NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2019019
©American Institute of Mathematical Sciences
Volume **14**, Number **3**, September **2019** pp. **471–487**

ON THE LOCAL AND GLOBAL EXISTENCE OF SOLUTIONS TO 1D TRANSPORT EQUATIONS WITH NONLOCAL VELOCITY

Hantaek Bae*

Department of Mathematical Sciences
Ulsan National Institute of Science and Technology (UNIST)
Ulsan, Republic of Korea

Rafael Granero-Belinchón

Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria
Santander, Spain

Omar Lazar

Departamento de Análisis Matemático & IMUS
Universidad de Sevilla
Sevilla, Spain

(Communicated by Seung-Yeal Ha)

Abstract. We consider the 1D transport equation with nonlocal velocity field:

$$\theta_t + u\theta_x + \nu\Lambda^\gamma\theta = 0,$$
$$u = \mathcal{N}(\theta),$$

where $\mathcal{N}$ is a nonlocal operator and Λ^γ is a Fourier multiplier defined by $\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \widehat{f}(\xi)$. In this paper, we show the existence of solutions of this model locally and globally in time for various types of nonlocal operators.

1. **Introduction.** In this paper, we study transport equations with nonlocal velocity. One of the most well-known equation is the two dimensional Euler equation in vorticity form,

$$\omega_t + u \cdot \nabla\omega = 0,$$

where the velocity u is recovered from the vorticity ω through

$$u = \nabla^\perp(-\Delta)^{-1}\omega \quad \text{or equivalently} \quad \widehat{u}(\xi) = \frac{i\xi^\perp}{|\xi|^2}\widehat{\omega}(\xi).$$

Other nonlocal and quadratically nonlinear equations, such as the surface quasi-geostrophic equation, the incompressible porous medium equation, Stokes equations, magneto-geostrophic equation in multi-dimensions, have been studied intensively as one can see in [1, 2, 5, 6, 7, 8, 9, 13, 14, 15, 16, 19, 20, 23, 24, 25] and references therein.

2010 *Mathematics Subject Classification.* Primary: 35A01; Secondary: 35D30, 35Q35, 35Q86.
Key words and phrases. Fluid equations, 1D models, global weak solution.
* Corresponding author.

We here consider the 1D transport equations with nonlocal velocity field of the form

$$\theta_t + u\theta_x + \nu\Lambda^\gamma\theta = 0, \quad x \in \mathbb{R}, \tag{1a}$$

$$u = \mathcal{N}(\theta), \tag{1b}$$

where $\mathcal{N}$ is typically expressed by a Fourier multiplier. The differential operator $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$ is defined by the action of the following kernels [10]:

$$\Lambda^\gamma f(x) = c_\gamma \text{p.v.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x - y|^{1+\gamma}} dy, \tag{2}$$

where $c_\gamma > 0$ is a normalized constant. Alternatively, we can define $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$ as a Fourier multiplier: $\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \widehat{f}(\xi)$. The study of 1 is mainly motivated by [11] where Córdoba, Córdoba, and Fontelos proposed the following 1D model

$$\theta_t + u\theta_x = 0, \tag{3a}$$

$$u = -\mathcal{H}\theta, \quad (\mathcal{H}\text{: the Hilbert transform}) \tag{3b}$$

for the 2D surface quasi-geostrophic equation and proved the finite time blow-up of smooth solutions. In this paper, we deal with 3a-3b and its variations with the following objectives.

(1) The existence of weak solution with *rough initial data*. The existence of global-in-time solutions is possible even if strong solutions blow up in finite time, as in the case of the Burgers' equation.
(2) The existence of strong solution when the velocity u is more singular than θ. We intend to see the competitive relationship between nonlinear terms and viscous terms.

More specifically, the topics covered in this paper can be summarized as follows.

• **The model 1:** $\mathcal{N} = -\mathcal{H}$ **and** $\nu = 0$**.** We first show the existence of local-in-time solution in a critical space under the scaling $\theta_0(x) \mapsto \theta_0(\lambda x)$. We then introduce the notion of a weak super-solution and obtain a global-in-time weak super-solution with $\theta_0 \in L^1 \cap L^\infty$ and $\theta_0 \geq 0$.

• **The model 2:** $\mathcal{N} = -\mathcal{H}(\partial_{xx})^{-\alpha}$, $\alpha > 0$, $\nu = 1$, **and** $\gamma > 0$**.** This is a regularized version of 3a-3b which is also closely related to many equations as mentioned in [3]. In this case, we show the existence of weak solutions globally in time under weaker conditions on α and γ compared to [3].

• **The model 3:** $\mathcal{N} = -\mathcal{H}(\partial_{xx})^{\beta}$, $\beta > 0$, $\nu = 1$, **and** $\gamma > 0$**.** Since $\beta > 0$, the velocity field is more singular than the previous two models. In this case, we show the existence of strong solutions locally in time in two cases: (1) $0 < \beta \leq \frac{\gamma}{4}$ when $0 < \gamma < 2$ and (2) $0 < \beta < 1$ when $\gamma = 2$. We also show the existence of strong solutions for $0 < \beta < \frac{1}{2}$ and $\gamma = 2$ with rough initial data. We finally show the existence of strong solutions globally in time with $0 < \beta < \frac{1}{4}$ and $\gamma = 2$.

We will give detailed statements and proofs of our results in Section 3–5.

2. **Preliminaries.** All constants will be denoted by C that is a generic constant. In a series of inequalities, the value of C can vary with each inequality. We use following notation: for a Banach space X,

$$C_T X = C([0,T] : X), \quad L^p_T X = L^p(0,T : X).$$

The spatial derivatives are defined as

$$\partial^l f(t,x) = \frac{\partial^l f}{\partial x^l}(t,x), \quad l \in \mathbb{N}.$$

For $l = 1, 2, 3$, we also use the followings:

$$f_x, \quad f_{xx} = \partial_{xx} f, \quad f_{xxx}.$$

2.1. **Hilbert transform.** We now give some properties of the Hilbert transform and related function spaces. The Hilbert transform is defined by

$$\mathcal{H}f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

We will use the BMO space and its dual which is the Hardy space $\mathcal{H}^1$ which consists of those f such that f and $\mathcal{H}f$ are integrable [17, Chapter 6]. By the following Cotlar formula [12]

$$2\mathcal{H}(f\mathcal{H}f) = (\mathcal{H}f)^2 - f^2, \tag{4}$$

we have $f\mathcal{H}f \in \mathcal{H}^1$ and for any $f \in L^2$,

$$\|f\mathcal{H}f\|_{\mathcal{H}^1} \leq \|f\|_{L^2}^2. \tag{5}$$

We have $\Lambda f(x) = \mathcal{H}f_x(x)$ by using $\widehat{Hf}(\xi) = -\text{sgn}(\xi)\widehat{f}(\xi)$, where Λ is defined in 2.

2.2. **Function spaces.** Since we are dealing with equations on $\mathbb{R}$, we state some definitions and function spaces on $\mathbb{R}$.

Let $f \in \mathcal{S}'$, a tempered distribution. Then, its Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx.$$

Let $s \in \mathbb{R}$. The energy space H^s is defined by

$$H^s(\mathbb{R}) = \left\{ f \in \mathcal{S}' : \|f\|_{H^s}^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s \left|\widehat{f}(\xi)\right|^2 d\xi < \infty \right\}.$$

We also define homogeneous spaces:

$$\dot{H}^s(\mathbb{R}) = \left\{ f \in \mathcal{S}' : \|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}} |\xi|^{2s} \left|\widehat{f}(\xi)\right|^2 d\xi < \infty \right\}.$$

We note that for $s > 0$ and $\sigma > 0$,

$$\dot{H}^{-s} \subset H^{-s-\sigma} \tag{6}$$

because

$$\|f\|_{H^{-(s+\sigma)}}^2 = \int_{\mathbb{R}} \frac{\left|\widehat{f}(\xi)\right|^2}{(1+|\xi|^2)^{s+\sigma}} d\xi = \int_{\mathbb{R}} \frac{|\xi|^{2s}}{(1+|\xi|^2)^{s+\sigma}} \frac{\left|\widehat{f}(\xi)\right|^2}{|\xi|^{2s}} d\xi \leq C \|f\|_{\dot{H}^{-s}}^2.$$

In this paper, we also use two estimations in [18].

