{I_{wo}k_1}{\varepsilon_w k_0} + \frac{I_{wo}k_1}{\varepsilon_o k_0}\right)\int_0^t e^{\bar{C}_2(t-s)}g^k(s)\,ds. \tag{4.115}$$

Integrating (4.114) over $[0, T_0]$, and using (4.115), we have

$$\int_0^{T_0} g^{k+1}(s)\,ds \leq \left(\bar{C}_2 T_0 e^{\bar{C}_2 T_0} + 1\right)\left(\frac{I_{wo}k_1}{\varepsilon_w k_0} + \frac{I_{wo}k_1}{\varepsilon_o k_0}\right)\int_0^{T_0} g^k(s)\,ds. \tag{4.116}$$

Letting

$$
\begin{cases}
\bar{C}_2 T_0 e^{\bar{C}_2 T_0} \le 1, \\
\frac{I_{wo} k_1}{\varepsilon_w k_0} + \frac{I_{wo} k_1}{\varepsilon_o k_0} \le \frac{1}{4}
\end{cases}
\tag{4.117}
$$

for small T_0 and large ε_o, ε_w. We note that (4.116) and (4.115) imply that

$$
\begin{cases}
\Sigma_{k=1}^\infty \int_0^{T_0} g^{k+1}(t)\, dt < \infty, \\
\Sigma_{k=1}^\infty \max_{t \in [0,T_0]} f^{k+1}(t) < \infty.
\end{cases}
\tag{4.118}
$$

Thus, (4.118) combined with $(4.102)_1$ implies that

$$
(u_w^{k_i-1}, u_o^{k_i-1}) \rightharpoonup (u_w, u_o) \text{ weak-* in } L^\infty([0,T_0]; H_0^1 \cap H^2) \tag{4.119}
$$

as $k_i \to \infty$.

Step 5. Conclusion.

Based on (4.102), (4.103), and (4.119), it can be verified that (m, n, u_w, u_o) is a unique solution of (2.14)-(2.16). See [16] for more details. Thus the proof of Theorem 2.1 is complete.

Appendix B: Some issues related to the general model (1.1)**.** The purpose of this section is to elaborate on some of the differences between the model (1.1) and other Brinkman-type of two-phase models like the one mentioned in Remark 1.2.

The incompressible, viscous version of the generalized two-phase model (1.1)**.** It seems useful to impose some simplifying assumptions on the model (1.1) and derive a reduced version of it. We may impose the following assumptions:

- incompressible fluid, i.e., ρ_w and ρ_o are constant.
- porosity ϕ is constant

We may rescale the time such that the porosity disappears in the time derivative terms. The mass and momentum equations in (1.1) now take the form

$$
(s_o)_t + \nabla \cdot (\phi s_o \mathbf{u}_o) = \frac{Q_o}{\rho_o}, \qquad \mathbf{U}_o = \phi s_o \mathbf{u}_o
$$

$$
(s_w)_t + \nabla \cdot (\phi s_w \mathbf{u}_w) = \frac{Q_w}{\rho_w}, \qquad \mathbf{U}_w = \phi s_w \mathbf{u}_w \tag{4.120}
$$

$$
s_o[\nabla P_w + \nabla P_c + \rho_o \mathbf{g}] - \varepsilon_o \rho_o \nabla \cdot (s_o \nabla \mathbf{u}_o) = +\hat{k}\mathbf{u}_w - [\hat{k}_o + \hat{k}]\mathbf{u}_o
$$

$$
s_w[\nabla P_w + \rho_w \mathbf{g}] - \varepsilon_w \rho_w \nabla \cdot (s_w \nabla \mathbf{u}_w) = -[\hat{k}_w + \hat{k}]\mathbf{u}_w + \hat{k}\mathbf{u}_o,
$$

where we have used that $P_c = P_o - P_w$. Note that in the rest of this paper we will use $\hat{k}_{ow} = \hat{k}$. We can solve for $\mathbf{u}_w$ and $\mathbf{u}_o$ from the two momentum balance equations in (4.120) and find that

$$
\mathbf{u}_w =
$$

$$
-\frac{[s_w \hat{k}_o] + \hat{k}}{\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla P_w - \frac{s_o \hat{k}}{\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla P_c - \frac{(s_w \rho_w + s_o \rho_o)\hat{k} + s_w \rho_w \hat{k}_o}{\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\mathbf{g}
$$

$$
+ (\varepsilon_w \rho_w)\frac{[\hat{k}_o + \hat{k}]}{\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_w \nabla \mathbf{u}_w) + (\varepsilon_o \rho_o)\frac{\hat{k}}{\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_o \nabla \mathbf{u}_o),
$$

$$\mathbf{u}_o =$$

$$-\frac{[s_o\hat{k}_w] + \hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla P_w - \frac{s_o[\hat{k}_w + \hat{k}]}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla P_c - \frac{(s_w\rho_w + s_o\rho_o)\hat{k} + s_o\rho_o\hat{k}_w}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\mathbf{g}$$

$$+ (\varepsilon_w\rho_w)\frac{\hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_w\nabla\mathbf{u}_w) + (\varepsilon_o\rho_o)\frac{[\hat{k}_w + \hat{k}]}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_o\nabla\mathbf{u}_o).$$

$$(4.121)$$

That is, we find for $\mathbf{U}_i = \phi s_i\mathbf{u}_i$:

$$\mathbf{U}_w =$$

$$-\hat{\lambda}_w\nabla P_w - \frac{\phi s_o s_w\hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla P_c - \hat{\lambda}_w\rho_w\mathbf{g} + \frac{\phi s_o s_w\hat{k}\Delta\rho}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\mathbf{g}$$

$$+ (\varepsilon_w\phi\rho_w)\frac{s_w[\hat{k}_o + \hat{k}]}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_w\nabla\mathbf{u}_w)$$

$$+ (\varepsilon_o\phi\rho_o)\frac{s_w\hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_o\nabla\mathbf{u}_o),$$