(1) *Fractional product rule.* For $s > 0$ and p, p_i, and q_i such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}, \quad 1 \leq p < \infty, \quad p_i, q_i \neq 1,$$

we have the following estimations:

$$\|\Lambda^s(fg)\|_{L^p} \leq C\Big[\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^s g\|_{L^{q_2}} \Big], \tag{7}$$

and

$$\|(I-\Lambda)^s(fg)\|_{L^p} \le C\Big[\, \|(I-\Lambda)^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|(I-\Lambda)^s g\|_{L^{q_2}} \,\Big], \tag{8}$$

where $(I-\Delta)$ is defined as the Fourier multiplier whose symbol is $1+|\xi|^2$.

(2) *Commutator estimate.*

$$\sum_{|l|\le 2} \left\|\partial^l (fg) - f\partial^l g\right\|_{L^2} \le C\left(\|f_x\|_{L^\infty} \|g_x\|_{L^2} + \|f_{xx}\|_{L^2} \|g\|_{L^\infty}\right). \tag{9}$$

2.3. **Littlewood-Paley theory.** We here briefly introduce the Littlewood-Paley theory based on [4]. We first provide notation and definitions in the Littlewood-Paley theory. Let $\mathcal{C}$ be the ring of center 0, of small radius $\frac{3}{4}$ and great radius $\frac{8}{3}$. We take smooth radial functions (χ,ϕ) with values in $[0,1]$ that are supported on the ball $B_{\frac{4}{3}}(0)$ and $\mathcal{C}$, respectively, and satisfy

$$\begin{aligned}
&\chi(\xi) + \sum_{j=0}^{\infty} \phi\left(2^{-j}\xi\right) = 1 \quad \forall\, \xi \in \mathbb{R}^d,\\
&\sum_{j=-\infty}^{\infty} \phi\left(2^{-j}\xi\right) = 1 \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\},\\
&\left|j-j^{'}\right| \ge 2 \implies \text{supp } \phi\left(2^{-j}\cdot\right) \bigcap \text{supp } \phi\left(2^{-j^{'}}\cdot\right) = \emptyset,\\
&j \ge 1 \implies \text{supp } \chi \bigcap \text{supp } \phi\left(2^{-j}\cdot\right) = \emptyset.
\end{aligned} \tag{10}$$

From now on, we use the notation

$$\phi_j(\xi) = \phi\left(2^{-j}\xi\right).$$

We define dyadic blocks and lower frequency cut-off functions.

$$\begin{aligned}
&h = \mathcal{F}^{-1}\phi, \quad \widetilde{h} = \mathcal{F}^{-1}\chi,\\
&\Delta_j f = \phi_j(D) f = 2^{jd} \int_{\mathbb{R}^d} h\left(2^j y\right) f(x-y)dy,\\
&S_j f = \chi\left(2^{-j}D\right) f = 2^{jd} \int_{\mathbb{R}^d} \widetilde{h}\left(2^j y\right) f(x-y)dy,\\
&\Delta_{-1} f = \chi(D) f = \int_{\mathbb{R}^d} \widetilde{h}(y) f(x-y)dy.
\end{aligned} \tag{11}$$

Then, the homogeneous Littlewood-Paley decomposition is given by

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f \quad \text{in} \quad \mathcal{S}_h^{'},$$

where $\mathcal{S}_h^{'}$ is the space of tempered distributions $u \in \mathcal{S}^{'}$ such that

$$\lim_{j\to-\infty} S_j u = 0 \quad \text{in } \mathcal{S}'.$$

We now define the homogeneous Besov spaces:

$$\dot{B}^s_{p,q} = \left\{ f \in \mathcal{S}_h^{'} :\ \|f\|_{\dot{B}^s_{p,q}} = \left\|2^{js} \|\Delta_j f\|_{L^p}\right\|_{l^q(\mathbb{Z})} < \infty \right\}.$$

We recall Bernstein's inequality in 1D : for $1 \le p \le q \le \infty$ and $k \in \mathbb{N}$,

$$\sup_{|\alpha|=k} \|\partial^\alpha \Delta_j f\|_{L^p} \le C 2^{jk} \|\Delta_j f\|_{L^p}, \quad \|\Delta_j f\|_{L^q} \le C 2^{j\left(\frac{1}{p}-\frac{1}{q}\right)} \|\Delta_j f\|_{L^p}. \tag{12}$$

Moreover, the Besov spaces enjoy nice scaling properties. Let $f_\lambda(x) = f(\lambda x)$. Then, there exists a constant C such that

$$C^{-1} \|f_\lambda\|_{\dot{B}^s_{p,r}} \leq \lambda^{s-\frac{3}{p}} \|f\|_{\dot{B}^s_{p,r}} \leq C \|f_\lambda\|_{\dot{B}^s_{p,r}}. \tag{13}$$

We also have the following commutator estimate.

Lemma 2.1 (Commutator estimate)**.** *For $f, g \in \mathcal{S}$ (Schwarz class)*

$$\|[f, \Delta_j] g_x\|_{L^2} \leq C c_j 2^{-\frac{3}{2}j} \|f_x\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \|g\|_{\dot{B}^{\frac{3}{2}}_{2,1}}, \quad \sum_{j=-\infty}^{\infty} c_j \leq 1.$$

We finally introduce Simon's compactness.

Lemma 2.2. [26] *Let X_0, X_1, and X_2 be Banach spaces such that X_0 is compactly embedded in X_1 and X_1 is a subset of X_2. Then, for $1 \leq p < \infty$, the set $\left\{v \in L^p_T X_0 : \frac{\partial v}{\partial t} \in L^1_T X_2\right\}$ is compactly embedded in $L^p_T X_1$.*

3. The model 1.

We now study 1a-1b with $\mathcal{N} = -\mathcal{H}$ and $\nu = 0$ which is nothing but 3a-3b:

$$\theta_t - (\mathcal{H}\theta)\,\theta_x = 0, \tag{14a}$$

$$\theta(0, x) = \theta_0(x). \tag{14b}$$

3.1. Local well-posedness.

The local well-posedness of 14a-14b is established in H^2 ([2]) and $H^{\frac{3}{2}-\gamma}$ with the viscous term $\Lambda^\gamma\theta$ ([14]). To improve these results, we notice that 14a-14b has the following scaling invariant property: if $\theta(t, x)$ is a solution of 14a-14b, then so is $\theta_\lambda(t, x) = \theta(\lambda t, \lambda x)$. So, we take initial data in a space whose norm is closely invariant under the scaling:

$$\theta_0(x) \mapsto \theta_{\lambda 0}(x) = \theta_0(\lambda x).$$

In this paper, we take the space $\dot{B}^{\frac{3}{2}}_{2,1}$ because there is a constant C such that

$$C^{-1} \|\theta_{\lambda 0}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq \|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq C \|\theta_{\lambda 0}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}$$

by taking $s = \frac{3}{2}$, $p = 2$, and $r = 1$ in 13. The first result in this paper is the following theorem.

Theorem 3.1. *For any $\theta_0 \in \dot{B}^{\frac{3}{2}}_{2,1}$, there exists $T = T(\|\theta_0\|)$ such that a unique solution of 14a-14b exists in $C_T \dot{B}^{\frac{3}{2}}_{2,1}$.*

Proof. We only provide a priori estimates of θ in the space stated in Theorem 3.1. The other parts, including the approximation procedure, are rather standard.