$$\mathbf{U}_o =$$

$$-\hat{\lambda}_o\nabla P_w - \hat{\lambda}_o\nabla P_c + \frac{\phi s_o s_w\hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla P_c - \hat{\lambda}_o\rho_o\mathbf{g} - \frac{\phi s_o s_w\hat{k}\Delta\rho}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\mathbf{g}$$

$$+ (\varepsilon_w\phi\rho_w)\frac{s_o\hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_w\nabla\mathbf{u}_w)$$

$$+ (\varepsilon_o\phi\rho_o)\frac{s_o[\hat{k}_w + \hat{k}]}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_o\nabla\mathbf{u}_o),$$

$$(4.122)$$

where $\Delta\rho = \rho_w - \rho_o$ and with generalized mobility functions $\hat{\lambda}_i$ $(i = w, o, T)$ of the form

$$\hat{\lambda}_w = \hat{\lambda}_w(s_w) = \frac{[s_w^2\hat{k}_o] + s_w\hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\phi,$$

$$\hat{\lambda}_o = \hat{\lambda}_o(s_w) = \frac{[s_o^2\hat{k}_w] + s_o\hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\phi, \qquad (4.123)$$

$$\hat{\lambda}_T = \hat{\lambda}_T(s_w) = \hat{\lambda}_w(s_w) + \hat{\lambda}_o(s_w).$$

Note that the resistance force coefficients $\hat{k}_w(s_w), \hat{k}_o(s_w), \hat{k}(s_w)$ typically are functions of water saturation s_w. From (4.122) we find

$$\mathbf{U}_T = \mathbf{U}_w + \mathbf{U}_o = -\hat{\lambda}_T\nabla P_w - \hat{\lambda}_o\nabla P_c - [\hat{\lambda}_w\rho_w + \hat{\lambda}_o\rho_o]\mathbf{g}$$

$$+ (\varepsilon_w\phi\rho_w)\frac{s_w\hat{k}_o + \hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_w\nabla\mathbf{u}_w) + (\varepsilon_o\phi\rho_o)\frac{s_o\hat{k}_w + \hat{k}}{\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]}\nabla \cdot (s_o\nabla\mathbf{u}_o).$$

$$(4.124)$$

Remark 4.1. A fundamental difference between the above expression (4.124) for the total superficial velocity $\mathbf{U}_T$ for the mixture and the model described in Remark 1.2 and expressed by the mixture momentum balance equation (1.12), is the viscous

terms. Taking the divergence of (4.124) combined with the sum of the two first equations of (4.120) and (1.3), we arrive at

$$\frac{Q_o}{\rho_o} + \frac{Q_w}{\rho_w} = -\nabla \cdot (\hat{\lambda}_T \nabla P_w) - \nabla \cdot (\hat{\lambda}_o \nabla P_c) - \nabla \cdot ([\hat{\lambda}_w \rho_w + \hat{\lambda}_o \rho_o] \mathbf{g})$$

$$+ \nabla \cdot \left((\varepsilon_w \phi \rho_w) \frac{s_w \hat{k}_o + \hat{k}}{\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]} \nabla \cdot (s_w \nabla \mathbf{u}_w) \right) \tag{4.125}$$

$$+ \nabla \cdot \left((\varepsilon_o \phi \rho_o) \frac{s_o \hat{k}_w + \hat{k}}{\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w]} \nabla \cdot (s_o \nabla \mathbf{u}_o) \right).$$

This gives an elliptic pressure equation for P_w. However, due to the appearance of complex viscous terms, it seems not easy to obtain H^1-estimate of P_w needed for compactness of the non-conservative pressure term $s_w \nabla P_w$ in the momentum balance equation.

We may conclude that it does not seem straightforward to analyse the incompressible, viscous model (4.120) by relying on an approach similar to the one used in [9]. The main reason is the use of interstitial fluid velocity $\mathbf{u}_i$ instead of Darcy velocity $\mathbf{U}_i = \phi s_i \mathbf{u}_i$.

The incompressible, inviscid case $\varepsilon_w = \varepsilon_o = 0$. It is instructive to also derive the model where viscous terms are set to zero. Hence, in the following we consider the incompressible model (4.120) where we assume that viscous terms in the momentum equations are ignored by setting $\varepsilon_w = \varepsilon_o = 0$. We observe that (4.124) gives

$$\nabla P_w = -\frac{\mathbf{U}_T}{\hat{\lambda}_T} - \frac{\hat{\lambda}_o}{\hat{\lambda}_T} \nabla P_c - [\hat{f}_w \rho_w + \hat{f}_o \rho_o] \mathbf{g}, \tag{4.126}$$

where

$$\hat{f}_w = \frac{\hat{\lambda}_w}{\hat{\lambda}_T} = \hat{f}_w(s_w), \qquad \hat{f}_o = \frac{\hat{\lambda}_o}{\hat{\lambda}_T} = \hat{f}_o(s_w) \tag{4.127}$$

are the standard fractional flow functions which satisfy that $\hat{f}_w + \hat{f}_o = 1$. Using (4.126) in (4.122) we get

$$\mathbf{U}_w = -\hat{\lambda}_w \nabla P_w - \frac{\phi s_o s_w \hat{k}}{(\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w])} \nabla P_c - \rho_w \hat{\lambda}_w \mathbf{g} + \frac{\phi s_o s_w \Delta \rho \hat{k}}{(\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w])} \mathbf{g}$$

$$= \mathbf{U}_T \frac{\hat{\lambda}_w}{\hat{\lambda}_T} + \frac{\hat{\lambda}_w [\hat{\lambda}_w \rho_w + \hat{\lambda}_o \rho_o]}{\hat{\lambda}_T} \mathbf{g} + \frac{\hat{\lambda}_w \hat{\lambda}_o}{\hat{\lambda}_T} \nabla P_c - \rho_w \hat{\lambda}_w \mathbf{g}$$

$$- \frac{\phi s_o s_w \hat{k}}{(\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w])} \nabla P_c + \frac{\phi s_o s_w \Delta \rho \hat{k}}{(\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w])} \mathbf{g}$$