We apply Δ_j to 14a, multiply by $\Delta_j\theta$, and integrate the resulting equation over $\mathbb{R}$ to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta\|^2_{L^2} &= \int_{\mathbb{R}} \Delta_j\left((\mathcal{H}\theta)\theta_x\right)\Delta_j\theta dx \\
&= \int_{\mathbb{R}} \left((\mathcal{H}\theta)\Delta_j\theta_x\right)\Delta_j\theta dx + \int_{\mathbb{R}} \Delta_j\left((\mathcal{H}\theta)\theta_x\right)\Delta_j\theta dx - \int_{\mathbb{R}} \left((\mathcal{H}\theta)\Delta_j\theta_x\right)\Delta_j\theta dx \\
&= \int_{\mathbb{R}} \left((\mathcal{H}\theta)\Delta_j\theta_x\right)\Delta_j\theta dx + \int_{\mathbb{R}} \left\{[\Delta_j, \mathcal{H}\theta]\,\Delta_j\theta_x\right\}\Delta_j\theta dx \\
&= -\frac{1}{2}\int_{\mathbb{R}} (\mathcal{H}\theta)_x\,|\Delta_j\theta|^2\,dx + \int_{\mathbb{R}} \left\{[\Delta_j, \mathcal{H}\theta]\,\Delta_j\theta_x\right\}\Delta_j\theta dx.
\end{aligned} \tag{15}$$

By the Bernstein inequality, we have

$$\|\mathcal{H}\theta_x\|_{L^\infty} \leq C\|\theta\|_{\dot{B}^{\frac{3}{2}}_{2,1}}. \tag{16}$$

We then apply Lemma 2.1 to the second term in the right-hand side of 15 to obtain

$$\int_{\mathbb{R}} [\Delta_j, \mathcal{H}\theta]\, \Delta_j\theta_x \Delta_j\theta dx \leq Cc_j 2^{-\frac{3}{2}j} \|\theta\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\Delta_j\theta\|_{L^2}. \tag{17}$$

By 15, 16, and 17, we have

$$\frac{d}{dt}\|\theta\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq C\|\theta\|^3_{\dot{B}^{\frac{3}{2}}_{2,1}},$$

from which we deduce

$$\|\theta(t)\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq \frac{\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}}{1 - Ct\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}} \leq 2\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \quad \text{for all } t \leq T = \frac{1}{2C\|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}}.$$

This completes the proof. □

3.2. **Global weak super-solution.** We next consider 14a-14b with rough initial data. More precisely, we assume that θ_0 satisfies the following conditions

$$\theta_0 \geq 0, \quad \theta_0 \in L^1 \cap L^\infty. \tag{18}$$

Since θ satisfies the transport equation, we have

$$\theta(t,x) \geq 0, \quad \theta \in L^\infty(\mathbb{R}) \quad \text{for all time.} \tag{19}$$

If we follow the usual weak formulation of 14a-14b, for all $\psi \in C_c^\infty([0,T)\times\mathbb{R})$

$$\int_0^T \int_{\mathbb{R}} \left[-\theta\psi_t + (\mathcal{H}\theta)\,\theta\psi_x + (\Lambda\theta)\,\theta\psi\right] dxdt = \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx. \tag{20}$$

For $\theta_0 \geq 0$, there is gain of a half derivative from the structure of the nonlinearity, that is

$$\|\theta(t)\|_{L^1} + \int_0^t \left\|\Lambda^{\frac{1}{2}}\theta(s)\right\|^2_{L^2} ds = \|\theta_0\|_{L^1}. \tag{21}$$

So, we can rewrite the left-hand side of 20 as

$$\int_0^T \int_{\mathbb{R}} \left[-\theta\psi_t + (\mathcal{H}\theta)\,\theta\psi_x + \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, \psi\right]\theta + \left|\Lambda^{\frac{1}{2}}\theta\right|^2\psi\right] dxdt = \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx.$$

However, the $\dot{H}^{\frac{1}{2}}$ regularity derived from 21 is not enough to pass to the limit in

$$\int_0^T \int_{\mathbb{R}} \left|\Lambda^{\frac{1}{2}}\theta^\epsilon\right|^2 \psi dxdt$$

from the ϵ-regularized equations described below. So, we introduce a new notion of solution. Let

$$\mathcal{A}_T = L^\infty_T\left(L^1 \cap L^\infty\right) \cap L^2_T H^{\frac{1}{2}}.$$

Definition 3.2. We say θ is a weak super-solution of 14a-14b on the time interval $[0,T]$ if $\theta(t,x) \geq 0$ for all $t \in [0,T]$, $\theta \in \mathcal{A}_T$, and for each nonnegative $\psi \in C_c^\infty([0,T)\times\mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} \left[-\theta\psi_t + (\mathcal{H}\theta)\,\theta\psi_x + \Lambda^{\frac{1}{2}}\theta\left[\Lambda^{\frac{1}{2}}, \psi\right]\theta + \left|\Lambda^{\frac{1}{2}}\theta\right|^2\psi\right] dxdt \geq \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx. \tag{22}$$

To deal with the third term in 22, we use the following Lemma.

Lemma 3.3. [3] *For $f \in L^{\frac{3}{2}}$, $g \in L^{\frac{3}{2}}$ and $\psi \in W^{1,\infty}$, we have*

$$\left\| \left[\Lambda^{\frac{1}{2}}, \psi\right] f - \left[\Lambda^{\frac{1}{2}}, \psi\right] g \right\|_{L^6} \le C \|\psi\|_{W^{1,\infty}} \|f - g\|_{L^{\frac{3}{2}}}.$$

The second result in our paper is the following theorem.

Theorem 3.4. *For any θ_0 satisfying 18, there exists a weak super-solution of 14a-14b in $\mathcal{A}_T$.*

Proof. We first regularize initial data as $\theta_0^\epsilon = \rho_\epsilon * \theta_0$ where ρ_ϵ is a standard mollifier that preserve the positivity of the regularized initial data. We then regularize the equation by introducing the Laplacian term with a coefficient $\epsilon > 0$, namely

$$\theta_t^\epsilon - \mathcal{H}\theta^\epsilon \theta_x^\epsilon = \epsilon \theta_{xx}^\epsilon. \tag{23}$$

For the proof of the existence of a global-in-time smooth solution we refer to [21]. Moreover, θ^ϵ satisfies that $\theta^\epsilon \ge 0$ and

$$\|\theta^\epsilon(t)\|_{L^1} + \|\theta^\epsilon(t)\|_{L^\infty} + \int_0^t \left\|\Lambda^{\frac{1}{2}}\theta^\epsilon(s)\right\|_{L^2}^2 ds \le \|\theta_0\|_{L^1} + \|\theta_0\|_{L^\infty}. \tag{24}$$

Therefore, (θ^ϵ) is bounded in $\mathcal{A}_T$ uniformly in $\epsilon > 0$.

The first two terms on the left-hand side of 24 imply

$$\|\theta^\epsilon\|_{L_T^p L^q} \le \|\theta_0\|_{L^1} + \|\theta_0\|_{L^\infty}$$

for any $p, q \in [1, \infty]$. In particular,

$$\mathcal{H}\theta^\epsilon \in L_T^4 L^2, \quad \theta^\epsilon \in L_T^2 L^\infty.$$

These two bounds imply

$$\left((\mathcal{H}\theta^\epsilon)\,\theta^\epsilon\right)_x \in L_T^{\frac{4}{3}} \dot{H}^{-1} \subset L_T^{\frac{4}{3}} H^{-2}$$

by the embedding 6. From $\theta^\epsilon \in L_T^2 \dot{H}^{\frac{1}{2}}$, we also have

$$\epsilon\theta_{xx}^\epsilon \in L_T^2 \dot{H}^{-\frac{3}{2}} \subset L_T^2 H^{-2}$$

by the embedding 6. Moreover, for any $\phi \in H^2$,

$$\int_{\mathbb{R}} |\theta^\epsilon \Lambda \theta^\epsilon \phi|\, dx \le \left\|\Lambda^{\frac{1}{2}}\theta^\epsilon\right\|_{L^2}^2 \|\phi\|_{L^\infty} + \left\|\Lambda^{\frac{1}{2}}\theta^\epsilon\right\|_{L^2} \|\theta^\epsilon\|_{L^\infty} \left\|\Lambda^{\frac{1}{2}}\phi\right\|_{L^\infty}$$

which implies that

$$\theta^\epsilon \Lambda \theta^\epsilon \in L_T^1 H^{-2}.$$

Combining all together, we obtain

$$\theta_t^\epsilon = \mathcal{H}\theta^\epsilon \theta_x^\epsilon + \epsilon\theta_{xx}^\epsilon = \left(\mathcal{H}\theta^\epsilon \theta^\epsilon\right)_x - \theta^\epsilon \Lambda \theta^\epsilon + \epsilon \theta_{xx}^\epsilon \in L_T^1 H^{-2}.$$