$$= \mathbf{U}_T \hat{f}_w + \left(-\hat{f}_w \hat{\lambda}_o + \frac{\phi s_o s_w \hat{k}}{(\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w])} \right) \Delta \rho \mathbf{g}$$

$$- \left(-\hat{f}_w \hat{\lambda}_o + \frac{s_o s_w \hat{k}}{(\hat{k}_o \hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w])} \right) \nabla P_c$$

$$= \mathbf{U}_T \hat{f}_w(s_w) - \hat{h}(s_w) \Delta \rho \mathbf{g} + \hat{h}(s_w) \nabla P_c(s_w), \tag{4.128}$$

where $\Delta\rho = \rho_w - \rho_o$ and $\hat{h}(s_w)$ is defined by

$$\hat{h} \stackrel{\text{def}}{=} \hat{f}_w\hat{\lambda}_o - \frac{\phi s_o s_w \hat{k}}{(\hat{k}_o\hat{k}_w + \hat{k}[\hat{k}_o + \hat{k}_w])} = \frac{s_w^2(1-s_w)^2}{s_w^2\hat{k}_o + s_o^2\hat{k}_w + \hat{k}}\phi = \hat{h}(s_w). \qquad (4.129)$$

From this, it also follows that

$$\mathbf{U}_o = \mathbf{U}_T - \mathbf{U}_w = \mathbf{U}_T(1 - \hat{f}_w(s_w)) + \hat{h}(s_w)\Delta\rho\mathbf{g} - \hat{h}(s_w)\nabla P_c(s_w). \qquad (4.130)$$

Consequently, the model (4.120) has been reduced to solving the following conservation law

$$\partial_t s_w + \nabla \cdot \mathbf{U}_w = \frac{Q_w}{\rho_w}, \qquad (4.131)$$

where $\mathbf{U}_w = \mathbf{U}_w(s_w)$ is given by (4.128).

Remark 4.2. The model (4.131) combined with (4.128) (last line), (4.127), (4.129), and the generalized mobility functions as defined by (4.123), recovers the classical incompressible immiscible model except that we now also include for an additional water-oil drag force effect through the term $\hat{k}$. Setting this term to zero reproduces exactly the classical formulation. To obtain closed expressions for the generalized mobility functions $\hat{\lambda}_i$ as well as for $\hat{h}(s_w)$, we must specify appropriate functional correlations for (i) the water-rock resistance force $\hat{k}_w$; (ii) the oil-rock resistance force $\hat{k}_o$; (iii) the water-oil drag force effect $\hat{k}$. In the recent work [31] we used correlations of the following form:

$$\hat{k}_w = I_w\frac{\mu_w}{K}\phi s_w^\alpha, \qquad \hat{k}_o = I_o\frac{\mu_o}{K}\phi s_w^\beta, \qquad \hat{k} = I\frac{\mu_w\mu_o}{K}\phi s_w(1 - s_w), \qquad (4.132)$$

where μ_i is fluid viscosity, K absolute permeability, I_w, I_o, I are parameters that characterize the strength of the resistance force (similar to the end points of relative permeability in classical formulation). The generalized mobility functions introduced above account for two different resistance forces, instead of only one, as in standard Darcy's equation-based formulation. Mobility functions that are measured experimentally are known to generally depend on the flow configuration. Two main flow regimes are present in the expression for $\mathbf{U}_w$ given by (4.128) and $\mathbf{U}_o$ given by (4.130): Co-current flow (i.e., flow of water and oil in the same direction) represented by the first component $\mathbf{U}_T\hat{f}_w$ and counter-current flow (i.e., flow in the opposite direction) represented by $-\hat{h}(s_w)\Delta\rho\mathbf{g}$ and $\hat{h}(s_w)\nabla P_c(s_w)$ (compare with (4.130)). The fact that the fluid-fluid interaction term $\hat{k}$ is explicitly accounted for and appears in $\hat{h}$ (4.129) and mobility functions (4.123) implies that a more accurate understanding of water-oil flow mechanisms can be sought [31].

Appendix C: A pressure evolution equation. From the two mass balance equations we get after multiplying the n mass balance with ρ_w and the m mass balance with ρ_o and summing the two resulting equations

$$\frac{\rho_w s_o}{C_o}P_{ot} + \frac{\rho_o s_w}{C_w}P_{wt} + \rho_w(s_o\rho_o u_o)_x + \rho_o(s_w\rho_w u_w)_x$$
$$= -\rho_w\rho_o s_o Q_p - \rho_w\rho_o s_w Q_p + \rho_w\rho_o Q_I \qquad (4.133)$$

or

$$\kappa P_{ot} - \frac{\rho_o s_w}{C_w}P_c's_{wt} + \rho_w(s_o\rho_o u_o)_x + \rho_o(s_w\rho_w u_w)_x = \rho_w\rho_o(Q_I - Q_p), \qquad (4.134)$$

with

$$\kappa = s_o\rho_w\frac{1}{C_o} + s_w\rho_o\frac{1}{C_w} = s_o\rho_w\frac{\partial\rho_o}{\partial P_o} + s_w\rho_o\frac{\partial\rho_w}{\partial P_w}. \qquad (4.135)$$

Clearly,

$$s_{wt} = -s_{ot} = -\left(\frac{n}{\rho_o}\right)_t = -\frac{1}{\rho_o}n_t + \frac{n}{\rho_o^2}\rho_{ot} = -\frac{1}{\rho_o}n_t + \frac{n}{C_o\rho_o^2}P_{ot}.$$

Consequently,

$$-\frac{\rho_o s_w}{C_w}P_c's_{wt} = \frac{\rho_o s_w}{C_w}P_c'\left[\frac{1}{\rho_o}n_t - \frac{n}{C_o\rho_o^2}P_{ot}\right] = \frac{s_w P_c'}{C_w}n_t - \frac{s_o s_w P_c'}{C_w C_o}P_{ot}.$$

Using this in (4.134) we get

$$\kappa P_{ot} + \frac{s_w P_c'}{C_w}n_t - \frac{s_o s_w P_c'}{C_w C_o}P_{ot} + \rho_w(s_o\rho_o u_o)_x + \rho_o(s_w\rho_w u_w)_x = \rho_w\rho_o(Q_I - Q_p), \quad (4.136)$$

that is,

$$\left[\kappa - \frac{s_o s_w P_c'}{C_w C_o}\right]P_{ot} + \left[\rho_w - \frac{s_w P_c'}{C_w}\right](s_o\rho_o u_o)_x + \rho_o(s_w\rho_w u_w)_x$$
$$= \rho_w\rho_o(Q_I - Q_p) + \frac{s_w P_c'}{C_w}nQ_p. \quad (4.137)$$

Note that

$$\left[\kappa - \frac{s_o s_w P_c'}{C_w C_o}\right] = \frac{s_o\tilde{\rho}_w}{C_o} + \frac{s_w\rho_o}{C_w} := \tilde{\kappa}, \qquad \tilde{\rho}_w = \rho_w - \frac{s_w P_c'}{C_w}$$

so that

$$\tilde{\kappa}P_{ot} + \tilde{\rho}_w(nu_o)_x + \rho_o(mu_w)_x = \rho_w\rho_o(Q_I - Q_p) + \frac{s_w P_c'}{C_w}nQ_p \quad (4.138)$$

or

$$P_{ot} + \tilde{\eta}\tilde{\rho}_w(nu_o)_x + \tilde{\eta}\rho_o(mu_w)_x = \tilde{\eta}\rho_w\rho_o(Q_I - Q_p) + \tilde{\eta}\frac{s_w P_c'}{C_w}nQ_p,$$
$$\tilde{\eta} = \frac{1}{\tilde{\kappa}} = \frac{C_w C_o}{s_o\tilde{\rho}_w C_w + s_w\rho_o C_o}. \quad (4.139)$$

Similarly, we have for P_w:

$$\kappa P_{wt} + \frac{\rho_w s_o P_c'}{C_o}s_{wt} + \rho_w(s_o\rho_o u_o)_x + \rho_o(s_w\rho_w u_w)_x = \rho_w\rho_o(Q_I - Q_p). \quad (4.140)$$

Clearly,

$$s_{wt} = \left(\frac{m}{\rho_w}\right)_t = \frac{1}{\rho_w}m_t - \frac{m}{\rho_w^2}\rho_{wt} = \frac{1}{\rho_w}m_t - \frac{m}{C_w\rho_w^2}P_{wt}.$$

Consequently,

$$\frac{\rho_w s_o}{C_o}P_c's_{wt} = \frac{\rho_w s_o}{C_o}P_c'\left[\frac{1}{\rho_w}m_t - \frac{m}{C_w\rho_w^2}P_{wt}\right] = \frac{s_o P_c'}{C_o}m_t - \frac{s_w s_o P_c'}{C_w C_o}P_{wt}.$$

Using this in (4.140) we get

$$\kappa P_{wt} + \frac{s_o P_c'}{C_o}m_t - \frac{s_o s_w P_c'}{C_w C_o}P_{wt} + \rho_w(s_o\rho_o u_o)_x + \rho_o(s_w\rho_w u_w)_x = \rho_w\rho_o(Q_I - Q_p),$$
$$(4.141)$$

that is,

$$\left[\kappa - \frac{s_o s_w P_c'}{C_w C_o}\right]P_{wt} + \rho_w(s_o\rho_o u_o)_x + \left[\rho_o - \frac{s_o P_c'}{C_o}\right](s_w\rho_w u_w)_x$$
$$= \rho_w\rho_o(Q_I - Q_p) - \frac{s_o P_c'}{C_o}(\rho_w Q_I - mQ_p). \quad (4.142)$$

Note that

$$\left[\kappa - \frac{s_o s_w P_c'}{C_w C_o}\right] = \frac{s_o \rho_w}{C_o} + \frac{s_w \tilde{\rho}_o}{C_w} := \tilde{\kappa}, \qquad \tilde{\rho}_o = \rho_o - \frac{s_o P_c'}{C_o},$$

so that

$$\tilde{\kappa} P_{wt} + \rho_w (nu_o)_x + \tilde{\rho}_o (mu_w)_x = \rho_w \rho_o (Q_I - Q_p) - \frac{s_o P_c'}{C_o}(\rho_w Q_I - mQ_p), \quad (4.143)$$

or

$$P_{wt} + \tilde{\eta}\rho_w(nu_o)_x + \tilde{\eta}\tilde{\rho}_o(mu_w)_x = \tilde{\eta}\rho_w\rho_o(Q_I - Q_p) - \tilde{\eta}\frac{s_o P_c'}{C_o}(\rho_w Q_I - mQ_p),$$

$$\tilde{\eta} = \frac{1}{\tilde{\kappa}} = \frac{C_w C_o}{s_o \rho_w C_w + s_w \tilde{\rho}_o C_o}.$$

$$(4.144)$$

Appendix D: Discrete schemes for the compressible/incompressible version of (1.1).

A semi-discrete scheme for the compressible model. We consider discrete schemes for both the compressible and incompressible version of (2.14). For that purpose we introduce a reformulation that brings the compressible model closer to the incompressible model. In particular, we solve explicitly only for the mass transport of $m = s_w \rho_w$ whereas the mass n is computed implicitly. This will be in the spirit of the incompressible approach where we solve the mass balance equation for s_w and computes $s_o = 1 - s_w$. Details are given below.