To pass to the limit into the weak super-solution formulation, we extract a subsequence of (θ^ϵ), using the same index ϵ for simplicity, and a function $\theta \in \mathcal{A}_T$ such that

$$\begin{aligned} \theta^\epsilon &\rightharpoonup \theta \quad \text{in} \quad L_T^p L^q \quad \text{for all } p, q \in (1, \infty), \\ \theta^\epsilon &\rightharpoonup \theta \quad \text{in} \quad L_T^2 H^{\frac{1}{2}}, \\ \theta^\epsilon &\to \theta \quad \text{in} \quad L_T^2 L_{\text{loc}}^p \text{ for all } 1 < p < \infty, \end{aligned} \tag{25}$$

where we use Lemma 2.2 for the strong convergence with

$$X_0 = L_T^2 H^{\frac{1}{2}}, \quad X_1 = L_T^2 L_{\text{loc}}^p, \quad X_2 = L_T^1 H^{-2}.$$

We now multiply 23 by a nonnegative test function $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R})$ and integrate over $\mathbb{R}$. Then,

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\Big[-\theta^\epsilon\psi_t+\underbrace{(\mathcal{H}\theta^\epsilon)\,\theta^\epsilon\psi_x}_{\text{I}}+\epsilon\theta^\epsilon\psi_{xx}\Big]dxdt-\int_{\mathbb{R}}\theta_0^\epsilon(x)\psi(0,x)dx\\&=-\int_0^T\int_{\mathbb{R}}\underbrace{\Lambda^{\frac12}\theta^\epsilon\left[\Lambda^{\frac12},\psi\right]\theta^\epsilon}_{\text{II}}\,dxdt-\int_0^T\int_{\mathbb{R}}\underbrace{\left|\Lambda^{\frac12}\theta^\epsilon\right|^2\psi}_{\text{III}}\,dxdt.\end{aligned}\tag{26}$$

We note that we are able to rearrange terms in the usual weak formulation into 26 since θ^ϵ is smooth. By the strong convergence in 25, we can pass to the limit to I. Moreover, since

$$\left[\Lambda^{\frac12},\psi\right]\theta^\epsilon\to\left[\Lambda^{\frac12},\psi\right]\theta$$

strongly in $L_T^2L^6$ by Lemma 3.3 and the weak convergence in 25, we can pass to the limit to II. Lastly, define

$$g^\epsilon=\Lambda^{\frac12}\theta^\epsilon\sqrt{\psi}\quad\text{and}\quad g=\Lambda^{\frac12}\theta\sqrt{\psi}.$$

We then have that $g^\epsilon\rightharpoonup g$ in $L^2([0,T]\times\mathbb{R})$. Since the L^2 norm is weakly lower semicontinuous, we find that

$$\liminf_{\epsilon\to0}\|g^\epsilon\|_{L^2([0,T]\times\mathbb{R})}\geq\|g\|_{L^2([0,T]\times\mathbb{R})},$$

or, equivalently,

$$\liminf_{\epsilon\to0}\int_0^T\int_{\mathbb{R}}\left|\Lambda^{\frac12}\theta^\epsilon\right|^2\psi dxdt\geq\int_0^T\int_{\mathbb{R}}\left|\Lambda^{\frac12}\theta\right|^2\psi dxdt.$$

Combining all the limits together, we obtain that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left[-\theta\psi_t+(\mathcal{H}\theta)\,\theta\psi_x+\Lambda^{\frac12}\theta\left[\Lambda^{\frac12},\psi\right]\theta+\left|\Lambda^{\frac12}\theta\right|^2\psi\right]dxdt\\&\geq\int_{\mathbb{R}}\theta_0(x)\psi(x,0)dx.\end{aligned}\tag{27}$$

This completes the proof. □

4. **The model 2.**

We now consider the following equation:

$$\theta_t-\left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta\right)\theta_x+\Lambda^\gamma\theta=0,\tag{28a}$$

$$\theta(0,x)=\theta_0(x),\tag{28b}$$

where $\alpha,\gamma>0$. In this case, we focus on the existence of weak solutions under some conditions of (α,γ). As before, we assume that θ_0 satisfies the following conditions

$$\theta_0\geq0,\quad\theta_0\in L^1\cap L^\infty.\tag{29}$$

Let

$$\mathcal{B}_T=L_T^\infty\left(L^1\cap L^\infty\right)\cap L_T^2H^{\frac{\gamma}{2}}.$$

Definition 4.1. We say θ is a weak solution of 28a-28b on the time interval $[0,T]$ if $\theta(t,x)\geq0$ for all $t\in[0,T]$, $\theta\in\mathcal{B}_T$, and for each $\psi\in C_c^\infty([0,T)\times\mathbb{R})$,

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left[\theta\psi_t-\left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta\right)\theta\psi_x-\Lambda^{1-\frac{\gamma}{2}}(\partial_{xx})^{-\alpha}\theta\Lambda^{\frac{\gamma}{2}}(\theta\psi)-\theta\Lambda^\gamma\psi\right]dxdt\\&=\int_{\mathbb{R}}\theta_0(x)\psi(x,0)dx.\end{aligned}$$

The third result in the paper is the following.

Theorem 4.2. *Suppose that two positive numbers α and γ satisfy*

$$0<\gamma<1, \quad \frac{1}{2}-\frac{\gamma}{2}<\alpha<\frac{1}{2}. \tag{30}$$

Then, for any θ_0 satisfying 29, there exists a weak solution of 28a-28b in $\mathcal{B}_T$ for all $T>0$.

Proof. As in the proof of Theorem 3.4, we regularize θ_0 and the equation as

$$\theta_0^\epsilon=\rho_\epsilon*\theta_0, \quad \theta_t^\epsilon-\left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\right)\theta_x^\epsilon+\Lambda^\gamma\theta^\epsilon=\epsilon\theta_{xx}^\epsilon. \tag{31}$$

Then, the corresponding θ^ϵ satisfies

$$\theta^\epsilon(t,x)\geq 0, \quad \|\theta^\epsilon(t)\|_{L^\infty}\leq\|\theta_0\|_{L^\infty} \quad \text{for all time} \tag{32}$$

and

$$\|\theta^\epsilon(t)\|_{L^1}+\int_0^t\left\|\Lambda^{\frac{1}{2}}(\partial_{xx})^{-\frac{\alpha}{2}}\theta^\epsilon(s)\right\|_{L^2}^2 ds\leq\|\theta_0\|_{L^1}. \tag{33}$$

We next multiply 31 by θ^ϵ and integrate over $\mathbb{R}$. Then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta^\epsilon(t)\|_{L^2}^2+\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|_{L^2}^2+\epsilon\|\theta_x^\epsilon\|_{L^2}^2=-\frac{1}{2}\int_{\mathbb{R}}\left\{\Lambda(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\}(\theta^\epsilon(t))^2dx\\
&=-\frac{1}{2}\int_{\mathbb{R}}\left\{(1-\Delta)^{-\frac{\gamma}{4}}\Lambda(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\}(1-\Delta)^{\frac{\gamma}{4}}\Big(\theta^\epsilon(t)\Big)^2dx\\
&\leq C\left\|(1-\Delta)^{-\frac{\gamma}{4}}\Lambda(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\|_{L^2}\left\|(1-\Delta)^{\frac{\gamma}{4}}\theta^\epsilon(t)\right\|_{L^2}\|\theta^\epsilon(t)\|_{L^\infty},
\end{aligned}$$

where we use the fractional product rule 8 to obtain

$$\left\|(1-\Delta)^{\frac{\gamma}{4}}(\theta^\epsilon(t))^2\right\|_{L^2}\leq C\|\theta^\epsilon(t)\|_{L^\infty}\left\|(1-\Delta)^{\frac{\gamma}{4}}\theta^\epsilon(t)\right\|_{L^2}.$$

By this bound and 32, we have

$$\left\|(1-\Delta)^{\frac{\gamma}{4}}(\theta^\epsilon(t))^2\right\|_{L^2}\leq C\|\theta_0\|_{L^\infty}\left(\|\theta^\epsilon(t)\|_{L^2}+\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|_{L^2}\right). \tag{34}$$