The starting point is the model (2.14) with (n, m, u_w, u_o) as the main (unknown) variables. We rewrite the model in the following equivalent form with (m, P_w, u_w, u_o) as the main variables:

$$(m)_t + (mu_w)_x = -mQ_p + \rho_w Q_I$$

$$P_{wt} + \tilde{\eta}\rho_w(nu_o)_x + \tilde{\eta}\tilde{\rho}_o(mu_w)_x = \tilde{\eta}\rho_w\rho_o(Q_I - Q_p) - \tilde{\eta}\frac{s_o P_c'}{C_o}(\rho_w Q_I - mQ_p)$$

$$s_o(P_w + P_c)_x = -(\hat{k}_o + \hat{k})u_o + \hat{k}u_w + ng + \varepsilon_o(nu_{ox})_x$$

$$s_w(P_w)_x = -(\hat{k}_w + \hat{k})u_w + \hat{k}u_o + mg + \varepsilon_w(mu_{wx})_x$$

$$(4.145)$$

with

$$\tilde{\eta} = \frac{C_w C_o}{s_o \rho_w C_w + s_w \tilde{\rho}_o C_o}, \qquad \tilde{\rho}_o = \rho_o - \frac{s_o P_c'}{C_o}. \qquad (4.146)$$

We refer to Appendix C and (4.144) which gives the pressure evolution equation $(4.145)_2$. Note that s_w, s_o, n, P_o are determined by

$$s_w = \frac{m}{\rho_w(P_w)}, \qquad s_o = 1 - s_w$$

$$n = s_o \rho_o(P_o) = \left(1 - \frac{m}{\rho_w(P_w)}\right)\rho_o(P_o) = n(m, P_w) \qquad (4.147)$$

$$P_o = P_c(s_w) + P_w = P_o(m, P_w)$$

We solve (4.145) on our domain Ω with boundary conditions

$$u_w|_{\partial\Omega} = u_o|_{\partial\Omega} = 0 \qquad (4.148)$$

and initial condition

$$m(x, t = 0) = m_0(x), \qquad P_w(x, t = 0) = P_w(m_0(x), n_0(x)). \qquad (4.149)$$

System of ODEs. We consider the domain $\Omega = [0, 1]$ and introduce a grid of N_x cells with nodes x_j placed at the center of the cells

$$x_1 = \frac{1}{2}\Delta x, \quad x_2 = (1+\frac{1}{2})\Delta x, \quad \ldots, \quad x_j = (j-\frac{1}{2})\Delta x, \quad \ldots, \quad x_{N_x} = (N_x-\frac{1}{2})\Delta x$$

and cell interfaces $x_{j+1/2}$ at the cell interfaces

$$x_{1/2} = 0, \quad x_{3/2} = \Delta x, \quad \ldots, \quad x_{j+1/2} = j\Delta x, \quad \ldots, \quad x_{N_x+1/2} = N_x\Delta x = 1,$$

where $\Delta x = 1/N_x$. We introduce the approximate mass and pressure $\{m_j(t)\}_{j=1}^{N_x}$ and $\{P_{w,j}(t)\}_{j=1}^{N_x}$ associated with the nodes $\{x_j\}_{j=1}^{N_x}$ whereas the approximate velocities $\{u_{w,j+1/2}\}_{j=0}^{N_x}$ and $\{u_{o,j+1/2}\}_{j=0}^{N_x}$ are associated with the cell interfaces $\{x_{j+1/2}\}_{j=0}^{N_x}$. In the following we describe a semi-discrete version of (4.145).

Step 1: Mass transport. We solve for $m_j(t)$ by considering the following ODE corresponding to $(4.145)_1$:

$$\frac{dm_j}{dt} + \frac{1}{\Delta x}([mu_w]_{j+1/2} - [mu_w]_{j-1/2}) = -m_j Q_{p,j} + \rho_{wj} Q_{I,j}, \qquad m = s_w \rho_w \tag{4.150}$$

where

$$[mu_w]_{j+1/2} = \begin{cases} m_j u_{w,j+1/2}, & \text{if } u_{w,j+1/2} \geq 0; \\ m_{j+1} u_{w,j+1/2}, & \text{if } u_{w,j+1/2} < 0. \end{cases} \tag{4.151}$$

This can also be expressed as

$$[mu_w]_{j+1/2} = \frac{m_j + m_{j+1}}{2} u_{w,j+1/2} - \frac{1}{2}(m_{j+1} - m_j)|u_{w,j+1/2}|.$$

Step 2: Computation of velocities and pressure. Next, we solve for $P_{w,j}(t)$ and $u_{w,j+1/2}(t)$ and $u_{o,j+1/2}(t)$ by considering the following ODE system corresponding to $(4.145)_{2,3,4}$:

$$\begin{aligned}
\frac{dP_{wj}}{dt} &+ \tilde{\eta}_j \rho_{wj} \frac{1}{\Delta x}([nu_o]_{j+1/2} - [nu_o]_{j-1/2}) \\
&+ \tilde{\eta}_j \tilde{\rho}_{oj} \frac{1}{\Delta x}([mu_w]_{j+1/2} - [mu_w]_{j-1/2}) \\
&= [\tilde{\eta}\rho_w\rho_o]_j (Q_{I,j} - Q_{p,j}) - \left[\tilde{\eta}\frac{s_o P_c'}{C_o}\right]_j (\rho_{wj} Q_{I,j} - m_j Q_{p,j}),
\end{aligned} \tag{4.152}$$

which is combined with the momentum balance equations

$$\begin{aligned}
s_{o,j+1/2}\frac{1}{\Delta x}(P_{w,j+1} - P_{w,j}) &= -s_{o,j+1/2}\frac{1}{\Delta x}(P_{c,j+1} - P_{c,j}) \\
&\quad -\hat{k}_{o,j+1/2} u_{o,j+1/2} + \hat{k}_{j+1/2}\left(u_{w,j+1/2} - u_{o,j+1/2}\right) + n_{j+1/2} g \\
&\quad +\varepsilon_o \frac{1}{\Delta x^2}\left(n_{j+1}[u_{o,j+3/2} - u_{o,j+1/2}] - n_j[u_{o,j+1/2} - u_{o,j-1/2}]\right), \\
s_{w,j+1/2}\frac{1}{\Delta x}(P_{w,j+1} - P_{w,j}) &= \\
&\quad -\hat{k}_{w,j+1/2} u_{w,j+1/2} - \hat{k}_{j+1/2}\left(u_{w,j+1/2} - u_{o,j+1/2}\right) + m_{j+1/2} g \\
&\quad +\varepsilon_w \frac{1}{\Delta x^2}\left(m_{j+1}[u_{w,j+3/2} - u_{w,j+1/2}] - m_j[u_{w,j+1/2} - u_{w,j-1/2}]\right).
\end{aligned} \tag{4.153}$$

Here we note that the average $s_{w,j+1/2}$ in (4.153) is based on upwind relatively $u_{w,j+1/2}$

$$s_{w,j+1/2} = \begin{cases} s_{w,j}, & \text{if } u_{w,j+1/2} > 0; \\ \frac{s_{w,j}+s_{w,j+1}}{2}, & \text{if } u_{w,j+1/2} = 0; \\ s_{w,j+1}, & \text{if } u_{w,j+1/2} < 0. \end{cases} \tag{4.154}$$

Similarly, for $s_{o,j+1/2}$ and for the interaction terms $\hat{k}_{w,j+1/2}$ and $\hat{k}_{o,j+1/2}$. In addition, $\hat{k}_{j+1/2}$ is based on upwind relatively $u_{w,j+1/2}$ and $u_{o,j+1/2}$ as follows:

$$\hat{k}_{j+1/2} = \begin{cases} \hat{k}_j, & \text{if } u_{w,j+1/2} > 0 \ \& \ u_{o,j+1/2} > 0 \ ; \\ \frac{\hat{k}_j+\hat{k}_{j+1}}{2}, & \text{if } u_{w,j+1/2}u_{o,j+1/2} \leq 0; \\ \hat{k}_{j+1}, & \text{if } u_{w,j+1/2} < 0 \ \& \ u_{o,j+1/2} < 0. \end{cases} \tag{4.155}$$

Moreover, $[nu_o]_{j+1/2}$ and $[mu_w]_{j+1/2}$ appearing in (4.152) employ upwind as described in (4.151). Now, we are in a position where we can describe a fully discrete model.

A fully discrete scheme. We assume that we have given $(m_j^k, P_{w,j}^k, u_{w,j}^k, u_{o,j}^k)$. We then compute the approximate solution at time t^{k+1} expressed by $(m_j^{k+1}, P_{w,j}^{k+1}, u_{w,j}^{k+1}, u_{o,j}^{k+1})$ as follows:

Step 1: Mass transport.

$$\frac{m_j^{k+1} - m_j^k}{\Delta t} + \frac{1}{\Delta x}([mu_w]_{j+1/2}^k - [mu_w]_{j-1/2}^k) = -m_j^k Q_{p,j} + \rho_{wj}^k Q_{I,j}, \tag{4.156}$$

where

$$[mu_w]_{j+1/2}^k = \begin{cases} m_j^k u_{w,j+1/2}^k, & \text{if } u_{w,j+1/2}^k \geq 0; \\ m_{j+1}^k u_{w,j+1/2}^k, & \text{if } u_{w,j+1/2}^k < 0. \end{cases} \tag{4.157}$$

Having computed m_j^{k+1} we can compute an updated water saturation $s_{w,j}^{k+1/2}$ given by

$$s_{w,j}^{k+1/2} = \frac{m_j^{k+1}}{\rho_w(P_{w,j}^k)}. \tag{4.158}$$

Similarly, we compute updated mass $n_j^{k+1/2} = (1-s_{w,j}^{k+1/2})\rho_o(P_{o,j}^{k+1/2})$ and $P_{o,j}^{k+1/2} = P_{w,j}^k + P_c(s_{w,j}^{k+1/2})$, according to (4.147), which are needed to evaluate coefficients in the next step.

Step 2: Computation of velocities and pressure. Next, we solve simultaneously for $P_{w,j}^{k+1}$ and $u_{w,j+1/2}^{k+1}$ and $u_{o,j+1/2}^{k+1}$ by considering the following algebraic system

$$\frac{P_{wj}^{k+1} - P_{wj}^k}{\Delta t} + [\tilde{\eta}\rho_w]_j^{k+1/2}\frac{1}{\Delta x}\Big([n^{k+1/2}u_o^{k+1}]_{j+1/2} - [n^{k+1/2}u_o^{k+1}]_{j-1/2}\Big)$$
$$+ [\tilde{\eta}\tilde{\rho}_o]_j^{k+1/2}\frac{1}{\Delta x}\Big([m^{k+1}u_w^{k+1}]_{j+1/2} - [m^{k+1}u_w^{k+1}]_{j-1/2}\Big)$$
$$= [\tilde{\eta}\rho_w\rho_o]_j^{k+1/2}(Q_{I,j} - Q_{p,j}) - \Big[\tilde{\eta}\frac{s_o P_c'}{C_o}\Big]_j^{k+1/2}(\rho_{wj}^k Q_{I,j} - m_j^{k+1}Q_{p,j}),$$
$$\tag{4.159}$$

which is combined with the momentum balance equations

$$
s_{o,j+1/2}^{k+1/2}\frac{1}{\Delta x}(P_{w,j+1}^{k+1} - P_{w,j}^{k+1}) = -s_{o,j+1/2}^{k+1/2}\frac{1}{\Delta x}(P_{c,j+1}^{k+1/2} - P_{c,j}^{k+1/2})
$$
$$
-\hat{k}_{o,j+1/2}^{k+1/2}u_{o,j+1/2}^{k+1} + \hat{k}_{j+1/2}^{k+1/2}\left(u_{w,j+1/2}^{k+1} - u_{o,j+1/2}^{k+1}\right) + n_{j+1/2}^{k+1/2}g
$$
$$
+\varepsilon_o\frac{1}{\Delta x^2}\left(n_{j+1}^{k+1/2}[u_{o,j+3/2}^{k+1} - u_{o,j+1/2}^{k+1}] - n_j^{k+1/2}[u_{o,j+1/2}^{k+1} - u_{o,j-1/2}^{k+1}]\right),
$$
$$
s_{w,j+1/2}^{k+1/2}\frac{1}{\Delta x}(P_{w,j+1}^{k+1} - P_{w,j}^{k+1}) =
$$
$$
-\hat{k}_{w,j+1/2}^{k+1/2}u_{w,j+1/2}^{k+1} - \hat{k}_{j+1/2}^{k+1/2}\left(u_{w,j+1/2}^{k+1} - u_{o,j+1/2}^{k+1}\right) + m_{j+1/2}^{k+1}g
$$
$$
+\varepsilon_w\frac{1}{\Delta x^2}\left(m_{j+1}^{k+1}[u_{w,j+3/2}^{k+1} - u_{w,j+1/2}^{k+1}] - m_j^{k+1}[u_{w,j+1/2}^{k+1} - u_{w,j-1/2}^{k+1}]\right).
$$
$$
(4.160)
$$