We now consider $\left\|(1-\Delta)^{-\frac{\gamma}{4}}\Lambda(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\|_{L^2}$. For $|\xi|\leq 1$,

$$\int_{|\xi|\leq 1}\frac{|\xi|^{2(1-2\alpha)}\left|\widehat{\theta^\epsilon}(t,\xi)\right|^2}{(1+|\xi|^2)^{\frac{\gamma}{2}}}d\xi\leq\|\theta^\epsilon(t)\|_{L^2}^2 \quad \text{when } \alpha<\frac{1}{2}.$$

For $|\xi|\geq 1$,

$$\int_{|\xi|\geq 1}\frac{|\xi|^{2(1-2\alpha)}\left|\widehat{\theta^\epsilon}(t,\xi)\right|^2}{(1+|\xi|^2)^{\frac{\gamma}{2}}}d\xi\leq C\int_{|\xi|\geq 1}|\xi|^{2\left(\frac{1}{2}-\alpha\right)}\left|\widehat{\theta^\epsilon}(t,\xi)\right|^2d\xi \quad \text{when } \alpha>\frac{1}{2}-\frac{\gamma}{2}.$$

So,

$$\left\|(1-\Delta)^{-\frac{\gamma}{4}}\Lambda(\partial_{xx})^{-\alpha}\theta^\epsilon(t)\right\|_{L^2}\leq C\left(\|\theta^\epsilon(t)\|_{L^2}+\left\|\Lambda^{\frac{1}{2}}(\partial_{xx})^{-\frac{\alpha}{2}}\theta^\epsilon(t)\right\|_{L^2}\right). \tag{35}$$

By 34 and 35, we obtain

$$\begin{aligned} &\frac{d}{dt}\|\theta^\epsilon(t)\|_{L^2}^2+\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|_{L^2}^2+\epsilon\|\theta^\epsilon_x\|_{L^2}^2 \\ &\leq C\|\theta_0\|_{L^\infty}\left(\|\theta^\epsilon(t)\|_{L^2}+\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|_{L^2}\right)\left(\|\theta^\epsilon(t)\|_{L^2}+\left\|\Lambda^{\frac{1}{2}}(\partial_{xx})^{-\frac{\alpha}{2}}\theta^\epsilon(t)\right\|_{L^2}\right) \\ &\leq C\left(\|\theta_0\|_{L^\infty}+\|\theta_0\|_{L^\infty}^2\right)\|\theta^\epsilon(t)\|_{L^2}^2+C(1+\|\theta_0\|_{L^\infty}^2)\left\|\Lambda^{\frac{1}{2}}(\partial_{xx})^{-\frac{\alpha}{2}}\theta^\epsilon(t)\right\|_{L^2}^2 \\ &+\frac{1}{2}\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|_{L^2}^2 \end{aligned} \tag{36}$$

and so

$$\begin{aligned} &\frac{d}{dt}\|\theta^\epsilon(t)\|_{L^2}^2+\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(t)\right\|_{L^2}^2+\epsilon\|\theta^\epsilon_x\|_{L^2}^2 \\ &\leq C\left(\|\theta_0\|_{L^\infty}+\|\theta_0\|_{L^\infty}^2\right)\|\theta^\epsilon(t)\|_{L^2}^2+C(1+\|\theta_0\|_{L^\infty}^2)\left\|\Lambda^{\frac{1}{2}}(\partial_{xx})^{-\frac{\alpha}{2}}\theta^\epsilon(t)\right\|_{L^2}^2. \end{aligned}$$

By Gronwall's inequality,

$$\|\theta^\epsilon(t)\|_{L^2}^2\leq\left(\|\theta_0\|_{L^2}^2+C(1+\|\theta_0\|_{L^\infty}^2)\|\theta_0\|_{L^1}\right)e^{C\left(\|\theta_0\|_{L^\infty}+\|\theta_0\|_{L^\infty}^2\right)t},$$

where we use 33 to bound the time integral of $\left\|\Lambda^{\frac{1}{2}}(\partial_{xx})^{-\frac{\alpha}{2}}\theta^\epsilon(t)\right\|_{L^2}^2$. Hence we finally derive the following

$$\begin{aligned} &\|\theta^\epsilon(t)\|_{L^2}^2+\int_0^t\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon(s)\right\|_{L^2}^2ds+\epsilon\int_0^t\|\theta^\epsilon_x(s)\|_{L^2}^2ds \\ &\leq\|\theta_0\|_{L^2}^2+\frac{\left(\|\theta_0\|_{L^2}^2+C(1+\|\theta_0\|_{L^\infty}^2)\|\theta_0\|_{L^1}\right)e^{C\left(\|\theta_0\|_{L^\infty}+\|\theta_0\|_{L^\infty}^2\right)t}}{C\left(\|\theta_0\|_{L^\infty}+\|\theta_0\|_{L^\infty}^2\right)}. \end{aligned} \tag{37}$$

Therefore, (θ^ϵ) is bounded in $\mathcal{B}_T$ uniformly in $\epsilon>0$.

By 32 and 33,

$$\theta^\epsilon\in L^\infty_T(L^1\cap L^\infty). \tag{38}$$

We next consider $\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon$. We first choose $\beta\in\left[0,\frac{1}{2}\right)$ also satisfying

$$2\alpha-\frac{1}{2}<\beta\leq 2\alpha+\frac{\gamma}{2}. \tag{39}$$

Then,

$$\begin{aligned} \int_{\mathbb{R}}|\xi|^{2(\beta-2\alpha)}\left|\widehat{\theta^\epsilon}(\xi)\right|^2d\xi &= \int_{|\xi|\leq 1}|\xi|^{2(\beta-2\alpha)}\left|\widehat{\theta^\epsilon}(\xi)\right|^2d\xi+\int_{|\xi|\geq 1}|\xi|^{2(\beta-2\alpha)}\left|\widehat{\theta^\epsilon}(\xi)\right|^2d\xi \\ &\leq\left\|\widehat{\theta^\epsilon}\right\|_{L^\infty}^2+\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon\right\|_{L^2}^2\leq\|\theta^\epsilon\|_{L^1}^2+\left\|\Lambda^{\frac{\gamma}{2}}\theta^\epsilon\right\|_{L^2}^2 \end{aligned}$$

and so

$$\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\in L^2_T\dot{H}^\beta.$$

Moreover, by Sobolev embedding,

$$\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\in L^2_TL^p,\quad\frac{1}{p}=\frac{1}{2}-\beta \tag{40}$$

where β is defined in 39. By 37, we also have

$$\Lambda^\gamma\theta^\epsilon+\epsilon\theta^\epsilon_{xx}\in L^2_TH^{-2}.$$

Combining all together, we derive that

$$\theta^\epsilon_t\in L^1_TH^{-2}.$$

Finally, 7 and 38 imply that

$$\Lambda^{\frac{\gamma}{2}}(\theta^\epsilon\psi) \in L_T^2 L^2. \tag{41}$$

To pass the limit to this formulation, we extract a subsequence of (θ^ϵ), using the same index ϵ for simplicity, and a function $\theta \in \mathcal{B}_T$ such that

$$\theta^\epsilon \rightharpoonup \theta \quad \text{in} \quad L_T^2 H^{\frac{\gamma}{2}}, \tag{42a}$$

$$\theta^\epsilon \to \theta \quad \text{in } L_T^2 L_{\text{loc}}^p \text{ for all } 1 < p < \frac{2}{1-\gamma}, \tag{42b}$$

$$\theta^\epsilon \to \theta \quad \text{in } L_T^2 H^{1-\frac{\gamma}{2}-2\alpha},. \tag{42c}$$

Here, we use Lemma 2.2 with

$$X_0 = L_T^2 H^{\frac{\gamma}{2}}, \quad X_1 = L_T^2 L_{\text{loc}}^p, \quad X_2 = L_T^1 H^{-2}$$

to obtain 42b. Similarly, we use Lemma 2.2 with the condition 30 and

$$X_0 = L_T^2 H^{\frac{\gamma}{2}}, \quad X_1 = L_T^2 H^{1-\frac{\gamma}{2}-2\alpha}, \quad X_2 = L_T^1 H^{-2}$$

to obtain 42c.

We now multiply 31 by a test function $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R})$ and integrate over $\mathbb{R}$. Then,

$$\begin{aligned}
&\int_0^T \int \Big[\theta^\epsilon\psi_t - \underbrace{\left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\right)\theta^\epsilon\psi_x}_{\text{I}} + \Lambda^\gamma\theta^\epsilon\psi + \epsilon\theta^\epsilon\psi_{xx}\Big]dxdt \\
&- \int \theta_0^\epsilon(x)\psi(0,x)dx \\
&= \int_0^T \int \underbrace{\Lambda^{1-\frac{\gamma}{2}}\mathcal{H}(\partial_{xx})^{-\alpha}\theta^\epsilon\Lambda^{\frac{\gamma}{2}}(\theta^\epsilon\psi)}_{\text{II}}\,dxdt.
\end{aligned} \tag{43}$$

By 40 and the strong convergence in 42b, we can pass to the limit to I. By 41 and the strong convergence in 42c, we can also pass to the limit to II. Therefore, we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \Big[\theta\psi_t - \left(\mathcal{H}(\partial_{xx})^{-\alpha}\theta\right)\theta\psi_x - \Lambda^{1-\frac{\gamma}{2}}(\partial_{xx})^{-\alpha}\theta\Lambda^{\frac{\gamma}{2}}(\theta\psi) - \theta\Lambda^\gamma\psi\Big]\,dxdt \\
&= \int_{\mathbb{R}} \theta_0(x)\psi(x,0)dx.
\end{aligned}$$

This completes the proof of Theorem 4.2. □

Remark 1. Theorem 4.2 improves Theorem 1.4 in [3], where (α,γ) is assumed to satisfy $\alpha \geq \frac{1}{2} - \frac{\gamma}{4}$. The main idea of taking weaker regularization in 28a-28b is that the Hilbert transform in front of $(1-\partial_{xx})^{-\alpha}$ gives 33 which makes to obtain 37. We choose $\alpha > \frac{1}{2} - \frac{\gamma}{2}$ instead of $\alpha \geq \frac{1}{2} - \frac{\gamma}{2}$ to apply compactness argument when we pass to the limit to ϵ-regularized equations.

5. **The model 3.**

In this section, we consider the following equation

$$\theta_t - \left(\mathcal{H}(\partial_{xx})^\beta\theta\right)\theta_x + \Lambda^\gamma\theta = 0, \tag{44a}$$

$$\theta(0,x) = \theta_0(x) \tag{44b}$$

where $\beta,\gamma > 0$. Depending on the range of β and γ, we will have four different results.

5.1. **Local well-posedness.** We begin with the local well-posedness result.

Theorem 5.1. *Let $0 < \gamma < 2$ and $0 < \beta \leq \frac{\gamma}{4}$. For $\theta_0 \in H^2(\mathbb{R})$ there exists $T = T(\|\theta_0\|_{H^2})$ such that a unique solution of 44a-44b exists in $C_T H^2$. Moreover, we have the following blow-up criterion:*

$$\limsup_{t\nearrow T^*} \|\theta(t)\|_{H^2} = \infty \quad \textit{if and only if} \quad \int_0^{T^*} \left(\|u_x(s)\|_{L^\infty} + \|\theta_x(s)\|_{L^\infty}^2 \right) ds = \infty, \tag{45}$$

where $u = -\mathcal{H}(\partial_{xx})^\beta \theta$.

Proof. Operating ∂^l on 44a, taking its L^2 inner product with $\partial^l \theta$, and summing over $l = 0, 1, 2$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{H^2}^2 + \left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{H^2}^2 &= -\sum_{l=0}^{2}\int \partial^l(u\theta_x)\partial^l\theta dx \\ = -\sum_{l=0}^{2}\int \left(\partial^l(u\theta_x) - u\partial^l\theta_x\right)\partial^l\theta dx &- \sum_{l=0}^{2}\int u\partial^l\theta_x\partial^l\theta dx = \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{46}$$

Using the commutator estimate 9, we have

$$\begin{aligned} \mathrm{I}_1 &\leq \sum_{l=0}^{2}\left\|\partial^l(u\theta_x) - u\partial^l\theta_x\right\|_{L^2}\|\theta\|_{H^2} \\ &\leq C\left(\|u_x\|_{L^\infty}\|\theta\|_{H^2} + \|u\|_{H^2}\|\theta_x\|_{L^\infty}\right)\|\theta\|_{H^2} \\ &\leq C_\kappa\left(\|u_x\|_{L^\infty} + \|\theta_x\|_{L^\infty}^2\right)\|\theta\|_{H^2}^2 + \kappa\|u\|_{H^2}^2. \end{aligned} \tag{47}$$

And by integration by parts,

$$\mathrm{I}_2 = -\frac{1}{2}\sum_{l=0}^{2}\int u\partial_x\left|\partial^l\theta\right|^2 dx = \frac{1}{2}\sum_{l=0}^{2}\int u_x\left|\partial^l\theta\right|^2 dx \leq C\|u_x\|_{L^\infty}\|\theta\|_{H^2}^2. \tag{48}$$

Since $\beta \leq \frac{\gamma}{4}$, for a sufficiently small $\kappa > 0$

$$\kappa\|u\|_{H^2}^2 \leq \frac{1}{2}\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{H^2}^2.$$

By 47 and 48, we obtain

$$\begin{aligned} \frac{d}{dt}\|\theta\|_{H^2}^2 + \left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{H^2}^2 &\leq C\left(\|u_x\|_{L^\infty} + \|\theta_x\|_{L^\infty}^2\right)\|\theta\|_{H^2}^2 \\ &\leq C\|\theta\|_{H^2}^3 + C\|\theta\|_{H^2}^4,\ \beta \leq \frac{\gamma}{4} \end{aligned} \tag{49}$$

from which we deduce that there is $T = T(\|\theta_0\|_{H^2})$ such that

$$\|\theta(t)\|_{H^2} \leq 2\|\theta_0\|_{H^2} \quad \text{for all } t < T.$$

49 also implies 45.

To show the uniqueness, let θ_1 and θ_2 be two solutions of 44a-44b, and let $\theta = \theta_1 - \theta_2$ and $u = u_1 - u_2 = -\mathcal{H}(\partial_{xx})^\beta\theta_1 + -\mathcal{H}(\partial_{xx})^\beta\theta_2$. Then, (θ, u) satisfies the following equations

$$\theta_t + u_1\theta_x - u\theta_{2x} = -\Lambda^\gamma\theta, \quad u = -\mathcal{H}(\partial_{xx})^\beta\theta, \quad \theta(0,x) = 0.$$

By taking the L^2 product of the equation with θ,

$$\begin{aligned}\frac{d}{dt}\|\theta\|_{L^2}^2 + 2\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|_{L^2}^2 &\le C\left(\|u_{1x}\|_{L^\infty} + \|\theta_{2x}\|_{L^\infty}^2\right)\|\theta\|_{L^2}^2 \\ &\le C\left(\left\|\Lambda^{\frac{\gamma}{2}}\theta_1\right\|_{H^2} + \|\theta_2\|_{H^2}^2\right)\|\theta\|_{L^2}^2.\end{aligned}$$

So, $\theta = 0$ in L^2 and thus a solution is unique. This completes the proof of Theorem 5.1. □

Theorem 5.1 provides a local existence result for $\beta \nearrow \frac{1}{2}$ as $\gamma \nearrow 2$. But, we can increase the range of β when we deal with 44a-44b directly with $\gamma = 2$ because we can do the integration by parts.