Equipped with $(P_{w,j}^{k+1}, u_{w,j+1/2}^{k+1}, u_{o,j+1/2}^{k+1})$ we can now update the saturation $s_{w,j}$ by

$$
s_{w,j}^{k+1} = \frac{m_j^{k+1}}{\rho_w(P_{w,j}^{k+1})}, \qquad (4.161)
$$

from which we also compute the updated oil mass n_j^{k+1} and pressure $P_{o,j}^{k+1}$ via (4.147). If necessary, we may repeat step 2 to improve the accuracy before we proceed to next time level.

Remark 4.3. The upwind discretization of $[n^{k+1/2}u_o^{k+1}]_{j+1/2}$ and $[m^{k+1}u_w^{k+1}]_{j+1/2}$ appearing in (4.159) are based on "old" velocities $u_{o,j+1/2}^k$ and $u_{w,j+1/2}^k$.

A semidiscrete scheme for the incompressible model. When fluids are incompressible the model (2.14) takes the following form with unknown variables (s_w, P_w, u_w, u_o)

$$
(s_w)_t + (s_w u_w)_x = -s_w Q_p + Q_I
$$
$$
(s_o)_t + (s_o u_o)_x = -s_o Q_p
$$
$$
s_w(P_w)_x = -\hat{k}_w u_w - \hat{k}(u_w - u_o) + s_w\rho_w g + \varepsilon_w\rho_w(s_w u_{wx})_x
$$
$$
s_o(P_w + P_c)_x = -\hat{k}_o u_o + \hat{k}(u_w - u_o) + s_o\rho_o g + \varepsilon_o\rho_o(s_o u_{ox})_x,
$$
$$
(4.162)
$$

subject to the boundary condition

$$
u_w|_{\partial\Omega} = u_o|_{\partial\Omega} = 0 \qquad (4.163)
$$

and initial condition

$$
s_w(x, t = 0) = s_{w0}(x). \qquad (4.164)
$$

Note that we can only determine P_w up to a constant and a reference pressure P^* at some point in the domain may be specified. An equivalent formulation of (4.162) is given by (after a summation of the two mass balance equation)

$$
(s_w)_t + (s_w u_w)_x = -s_w Q_p + Q_I
$$
$$
(s_w u_w + s_o u_o)_x = -Q_p + Q_I
$$
$$
s_w(P_w)_x = -\hat{k}_w u_w - \hat{k}(u_w - u_o) + s_w\rho_w g + \varepsilon_w\rho_w(s_w u_{wx})_x
$$
$$
s_o(P_w + P_c)_x = -\hat{k}_o u_o + \hat{k}(u_w - u_o) + s_o\rho_o g + \varepsilon_o\rho_o(s_o u_{ox})_x.
$$
$$
(4.165)
$$

This formulation is consistent with and follows directly from (4.145) by letting $C_w, C_o \to \infty$ (i.e., the fluids become incompressible). This is a consequence of the fact that $\tilde{\eta} \to \infty$ and $\tilde{\rho}_o \to \rho_o$, see (4.146).

Step 1: Mass transport. We solve the following ODE for $s_{w,j}(t)$ corresponding to $(4.165)_1$:

$$\frac{ds_{w,j}}{dt} + \frac{1}{\Delta x}([s_w u_w]_{j+1/2} - [s_w u_w]_{j-1/2}) = -s_{w,j}Q_{p,j} + Q_{I,j} \qquad (4.166)$$

where

$$[s_w u_w]_{j+1/2} = \begin{cases} s_{w,j}u_{w,j+1/2}, & \text{if } u_{w,j+1/2} \geq 0; \\ s_{w,j+1}u_{w,j+1/2}, & \text{if } u_{w,j+1/2} < 0. \end{cases} \qquad (4.167)$$

Step 2: Computation of velocities and pressure. Next, we solve for $P_{w,j}(t)$ and $u_{w,j+1/2}(t)$ and $u_{o,j+1/2}(t)$ by considering the algebraic system corresponding to $(4.165)_{2,3,4}$:

$$\frac{1}{\Delta x}([s_w u_w]_{j+1/2} - [s_w u_w]_{j-1/2}) + \frac{1}{\Delta x}([s_o u_o]_{j+1/2} - [s_o u_o]_{j-1/2}) = Q_{I,j} - Q_{p,j} \qquad (4.168)$$

which is combined with the momentum balance equations

$$\begin{aligned}
s_{w,j+1/2}\frac{1}{\Delta x}(P_{w,j+1} - P_{w,j}) = {} & \\
& -\hat{k}_{w,j+1/2}u_{w,j+1/2} - \hat{k}_{j+1/2}\Big(u_{w,j+1/2} - u_{o,j+1/2}\Big) + s_{w,j+1/2}\rho_w g \\
& +\varepsilon_w \frac{\rho_w}{\Delta x^2}\Big(s_{w,j+1}[u_{w,j+3/2} - u_{w,j+1/2}] - s_{w,j}[u_{w,j+1/2} - u_{w,j-1/2}]\Big) \\
s_{o,j+1/2}\frac{1}{\Delta x}(P_{w,j+1} - P_{w,j}) = {} & -s_{o,j+1/2}\frac{1}{\Delta x}(P_{c,j+1} - P_{c,j}) \\
& -\hat{k}_{o,j+1/2}u_{o,j+1/2} + \hat{k}_{j+1/2}\Big(u_{w,j+1/2} - u_{o,j+1/2}\Big) + s_{o,j+1/2}\rho_o g \\
& +\varepsilon_o \frac{\rho_o}{\Delta x^2}\Big(s_{o,j+1}[u_{o,j+3/2} - u_{o,j+1/2}] - s_{o,j}[u_{o,j+1/2} - u_{o,j-1/2}]\Big).
\end{aligned} \qquad (4.169)$$