Theorem 5.2. *Let $\gamma = 2$ and $0 < \beta < 1$. For $\theta_0 \in H^2(\mathbb{R})$ there exists $T = T(\|\theta_0\|_{H^2})$ such that a unique solution of 44a-44b exists in $C_T H^2$.*

Proof. We begin the L^2 bound:

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 \le \|\theta\|_{L^\infty}\left\|\mathcal{H}(\partial_{xx})^\beta\theta\right\|_{L^2}\|\theta_x\|_{L^2} \le C\|\theta\|_{H^2}^3.$$

We next estimate θ_{xx}. Indeed, after several integration parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{\dot{H}^2}^2 + \|\theta\|_{\dot{H}^3}^2 &= -\int\left\{\mathcal{H}(\partial_{xx})^\beta\theta_x\right\}\theta_x\theta_{xxx}dx + \frac{1}{2}\int\left\{\mathcal{H}(\partial_{xx})^\beta\theta_x\right\}\theta_{xx}\theta_{xx}dx \\ &= \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

When $0 < \beta < 1$,

$$\begin{aligned}|\mathrm{I}_1| &\le \|\theta_x\|_{L^\infty}\left\|\mathcal{H}(\partial_{xx})^\beta\theta_x\right\|_{L^2}\|\theta_{xxx}\|_{L^2} = \|\theta_x\|_{L^\infty}\left\|\Lambda^{2\beta+1}\theta\right\|_{L^2}\|\theta_{xxx}\|_{L^2} \\ &\le C\|\theta\|_{H^2}\|\theta_x\|_{L^2}^{1-\beta}\|\theta_{xxx}\|_{L^2}^{1+\beta} \le C\|\theta\|_{H^2}^4 + C\|\theta\|_{H^2}^{\frac{4-2\beta}{1-\beta}} + \frac{1}{4}\|\theta_{xxx}\|_{L^2}^2.\end{aligned}$$

And

$$\begin{aligned}|\mathrm{I}_2| &\le \left\|\mathcal{H}(\partial_{xx})^\beta\theta_x\right\|_{L^2}\|\theta_{xx}\|_{L^4}^2 \le C\left\|\mathcal{H}(\partial_{xx})^\beta\theta_x\right\|_{L^2}\|\theta_{xx}\|_{L^2}^{\frac{3}{2}}\|\theta_{xxx}\|_{L^2}^{\frac{1}{2}} \\ &\le C\|\theta\|_{H^2}^4 + \frac{1}{4}\|\theta_{xxx}\|_{L^2}^2.\end{aligned}$$

Therefore, we obtain

$$\frac{d}{dt}\|\theta\|_{H^2}^2 + \|\theta_x\|_{H^2}^2 \le C\|\theta\|_{H^2}^4 + C\|\theta\|_{H^2}^{\frac{4-2\beta}{1-\beta}}. \tag{50}$$

This implies that there exists $T = T(\|\theta_0\|_{H^2})$ such that there exists a unique solution of 44a-44b in $C_T H^2$. □

We may lower the regularity of the initial data to prove a local existence result of a weak solution by considering initial data in $\dot{H}^{\frac{1}{2}}$. The main tools to achieve this will be the use of the Hardy-BMO duality together with interpolation arguments. However, in order to simplify the computation, we consider an equivalent equation by changing the sign of the nonlinearity:

$$\theta_t + \left(\mathcal{H}(-\partial_{xx})^\beta\theta\right)\theta_x + \Lambda^\gamma\theta = 0, \tag{51a}$$

$$\theta(0,x) = \theta_0(x). \tag{51b}$$

This can be obtained from 51a-51b via $\theta \mapsto -\theta$. For this equation, we do $\dot{H}^{\frac{1}{2}}$ estimates and prove that there exists a local existence of a unique solution in that special case.

Theorem 5.3. *Let $\gamma = 2$ and $0 < \beta < \frac{1}{2}$. For any $\theta_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R})$, there exists $T = T(\|\theta_0\|_{\dot{H}^{\frac{1}{2}}})$ such that there exists a unique local-in-time solution in $C_T\dot{H}^{\frac{1}{2}} \cap L^2_T\dot{H}^{\frac{3}{2}}$.*

Proof. By recalling that $\Lambda^{2\beta} = (-\partial_{xx})^\beta$ we get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{\dot{H}^{\frac{1}{2}}} + \left\|\Lambda^{\frac{1+\gamma}{2}}\theta\right\|^2_{L^2} = -\int \Lambda^{\frac{1}{2}}\theta\Lambda^{\frac{1}{2}}\left\{\left(\mathcal{H}(-\partial_{xx})^\beta\theta\right)\theta_x\right\}dx$$
$$= -\int \theta_x\Lambda\theta\ \mathcal{H}(-\partial_{xx})^\beta\theta dx = -\int \theta_x\mathcal{H}\theta_x\mathcal{H}(-\partial_{xx})^\beta\theta dx.$$

We now use the $\mathcal{H}^1$-BMO duality to estimate the right hand side of the last equality. By using the estimate 5 and $\dot{H}^{\frac{1}{2}} \hookrightarrow BMO$, we obtain

$$\|\theta_x\mathcal{H}\theta_x\|_{\mathcal{H}^1} \le \|\theta\|^2_{\dot{H}^1}, \quad \left\|\mathcal{H}(-\partial_{xx})^\beta\theta\right\|_{BMO} \le C\|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}}$$

and thus we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{\dot{H}^{\frac{1}{2}}} + \left\|\Lambda^{\frac{1+\gamma}{2}}\theta\right\|^2_{L^2} \le C\|\theta\|^2_{\dot{H}^1}\|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}}.$$

By fixing $\gamma = 2$ and by using the interpolation inequalities

$$\|\theta\|^2_{\dot{H}^1} \le \|\theta\|_{\dot{H}^{\frac{3}{2}}}\|\theta\|_{\dot{H}^{\frac{1}{2}}}, \quad \|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}} \le \|\theta\|^{2\beta}_{\dot{H}^{\frac{3}{2}}}\|\theta\|^{1-2\beta}_{\dot{H}^{\frac{1}{2}}},$$

where we use $\frac{1}{2} \le 2\beta + \frac{1}{2} \le \frac{3}{2}$ for $\beta \in \left(0, \frac{1}{2}\right)$ to get the second inequality. Hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{\dot{H}^{\frac{1}{2}}} + \left\|\Lambda^{\frac{3}{2}}\theta\right\|^2_{L^2} \le \|\theta\|^2_{\dot{H}^1}\|\theta\|_{\dot{H}^{2\beta+\frac{1}{2}}}$$
$$\le \|\theta\|^{1+2\beta}_{\dot{H}^{\frac{3}{2}}}\|\theta\|^{2-2\beta}_{\dot{H}^{\frac{1}{2}}} \le \frac{1}{2}\|\theta\|^2_{\dot{H}^{\frac{3}{2}}} + 2\|\theta\|^{4\frac{1-\beta}{1-2\beta}}_{\dot{H}^{\frac{1}{2}}},$$

where we use the condition $\beta \in \left(0, \frac{1}{2}\right)$ again to derive the inequality. This implies local existence of a unique solution up to some time $T = T(\|\theta_0\|_{\dot{H}^{\frac{1}{2}}})$. □

Remark 2. In the case $\beta = 1/2$, the equation reduces to the following Hamilton-Jacobi equation (or primitive Burgers equation)

$$\theta_t - \theta_x^2 + \Lambda^\gamma\theta = 0.$$

For this equation, it seems that a naive approach based in energy methods cannot work. Indeed, if we multiply by $\Lambda\theta$ and integrate by parts the nonlinearity takes a commutator structure

$$\int \theta_x^2\Lambda\theta dx = \int \theta_x^2\mathcal{H}\theta_x dx = -\frac{1}{2}\int \theta_x[\mathcal{H},\theta_x]\theta_x\ dx.$$

However, it seems that, at this level of regularity, this commutator is comparable to an energy estimate:

$$\int \theta_x^2\Lambda\theta dx \le c\|\theta_x\|^3_{L^3} \le c\|\theta\|^3_{H^{1+\frac{1}{6}}} \le c\|\theta\|^2_{H^1}\|\theta\|_{H^{\frac{3}{2}}}$$

which is also equivalent to the use of Hardy-BMO duality:

$$\int \theta_x^2\mathcal{H}\theta_x dx \le \|\theta_x\|_{BMO}\|\theta_x\mathcal{H}\theta_x\|_{\mathcal{H}^1} \le \|\theta\|^2_{H^1}\|\theta\|_{H^{\frac{3}{2}}}.$$

Also, the best estimate that one has for the commutator $[\mathcal{H},\theta_x]\theta_x$ in L^2 is that it is controlled by $\|\theta_x\|_{BMO}\|\theta\|_{\dot{H}^1}$ (see e.g. [22]) which is, once again, similar to the use of the Hardy-BMO duality. So the commutator structure is not that useful in this special case.

Remark 3. It is also unclear whether the local solution *starting from an arbitrary initial data* becomes smooth. However, for smooth initial data satisfying size restriction in appropriate spaces, one can prove the desired smoothing effect.

5.2. **Global well-posedness.** We finally deal with 51a-51b with $\gamma = 2$.

Theorem 5.4. *Let $\gamma = 2$ and $\beta < \frac{1}{4}$. For any $\theta_0 \in H^2(\mathbb{R})$, there exists a unique global-in-time solution in $C_T H^2$.*

Proof. By Theorem 5.1, we only need to control the quantities in 45. Let $u = -\mathcal{H}(\partial_{xx})^\beta \theta$. We first note that 51a-51b satisfies the maximum principle and so

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty} \le C\|\theta_0\|_{H^2}.$$

We take the L^2 inner product of 51a with θ. Then,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 = -\int u\theta_x\theta dx \le \|\theta_0\|_{L^\infty}\|u\|_{L^2}\|\theta_x\|_{L^2}. \tag{52}$$

Since

$$\|u\|_{L^2} \le C\|\theta\|_{L^2}^{1-2\beta}\|\theta_x\|_{L^2}^{2\beta} \quad \text{for } \beta < \frac{1}{2},$$

we have

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\theta_x(s)\|_{L^2}^2 ds \le C\left(t, \|\theta_0\|_{H^2}\right). \tag{53}$$

We next take ∂_x to 51a, take its L^2 inner product with θ_x, and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta_x\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2 = \int u\theta_x\theta_{xx}dx \le 2\|u\|_{L^\infty}^2\|\theta_x\|_{L^2}^2 + \frac{1}{2}\|\theta_{xx}\|_{L^2}^2.$$

Since

$$\|u\|_{L^\infty}^2 \le C\|\theta\|_{L^2}^2 + C\|\theta_x\|_{L^2}^2 \quad \text{when } \beta < \frac{1}{4},$$

we obtain

$$\|\theta_x(t)\|_{L^2}^2 + \int_0^t \|\theta_{xx}(s)\|_{L^2}^2 ds \le C\left(t, \|\theta_0\|_{L^1}, \|\theta_0\|_{H^2}\right) \quad \text{when } \beta < \frac{1}{4}. \tag{54}$$

We also obtain

$$\begin{aligned} &\|\theta_x\|_{L^\infty}^2 \le C\left(\|\theta_x\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2\right), \\ &\|u_x\|_{L^\infty} \le C\left(\|\theta_x\|_{L^2} + \|\theta_{xx}\|_{L^2}\right) \quad \text{when } \beta < \frac{1}{4} \end{aligned} \tag{55}$$

By 53, 54 and 55 , we finally obtain

$$\begin{aligned} &\int_0^t \left(\|\theta_x(s)\|_{L^\infty}^2 + \|u_x(s)\|_{L^\infty}\right) ds \\ &\le C\int_0^t \left(\|\theta_x(s)\|_{L^2}^2 + \|\theta_{xx}(s)\|_{L^2}^2 + \|\theta_x(s)\|_{L^2} + \|\theta_{xx}(s)\|_{L^2}\right) ds \le C\left(t, \|\theta_0\|_{L^1}, \|\theta_0\|_{H^2}\right) \end{aligned}$$

and so we complete the proof of Theorem 5.4. ∎

Acknowledgments. The authors acknowledge the referees for their valuable comments and suggestions that highly improved the manuscript.

H.B. was supported by NRF-2018R1D1A1B07049015.

R.G.B. was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR), and by the Universidad de Cantabria.

O.L. was partially supported by the Marie-Curie Grant, acronym: TRANSIC, from the FP7-IEF program and by the ERC through the Starting Grant project H2020-EU.1.1.-63922.

Both O. L. and R.G.B. were partially supported by the Grant MTM2014-59488-P from the former Ministerio de Economía y Competitividad (MINECO, Spain).

REFERENCES

[1] H. Bae, D. Chae and H. Okamoto, On the well-posedness of various one-dimensional model equations for fluid motion, *Nonlinear Anal.*, **160** (2017), 25–43.

[2] H. Bae and R. Granero-Belinchón, Global existence for some transport equations with non-local velocity, *Adv. Math.*, **269** (2015), 197–219.

[3] H. Bae, R. Granero-Belinchón and O. Lazar, Global existence of weak solutions to dissipative transport equations with nonlocal velocity, *Nonlinearity*, **31** (2018) 1484–1515.

[4] H. Bahouri, J-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, 343, Springer, 2011.

[5] G. R. Baker, X. Li and A. C. Morlet, Analytic structure of 1D transport equations with nonlocal fluxes, *Physica D.*, **91** (1996), 349–375.

[6] J. A. Carrillo, L. C. F. Ferreira and J. C. Precioso, A mass-transportation approach to a one dimensional fluid mechanics model with nonlocal velocity, *Adv. Math.*, **231** (2012), 306–327.

[7] A. Castro and D. Córdoba, Global existence, singularities and ill-posedness for a nonlocal flux, *Adv. Math.*, **219** (2008), 1916–1936.

[8] A. Castro and D. Córdoba, Self-similar solutions for a transport equation with non-local flux, *Chinese Annals of Mathematics, Series B*, **30** (2009), 505–512.

[9] D. Chae, A. Cordoba, D. Cordoba and M. Fontelos, Finite time singularities in a 1D model of the quasi-geostrophic equation, *Adv. Math.*, **194** (2005), 203–223.

[10] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, *Comm. Math. Phys.*, **249** (2004), 511–528.

[11] A. Córdoba, D. Córdoba and M. Fontelos, Formation of singularities for a transport equation with nonlocal velocity, *Ann. of Math.*, **162** (2005), 1–13.

[12] M. Cotlar, A unified theory of Hilbert transforms and ergodic theorems, *Rev. Mat. Cuyana*, **1** (1955), 105–167.

[13] S. De Gregorio, On a one-dimensional model for the 3D vorticity equation, *J. Statist. Phys.*, **59** (1990), 1251–1263.

[14] H. Dong, Well-posedness for a transport equation with nonlocal velocity, *J. Funct. Anal.*, **255**, (2008), 3070–3097.

[15] H. Dong, On a multi-dimensional transport equation with nonlocal velocity, *Adv. Math.*, **264** (2014), 747–761.

[16] H. Dong and D. Li, On a one-dimensional α-patch model with nonlocal drift and fractional dissipation, *Trans. Amer. Math. Soc.*, **366** (2014), 2041–2061.

[17] J. Duoandikoetxea, *Fourier Analysis*, Translated and revised from the 1995 Spanish original by David Cruz-Uribe, Graduate Studies in Mathematics, 29, American Mathematical Society, 2000.

[18] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, **41** (1988), 891–907.

[19] A. Kiselev, Regularity and blow up for active scalars, *Math. Model. Math. Phenom.*, **5** (2010), 225–255.

[20] O. Lazar, On a 1D nonlocal transport equation with nonlocal velocity and subcritical or supercritical diffusion, *Journal of Diff. Eq.*, **261** (2016), 4974–4996.

[21] O. Lazar and P.-G. Lemarié-Rieusset, Infinite energy solutions for a 1D transport equation with nonlocal velocity, *Dynamics of PDEs*, **13** (2016), 107-131.
[22] D. Li, On Kato-Ponce and fractional Leibniz, arXiv:1609.01780.
[23] D. Li and J. Rodrigo, Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation, *Adv. Math.*, **217** (2008), 2563–2568.
[24] D. Li and J. Rodrigo, On a one-dimensional nonlocal flux with fractional dissipation, *SIAM J. Math. Anal.*, **43** (2011), 507–526.
[25] A. Morlet, Further properties of a continuum of model equations with globally defined flux, *J. Math. Anal. Appl.*, **221** (1998), 132–160.
[26] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65–96.

Received June 2018; revised February 2019.

E-mail address: hantaek@unist.ac.kr
E-mail address: rafael.granero@unican.es
E-mail address: omarlazar@us.es