Here we note that the average $s_{w,j+1/2}$ in (4.169) is based on upwind relatively $u_{w,j+1/2}$

$$s_{w,j+1/2} = \begin{cases} s_{w,j}, & \text{if } u_{w,j+1/2} > 0; \\ \frac{s_{w,j}+s_{w,j+1}}{2}, & \text{if } u_{w,j+1/2} = 0; \\ s_{w,j+1}, & \text{if } u_{w,j+1/2} < 0. \end{cases} \qquad (4.170)$$

Similarly, for $s_{o,j+1/2}$ and for the interaction terms $\hat{k}_{w,j+1/2}$ and $\hat{k}_{o,j+1/2}$. In addition, $\hat{k}_{j+1/2}$ is based on upwind relatively $u_{w,j+1/2}$ and $u_{o,j+1/2}$

$$\hat{k}_{j+1/2} = \begin{cases} \hat{k}_j, & \text{if } u_{w,j+1/2} > 0 \ \& \ u_{o,j+1/2} > 0 \ ; \\ \frac{\hat{k}_j+\hat{k}_{j+1}}{2}, & \text{if } u_{w,j+1/2}u_{o,j+1/2} \leq 0; \\ \hat{k}_{j+1}, & \text{if } u_{w,j+1/2} < 0 \ \& \ u_{o,j+1/2} < 0. \end{cases} \qquad (4.171)$$

Moreover, $[s_o u_o]_{j+1/2}$ and $[s_w u_w]_{j+1/2}$ appearing in (4.168) employ upwind as described in (4.170).

A fully discrete scheme for the incompressible model. We can now proceed with a description of a full discrete scheme for the incompressible case which bears clear similarity to the scheme for the compressible model.

Step 1: Mass transport.

$$\frac{s_{w,j}^{k+1} - s_{w,j}^k}{\Delta t} + \frac{1}{\Delta x}([s_w u_w]_{j+1/2}^k - [s_w u_w]_{j-1/2}^k) = -s_{w,j}^k Q_{p,j} + Q_{I,j} \qquad (4.172)$$

where

$$[s_w u_w]_{j+1/2}^k = \begin{cases} s_{w,j}^k u_{w,j+1/2}^k, & \text{if } u_{w,j+1/2}^k \geq 0; \\ s_{w,j+1}^k u_{w,j+1/2}^k, & \text{if } u_{w,j+1/2}^k < 0. \end{cases} \qquad (4.173)$$

Having computed $s_{w,j}^{k+1}$ we can compute pressure and velocities simultaneously at time level $k+1$.

Step 2: Computation of velocities and pressure. We solve for $P_{w,j}^{k+1}$ and $u_{w,j+1/2}^{k+1}$ and $u_{o,j+1/2}^{k+1}$ by considering the following algebraic system

$$\frac{1}{\Delta x}([s_w^{k+1} u_w^{k+1}]_{j+1/2} - [s_w^{k+1} u_w^{k+1}]_{j-1/2}) + \frac{1}{\Delta x}([s_o^{k+1} u_o^{k+1}]_{j+1/2} - [s_o^{k+1} u_o^{k+1}]_{j-1/2})$$
$$= Q_{I,j} - Q_{p,j}$$
$$(4.174)$$

which is combined with the momentum balance equations

$$s_{w,j+1/2}^{k+1} \frac{1}{\Delta x}(P_{w,j+1}^{k+1} - P_{w,j}^{k+1}) =$$
$$-\hat{k}_{w,j+1/2}^{k+1} u_{w,j+1/2}^{k+1} - \hat{k}_{j+1/2}^{k+1}\left(u_{w,j+1/2}^{k+1} - u_{o,j+1/2}^{k+1}\right) + s_{w,j+1/2}^{k+1}\rho_w g$$
$$+\varepsilon_w \frac{\rho_w}{\Delta x^2}\left(s_{w,j+1}^{k+1}[u_{w,j+3/2}^{k+1} - u_{w,j+1/2}^{k+1}] - s_{w,j}^{k+1}[u_{w,j+1/2}^{k+1} - u_{w,j-1/2}^{k+1}]\right),$$
$$s_{o,j+1/2}^{k+1} \frac{1}{\Delta x}(P_{w,j+1}^{k+1} - P_{w,j}^{k+1}) = -s_{o,j+1/2}^{k+1} \frac{1}{\Delta x}(P_{c,j+1}^{k+1} - P_{c,j}^{k+1})$$
$$-\hat{k}_{o,j+1/2}^{k+1} u_{o,j+1/2}^{k+1} + \hat{k}_{j+1/2}^{k+1}\left(u_{w,j+1/2}^{k+1} - u_{o,j+1/2}^{k+1}\right) + s_{o,j+1/2}^{k+1}\rho_o g$$
$$+\varepsilon_o \frac{\rho_o}{\Delta x^2}\left(s_{o,j+1}^{k+1}[u_{o,j+3/2}^{k+1} - u_{o,j+1/2}^{k+1}] - s_{o,j}^{k+1}[u_{o,j+1/2}^{k+1} - u_{o,j-1/2}^{k+1}]\right).$$
$$(4.175)$$

Remark 4.4. The upwind discretization of $[s_w^{k+1} u_w^{k+1}]_{j+1/2}$ and $[s_o^{k+1} u_o^{k+1}]_{j+1/2}$ appearing in (4.172) are based on "old" velocities $u_{w,j+1/2}^k$ and $u_{o,j+1/2}^k$.

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